Modeling A Player's Perspective II: Inductive Derivation of an Individual View* †

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Abstract

This is a sequel to our investigation of a player's perspective. We consider the inductive derivation of a player's individual view from his accumulated memory. Formally, an individual view is a pair of a subjective game description (information protocol) and a memory function. We first give an inductive derivation of this view based strictly on his accumulated memories. However, this derived view may fail to satisfy certain basic axioms. We give conditions on the memory function to guarantee that the derived view satisfies these basic axioms. Next, we proceed in another direction by allowing a player to add new elements in the construction of his view. We show that this general procedure results in an individual view that satisfies the basic axioms and respects his accumulated memories. There is no guarantee, however, that the constructed view will match the truth, even if the player's memory is perfect. We comment on the behavioral use of an inductively derived protocol and also on the famous examples of Plato's "Analogy of the Cave" and Piccione-Rubinstein's "Absentminded Driver".

1. Introduction

This is the sequel to our investigation of a player's perspective, and constitutes the central part of this two-part paper. In a sense, the development in Part II is very new, but in a different sense, it deals with a long-standing problem. In Section 1.1, we

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will explain the subject that we target in Part II while relating it to the traditions in philosophy and the sciences. Also, we will clarify how the formulation of an info-memory protocol from Part I matters for our subject. In Section 1.2, we summarize the actual development undertaken in Part II.

1.1. Inductive Derivation of an Individual View on a Social Situation

We consider the situation where a player has been part of a society for some time. In that time, he has varied his courses of actions and accumulated memories of his experiences. At some point, he wonders about the structure of the society he lives in. He constructs his view of the society based on his experiences. This is what we call an inductive derivation of a view by an individual player and it is the subject of Part II.

In the literature of game theory and economics, we can find little work on inductive derivations – a recent exception is Kaneko-Matsui [8]. In philosophy, however, induction has a long tradition, e.g., it was already well described in Plato's [11] "Analogy of the Cave". Modern sciences have adopted the inductive method by and large. We remark that scientists' inductive method differs from ordinary people's inductive derivation since scientists should follow statistical methods carefully, but ordinary people use induction boldly to derive their opinions based on only a few observations. The latter is our target.

Since Plato's "the Analogy of the Cave", mentioned in Part I, best illustrates our problem, we repeat it here. In the cave, prisoners have been chained from their birth, and their faces have been fixed to see only the wall of the bottom of the cave. From time to time, they have been observing shadows projected from the outside. Each prisoner has developed his view on the world based on his memories of experiencing those shadows. This analogy contains even a game theoretical element: Sometimes, those prisoners compete by predicting which shadow will come next. We emphasize that a player's view must be based on the shadows he noticed, without any direct knowledge about the outside of the cave.

Our theory of info-memory protocols was constructed to capture the above aspect of inductive derivation. In the introduction of Part I, we motivated the development of our theory with

(a): the consideration of a recurrent situation of an info-memory protocol;

(b): the observable elements as primitives of the theory.

In the Analogy of the Cave, a day with more or less the same shadows has been repeated from birth. Shadows are observables and correspond to information pieces in our theory. Some accumulation of observations in a recurrent situation may be needed to construct an individual view, though we do not need to assume a lot of accumulations since we target ordinary people's induction.

Technically speaking, (a) was irrelevant in Part I in that our theory could be neutrally applied to both a recurrent situation and a one-shot situation. In Part II, (a) becomes substantial, since we consider the accumulation of experiences. Motivation (b) must be clear since a player has no knowledge on the structure behind observables. The inductive derivation of a personal protocol uses (a) as well as (b).

The theory of info-memory protocols facilitates the consideration of the objective description as well as the construction of subjective views. An individual player depicts his view on the objective protocol by a personal subjective protocol. Methodologically, the distinction between the objective protocol and a personal protocol is made by certain choices of axioms. We required all the basic and nonbasic axioms for the objective protocol in Part I, and then obtained the equivalence of a protocol to an extensive game in Kuhn's [9] sense. We require only the basic axioms for a personal protocol. Nevertheless, if a player takes a viewpoint of an analyst, he may require his personal protocol to have nonbasic axioms.

The methodology of our theory is considerably more general than the inductive method discussed in Kaneko-Matsui [8]. Their method is specific to the game they adopted. The objective situation is described by an extensive game but a personal view is described by a state-space model. Our unified development based on (b) enables us to have a much wider scope than that paper had.

Nevertheless, this paper concentrates mainly on a theoretical development together with various examples. We refer to Kaneko-Matsui [8] as one possible application to the problem of a racial discrimination and emergence of prejudices. We remark that since their game is a simultaneous move form, our theory would need to be modified for a practical application to their game.

In the respect that a player has a subjective view on the objective situation, the explanation of our theory up to now may sound similar to the "subjective equilibrium" of Kalai-Lehrer [5], [6] and "self-confirming equilibrium" in Fudenberg-Levine [1]. In [5] and [6], they describe the situation where each player has "subjective probabilistic beliefs" that are stable while playing. A similar idea was discussed in a learning game in [1]. Differences from our theory are first that they do not touch the problem of induction, and second that in their theories a player revises (or not) his subjective probabilities on given possible types. The difference between those works and ours will be clearer in Section 1.2.[1] In the respect that the player considers his memories of past experiences, this paper may also appear to be related to the case-based decision theory of Gilboa-Schmeidler [2]. However, their theory is not concerned with the development of a personal (structural) view on the society, but rather with a case-based decision rule to be used in new situations.

[1] Broadly speaking, our theory is also related to the literature of belief revision or theory revision (cf. Schulte [13] for a recent survey). Ours can be separated from this literature in that our theory is about belief (theory) formation but not about revision.

1.2. Developments in Part II

Let us start with a general remark on induction. As claimed by Hume [4]: "We may draw inferences from the coherence of our perceptions, whether they be true or false; whether they represent nature justly, or be mere illusions of our senses (Section V of Book I, 2nd paragraph)." We adopt this position as a starting point, which implies that an inductively constructed view might involve false elements. Our aim is not to show such falsities, but rather to examine the structure of an inductive derivation, and to evaluate how close or far the view derived by induction is from the objective truth.

We consider a representative player, i, in the objective info-memory protocol $(\Pi^o, \pi^o, \mathtt{m}^o) = (\Pi^o, \pi^o, (\mathtt{m}_1^o, ..., \mathtt{m}_n^o))$. He has no *a priori* knowledge on $(\Pi^o, \pi^o, \mathtt{m}^o)$; his memory function $\mathtt{m}_i^o$ connects his mental state to the objective world. The entire structure of an inductive derivation is illustrated in Diagram 1.1 and Diagram 1.2.

info-memory protocol	$\rightarrow$	local memory	$\rightarrow$	accumulation of local memories	$\rightarrow$	personal protocol	$\rightarrow$
$(\Pi^o, \pi^o, \mathtt{m}^o)$		$\mathtt{m}_i^o\langle \xi, w\rangle$		$\bigcup_{\langle \xi, w\rangle \in \Xi_i^o} \mathtt{m}_i^o\langle \xi, w\rangle$		$(\Pi^i, \pi^i, \mathtt{m}_i^i)$	

Objective world $\rightarrow$ Subjective world $\rightarrow$ Internal Mental Activity $\rightarrow$

Diagram 1.1

In the last step of Diagram 1.1, a personal view $(\Pi^i, \pi^i, \mathtt{m}_i^i)$ is derived from the accumulated memories $\bigcup_{\langle \xi, w\rangle \in \Xi_i^o} \mathtt{m}_i^o\langle \xi, w\rangle$, where Ξ_i^o is the set of all positions for player i in the objective protocol (Π^o, π^o). Narrowly speaking, this step is called induction. Nevertheless, the other parts in Diagram 1.1 affect the results of inductive derivations and need some explanation.

The interface between the objective world and the subjective world of player i is his memory function $\mathtt{m}_i^o$. Experiences are accumulated in his mind in the form of memories. However, notice that the memory function $\mathtt{m}_i^o$ at each position $\langle \xi, w\rangle$ gives only a temporary memory $\mathtt{m}_i^o\langle \xi, w\rangle$, which we call a *local memory*. A local memory typically has insufficient information to consider the entire situation Π^o. When player i constructs a view on Π^o, he needs to reflect on the local memories he has had in the past. In other words, he needs to have a *global memory*. We give two comments on our treatment of global memory.

First, we consider a player i who accumulates all elements μ, called memory threads, in $\mathtt{m}_i^o\langle \xi, w\rangle$. This accumulation is expressed as $\bigcup_{\langle \xi, w\rangle \in \Xi_i^o} \mathtt{m}_i^o\langle \xi, w\rangle$ in Diagram 1.1. Here, it is our basic assumption that player i recalls each memory thread μ he ever had in his local memory. This is a kind of perfect recall of local memories. This does not mean perfect recall of what was experienced, since local memories can be subject to imperfections as we saw in Part I.

The other comment is about the union $\bigcup_{\langle \xi, w\rangle \in \Xi_i^o} \mathtt{m}_i^o\langle \xi, w\rangle$ being taken over the entire set Ξ_i^o of positions for player i. This requires sufficient repetitions and variations of actions taken by all players in order to reach each position. In large games, it might be too much to expect a player to have experienced all his positions. In a separate paper, we will discuss the accumulation of memories on some subset of a player's positions.

We presume that in addition to being able to accumulate local memories, player i can also analyze the accumulated memory set $\bigcup_{\langle \xi, w\rangle \in \Xi_i^o} \mathtt{m}_i^o\langle \xi, w\rangle$. This set may contain some information about historical events and their chronological order. Ultimately, player i will derive his personal view by cooking his accumulated memories $\bigcup_{\langle \xi, w\rangle \in \Xi_i^o} \mathtt{m}_i^o\langle \xi, w\rangle$.

More concretely, he will take two steps for inductive derivations in Part II:

(i): Inductive derivation of a view *purely* based on the accumulated set $\bigcup_{\langle \xi, w\rangle \in \Xi_i^o} \mathtt{m}_i^o\langle \xi, w\rangle$.

(ii): Inductive derivation of a view based on $\bigcup_{\langle \xi, w\rangle \in \Xi_i^o} \mathtt{m}_i^o\langle \xi, w\rangle$ and some additional elements.

These steps will be taken, respectively, in Section 3 and Section 5.

In (i), the constructed protocol is purely based on $\bigcup_{\langle \xi, w\rangle \in \Xi_i^o} \mathtt{m}_i^o\langle \xi, w\rangle$ without having any additional elements. Thus, the view is purely experiential. However, it might not satisfy even the basic axioms B2 (All Actions Used) and B4 (Weak Extension). We give sufficient conditions on the memory function $\mathtt{m}_i^o$ for the basic and nonbasic axioms to be met.

When the constructed protocol in (i) violates some basic axiom, player i might try to extend his protocol by supplementing the casual relations and/or adding new symbols. This extension is what is described by (ii). We show the existence of such a personal protocol in this generalized sense which satisfies all the basic axioms. In fact, there are a denumerable number of such extensions.[2] However, we are interested in minimal extensions. We also exemplify minimal personal protocols in various examples.

We will give one small section about the behavioral use of a personal protocol. Now, Diagram 1.1 is extended by adding Diagram 1.2. Once player i constructs a personal protocol $(\Pi^i, \pi^i, \mathtt{m}_i^i)$, he considers a behavioral plan in it, which is a personal strategy. He behaves according to this personal strategy. For this, we need to show that his personal strategy induces a strategy in the objective sense. Then, we can talk about optimal behavior. We give these in the 1-person case, and show, by example, that a

[2] under the assumption that there are denumerable number of new symbolic expressions.

subjectively optimal strategy may not be objectively optimal.

$\rightarrow$	personal protocol	$\rightarrow$	personal strategy	$\rightarrow$	optimization subjectively optimal	$\rightarrow$	induced strategy objectively optimal?
	$(\Pi^i, \pi^i, \mathtt{m}_i^i)$		σ_i^i		$\hat{\sigma}_i^i$		$\hat{\sigma}_i^o$
$\rightarrow$	Internal Mental Activity					$\rightarrow$	Objective World

Diagram 1.2

The above problem is related to the player's actual experiences. In a separate paper, we will investigate the generation of experiences together with the resulting equilibrium behavior in addition to the derived view.

Part II is written as follows. Section 2 reviews the theory of info-memory protocols developed in Part I. Section 3.1 gives an inductive derivation of a personal protocol purely based on accumulated memories of individual experiences. Then, we consider some inductive derivations for the analogy of the cave of Plato [11] and the absentminded driver of Piccione-Rubinstein [12]. In Section 3.4, we show that the derived personal protocol might not satisfy even the basic axioms. Section 4 lists conditions on the memory function to obtain each of the axioms, and shows that the classical memory function implies full recoverability of the original protocol. In Section 5 we provide a general version of an inductive derivation as an alternative method of obtaining a (basic) personal info-memory protocol. In Section 6, we give one theorem to show that a personal protocol can be extended to a protocol satisfying all the basic and nonbasic axioms. Section 7 is a small section about behavioral uses of a personal protocol, and Section 8 concludes Part II.

2. Info-memory Protocols

We summarize the concepts introduced in Part I. In Section 2.1, we define an information protocol *without* a specification of payoff functions, and in Section 2.2, we provide axioms for it. In Section 2.3, we give the definition of a memory function. Since payoff functions will be relevant only in Section 7, we introduce them there.

In this section, an info-memory protocol is introduced without specifying its objective or subjective use. After this section, the objective protocol is signified with the superscript $\circ$, i.e., $(\Pi^o, \pi^o, \mathtt{m}^o) = ((W^o, A^o, \prec^o), \pi^o, (\mathtt{m}_1^o, ..., \mathtt{m}_n^o))$, and the subjective one is with the superscript i, i.e., $(\Pi^i, \pi^i, \mathtt{m}_i^i) = ((W^i, A^i, \prec^i), \pi^i, \mathtt{m}_i^i)$. Since we consider only the personal memory function $\mathtt{m}_i^i$ of player i in $(\Pi^i, \pi^i, \mathtt{m}_i^i)$, the expression $\mathtt{m}_i^i$ looks redundant. However, we follow this rule to avoid unnecessary confusions.

2.1. An Information Protocol

An *information protocol* is given as a triple $\Pi = (W, A, \prec)$:

(IP1): W is a nonempty set of *information pieces*;

(IP2): A is a nonempty set of *actions*;

(IP3): $\prec$ is a subset of $\bigcup_{m=0}^{\infty}((W \times A)^m \times W)$.

Here $(W \times A)^0 \times W$ is stipulated as W. Throughout the paper, we assume that $W \cap A = \emptyset$ to avoid unnecessary complications. IP3 means that $\prec$ is the union of a unary relation on $(W \times A)^0 \times W = W$, a binary relation on $(W \times A)^1 \times W$, a trinary relation on $(W \times A)^2 \times W$,..., etc. In this paper, we consider only *finite* information protocols, i.e., those W, A and $\prec$ are all finite sets.

The subset $\prec$ of $\bigcup_{m=0}^{\infty}((W \times A)^m \times W)$ is called a *causality relation*. We call each element $\langle (w_1, a_1), ..., (w_m, a_m), w \rangle \in \prec$ a *feasible sequence*. We often write $[(w_1, a_1), ..., (w_m, a_m)] \prec w$ for $\langle (w_1, a_1), ..., (w_m, a_m), w \rangle \in \prec$. We will use $\langle \xi, w \rangle$ to denote an generic element of $\bigcup_{m=0}^{\infty}((W \times A)^m \times W)$. When $\langle \xi, w \rangle \in \prec \cap ((W \times A)^0 \times W)$, the feasible sequence is just $\langle w \rangle$ and is denoted by $\prec w$.

Let $\Pi = (W, A, \prec)$ be an information protocol. We partition W into:

(Mv1): $W^D = \{w \in W : [(w, a)] \prec v$ for some $a \in A$ and $v \in W\}$;

(Mv2): $W^E = W - W^D$.

These are the sets of *decision pieces* and *endpieces*. The counterparts in an extensive game are the information sets and endnodes. As was shown in Part I, a correspondence between an extensive game and an information protocol holds exactly when all the basic and nonbasic axioms of Section 2.2 are satisfied.

We next introduce the player set $N = \{0, 1, ..., n\}$ to the information protocol Π. Player 0 is called the *chance* player, and players $1, ..., n$ are called *personal* players. A *player assignment* is a function $\pi : W \to 2^N$ satisfying the conditions (i) $\pi(w)$ consists of a single player for all $w \in W^D$ and (ii) $\pi(w) = \{1, ..., n\}$ for all $w \in W^E$. This means that only one player receives a decision piece, but all players receive an endpiece. We will sometimes denote the set of information pieces received by player i by $W_i = \{w \in W : i \in \pi(w)\}$. By the definition of a player assignment π, W_i includes W^E for each personal player i.

2.2. Basic and Nonbasic Axioms

We provide four basic axioms B1-B4 and two nonbasic axioms N1-N2 for an information protocol $\Pi = (W, A, \prec)$. The reason for this division of the axioms into two classes will be clearer shortly.

The first and second axioms require that all pieces and actions are potentially used in the protocol.

Axiom B1 (All Pieces Used): $\prec w$ for any $w \in W$.

Axiom B2 (All Actions Used): for any $a \in A$, $[(u,a)] \prec v$ for some $u, v \in W$.

We define a *subsequence* of a sequence $[(w_1, a_1), ..., (w_k, a_k)]$ by regarding each (w_t, a_t) as a component of the sequence. We say that $\langle (v_1, a_1), ..., (v_m, a_m), v_{m+1} \rangle$ is a *subsequence* of a sequence $\langle (u_1, b_1), ..., (u_k, b_k), u_{k+1} \rangle$ iff $[(v_1, a_1), ..., (v_m, a_m), (v_{m+1}, a)]$ is a subsequence of $[(u_1, b_1), ..., (u_k, b_k), (u_{k+1}, b)]$ for some a and b. A *supersequence* is defined likewise.

Axiom B3 (Contraction): Let $\langle \xi, v \rangle$ be a feasible sequence, and $\langle \xi', v' \rangle$ a subsequence of $\langle \xi, v \rangle$. Then $\langle \xi', v' \rangle$ is a feasible sequence.

The next axiom guarantees that a player has some available action at each of his decision pieces regardless of the history. Under this axiom, the player at a decision piece can move.

Axiom B4 (Weak Extension): If $\xi \prec w$ and $w \in W^D$ then there is $a \in A$ and $v \in W$ such that $[\xi, (w,a)] \prec v$.

We call an information protocol Π a *basic* protocol iff Π satisfies Axioms B1-B4.

We add the other two axioms from the viewpoint of the objective observer. For those axioms, we will use the notion of a position. We say that a feasible sequence $\langle \xi, v \rangle$ is *maximal* iff there is no proper feasible supersequence of $\langle \xi, v \rangle$. We say that $\langle (w_1, a_1), ..., (w_k, a_k), w_{k+1} \rangle$ for $k = 1, ..., m$ or $\langle w_1 \rangle$ is an *initial fragment* of $\langle \xi, w_{m+1} \rangle = \langle (w_1, a_1), ..., (w_m, a_m), w_{m+1} \rangle$. A *position* $\langle \xi, w \rangle$ is an initial fragment of some maximal feasible sequence. Each position $\langle \xi, w \rangle$ can be regarded as an exhaustive history to w. We denote the set of all positions by Ξ, and partition Ξ into the set of *end positions* $\Xi^E = \{\langle \xi, w \rangle \in \Xi : w \in W^E\}$ and the set of *decision positions* $\Xi^D = \{\langle \xi, w \rangle \in \Xi : w \in W^D\}$.

Now, we have the first nonbasic axiom.

Axiom N1 (Extension): If $\langle \xi, v \rangle$ is a position and $[(v,a)] \prec u$, then there is a $w \in W$ such that $\langle \xi, (v,a), w \rangle$ is a position.

Axiom N1 looks similar to Axiom B4. Axiom N1 states a possible extension of a position, while B4 states a possible extension of a feasible sequence. Here, we define the set of *actual* available actions A_v at information piece v as:

$$A_v = \{a \in A : [(v,a)] \prec u \text{ for some } u \in W\}. \tag{2.1}$$

Then $A_v \neq \emptyset$ for any decision piece $v \in W^D$ and $A_v = \emptyset$ for any endpiece $v \in W^E$. Using A_v, the difference between B4 and N1 is described as follows. The position is extended with every $a \in A_v$ in N1, while a feasible sequence can be extended with some $a \in A_v$. Actually, it was shown in Part I that with the other basic axioms, N1 implies B4.

The last axiom is more essential from the viewpoint of the objective observer. It states that a position is an exhaustive history determining the present piece. It makes an information protocol comparable with an extensive game in the sense of Kuhn [9]

Axiom N2 (Determination): Let $\langle \xi, v \rangle$ and $\langle \zeta, w \rangle$ be two positions. Then $\xi = \zeta$ implies $v = w$.

As stated in Part I, we stipulate that this axiom is applied to two positions $\langle v \rangle$ and $\langle w \rangle$, that is, if $\langle v \rangle$ and $\langle w \rangle$ are positions, then $v = w$. This implies that there is a distinguished piece $w^0 \in W$, called the *root*, such that every position starts with w^0.

Now we introduce anther set of available actions: Let $\langle \xi, v \rangle$ be a position, i.e., $\langle \xi, v \rangle \in \Xi$. We define

$$A_{\langle \xi, v \rangle} = \{a \in A : \langle \xi, (v, a), u \rangle \in \Xi \text{ for some } u \in W\}. \tag{2.2}$$

Under Axioms B3 and N1, $A_{\langle \xi, v \rangle}$ coincides with A_v of (2.1). Nevertheless, since we do not necessarily assume all the axioms, the set $A_{\langle \xi, v \rangle}$ may be relevant. See, e.g., footnote 2. In the information protocol of Figure 2.1, $A_w = \{a, b\}$ differs from $A_{\langle w \rangle} = \{a\}$. This protocol satisfies all axioms except N1, and therefore it is basic.

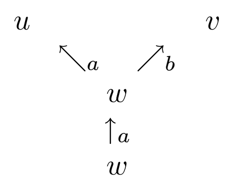

Figure 2.1

The main result of Part I is the "equivalence" between an information protocol satisfying Axioms B1-B3, N1-N2 and an extensive game in the sense of Kuhn [9], which was shown using the notion of an "infomorphism".

2.3. Memory Functions

Let $\Pi = (W, A, \prec)$ be an information protocol, and π a player assignment in Π. Here, we do not assume that Π satisfies any particular axioms.

A memory of a player at a point of time consists of a finite number of memory threads of perceived information pieces, available action sets and actions taken in the past in $\Pi = (W, A, \prec)$. In Part II we restrict attention to memory threads that are constructed only of actual pieces in W and A. This excludes the Orwell memory function discussed in Part I.

Formally, a *memory thread* is a finite sequence

$$\mu = \langle (v_1, B_1, b_1), ..., (v_m, B_m, b_m), (v_{m+1}, B_{m+1}) \rangle \tag{2.3}$$

where

$$v_t \in W, b_t \in B_t \text{ for all } t = 1, ..., m \text{ and } v_{m+1} \in W. \tag{2.4}$$

Each component (v_t, B_t, b_t) or (v_{m+1}, B_{m+1}) in μ is called a *memory knot*. A memory thread differs from a feasible sequence in that the perceived available action sets, B_t's, are included. Throughout Part II, we assume

$$B_t \subseteq A_{v_t} \text{ for } t = 1, ..., m+1. \tag{2.5}$$

The perceived set of available actions may be smaller than A_{v_t}, but it consists only of actual available actions at v_t.

We denote the ending pair (v_{m+1}, B_{m+1}) of μ of (2.3) by $\varepsilon(\mu) = (\varepsilon_1(\mu), \varepsilon_2(\mu))$, which is called the *tail* of μ. Also, we define $\Xi_i := \{\langle \xi, w \rangle \in \Xi : i \in \pi(w)\}$, i.e., the set of decision positions for player i including the endpositions. Now, we have the definition of a memory function.

Definition 2.1. We say that a function $\mathtt{m}_i$ is a *memory function* of player i iff for each position $\langle \xi, w \rangle \in \Xi_i$, $\mathtt{m}_i\langle \xi, w \rangle$ is a finite non-empty set of memory threads satisfying: for all $\mu, \eta \in \mathtt{m}_i\langle \xi, w \rangle$,

$$\varepsilon(\mu) = \varepsilon(\eta) \tag{2.6}$$

$$A_w \neq \emptyset \Longleftrightarrow \varepsilon_2(\mu) \neq \emptyset, \tag{2.7}$$

$$\varepsilon_1(\mu) \in W_i. \tag{2.8}$$

Condition (2.6) states that the tails of any memory threads at a position are identical, which means that player i has the uniquely determined perception $\varepsilon(\mu)$ at $\langle \xi, w \rangle$. Then (2.7) states that his perceived set of available actions is nonempty at a decision position and is empty at an endposition. The last condition (2.8) states that player i's perception $\varepsilon_1(\mu)$ of w must be in W_i.

Here, we mention two memory functions: the *perfect information memory function* $\mathtt{m}_i^{PI}$ and the *classical memory function* $\mathtt{m}_i^C$. The former is given as:

$$\mathtt{m}_i^{PI}\langle \xi, w \rangle = \{\langle \xi, w \rangle^*\} \text{ for each } \langle \xi, w \rangle \in \Xi_i, \tag{2.9}$$

where $\langle \zeta, v \rangle^* = \langle (v_1, a_1), ..., (v_m, a_m), v \rangle^*$ is defined by:

$$\langle \zeta, v \rangle^* = \langle (v_1, A_{v_1}, a_1), ..., (v_m, A_{v_m}, a_m), (v, A_v) \rangle. \tag{2.10}$$

He can recall the complete history of the position $\langle \xi, w \rangle$. The latter is given as:

$$\mathtt{m}_i^C\langle \xi, w \rangle = \{\langle \zeta, v \rangle^* \in \Xi : v = w\} \text{ for each } \langle \xi, w \rangle \in \Xi_i. \tag{2.11}$$

He can infer all complete memory threads having the same last w at the position $\langle \xi, w\rangle$, but he cannot determine which is correct. The memory functions $\mathtt{m}_i^{PI}$ and $\mathtt{m}_i^{C}$ satisfy conditions (2.5), (2.6), (2.7) and (2.8) even if $\Pi = (W, A, \prec)$ satisfies none of the basic axioms. They will play important roles in the subsequent analysis, especially as subjective memory functions.

3. Inductive Derivation of a Personal View I: Strict Form

Now, we start to consider the inductive derivation of a personal view by player i based on his accumulated memories. In this section, an inductive derivation is defined purely based on his accumulated memories.

3.1. Personal Info-Memory Protocols purely based on Experiences

Let $(\Pi^o, \pi^o, \mathtt{m}^o) = ((W^o, A^o, \prec^o), \pi^o, (\mathtt{m}_1^o, ..., \mathtt{m}_n^o))$ be an objective info-memory protocol satisfying Axioms B1-B3 and N1-N2. We focus on a particular player i. Recall that the domain of the memory function $\mathtt{m}_i^o$ of player i is $\Xi_i^o := \Xi_i(\Pi^o) = \{\langle \xi, w\rangle : \langle \xi, w\rangle$ is a position in Π^o with $i \in \pi^o(w)\}$.

Player i has accumulated the threads that have occurred in his mind. Now, let the accumulation set of threads be

$$T(\mathtt{m}_i^o) = \bigcup_{\langle \xi, w\rangle \in \Xi_i^o} \mathtt{m}_i^o\langle \xi, w\rangle. \tag{3.1}$$

Player i is going to cook these accumulated memory threads to construct his personal view on $(\Pi^o, \pi^o, \mathtt{m}^o)$. The adoption of $T(\mathtt{m}_i^o)$ is the *perfect accumulation assumption.* Since player i is assumed to have no *a priori* knowledge on the structure of the objective protocol, this assumption requires that it has been repeated enough to hit each $\langle \xi, w\rangle \in \Xi_i^o$. It would be natural to have some restriction on the domain for the union in (3.1). Such restrictions as well as the rationales for them will be provided in a separate paper. See comments on this assumption and a restricted form of (3.1) in Section 8.

Now, we define the following sets:

(i): $T^*(\mathtt{m}_i^o)$ is the set of all subthreads of memory threads in $T(\mathtt{m}_i^o)$;

(ii): W^{o*} is the set of all information pieces occurring in $T(\mathtt{m}_i^o)$;

(iii): A^{o*} is the set of all available actions occurring in B of a knot (v, B, b) or (v, B) of some memory threads μ in $T(\mathtt{m}_i^o)$.

We did not collect the actions b in knots (v, B, b) in memory threads, but, instead, we take the union of B's. That is, the perceived available actions are considered to be ingredients for the inductive derivation of a view. This difference may matter for some consideration, which will be discussed in Section 3.4.

In the inductive derivation of a view, player i regards a memory thread $\mu = \langle(v_1, B_1, b_1), ..., (v_m, B_m, b_m), (v_{m+1}, B_{m+1})\rangle$ in $T^*(\mathtt{m}_i^o)$ as a feasible sequence in his personal protocol. However, the sets of available actions in μ are redundant with as feasible sequences in an information protocol. Thus, we take one operation of eliminating those sets:

$$\mu_{-A} = \langle(v_1, b_1), ..., (v_m, b_m), v_{m+1}\rangle. \tag{3.2}$$

This operation is applied to $T^*(\mathtt{m}_i^o)$ as well as $\mathtt{m}_i^o\langle\xi, w\rangle$ in the following way:

$$T^*(\mathtt{m}_i^o)_{-A} = \{\mu_{-A} : \mu \in T^*(\mathtt{m}_i^o)\} \text{ and } \mathtt{m}_i^o\langle\xi, w\rangle_{-A} = \{\mu_{-A} : \mu \in \mathtt{m}_i\langle\xi, w\rangle\}.$$

Player i may regard $T^*(\mathtt{m}_i^o)_{-A}$ as a set of feasible sequences in a protocol.

We need one more concept before we define a personal protocol. It is the subjective player assignment. A *subjective player assignment* is a function $\pi^i : W^{o*} \to 2^{\{0,1,...,n\}} \backslash \{\emptyset\}$ satisfying:

$$\begin{aligned} \pi^i(w) \text{ is a singleton set if } \varepsilon_1(\mu) &= w \text{ and } \varepsilon_2(\mu) \neq \emptyset \\ \text{for some } \mu &\in T^*(\mathtt{m}_i^o). \end{aligned} \tag{3.3}$$

We say that π^i is *personally correct* iff for all $v \in W^{o*}$,

$$\text{if } v = \varepsilon_1(\mu) \text{ for some } \mu \in T(\mathtt{m}_i^o), \text{ then } i \in \pi^i(v). \tag{3.4}$$

Note that $T(\mathtt{m}_i^o)$ is used rather than $T^*(\mathtt{m}_i^o)$ in (3.4). The function π^i describes player i's perception of who has moved at each perceived piece. Personal correctness requires player i to assign himself to each tail of a memory thread at a position belonging to him, and condition (2.8) guarantees that each tail piece objectively is one of his own. It could be possible for him to assign only the tails to himself and assigns all other pieces to player 0.

We are now in a position to define a personal info-memory protocol in the strict sense.

Definition 3.1. $(\Pi^i, \pi^i, \mathtt{m}_i^i)$ is a *personal info-memory protocol* for player i in a *strict sense* iff

(P1$_s$): $\Pi^i = (W^i, A^i, \prec^i)$ is an information protocol with $W^i = W^{o*}$, $A^i = A^{o*}$ and $\prec^i = T^*(\mathtt{m}_i^o)_{-A}$;

(P2$_s$): π^i is a personally correct subjective player assignment;

(P3$_s$): $\mathtt{m}_i^i$ is a memory function over the domain $\Xi_i(\Pi^i)$ with the conditions:
(a) $T^*(\mathtt{m}_i^i)_{-A} = T^*(\mathtt{m}_i^o)_{-A}$ and (b) $\langle\xi, w\rangle \in \mathtt{m}_i^i\langle\xi, w\rangle_{-A}$ for $\langle\xi, w\rangle \in \Xi_i(\Pi^i)$,
where $\Xi_i(\Pi^i)$ is the set of positions for player i in (Π^i, π^i).

The above definition includes the inductive constructions of (i) information protocol Π^i, (ii) player assignment π^i, and (iii) memory function $\mathtt{m}^i_i$. Here, Π^i is uniquely defined, but π^i and the memory function $\mathtt{m}^i_i$ have multiple candidates.

Regarding π^i, there is some freedom for the other players and their moves as mentioned above, but the objective one π^o is always one candidate.

Lemma 3.2. Let π^i be the restriction of π^o to W^{o*}. Then π^i is a personally correct subjective player assignment.

Proof. Consider $v \in W^{o*}$. The only restrictions of π^i are (3.3) and (3.4). Consider (3.3). We prove the contrapositive. Let $w \in W^{o*}$ with $\varepsilon_1(\mu) = w$ and $\mu \in T^*(\mathtt{m}^o_i)$. Suppose that $\pi^o(v)$ is not singleton. Then, v is an endpiece in Π^o. Thus, $A^o_v = \emptyset$ and $i \in \pi^o(v)$. Also, there is a thread $\mu' \in T(\mathtt{m}^o_i)$ such that μ is a subthread of μ' but $\varepsilon(\mu') = \varepsilon(\mu')$. By (2.7), we have $\varepsilon_2(\mu') = \emptyset$, *a fortiori*, $\varepsilon_2(\mu) = \emptyset$.

Consider (3.4). Let $v = \varepsilon_1(\mu)$ for some $\mu \in T(\mathtt{m}^o_i)$. By (2.8), we have $v = \varepsilon_1(\mu) \in W^o_i$. Thus, $i \in \pi^o(v) = \pi^i(v)$. ■

We need to discuss the subjective memory function $\mathtt{m}^i_i$ of P3$_s$ more. Condition (a) requires only that the accumulation of subsequences of threads from the two functions $\mathtt{m}^o_i$ and $\mathtt{m}^i_i$ are the same, but the sets of available actions are excluded in this comparison. The condition (b), called the *internal veridicality*, means that the variable $\langle \xi, w\rangle$ of $\mathtt{m}^i_i$ should be in $\mathtt{m}^i_i\langle \xi, w\rangle$. This expresses the idea that when the player constructs $\mathtt{m}^i_i$, he must be aware of the variable of $\mathtt{m}^i_i$, and regards it as the true history relative to in his subjective world. On the other hand, this may not be true for the objective memory function $\mathtt{m}^o_i$ since player i may not have (Π^o, π^o) in his mind. Thus, (b) may exclude the true $\mathtt{m}^o_i$ as a candidate for $\mathtt{m}^i_i$.

The restriction of P3$_s$ does not appear to restrict the available action sets in memory threads. This is not true, however, since the subjective memory function $\mathtt{m}^i_i$ satisfies (2.5), that is, the perceived available actions must be actually available within his subjective information protocol Π^i.

Two candidates for $\mathtt{m}^i_i$ that are always included are the perfect information memory function $\mathtt{m}^{PI}_i$ in (Π^i, π^i) and the classical memory function $\mathtt{m}^C_i$ in (Π^i, π^i).

Lemma 3.3. Let Π^i be the personal information protocol defined by P1$_s$ and π^i any personally correct subjective player assignment in Π^i. Then either the perfect information memory function $\mathtt{m}^{PI}_i$ or classical memory function $\mathtt{m}^C_i$ in (Π^i, π^i) satisfies P3$_s$.

Proof. Let $\mathtt{m}^i_i = \mathtt{m}^{PI}_i$ in (Π^i, π^i). Essentially the same proof can be applied to the case of the classical memory function. Clearly $\mathtt{m}^i_i$ satisfies (b). We consider (a). Since $\mathtt{m}^i_i = \mathtt{m}^{PI}_i$, we have $T^*(\mathtt{m}^i_i)_{-A} \subseteq \prec^i$.

Let us prove the other direction. Let $\langle \xi, w\rangle$ be a maximal feasible sequence in $\prec^i$. We show $i \in \pi^i(w)$. Since $\langle \xi, w\rangle$ is a maximal feasible sequence, there is a memory thread $\mu \in T(\mathtt{m}^o_i)$ with $\mu_{-A} = \langle \xi, w\rangle$. Since $\varepsilon_1(\mu) = w$, we have $i \in \pi^i(w)$ by (3.4).

This means that $\Xi_i(\Pi^i)$ includes all maximal feasible sequences for Π^i. Hence, since $\mathtt{m}_i^i = \mathtt{m}_i^{PI}$, we have $\Xi_i(\Pi^i) = T(\mathtt{m}_i^i)_{-A}$. Thus $T(\mathtt{m}_i^i)_{-A}$ includes all the maximal feasible sequences in $\prec^i$. That means, $T^*(\mathtt{m}_i^i)_{-A} \supseteq \prec^i$.

In the above two paragraphs, we have seen that $\prec^i = T^*(\mathtt{m}_i^i)_{-A}$. By P1$_s$, we have $\prec^i = T^*(\mathtt{m}_i^o)_{-A}$. Thus, $T^*(\mathtt{m}_i^i)_{-A} = T^*(\mathtt{m}_i^o)_{-A}$. ■

Let us consider one basic example first and then we will look at the analogy of the cave and a twist on the absentminded driver.

Example 3.4. Consider the following 2-person protocol $\Pi^o = (W^o, A^o, \prec^o)$:

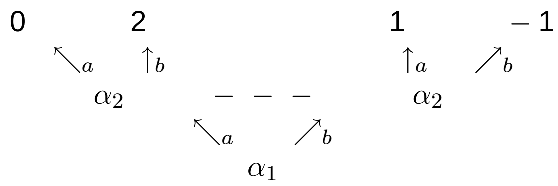

Figure 3.1

Player 1 moves at α_1 and player 2 moves at α_2. The positions for player 1 are the root position and the four endpositions, i.e.,

$$\begin{aligned}\Xi_1^o &= \{\langle\alpha_1\rangle, \langle(\alpha_1, a), (\alpha_2, a), \mathsf{0}\rangle, \langle(\alpha_1, a), (\alpha_2, b), \mathsf{2}\rangle,\\ &\quad \langle(\alpha_1, b), (\alpha_2, a), \mathsf{1}\rangle, \langle(\alpha_1, b), (\alpha_2, b), -\mathsf{1}\rangle\}.\end{aligned}$$

We assume that $\mathtt{m}_1^o$ is the (exact) perfect recall memory function described in Part I:

$$\mathtt{m}_1^o\langle\xi, w\rangle = \begin{cases} \{\langle(\alpha_1, \{a, b\})\rangle\} & \text{if } \langle\xi, w\rangle = \langle\alpha_1\rangle \\ \\ \{\langle(\alpha_1, \{a, b\}, x), (w, \emptyset)\rangle\} & \text{if } \langle\xi, w\rangle = \langle(\alpha_1, x), (\alpha_2, y), w\rangle, \end{cases}$$

where $x, y \in \{a, b\}$. With this memory function, player 1 perceives and recalls correctly all his own (but player 2's) pieces, actions, and available action sets.

The accumulated memory set $T(\mathtt{m}_1^o)$ is given as:

$$\begin{aligned}&\{\langle(\alpha_1, \{a, b\})\rangle, \langle(\alpha_1, \{a, b\}, a), (\mathsf{0}, \emptyset)\rangle, \langle(\alpha_1, \{a, b\}, a), (\mathsf{2}, \emptyset)\rangle,\\ &\langle(\alpha_1, \{a, b\}, b), (\mathsf{1}, \emptyset)\rangle, \langle(\alpha_1, \{a, b\}, b), (-\mathsf{1}, \emptyset)\rangle\}.\end{aligned}$$

From this set, $W^1 = W^{o*} = \{\alpha_1, -\mathsf{1}, \mathsf{0}, \mathsf{1}, \mathsf{2}\}$, $A^1 = A^{o*} = \{a, b\}$ and $\prec^1 = T^*(\mathtt{m}_1^o)_{-A}$ is given as

$$\begin{aligned}&\{\langle\alpha_1\rangle, \langle(\alpha_1, a), \mathsf{0}\rangle, \langle(\alpha_1, a), \mathsf{2}\rangle, \langle(\alpha_1, b), \mathsf{1}\rangle, \langle(\alpha_1, b), -\mathsf{1})\rangle,\\ &\langle\mathsf{0}\rangle, \langle\mathsf{2}\rangle, \langle\mathsf{1}\rangle, \langle-\mathsf{1}\rangle\}.\end{aligned} \quad (3.5)$$

Thus, the personal information protocol $\Pi^1 = (W^1, A^1, \prec^1)$ is depicted as Figure 3.2:

$$\begin{array}{cccccc} \mathsf{0} & & \mathsf{2} & & \mathsf{1} & & \mathsf{-1} \\ & \nwarrow_a & \uparrow_a & & \uparrow_b & \nearrow_b & \\ & & & \alpha_1 & & & \end{array}$$

Figure 3.2

This personal protocol satisfies Axioms B1-B4, N1, but not Axiom N2 (Determination). Player 1 may take this personal protocol as a satisfactory description. He may wonder if there is some cause for indeterminacy, in which case, he may extend his view. Such an extension will be considered in the Sections 5 and 6.

The protocol Π^1 involves player 2 if the subjective player assignment $\pi^1(w) = \{1, 2\}$ for an endpiece w. It would be purely personal if $\pi^1(w) = \{1\}$ for every endpiece w.

In the personal protocol Π^1 (not in the objective one Π^o), the classical memory function $\mathtt{m}_1^C$ and the perfect information memory function $\mathtt{m}_1^{PI}$ for the personal protocol coincide and are defined by:

$$\mathtt{m}_1^1\langle \xi, w\rangle = \begin{cases} \{\langle(\alpha_1, \{a, b\})\rangle\} & \text{if } \langle \xi, w\rangle = \langle \alpha_1\rangle \\ \\ \{\langle(\alpha_1, \{a, b\}, x), (w, \emptyset)\rangle\} & \text{if } \langle \xi, w\rangle = \langle(\alpha_1, x), w\rangle, \end{cases}$$

where $x \in \{a, b\}$. This $\mathtt{m}_1^1$ is essentially the same as $\mathtt{m}_1^o$, except for their domains.

3.2. The Analogy of the Cave

Here, we return to the Analogy of the Cave of Plato [11]. As described in Section 1.1, the prisoners have been chained from their birth in the cave and have ever looked only at the wall of the cave. At the wall, the shadows of various moving objects are projected from the mouth of the cave. Those shadows are only "real" for the prisoners. Each prisoner may construct a view on the world based on the memories of the shadows. Then, Plato continues:

"There was probably a certain honour and glory to be won among the prisoners, and prizes for keen-sightedness for those best able to remember the order of sequence among shadows and so be best able to divine their future appearances" (Plato [11], Book VII, p.243).

We consider one-person version with the objective information protocol $\Pi^o = (W^o, A^o, \prec^o)$ given as $W^o = \{m(orning), H(orse), C(art), e(vening)\}$, $A^o = \{p(ass)\}$, and having $\prec^o$ described by

$$m \underset{p}{\rightarrow} H \underset{p}{\rightarrow} C \underset{p}{\rightarrow} e$$

Figure 3.3

The day starts in the *m*orning, then player 1 sees that the shadow of *H*orse, then that of *C*art, and the day ends in the evening. Sometimes, in the morning, a question of what comes first is asked. This part is not formulated in the above protocol, but we can imagine a player competing with others to produce the most accurate protocol. Suppose that the memory function $\mathtt{m}_1^o$ is defined over the set of all positions.

Now, suppose that in the evening, he tries to recall the events of the day. If he recalls the events and their order, i.e., his memory function is expressed as the perfect information memory function, then his personal information protocol Π^1 is identical to the objective one Π^o described by Figure 3.3. With this protocol, he can predict what comes first and will win honour and glory.

Suppose, alternatively that he tries to recall the events of the day in the evening, but fails to recall the order of H and C. In this case, his memory function $\mathtt{m}_1^o$ might be described by

$$\mathtt{m}_1^o\langle \xi, w\rangle = \begin{cases} \{\langle (w,\{p\})\rangle\} & \text{if } w \neq e \\ \{\langle (m,\{p\},p),(H,\{p\},p),(C,\{p\},p),(e,\emptyset)\rangle, & \\ \langle (m,\{p\},p),(C,\{p\},p),(H,\{p\},p),(e,\emptyset)\rangle\} & \text{if } w = e \end{cases} \tag{3.6}$$

The memory $\mathtt{m}_1^o\langle \xi, w\rangle$ in the evening has two possible sequences of events. Since he cannot isolate one from the other, his personal protocol Π^1 in the strict sense is

$$\begin{array}{ccccccc} & & H & \underset{p}{\rightarrow} & C & & \\ & \nearrow_p & & & & \searrow^p & \\ m & & & & & & e \\ & \searrow_p & & & & \nearrow_p & \\ & & C & \underset{p}{\rightarrow} & H & & \end{array}$$

Figure 3.4

Here, honour and glory might elude him due to his poor subjective description.

3.3. Absentminded Driver Game

Next, we consider a perfect information version of the absentminded driver game of Piccione-Rubinstein [12] as the objective protocol, which is depicted in Figure 3.5.

$$\begin{array}{ccc} & & \mathsf{1} \\ & & \uparrow_b \\ \mathsf{2} & \longleftarrow_a & \alpha' \\ & & \uparrow_b \\ \mathsf{0} & \longleftarrow_a & \alpha \end{array}$$

Figure 3.5

In the objective protocol, the information pieces α and α' representing the two exits are distinguished. However, suppose that the driver does not distinguish those two exits when he perceives them, or anytime later in his memory. Actually, he doesn't recall anything at an exit, but when he arrives at an endpiece, he tries to recall what happened. This story could be described by the following memory function:

$$\mathtt{m}_1^o\langle\xi, w\rangle = \begin{cases} \{\langle(\alpha, \{a,b\})\rangle\} & \text{if } w = \alpha, \alpha' \\ \{\langle(\alpha, \{a,b\}, a), (\mathsf{0}, \emptyset)\rangle\} & \text{if } w = \mathsf{0} \\ \{\langle(\alpha, \{a,b\}, b), (\alpha, \{a,b\}, a), (\mathsf{2}, \emptyset)\rangle\} & \text{if } w = \mathsf{2} \\ \{\langle(\alpha, \{a,b\}, b), (\alpha, \{a,b\}, b), (\mathsf{1}, \emptyset)\rangle\} & \text{if } w = \mathsf{1} \end{cases} \tag{3.7}$$

The first line means that when he is at an exit, he recalls nothing about the past and regards an exit as an exit. Each other line describes that when he reaches an endposition, he recalls his actions accurately, and the number of times he arrived at an exit, but again, he does not distinguish the exits.

In this example with the memory function $\mathtt{m}_1^o$ of (3.7), the derived personal protocol Π^1 is the standard absentminded driver game described in Figure 3.6. In this case, his imperfect perception and/or memory described in the memory function $\mathtt{m}_1^o$ lead to an irreflexive information protocol.

$$\begin{array}{ccc} & & \mathsf{1} \\ & & \uparrow_b \\ \mathsf{2} & \longleftarrow_a & \alpha \\ & & \uparrow_b \\ \mathsf{0} & \longleftarrow_a & \alpha \end{array}$$

Figure 3.6

Now, his consciousness of his protocol of Figure 3.6 leads him to the classical memory function $\mathtt{m}_1^1 = \mathtt{m}_1^C$. However, with this memory function, he cannot still distinguish between the first exit α and the second exit α, having the same set $\{\langle(\alpha, \{a,b\})\rangle, \langle(\alpha, \{a,b\}, b),$

$(\alpha, \{a, b\})\rangle\}$ at these exits. Thus, the strict derivation of this personal information protocol alone may not help the driver to behave properly.

However, if he uses this protocol and puts a check on his hand when he passes the first exit to distinguish the two exits, then he can take a at the second exit. This ability could be described by a personal memory function, which then becomes the perfect information memory function $\mathtt{m}_1^1 = \mathtt{m}_1^{PI}$.

The above consideration suggests some possibility of dynamic development of a memory function and his behavior. But a general consideration should be in a future paper.

3.4. Are Derived Protocols Basic?

In all the examples discussed so far, the personal protocols derived are basic ones, though Axiom N2 is violated by those of Figures 3.2 and 3.4 . In the definition of a personal info-memory protocol $(\Pi^i, \pi^i, \mathtt{m}_i^i)$, we did not require Π^i to be a basic protocol, and in some cases, it might not be. We have examples of objective memory functions that never give a basic personal protocol for $\mathrm{P1}_s$. One case is the Markov memory function.

As in Part I, we define the *Markov* memory function $\mathtt{m}_i^M$ over Ξ_i^o by:

$$\mathtt{m}_i^M\langle \xi, w\rangle = \{\langle (w, A_w^o)\rangle\} \text{ for } \langle \xi, w\rangle \in \Xi_i^o. \tag{3.8}$$

Here, player i has the actual current piece w and actual action set A_w^o in his memory, but he does not recall the past at all. Thus, every memory thread in $T(\mathtt{m}_i^M)$ has length 1, i.e.,

$$T(\mathtt{m}_i^M) = \{\langle (w, A_w^o)\rangle : w \in W^o \text{ and } i \in \pi^o(w)\}. \tag{3.9}$$

Player i has a very limited local memory at each position, while the perfect accumulation assumption of his memories is still assumed. This assumption means that he is conscious about his forgetfulness and obtains $T(\mathtt{m}_i^M)$. This case is extreme and may sound strange, but it gives some insight for our analysis of the inductive derivation.

Notice that $\mathrm{P1}_s$ defines $\prec^i$ to be $T^*(\mathtt{m}_i^o)_{-A}$. In the case of the objective Markov memory function, by (3.8), the length of each sequence in $T^*(\mathtt{m}_i^o)_{-A}$ is also 1. Hence, $\prec^i$ gives all pieces in his memory as isolated points in his subjective protocol. It is a direct consequence, that Axiom B2 is violated by the player's subjective protocol, since Axiom B2 requires $\prec^i$ to have feasible sequences of at least length 2.

Proposition 3.5 (Failure). Suppose that $\mathtt{m}_i^o$ is the Markov. Then Π^i defined by $\mathrm{P1}_s$ violates Axiom B2.

This failure is not specific to the Markov memory function. Indeed, as far as at least one perceived available action is not actually taken in any memory thread in $T^*(\mathtt{m}_i^o)$, Axiom B2 would be violated. The source of this violation is our adoption of A^{o*} in

$P1_s$ as coming from the sets of available actions in memory knots. If we alternatively adopted A^{o*} as coming from perceived actions taken, i.e., b rather than B in a knot like (v, B, b), then Axiom B2 would be satisfied by Π^i for any objective memory function.

By sticking to our adoption the definition of A^{o*} as coming from the perceived available action sets, the B's, we are taking the position that a player's subjective view of actions in a protocol is based on the actions he perceived as being available, not necessarily on the actions he perceived as having been taken. In either case, i.e., under either of the possible definitions for A^{o*} discussed above, the derived protocol might still not be basic since it might violate Axiom B4.

Consider the one player objective information protocol describe in Figure 3.7 with the memory function $\mathtt{m}_1^o\langle w^0\rangle = \{\langle(w^0, \{a\})\rangle\}$ and $\mathtt{m}_1^o\langle(w^0, a), w\rangle = \{\langle(w^0, \{a\}, a), (w^0, \phi)\rangle\}$.

$$\begin{array}{ccc} w & \qquad\qquad & w^0 \\ \uparrow_a & & \uparrow_a \\ w^0 & & w^0 \end{array}$$

Figure 3.7 Figure 3.8

In this case, the derived personal protocol defined by $P1_s$ is described in Figure 3.8, which satisfies Axiom B2, but not Axiom B4. It will be seen in Section 4 that Axioms B1 and B3 are always satisfied by a personal protocol in the strict sense. In Section 4, we will give conditions on the objective memory function $\mathtt{m}_i^o$ to guarantee the personal protocol in the strict sense satisfies each of the basic and nonbasic axioms.

4. Conditions on Memory Functions for a Basic Protocol

We have seen that the derived personal protocol may not be even a basic one. There are two ways to proceed with our analysis. One is to relax the definition of a personal protocol, which is the subject of Sections 5 and 6. The other is to consider conditions over the memory function or the accumulated set of memories to guarantee some or all of the axioms. This section pursues the latter way. In Section 4.1, we consider specific conditions on a memory function to guarantee the corresponding axioms. In Section 4.2, we consider conditions on the memory function to guarantee that subjective and objective information protocols coincide (excluding the player assignment and memory function).

4.1. Memory Conditions and Axioms

First, we consider the following condition on the accumulated memory set $T^*(\mathtt{m}_i^o)$:

(*action-closedness*): for any $\mu = \langle\eta, (v, B)\rangle \in T^*(\mathtt{m}_i^o)$ and $b \in B$, there is a (u, C) such that $\langle \eta, (v, B, b), (u, C)\rangle \in T^*(\mathtt{m}_i^o)$.

This states that if b is perceived as an available action in a thread μ, it is actually used in μ. This condition guarantees Axioms B2 and B4. We can state *action-closedness* directly on the memory function $\mathtt{m}_i^o$, but the above form economizes symbolic expressions.

Proposition 4.1. Let Π^i be defined by $\mathrm{P1}_s$.

(1): Π^i satisfies Axioms B1 and B3.

(2): Π^i satisfies Axioms B2 and B4 under *action-closedness*.

Proof. (1): Since $T^*(\mathtt{m}_i^o)$ is subthread-closed, Axioms B1 and B3 hold.

(2):B2: Let $a \in A^i = A^{o*}$. Then $a \in B$ for some $\langle \eta, (v, B)\rangle \in T^*(\mathtt{m}_i^o)$. Then, by *action-closedness,* there is a (u, C) such that $\langle \eta, (v, B, a), (u, C)\rangle \in T^*(\mathtt{m}_i^o)$. Thus, $\langle (v, B, a), (u, C)\rangle \in T^*(\mathtt{m}_i^o)$ by taking a subthread. This implies $[v, a] \prec^i u$.

B4: Suppose $\xi \prec^i w$ and $A_w^i \neq \emptyset$. Let $\mu = \langle \xi', (w, B)\rangle \in T^*(\mathtt{m}_i^o)$ and $\mu_{-A} = \langle \xi, w\rangle$. If $\varepsilon(\mu) = \varepsilon(\eta)$ for some $\eta \in T(\mathtt{m}_i^o)$, then $B \neq \emptyset$ by $A_w^i \neq \emptyset$ and (2.7). If $\varepsilon(\mu) \neq \varepsilon(\eta)$ for any $\eta \in T(\mathtt{m}_i^o)$, then $(\varepsilon(\mu), b)$ occurs in some $\eta \in T(\mathtt{m}_i^o)$ for some b, which implies $B \neq \emptyset$.

We have $\langle \xi', (w, B) \in T^*(\mathtt{m}_i^o)$ and $b \in B$, since $B \neq \emptyset$. Hence,by*action-closedness*, there is a (u, C) such that $\langle \xi', (w, B, b), (u, C)) \in T^*(\mathtt{m}_i^o)$. Thus, $\langle \xi, (w, b), u\rangle \in \prec^i$. ■

The personal protocol Π^i defined by $\mathrm{P1}_s$ may satisfy all the axioms including N1 and N2. Then, the personal protocol Π^i is essentially an extensive game by the main result of Part I of this paper. Here, we give conditions on $T^*(\mathtt{m}_i^o)$ to guarantee these axioms. However, since N1 and N2 are stated in terms of positions, a parallel notion must be introduced into $T^*(\mathtt{m}_i^o)$. We say that μ is a *maximal* memory thread iff $\mu \in T^*(\mathtt{m}_i^o)$ and there is no thread $\mu' \in T^*(\mathtt{m}_i^o)$ such that μ is a proper subthread of μ'. A *positional* thread is an initial fragment of a maximal memory thread. Now, *action-closedness* is modified into

(*positional action-closedness*): for any positional thread $\mu = \langle \eta, (v, B)\rangle$ in $T^*(\mathtt{m}_i^o)$ and $b \in B$, there is a (u, C) such that $\langle \eta, (v, B, b), (u, C)\rangle$ is a positional thread in $T^*(\mathtt{m}_i^o)$.

In fact, Axiom N1 has also the other part that for any action a determined by v, any position with the last v can be extended with a. We formulate this as a separate condition.

(*separability*): for any $\mu, \mu' \in T^*(\mathtt{m}_i^o)$, $\varepsilon_1(\mu) = \varepsilon_1(\mu')$ implies $\varepsilon_2(\mu) = \varepsilon_2(\mu')$;

Under this condition, we have the following lemma.

Lemma 4.2. Suppose Π^i is defined by $\mathrm{P1}_s$ and $T^*(\mathtt{m}_i^o)$ satisfies separability.

(a): The function $f : T^*(\mathtt{m}_i^o) \to \prec^i$ defined by $f(\mu) = \mu_{-A}$ is a bijection.

(b): For any $\mu \in T^*(\mathtt{m}_i^o)$, μ is positional in $T^*(\mathtt{m}_i^o)$ if and only if μ_{-A} is a position in

Π^i.

Proof (a): By P1$_s$, f is an onto mapping. We show that it is also one-one function. Let $\mu = \langle (w_1, B_1, b_1), ..., (w_m, B_m, b_m),\ (w_{m+1}, B_{m+1})\rangle, \eta = \langle (w_1, C_1, b_1), ..., (w_m, C_m, b_m), (w_m, C_{m+1})\rangle \in T^*(\mathtt{m}_i^o)$ and $f(\mu) = f(\eta)$. Since, $\mu, \eta \in T^*(\mathtt{m}_i)$, all their subthreads of μ, η are in $T^*(\mathtt{m}_i)$. By Separability, we have $A_k = B_k$ for $i = 1, ..., m+1$. Hence, $\mu = \eta$.

(b) By separability and (a) of this lemma, for any $\mu, \mu' \in T^*(\mathtt{m}_i^o)$, μ is a subthread of μ' if and only if μ_{-A} is a subsequence of μ'_{-A}. Hence, μ is positional in $T^*(\mathtt{m}_i^o)$ if and only if μ_{-A} is a position in Π^i. ■

For Axiom N2, we consider the following condition:

(*determination*): for any positional threads $\mu = \langle \eta, (v, B)\rangle$ and $\mu' = \langle \eta', (v', B')\rangle$, if $\eta = \eta'$, then $(v, B) = (v', B')$.

This states that whenever two positional memory threads have the identical histories, the present perceived pieces and available actions are also identical.

Now, we can state the following proposition.

Proposition 4.3. Let Π^i be defined by P1$_s$.

(1): Π^i satisfies Axiom N1 under *separability* and *positional action-closedness.*

(2): Π^i satisfies Axiom N2 under *separability and determination*.

Proof.(1): Let $\langle \xi, v\rangle$ be a position in Π^i and $[(v, a)] \prec^i u$ for some $u \in W^i$. Then by Lemma 4.2.(a), there is a unique $\mu = \langle \eta, (v, B)\rangle \in T^*(\mathtt{m}_i^o)$ with $\mu_{-A} = \langle \xi, v\rangle$, and by Lemma 4.2.(b), it is a positional thread. Furthermore, for the unique $\mu' \in T^*(\mathtt{m}_i^o)$ with $\mu'_{-A} = \langle v\rangle$, we have $\varepsilon_1(\mu) = \varepsilon_1(\mu') = v$. Hence, we have $\varepsilon_2(\mu) = \varepsilon_2(\mu') = B$ by *separability.* Thus $a \in B$ by (2.4). By *positional action-closedness*, we have a positional thread $\langle \eta, (v, B, a), (u, C)\rangle \in T^*(\mathtt{m}_i^o)$. This means that $\langle \eta, (v, B, a), (u, C)\rangle$ is an initial fragment of a maximal thread $\eta' \in T^*(\mathtt{m}_i^o)$. Then, $\langle \xi, (v, a), u\rangle = \langle \eta, (v, B, a), (u, C)\rangle_{-A}$ is an initial fragment of η_{-A}. Thus, by Lemma 4.2.(b), $\langle \xi, (v, a), u\rangle$ is a position in Π^i.

(2): Let $\langle \xi, v\rangle$ and $\langle \eta, u\rangle$ be two positions in Π^i with $\xi = \eta$. Then, there are two threads $\mu = \langle \eta, (v, B)\rangle$ and $\mu' = \langle \eta', (v', B')\rangle$. Since $\xi = \eta$, we have $\eta = \eta'$ by *separability*. By Lemma 4.2.(b), μ and μ' are positional threads. Therefore, by *determination*, $(v, B) = (v, B')$, *a fortiori*, $v = v'$. ■

We note that the above derivations of axioms for the personal protocol did not use any of the axioms from the original protocol. We could use a combination of restrictions on memory and the axioms satisfied by the original protocol to derive axioms for the personal protocol. We given one such example. The next proposition gives a sufficient condition on the player's memory function, that together with the satisfaction of N2 on the original protocol guarantees that the personal protocol satisfies Axiom N2.

We say that the player's memory function $\mathtt{m}_i^o$ is *weakly positional* iff for all $\langle \xi, w \rangle \in \Xi^o$, $\mu \in \mathtt{m}_i^o\langle \xi, w \rangle$ implies $\mu_{-A} = \langle \eta, v \rangle$ for some $\langle \eta, v \rangle \in \Xi^o$.

Proposition 4.4. Let Π^i be defined by P1$_s$. If $\mathtt{m}_i^o$ is weakly positional, then Π^i satisfies Axiom N2.

Proof. Since $\mathtt{m}_i^o$ is positional, $\Xi(\Pi^i) \subseteq \Xi^o = \Xi(\Pi^o)$. Hence, Axiom N2 on Π^o implies Axiom N2 for Π^i. ■

4.2. Full Recoverability: The Classical Memory Function

Since Definition 3.1 of an inductive derivation is strict, it may appear to exclude the possibility that the player reaches the full understanding of the objective situation. However, this is not the case for the classical memory function. As was discussed in Part I, the classical memory function is, more or less, equivalent to the assumption that the player knows the full structure of the objective situation. In this subsection, we show this fact from the experiential point of view.

Before going to the classical memory function, we give a simple observation on full recoverability for any memory function $\mathtt{m}_i^o$.

Lemma 4.5 (Criterion for Full Recoverability (1)). The personal information protocol $\Pi^i = (W^i, A^i, \prec^i)$ in the strict sense coincides with the objective protocol $\Pi^o = (W^o, A^o, \prec^o)$ if and only if $T^*(\mathtt{m}_i^o)_{-A} = \prec^o$.

Proof. (Only-If): By P1$_s$, $\prec^o = \prec^i = T^*(\mathtt{m}_i^o)_{-A}$.

(If): Since $T^*(\mathtt{m}_i^o)_{-A} = \prec^o$, we have $W^{o*} = W^o$ and $A^{o*} \supseteq A^o$ using Axioms B1 and B2 for Π^o. Since $A^{o*} \subseteq A^o$ by (2.5), we have $A^{o*} = A^o$. Since we have $\Pi^i = (W^i, A^i, \prec^i) = (W^{o*}, A^{o*}, T^*(\mathtt{m}_i^o)_{-A}) = (W^o, A^o, T^*(\mathtt{m}_i^o)_{-A})$ by P1$_s$, Π^i coincides with $\Pi^o = (W^o, A^o, \prec^o)$. ■

The next proposition shows that the classical memory function leads to full recoverability.

Proposition 4.6 (Classical Full Recoverability). Let $\mathtt{m}_i^o = \mathtt{m}_i^C$. Then the personal information protocol $\Pi^i = (W^i, A^i, \prec^i)$ in the strict sense coincides with the objective information protocol $\Pi^o = (W^o, A^o, \prec^o)$.

Proof. Since Ξ_i^o contains all maximal feasible sequences in Π^o and since $T(\mathtt{m}_i^C) = \{\langle \xi, w \rangle^* : \langle \xi, w \rangle \in \Xi_i^o\}$, the set $T^*(\mathtt{m}_i^C)_{-A}$ coincides with the set $\prec^o$ of all feasible sequences in the objective protocol Π^o. By Lemma 4.5, $\Pi^i = (W^i, A^i, \prec^i)$ coincides with $\Pi^o = (W^o, A^o, \prec^o)$. ■

There are still other candidates for full recoverability. For example, the perfect information memory function $\mathtt{m}_i^o = \mathtt{m}_i^{PI}$ guarantees full recoverability. Nevertheless, in most cases we would not expect full recoverability. For example, the memory function $\mathtt{m}_1^o$ of (3.6) in the Analogy of the Cave violates the condition of Lemma 4.5, and thus, the objective protocol is eliminated from the candidates for his view. The memory function there involves false memories, and it is inevitable to have false feasible sequences in the personal protocol.

Finally, consider how much the personal player assignment π^i and memory function $\mathtt{m}_i^i$ are determined by P2$_s$ and P3$_s$ in the context of Proposition 4.6. In contrast to Π^i, neither π^i nor $\mathtt{m}_i^i$ is uniquely determined.

The subjective player assignment π^i coincides with π^o at every decision piece for player i, but might be different from $\pi^o(w)$ for other pieces. One possible additional assumption is:

$$\pi^i(w) \subseteq \pi^o(w) \text{ for all } w \in W^{o*}. \tag{4.1}$$

This implies that $\pi^i(w)$ coincides with $\pi^o(w)$ over decision pieces even for other players. In this paper, we focus rather on the inductive derivation of an information protocol and so, we keep the subjective player assignment still as somewhat arbitrary.

In the context of Proposition 4.6, the memory function $\mathtt{m}_i^i$ defined by P3$_s$ is not determined uniquely either. The choice of a particular subjective memory function corresponds to an assumption on how much he uses the derived information protocol. This was already pointed out in the absentminded driver game in Section 3.3. We leave the precise construction of the subjective memory function as well as the subjective player assignment to future research.

5. Inductive Derivation of an Info-memory Protocol in a General Sense

Definition 3.1 was purely based on individual experiences. This inductive derivation does not necessarily provide a basic protocol in the sense that Axioms B2 and B4 may be violated. In this section, we allow a player to have an extended causal relation $\prec^i$ as well as to add new information pieces, actions and maybe, some other players. Then, we show that a (basic) personal info-memory protocol always exists.

5.1. Personal Information Protocol in a General Sense

As in Section 3, let $(\Pi^o, \pi^o, \mathtt{m}^o) = ((W^o, A^o, \prec^o), \pi^o, (\mathtt{m}_1^o, ..., \mathtt{m}_n^o))$ be the objective info-memory protocol satisfying Axioms B1-B3 and N1-N2. Recall also that the domain of $\mathtt{m}_i^o$ is $\Xi_i^o = \Xi_i(\Pi^o)$. Let $T(\mathtt{m}_i^o) = \bigcup_{\langle \xi, w\rangle \in \Xi_i^o} \mathtt{m}_i^o\langle \xi, w\rangle$, and $T^*(\mathtt{m}_i^o)$ the set of all subthreads of memory threads in $T(\mathtt{m}_i^o)$. Then, W^{o*} is the set of information pieces occurring in $T^*(\mathtt{m}_i^o)$, and A^{o*} is the set of perceived available actions occurring in $T^*(\mathtt{m}_i^o)$.

We adopt the following principles for the inductive reconstruction of a personal info-memory protocol:

(i): it is based on the accumulated set $T^*(\mathtt{m}_i^o)$ of memories;

(ii): the constructed protocol is basic, i.e., it satisfies Axioms B1-B4.

In Section 3, we required (i) in the strict form, but here we only require a personal protocol include the accumulated memory set $T^*(\mathtt{m}_i^o)$. Instead, we require Axioms B1-B4.

Definition 5.1. We say that $(\Pi^i, \pi^i, \mathtt{m}_i^i)$ is a *(basic) personal info-memory protocol* for player i in the *general sense* iff

($\mathrm{P1}_g$): $\Pi^i = (W^i, A^i, \prec^i)$ is a basic protocol with $A^i \supseteq A^{o*}$ and $\prec^i \supseteq T^*(\mathtt{m}_i^o)_{-A}$;

($\mathrm{P2}_g$): $\pi^i : W^i \to 2^{\{0,1,\ldots,n\}} \backslash \{\emptyset\}$ satisfies the condition the restriction of π^i to W^{o*} is a personally correct subjective player assignment;

($\mathrm{P3}_g$): $\mathtt{m}_i^i$ is a memory function over its domain $\Xi_i(\Pi^i)$ with the conditions:
(a) $T^*(\mathtt{m}_i^i)_{-A} \supseteq T^*(\mathtt{m}_i^o)_{-A}$, and (b) $\langle \xi, w\rangle \in \mathtt{m}_i^i \langle \xi, w\rangle_{-A}$ for all $\langle \xi, w\rangle \in \Xi_i(\Pi^i)$.

In the following, we may call a personal info-memory protocol $(\Pi^i, \pi^i, \mathtt{m}_i^i)$ in the general sense, simply, a *personal protocol*.

First, $\mathrm{P1}_g$ requires Axioms B1-B4. This fact and $\prec^i \supseteq T^*(\mathtt{m}_i^o)_{-A}$ imply $W^i \supseteq W^{o*}$. Thus, the difference between $\mathrm{P1}_g$-$\mathrm{P3}_g$ and $\mathrm{P1}_s$-$\mathrm{P3}_s$ is the replacement of the equalities of $\mathrm{P1}_s$ by the set-inclusion relations. This replacement requires some other corresponding modifications in $\mathrm{P2}_g$ and $\mathrm{P3}_g$.

When $W^i \supsetneq W^{o*}$ or $A^i \supsetneq A^{o*}$, W^i or A^i have already some new elements. In this case, $\prec^i$ is also a strictly larger than $T^*(\mathtt{m}_i^o)_{-A}$. This possible addition of new symbols enables us to find a larger protocol with more axioms. Indeed, we will add new symbols to in the proofs of Theorems 5.2 and 6.1.

The following lemma guarantees that once we have a basic personal protocol satisfying $\mathrm{P1}_g$,we can find a subjective player assignment satisfying $\mathrm{P2}_g$ and a subjective memory function satisfying $\mathrm{P3}_g$. It uses the objective player assignment π^o for experienced pieces, and the full player set to end pieces. The perfect information memory function or the classical memory function are used.

Lemma 5.2. Let $\Pi^i = (W^i, A^i, \prec^i)$ be a basic protocol satisfying $\mathrm{P1}_g$.

(a): Let $\pi^i : W^i \to 2^{\{0,1,\ldots,n\}} \backslash \{\emptyset\}$ be a player assignment satisfying: (i) $\pi^i(w) = \pi^o(w)$ for $w \in W^{o*}$ and (ii) $i \in \pi^i(w)$ for all $w \in W^{iE} - W^{o*}$. Such a π^i exists and it satisfies $\mathrm{P2}_g$.

(b): Let $\mathtt{m}_i^i$ be the perfect information function $\mathtt{m}_i^{PI}$ or the classical memory function $\mathtt{m}_i^C$ for (Π^i, π^i) where π^i satisfies (a). Then $\mathtt{m}_i^i$ satisfies $\mathrm{P3}_g$.

Proof (a): By Lemma 3.1, the restriction of π^i to W^{o*} is a personally correct subjective player assignment since $\pi^i(w) = \pi^o(w)$ on W^{o*}. Hence, $\mathrm{P2}_g$ is satisfied. There is no problem with existence; one possibility is to assign $\pi^i(w) = \{0\}$ for all $w \in W^{iD} - W^{0*}$ and $\pi^i(w) = \{1, ..., n\}$ for all $w \in W^{iE}$, i.e., all unexperienced decision pieces are assigned to player 0 and all unexperienced end-pieces are assigned to the full player set.

(b): Let $\mathtt{m}_i^i = \mathtt{m}_i^{PI}$, i.e., consider the perfect information memory function. The proof for the classical memory function is essentially the same. We will show that $T^*(\mathtt{m}_i^i)_{-A} \supseteq \prec^i$. Since π^i satisfies (a), $i \in \pi^i(w)$ for all $w \in W^{iE}$. Hence, $\Xi_i(\Pi^i) \supseteq \Xi^E(\Pi^i)$, i.e., player i''s positions in Π^i include all the endpositions. Since Π^i is basic, the endpositions are the maximal feasible sequences which include all feasible sequences of Π^i as subsequences. Since $\mathtt{m}_i^i = \mathtt{m}_i^{PI}$, $T^*(\mathtt{m}_i^i)_{-A} \supseteq \prec^i$. By P1$_g$ for Π^i, $\prec^i \supseteq T^*(\mathtt{m}_i^o)_{-A}$. Hence, $T^*(\mathtt{m}_i^i)_{-A} \supseteq T^*(\mathtt{m}_i^o)_{-A}$. ■

Next, we show how to construct a protocol satisfying P1$_g$.

Theorem 5.3 (Existence of a (basic) Personal Protocol). There is a (basic) personal info-memory protocol $(\Pi^i, \pi^i, \mathtt{m}_i^i)$ in the general sense (in fact, satisfying Axiom N1 (Extension)).

In the following proof, we use new information pieces as new endpieces. We can prove this theorem without using new symbols under the assumption that there is an piece $w \in W^{o*}$ and $\varepsilon_1(\mu) = w$ for some $\mu \in T^*(\mathtt{m}_i^o)$ such that $\varepsilon_2(\mu) = \emptyset$ for any $\mu \in T^*(\mathtt{m}_i^o)$ with $\varepsilon_1(\mu) = w$. This single w can be used in the same as new information pieces in the following proof. The information protocol and memory function described by Figure 3.7 is an example when we would need new pieces.

Proof of Theorem 5.3. First, we consider a construction of $\Pi^i = (W^i, A^i, \prec^i)$. If $W^i = W^{o*}$, $A^i = A^{o*}$ and $\prec^i = T^*(\mathtt{m}_i^o)_{-A}$ satisfy Axioms B1-B4, there is nothing to be proved. Suppose that this is not the case. For each $w \in W^{o*}$, let $B(w)$ be the set of all available actions that occur in B of knot (w, B, b) or (w, B) in some memory threads μ in $T^*(\mathtt{m}_i^o)$.

Since $T^*(\mathtt{m}_i^o)_{-A}$ is a finite set, each sequence is included in a maximal sequence. In the same way as before, we call an initial fragment of a maximal sequence a position. We denote the set of positions in $T^*(\mathtt{m}_i^o)_{-A}$ by $\Xi(T^*(\mathtt{m}_i^o)_{-A})$. Axiom N1 may be violated in that for a position $\langle \xi, w \rangle \in \Xi(T^*(\mathtt{m}_i^o)_{-A})$ and for an action $a \in B(w)$, there is no position $\langle \xi, (w, a), v \rangle \in \Xi(T^*(\mathtt{m}_i^o)_{-A})$ and $v \in W^{o*}$. This difficulty would be solved if we add a new piece $\mathsf{v}_{[\xi,(w,a)]}$ so that $\langle \xi, (w, a), \mathsf{v}_{[\xi,(w,a)]} \rangle$ is an endposition.

Let $\langle \xi, w \rangle$ be an arbitrary position in $\Xi(T^*(\mathtt{m}_i^o)_{-A})$. Take an arbitrary action $a \in B(w)$ so that $\langle \xi, (w, a), v \rangle \notin \Xi(T^*(\mathtt{m}_i^o)_{-A})$ for any $v \in W^{o*}$. Let v be a new symbol. Then, we call

$$\langle \xi, (w, a), \mathsf{v}_{[\xi,(w,a)]} \rangle \tag{5.1}$$

a new *endposition,* and also $\mathsf{v}_{[\xi,(w,a)]}$ a *new endpiece*. Note that the addition is done over all positions and actions. Then, let W^i be the set consisting the pieces in W^{o*}

and the new endpieces. Let A^i be the same as A^{o*}. Let Σ be the set consisting of the positions in $\Xi(T^*(\mathtt{m}_i^o)_{-A})$ and the newly introduced endpositions. Finally, Σ^* is the set of all subsequences of Σ, and let $\prec^i = \Sigma^*$. Clearly, $(W^i, A^i, \prec^i)$ satisfies Axiom N1 by construction.

We now prove that it satisfies Axioms B1-B3. Axiom B1 (All Pieces Used) and Axiom B3 (Contraction) follow from the fact that we have taken the set Σ^* of all subsequences of Σ. Axiom B2 holds since every action a is used in (5.1) or else $\langle \xi, (w, a), v \rangle \in \Xi(T^*(\mathtt{m}_i^o)_{-A})$ for some $v \in W^{o*}$. ■

There are an infinite (denumerable[3]) number of personal info-memory protocols for each objective $(\Pi^o, \pi^o, \mathtt{m}^o)$, since a protocol of any finite large size can be constructed by adding new information pieces and actions. Nevertheless, we are not interested in such a large protocol including irrelevant pieces and actions. A "small" protocol is more relevant. One criterion for a "small" protocol is to choose a minimal one in the sense of the set-inclusion relation: That is, if

$$W^i \subseteq W^{i\prime}, A^i \subseteq A^{i\prime} \text{ and } \prec^i \ \subseteq \ \prec^{i\prime}, \tag{5.2}$$

then $(W^i, A^i, \prec^i)$ is (weakly) *smaller than* $(W^{i\prime}, A^{i\prime}, \prec^{i\prime})$. By this relation, we always have a smallest personal protocol. Nevertheless, there may be multiple smallest ones. This will be seen in the next subsection.

When Π^i of the personal protocol $(\Pi^i, \pi^i, \mathtt{m}_i^i)$ in the strict sense satisfies Axioms B1-B4, $(\Pi^i, \pi^i, \mathtt{m}_i^i)$ is also a personal protocol in the general sense, and Π^i is the smallest information protocol.

Theorem 5.4 (Condition for the Smallest Personal Protocol). Assume that *action-closedness* holds for $T^*(\mathtt{m}_i^o)$. Then the personal protocol in the strict sense is a smallest (basic) personal protocol in the general sense.

Proof. By Lemma 3.5, $\Pi^i = (W^i, A^i, \prec^i)$ defined by $P1_s$ satisfies Axioms B1-B4. By definition, this Π^i is the smallest personal information protocol. ■

The proof of Theorem 5.3 gives a procedure to construct a basic personal protocol by adding new endpieces. However, the addition of new symbols may not be needed as was mentioned in the paragraph before the proof of the theorem if there is an appropriate endpiece in W^{o*}. We see this possibility as well as the multiplicity of minimal personal protocols in the case of the Markov memory function.

Consider the 2-person information protocol $\Pi^o = (W^o, A^o, \prec^o)$ of Example 3.2. Suppose that the objective memory function $\mathtt{m}_1^o$ of player 1 is the Markov $\mathtt{m}_1^M$. In this case, $T^*(\mathtt{m}_1^M) = \{\langle(\alpha_1, \{a, b\})\rangle, \langle(\mathtt{0}, \emptyset)\rangle, \langle(\mathtt{2}, \emptyset)\rangle, \langle(\mathtt{1}, \emptyset)\rangle, \langle(\mathtt{-1}, \emptyset)\rangle\}$. Hence, $W^{o*} = \{\alpha_1, \mathtt{0}, \mathtt{2}, \mathtt{1}, \mathtt{-1}\}$, $A^{o*} = \{a, b\}$ and $T^*(\mathtt{m}_1^M)_{-A} = \{\langle\alpha_1\rangle, \langle\mathtt{0}\rangle, \langle\mathtt{2}\rangle, \langle\mathtt{1}\rangle, \langle\mathtt{-1}\rangle\}$. We can find

[3] under the assumption that there are a denumerable number of new symbols.

a personal information protocol in the general sense without adding new symbols. Indeed, the protocol given in Figure 3.2 is one. Moreover, it is a minimal one.

In the same example, the proof of Theorem 5.3 gives $W^{1\prime} = \{\alpha_1, \mathsf{0}, \mathsf{2}, \mathsf{1}, -\mathsf{1}\} \cup \{\mathsf{v}_{[(\alpha_1,a)]}, \mathsf{v}_{[(\alpha_1,b)]}\}$, $A^{1\prime} = \{a, b\}$ and $\prec^{1\prime}$ is described as Figure 5.1 with $\mathsf{0}, \mathsf{2}, \mathsf{1}, -\mathsf{1}$ as isolated points. This is a basic protocol. While $W^{1\prime}$ is strictly larger than the one in Figure 3.2 , $\prec^{1\prime}$ is a set-theoretically minimal. Hence, $(W^{1\prime}, A^{1\prime}, \prec^{1\prime})$ is also a minimal one.

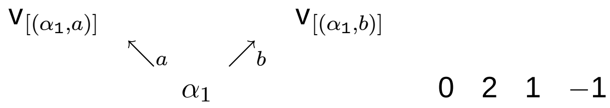

Figure 5.1

In the case of the Markov memory function, we can always construct a personal information protocol in the general sense without adding new information pieces.

5.2. Condition for the Exclusion of the Objective Info-Memory Protocol

In the case of personal info-memory protocols in the strict sense, Lemma 4.5 gave a condition for full recoverability. It was quite stringent condition, though the classical case allows full recoverability. For personal protocols in the general sense, the situation is different, since the requirements of $P1_g$,$P2_g$ and $P3_g$ allow a lot of candidates. Therefore, we may allow even the objective protocol to be a personal one. A condition for it is given by the next lemma.

Lemma 5.5 (Criterion for Full Recoverability (2)). The objective protocol $\Pi^o = (W^o, A^o, \prec^o)$ is a personal protocol in the sense of $P1_g$ if and only if $A^{o*} \subseteq A^o$ and $T^*(\mathtt{m}_i^o)_{-A} \subseteq \prec^o$.

Proof. Condition $P1_g$ for $\Pi^o = (W^o, A^o, \prec^o)$ is that $A^{o*} \subseteq A^o$ and $T^*(\mathtt{m}_i^o)_{-A} \subseteq \prec^o$. ■

The significance of this lemma is its negative form rather than the positive form. The positive form states only that when $A^{o*} \subseteq A^o$ and $T^*(\mathtt{m}_i^o)_{-A} \subseteq \prec^o$, the possibility to have the objective protocol as a personal one is just logically possible. On the other hand, when A^{o*} * A^o or $T^*(\mathtt{m}_i^o)_{-A}$ * $\prec^o$, this possibility is excluded.

The condition $T^*(\mathtt{m}_i^o)_{-A} \subseteq \prec^o$ means that partial observations and pure forgetfulness are allowed but no falsities should be involved. Some forgetfulness involves false elements, which will be seen in the next subsection.

6. Construction of a Full Protocol

In this section, an individual player takes a position of an analyst, and consider an extension of a personal info-memory protocol so that it satisfies all the axioms B1-B3 and N1-N2. We give a theorem providing a procedure to have such an extension.

6.1. The Personal Full Protocol Theorem

It was shown in Section 5 that there exists always a (basic) personal info-memory protocol $(\Pi^i, \pi^i, \mathtt{m}_i^i)$ constructed using player i's memory function $\mathtt{m}_i^o$. It may, however, violate the nonbasic axiom N2 (Determination). The personal protocols depicted in Figure 3.2 and Figure 5.1 violate N2. In either case, the description of the situation is insufficient so that it lacks some causality. Once a player notices the indeterminacy, he may extend his personal protocol to have a more precise explanation. In this section, we show that there is a personal info-memory protocol which satisfies all the basic and nonbasic axioms.

The above consideration of constricting a full protocol is not purely a question about the inductive derivation of an individual view, but it is viewed as a methodological question from the viewpoint of an analyst. When a view does not give a deterministic explanation for relationships between causes and effects, the analyst thinks about new reasons. A player may assume the role of the analyst. The following theorem can be regarded as one result on this problem. It will be proved in Section 6.2.

Theorem 6.1 (Personal Full Protocol). Let $(\Pi^o, \pi^o, \mathtt{m}^o)$ be the objective info-memory protocol. There is a personal info-memory protocol $(\Pi^i, \pi^i, \mathtt{m}_i^i)$ in the general sense such that Π^i is a full protocol, i.e., it satisfies Axioms B1-B3 and N1-N2.

Before going to the proof of the theorem, we illustrate our procedure using an example. Consider the personal protocol $\Pi^1 = (W^1, A^1, \prec^1)$ described by Figure 3.2. In this example, the endpieces $\mathsf{0}$ and $\mathsf{2}$ have the same histories, in which sense Axiom N2 is violated. This protocol is extended by adding new information pieces so that the extended one satisfies all the axioms B1-B3 and N1-N2. The procedure of adding new pieces is described in Figure 6.1.

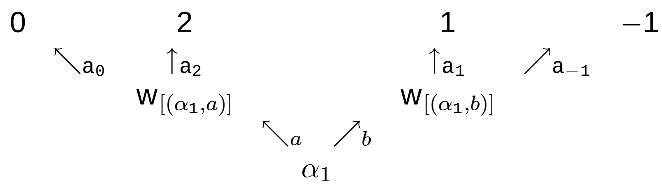

Figure 6.1

Here, note that a and w are new (object) symbols. For example, $\mathsf{w}_{[(\alpha_1,a)]}$ is the symbol w indexed with $[(\alpha_1, a)]$.

The procedure to be given does not talk about who move at newly introduced pieces. Perhaps, nature "player 0" is assigned to those pieces. This possibility was mentioned in the proof of Lemma 5.2 where we showed we could always find a subjective player assignment and use the classical or perfect information memory function to complete the info-memory protocol.

6.2. Proof of Theorem 6.1

Theorem 5.3 states that for the objective protocol $(\Pi^o, \pi^o, \mathtt{m}^o)$, there is a personal protocol $(\Pi^i, \pi^i, \mathtt{m}^i_i)$ in the general sense such that $(\Pi^i, \pi^i, \mathtt{m}^i_i)$ satisfies Axioms B1-B3 and N1. We extend $(\Pi^i, \pi^i, \mathtt{m}^i_i)$ to have a full protocol. In the following, we denote the information protocol Π^i by $\Pi = (W, A, \prec)$ for the economy of symbols.

Suppose that $\Pi = (W, A, \prec)$ is a (finite) information protocol satisfying Axioms B1-B3 and N1. Let Σ be the set of maximal feasible sequences. Each sequence in Σ has a finite length and the cardinality of Σ is finite, too. Each sequence $\langle (w_1, a_1), ..., (w_m, a_m), w_{m+1} \rangle$ in Σ has the natural order from the initial piece w_1 to the ending piece w_{m+1}.

Since each position in Ξ is an initial fragment of some sequence in Σ, we can regard Σ as representing Ξ. We would like to construct a tree based on Ξ, but Axiom N2 may not be satisfied. That is, Ξ does not satisfy the determination property (Axiom N2) as illustrated in Section 6.1. Therefore, we add new information pieces and actions to overcome the indeterminacy. Specifically, we construct a new information protocol $(\mathsf{W}, \mathsf{A}, <)$ by induction on the structure of Σ.

For the inductive construction of $(\mathsf{W}, \mathsf{A}, <)$, we need to use the following facts on Ξ:

(Σ1): For any $w \in W$, there is a position $\langle (w_1, a_1), ..., (w_m, a_m), w_{m+1} \rangle$ in Ξ with $w_{m+1} = w$.

(Σ2): Let $\langle \xi, w \rangle$ be a position and $a \in A_w$. Then, there is a $u \in W$ such that $\langle \xi, (w, a), u \rangle$ is a position.

Property Σ1 is proved in Section 2.2 of Part I, and Σ2 is Axiom N1 itself. By Σ2, we can define the set of immediate successors of a position $\langle \xi, w \rangle$ and $a \in A_w$:

$$I[\xi, (w, a)] = \{u \in W : \langle \xi, (w, a), u \rangle \in \Xi\}. \tag{6.1}$$

This set is nonempty. If Axiom N2 holds, this set is singleton.

One more important remark is that any position in Ξ should be captured by our inductive argument. For this purpose, we mention the following property due to Axiom B3:

(Σ3): Let $\langle (w_1, a_1), ..., (w_m, a_m), w_{m+1} \rangle$ be a sequence in Σ. Then $a_t \in A_{w_t}$ for $t = 1, ..., m$.

Now, we define the ν-sequences by induction. The set of ν-sequences will be W, and the set of available actions A will be defined in the inductive steps. In the following, we use the two pure symbols w and a, which will be used to have new information pieces by associating some subscripts with them.

First, we define *ν-sequences* by induction on the length of each maximal feasible sequence in Σ from its initial pair:

(I)(0): $\langle \mathsf{w}_\emptyset \rangle$ is a ν-sequence.

(1): If $\langle w \rangle$ is a position in Ξ, then $\langle (\mathsf{w}_\emptyset, \mathsf{a}_w), w \rangle$ is a ν-sequence. Here $\langle w \rangle$ is called the *associated position with* the ν-sequence $\langle (\mathsf{w}_\emptyset, \mathsf{a}_w), w \rangle$.

(II): Suppose that a ν-sequence $\langle \xi', w \rangle$ $(w \in W)$ and its associated position $\langle \xi, w \rangle \in \Xi$ are already defined. If $A_w = \emptyset$, this inductive definition terminates. Suppose $A_w \neq \emptyset$. Then, for any $a \in A_w$,

(1): $\langle \xi', (w, a), \mathsf{w}_{[\xi,(w,a)]} \rangle$ is a ν-sequence;

(2): $\langle \xi', (w, a), (\mathsf{w}_{[\xi,(w,a)]}, \mathsf{a}_u), u \rangle$ is a ν-sequence for any piece $u \in I[\xi, (w, a)]$. Here $\langle \xi, (w, a), u \rangle$ is called the *associated position with* $\langle \xi', (w, a), (\mathsf{w}_{[\xi,(w,a)]}, \mathsf{a}_u), u \rangle$.

Since Ξ is a finite set, the above inductive definition terminates. We define W to be the set of all ν-sequences. In step I, we add one piece $\mathsf{w}_\emptyset$, which is the root in W. If Ξ has the root $\langle w^0 \rangle$, $\mathsf{w}_\emptyset$ is still attached below $\langle w^0 \rangle$, in which case this addition is unnecessary. In step II, the essential case is that $I[\xi, (w, a)]$ consists of multiple pieces - indeterministic case. Then we add one new piece $\mathsf{w}_{[\xi,(w,a)]}$ to which the indeterminacy is attributed. The attached player at $\mathsf{w}_{[\xi,(w,a)]}$ has the action set $\{\mathsf{a}_u : u \in I[\xi, (w, a)]\}$.

It follows from $\Sigma 2$ and $\Sigma 3$ that every position in Ξ appears once in the above inductive definition.

Now, we define A by

$$\mathsf{A} = A \cup \{\mathsf{a}_u : u \in W\}. \tag{6.2}$$

Any $a \in \mathsf{A}$ appears in some ν-sequence in W since Π is basic. We also define W^{new} by

$$\mathsf{W}^{new} = \{\mathsf{w}_{[\xi,(w,a)]} : \langle \xi, w \rangle \in \Xi \text{ and } a \in A_w\}. \tag{6.3}$$

These are the sets of newly introduced symbols.

Now, we define the relation $<$ as a subset of $\bigcup_{m=0}^{\infty} (\mathsf{W} \times \mathsf{A})^m \times \mathsf{W}$. First, we notice that the natural binary relation $<_0$ in W can be defined, since each $\omega \in \mathsf{W}$ is a ν-sequence and W is constructed from $\langle \mathsf{w}_\emptyset \rangle$ by extending each ν-sequence. Specifically, the relation $<_0$ in W is defined as follows: for any $\omega, \omega' \in \mathsf{W}$,

$(i) : [\omega, \alpha] <_0 \omega' \Longleftrightarrow [\omega, \alpha]$ is an initial fragment of ω';

$(ii) : \omega <_0 \omega' \Longleftrightarrow [\omega, \alpha] <_0 \omega'$ for some α.

Then, we have the following lemma.

Lemma 6.1. The pair $(\mathsf{W}, <_0)$ is a tree with the root $\langle \mathsf{w}_\emptyset \rangle$.

Proof. By the above definitions (i) and (ii), the relation $<_0$ is a partial ordering over W. By definition and $\langle \mathsf{w}_\emptyset \rangle$ is the smallest element in W. It remains to show that for any $\omega \in \mathsf{W}$, the set $\{\omega' : \omega' <_0 \omega \text{ or } \omega' = \omega\}$ is a totally ordered set with $<_0$. Take two elements ω', ω'' from this set. Then both are initial fragments of ω. If ω' is longer than ω'', then ω'' is an initial fragment of ω', and if ω' has the same length as ω'', ω' is longer than ω' is identical to ω''. ■

Now, we extend $<_0$ to a relation in $\bigcup_{m=0}^{\infty}(\mathsf{W} \times \mathsf{A})^m \times \mathsf{W}$ as follows:

$$\begin{aligned} [(\omega_1, \alpha_1), ..., (\omega_m, \alpha_m)] \quad &< \quad \omega_{m+1} \\ &\iff \quad (\omega_t, \alpha_t) <_0 \omega_{t+1} \text{ for } t = 1, ..., m. \end{aligned} \tag{6.4}$$

Here, we complete the definition of the extended information protocol $\Pi^e := (\mathsf{W}, \mathsf{A}, <)$. This is a finite protocol.

In the protocol $\Pi^e = (\mathsf{W}, \mathsf{A}, <)$, we denote the set of all positions in $\Pi^e = (\mathsf{W}, \mathsf{A}, <)$ by $\Xi(\Pi^e)$. A position in $(\mathsf{W}, \mathsf{A}, <)$ can be described as

$$\langle (\omega_1, \alpha_1), ..., (\omega_k, \alpha_k), \omega_{k+1} \rangle$$

where $\omega_1, ..., \omega_k, \omega_{k+1} \in \mathsf{W}$ and $\alpha_1, ..., \alpha_k \in \mathsf{A}$. In fact, each position $\langle (\omega_1, \alpha_1), ...,$ $(\omega_k, \alpha_k), \omega_{k+1} \rangle$ can be represented as ω_{k+1} without loosing any information. Indeed, ω_{k+1} is a ν-sequence represented as $\langle (v_1, \alpha_1), ..., (v_k, \alpha_k), v_{k+1} \rangle$, and then it holds that for $t = 1, ..., k, k+1$,

$$\omega_t = \langle (v_1, \alpha_1), ..., (v_{t-1}, \alpha_{t-1}), v_t \rangle \text{ for } t = 1, ..., k+1. \tag{6.5}$$

Thus, $(\Xi(\Pi^e), <)$ and $(\mathsf{W}, <_0)$ are isomorphic with the mapping $\varphi : \Xi(\Pi^e) \to \mathsf{W}$ defined by

$$\varphi \langle (\omega_1, \alpha_1), ..., (\omega_k, \alpha_k), \omega_{k+1} \rangle = \omega_{k+1} \tag{6.6}$$

for any $\langle (\omega_1, \alpha_1), ..., (\omega_k, \alpha_k), \omega_{k+1} \rangle \in \Xi(\Pi^e)$.

In the following, we use W as the set positions rather than $\Xi(\Pi^e)$ for simplicity.

Lemma 6.3. The information protocol $\Pi^e = (\mathsf{W}, \mathsf{A}, <)$ satisfies Axioms B1-B3 and N1-N2.

Proof. Axioms B1 and B2 follow the inductive definition of ν-sequences and (6.2). Axiom B3 follows (6.4). Consider Axiom N1. By the above remark, we can represent a position by a ν-sequence in W. Let $\langle \xi', v \rangle = \langle (v_1, \alpha_1), ..., (v_k, \alpha_k), v \rangle$ be a ν-sequence in W and $[(v, \alpha)] <_0 u$. There are two cases to be considered: $v \in W$ or $v \in \mathsf{W}^{new}$.

First, consider the case where $v \in W$. Then, let $\langle \xi, v \rangle$ be the associated position in $\Xi = \Xi(\Pi)$. By II.(1), $\alpha \in A_v$. Also, $\langle \xi', (v, \alpha), \mathsf{w}_{[\xi,(v,\alpha)]} \rangle$ is a ν-sequence. Then, $\langle \xi', v \rangle <_0 \langle \xi', (v, \alpha), \mathsf{w}_{[\xi,(v,\alpha)]} \rangle$ and $\langle \xi', (v, \alpha), \mathsf{w}_{[\xi,(v,\alpha)]} \rangle$ is a position.

Second, consider the case where $v \in \mathsf{W}^{new}$. Then, $\langle \xi', v\rangle$ is written as $\langle \eta', (w, b), v\rangle$, and let $\langle \eta, w\rangle$ be the associated position in Ξ. By II.(2), $\alpha \in \mathsf{A}_v$. Also, $\langle \xi', v\rangle$ is written as $\langle \eta', (w, b), \mathsf{w}_{[\eta,(w,b)]}\rangle$. Also, α is written as a_u for some $u \in I[\eta, (w, b)]$. Then, $\langle \eta', (w, b), (\mathsf{w}_{[\eta,(w,b)]}, \mathsf{a}_u), u\rangle$ is the immediate next ν-sequence of $\langle \eta', (w, b), \mathsf{w}_{[\eta,(w,b)]}\rangle$ by II.(2) and it is a position.

It is easy to see that $\Pi^e = (\mathsf{W}, \mathsf{A}, <)$ satisfies Axiom N2 by looking at the ν-sequences. ■

7. Behavioral Use of a Personal Protocol

A personal info-memory protocol is not only used for an understanding of the objective situation but also for a player's behavior. The latter was already indicated in Sections 3.2 and 3.3. Here, we give a more formal argument. First, a strategy for a player is considered in his personal info-memory protocol. Next, this personal strategy may prescribe behavior in the original objective protocol. However, this does not imply that he can behave optimally in the objective protocol, which will be shown using some simple example. We keep this section as a simple illustration with some suggestions for further developments. A full treatment of the subject will be discussed in a separate paper.

7.1. Behavior induced from a Strategy in a Personal Protocol

Let $(\Pi^i, \pi^i, \mathsf{m}^i_i)$ be a personal info-memory protocol in the general sense derived from the objective protocol $(\Pi^o, \pi^o, \mathsf{m}^o)$. Here, we consider a strategy σ^i_i (behavioral plan) for player i in $(\Pi^i, \pi^i, \mathsf{m}^i_i)$, and show how this σ^i_i might prescribe a strategy σ^o_i in the objective protocol $(\Pi^o, \pi^o, \mathsf{m}^o)$.

Let $\Xi^D_i(\Pi^i)$ denote player i's *decision* positions in his personal information protocol Π^i. A *personal strategy* σ^i_i for the personal protocol $(\Pi^i, \pi^i, \mathsf{m}^i_i)$ is a function on $\Xi^D_i(\Pi^i)$ satisfying the following two conditions: for any $\langle \zeta, v\rangle, \langle \zeta', v'\rangle \in \Xi^D_i(\Pi^i)$,

$$\sigma^i_i\langle \zeta, v\rangle \in \varepsilon_2(\mu) \text{ for all } \mu \in \mathsf{m}^i_i\langle \zeta, v\rangle \tag{7.1}$$

$$\mathsf{m}^i_i\langle \zeta, v\rangle = \mathsf{m}^i_i\langle \zeta', v'\rangle \text{ implies } \sigma^i_i\langle \zeta, v\rangle = \sigma^i_i\langle \zeta', v'\rangle. \tag{7.2}$$

Condition (7.1) means that an action is taken from the set of available actions stated by the memory function m^i_i at position $\langle \zeta, v\rangle$ in Π^i. Condition (7.2) means that his strategy may depend on, up to, his perceived local memory.

A *personal strategy* σ^o_i for $(\Pi^o, \pi^o, \mathsf{m}^o_i)$ is defined in the parallel manner only with the replacements of $(\Pi^i, \pi^i, \mathsf{m}^i_i)$ by $(\Pi^o, \pi^o, \mathsf{m}^o_i)$ in (7.1) and (7.2). Later, we will refer to these corresponding conditions as (7.1) and (7.2) in the objective case.

A personal strategy σ_i^i is a behavioral plan written in $(\Pi^i, \pi^i, \mathtt{m}_i^i)$ but perhaps not in the objective protocol $(\Pi^o, \pi^o, \mathtt{m}^o)$. An immediate question is whether in $(\Pi^o, \pi^o, \mathtt{m}^o)$, player i can behave with his strategy σ_1^1. In this subsection, we consider this question only in the following case. First, we define $\mathtt{m}_i^o\langle\xi, w\rangle \sqsubseteq \mathtt{m}_i^i\langle\zeta, v\rangle$ iff any $\mu \in \mathtt{m}_i^o\langle\xi, w\rangle$ is a last fragment of some $\mu' \in \mathtt{m}_i^i\langle\zeta, v\rangle$. Here, a *last fragment* μ of $\mu' = \langle(v_1, B_1, b_1), ..., (v_m, B_m, b_m), (v_{m+1}, B_{m+1})\rangle$ is expressed as $\langle(v_k, B_k, b_k), ..., (v_m, B_m, b_m), (v_{m+1}, B_{m+1})\rangle$ $(k = 1, ..., m)$. Thus, $\varepsilon(\mu) = \varepsilon(\mu')$. We consider the following condition: for any $\langle\xi, w\rangle \in \Xi_i^{oD}$,

$$\text{there is a unique local memory } \mathtt{m}_i^i\langle\zeta, v\rangle \text{ with } \mathtt{m}_i^o\langle\xi, w\rangle \sqsubseteq \mathtt{m}_i^i\langle\zeta, v\rangle. \tag{7.3}$$

Here, we require $\mathtt{m}_i^i\langle\zeta, v\rangle$ to be uniquely determined for each $\mathtt{m}_i^o\langle\xi, w\rangle$, but $\langle\zeta, v\rangle$ may be multiple. This requires player i to be able effectively to identify the set $\mathtt{m}_i^o\langle\xi, w\rangle$ in his consideration in terms of $(\Pi^i, \pi^i, \mathtt{m}_i^i)$. This condition gives a connection between the subjective $(\Pi^i, \pi^i, \mathtt{m}_i^i)$ and the objective $(\Pi^o, \pi^o, \mathtt{m}^o)$.

Now, we give one more definition: σ_i^o is the *induced behavioral plan* from σ_i^i iff σ_i^o is given as follows: for all $\langle\xi, w\rangle \in \Xi_i^{oD}$,

$$\sigma_i^o\langle\xi, w\rangle = \sigma_i^i\langle\zeta, v\rangle \text{ if } \mathtt{m}_i^o\langle\xi, w\rangle \sqsubseteq \mathtt{m}_i^i\langle\zeta, v\rangle. \tag{7.4}$$

This means that when player i receives his actual local memory $\mathtt{m}_i^o\langle\xi, w\rangle \sqsubseteq \mathtt{m}_i^i\langle\zeta, v\rangle$, he would choose the action $\sigma_i^i\langle\zeta, v\rangle$ prescribed in $(\Pi^i, \pi^i, \mathtt{m}_i^i)$. The well-definedness is the first problem to be considered. In the following proposition, we show the well-definedness under (7.3) and that it is regarded as a strategy in the objective $(\Pi^o, \pi^o, \mathtt{m}^o)$.

Proposition 7.1 (Induced Objective Strategy). Assume (7.3). Let σ_i^i be a personal strategy in $(\Pi^i, \pi^i, \mathtt{m}_i^i)$. The behavioral plan σ_i^o induced from σ_i^i is a personal strategy for the objective personal protocol $(\Pi^o, \pi^o, \mathtt{m}_i^o)$.

Since (2.5) for $\mathtt{m}_i^o$ guarantees $\varepsilon_2(\mu) \subseteq A_w^o$ for all $\mu \in \mathtt{m}_i^o\langle\xi, w\rangle$, we have $\sigma_i^o\langle\xi, w\rangle \in A_w^o$. Thus, he can behave with this induced strategy σ_i^o in $(\Pi^o, \pi^o, \mathtt{m}^o)$.

Proof of Proposition 7.1. First, we look at the well-definedness of σ_i^o. Consider any $\langle\xi, w\rangle \in \Xi_i^{oD}$. Here, $A_w^o \neq \emptyset$. By (7.3), there is a unique $\langle\zeta, v\rangle \in \Xi_i(\Pi^i)$ such that $\mathtt{m}_i^o\langle\xi, w\rangle \sqsubseteq \mathtt{m}_i^i\langle\zeta, v\rangle$. By (2.6), it is sufficient to consider an arbitrary $\mu \in \mathtt{m}_i^o\langle\xi, w\rangle$ and $\mu' \in \mathtt{m}_i^i\langle\zeta, v\rangle$. Then, by (2.7) and $A_w^o \neq \emptyset$, we have $\varepsilon_2(\mu) = \varepsilon_2(\mu') \neq \emptyset$. By (7.1) for $\mathtt{m}_i^i$, we have $\sigma_i^i\langle\zeta, v\rangle \in \varepsilon_2(\mu')$. By (7.2), this $\sigma_i^i\langle\zeta, v\rangle$ is invariant under the same $\mathtt{m}_i^i\langle\zeta, v\rangle$. Hence, $\sigma_i^o\langle\xi, w\rangle = \sigma_i^i\langle\zeta, v\rangle$ is uniquely determined.

By (7.1) and (7.2) for $(\Pi^i, \pi^i, \mathtt{m}_i^i)$, we have the corresponding (7.1) and (7.2) for $(\Pi^o, \pi^o, \mathtt{m}_i^o)$. ■

To the extent that (7.3) is satisfied, every subjective strategy σ_i^i of player i defined on his subjective protocol $(\Pi^i, \pi^i, \mathtt{m}_i^i)$ induces a unique objective strategy σ_i^o in the objective protocol $(\Pi^o, \pi^o, \mathtt{m}^o)$. Whether or not this strategy is subjectively optimal or objectively optimal is another question. In the next subsection, we consider this problem.

7.2. Subjective Optimality vs. Objective Optimality

We have a natural distinction between subjective optimality and objective optimality, since player i makes his behavior plan in his personal protocol but not in the objective protocol. We only point out possible differences between these optimalities in the 1-person case. We will consider optimality in multiple player protocols in a future paper.

Let $(\Pi^o, \pi^o, \mathtt{m}^o)$ be the 1-person objective info-memory protocol, and $(\Pi^1, \pi^1, \mathtt{m}_1^1)$ a personal info-memory protocol in the general sense. We assume that the *objective payoff function* h_1^o of player 1 is given as a real valued-function h_1^o on the set W^{oE} of endpieces in Π^o. Here we require that the *personal payoff function* h_1^1 is also a real-valued function on the set W^{1E} of endpieces in Π^1 satisfying:

(P4$_g$): $h_1^1(w) = h_1^o(w)$ for all endpiece $w \in W^{oE}$.

This requires that the perceived payoff function h_1 coincides with the objective one h_1^o over W^{oE}.

Consider the one-person objective protocol $(\Pi^o, \pi^o, \mathtt{m}_1^o, h_1^o)$ of Figure 7.1. We define $h_1^o(w) = w$ for all $w \in W^{oE}$:

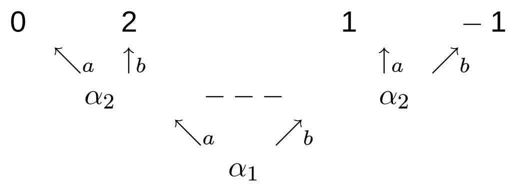

Figure 7.1

This is identical to Figure 3.1, except here it is objectively a one-person protocol. The endpieces express the objective payoff function h_1^o. We assume that the objective memory function $\mathtt{m}_1^o$ is the Markov $\mathtt{m}_1^M$ in (Π^o, π^o).

Let us see the optimal behavior from the objective viewpoint. The objectively optimal strategy for a player depends on his memory function since it is constrained by (7.1) and (7.2) for $(\Pi^o, \pi^o, \mathtt{m}_1^o)$. If the player's objective memory function is the Markov, i.e., $\mathtt{m}_1^o = \mathtt{m}_1^M$, then the *payoff maximizing strategy* $\hat{\sigma}_1^o$ in the objective sense is given as

$$\begin{aligned} \hat{\sigma}_1^o\langle \alpha_1 \rangle &= a \\ \hat{\sigma}_1^o\langle (\alpha_1, a), \alpha_2 \rangle &= \hat{\sigma}_1^o\langle (\alpha_1, b), \alpha_2 \rangle = b. \end{aligned} \tag{7.5}$$

If the player 1 takes the objective personal protocol $(\Pi^o, \pi^o, \mathtt{m}_1^o, h_1^o)$ in his mind, then he might conclude that (7.5) is optimal. As already discussed in Section 7.1, player 1 considers his strategy in his personal protocol $(\Pi^1, \pi^1, \mathtt{m}_1^1, h_1^1)$ which may differ from the objective one. In this case, his optimal behavior may not induce the objectively optimal strategy $\hat{\sigma}_1^o$.

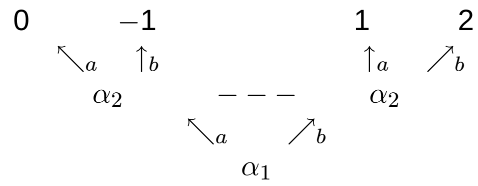

Figure 7.2

Here, suppose that player 1 constructs his personal protocol almost correctly but attaches payoffs incorrectly. That is, $\mathsf{2}$ and $-\mathsf{1}$ are switched. This personal protocol Π^1 is given as Figure 7.2, and π^1 assigns $\{1\}$ everywhere. His personal memory function $\mathtt{m}_1^1$ is assumed to be the classical one $\mathtt{m}_1^C$ in the protocol (Π^1, π^1). Using the assumption that $\mathtt{m}_1^o$ is the Markov memory function $\mathtt{m}_1^M$, we can verify that this information protocol $(\Pi^1, \pi^1, \mathtt{m}_1^1)$ is a personal one in the general sense. Finally, the personal payoff function h_1^1 is the same as h_1^o.

Observe that (7.3) holds between $\mathtt{m}_1^o = \mathtt{m}_1^M$ and $\mathtt{m}_1^1 = \mathtt{m}_1^C$. For example, for $\langle(\alpha_1, a), \alpha_2\rangle \in \Xi_1^{oD}$, $\mathtt{m}_i^i\langle(\alpha_1, b), \alpha_2\rangle = \mathtt{m}_1^C\langle(\alpha_1, a), \alpha_2\rangle = \{\langle(\alpha_1, a), \alpha_2\rangle, \langle(\alpha_1, b), \alpha_2\rangle\}$, and $\mathtt{m}_1^o\langle(\alpha_1, a), \alpha_2\rangle = \mathtt{m}_1^M\langle(\alpha_1, a), \alpha_2\rangle = \langle(\alpha_2, \{a, b\}\rangle$. Hence, $\varepsilon(\mu) = \varepsilon(\mu')$ if$\mu \in \mathtt{m}_1^o\langle(\alpha_1, a), \alpha_2\rangle$ and $\mu' \in \mathtt{m}_i^i\langle(\alpha_1, b), \alpha_2\rangle$.

In $(\Pi^1, \pi^1, \mathtt{m}_1^1, h_1^1)$, the optimal behavior is given as:

$$\begin{aligned} \sigma_1^1\langle\alpha_1\rangle &= b \\ \sigma_1^1\langle(\alpha_1, a), \alpha_2\rangle &= \sigma_1^1\langle(\alpha_1, a), \alpha_2\rangle = b. \end{aligned} \tag{7.6}$$

By Preposition 7.1, this subjective strategy σ_1^1 induces an objective strategy σ_1^o, but it differs from the objectively optimal strategy $\hat{\sigma}_1^o$ given by (7.5).

We can find an abundance of examples where subjective and objective optimalities differ. We leave a more general development of this subject to future research.

8. Conclusions of Part II and Further developments

In Part II of this paper, we have developed the theory of an inductive derivation of an individual view based on experienced memories. We have shown that an inductive derivation is possible theoretically. Nevertheless, a derived personal protocol is typically different from the objective protocol. Also, we briefly suggested its behavioral use, though it was in a very restricted form. A personal strategy in the derived protocol helps the behavior of the player in the sense that under some condition, it induces an objective strategy. However, the objective strategy induced by a subjectively optimal strategy may not be optimal objectively.

As we have been developing our theory, we are restricting our scope by assuming a lot of conditions and also are deviating the standard game theory more and more. Here, we give various remarks on the present and further developments.

(1) the Accumulated Memory Set $T^*(\mathsf{m}_i^o) = \bigcup_{\langle \xi, w \rangle \in \Xi_i^o} \mathsf{m}_i^o \langle \xi, w \rangle$: The adoption of this set as the accumulated experiences for player i means that he has experienced and recalled all possible courses in the game. In this sense we call this the perfect accumulation assumption. This could not be problematic when the objective information protocol $(\Pi^o, \pi^o, \mathsf{m}^o)$ is small, i.e., its length and width are small. However, this assumption is more problematic for a large objective protocol. To avoid this problem, we can replace the entire domain Ξ_i^o by a smaller restricted domain, on which the memory function m_i^o of player i is defined. To have such a restriction, we need to develop a theory of how player i has had experiences. This will be developed in a separate paper.

(2) Restricted Experiences: The polarly opposite case to the perfect accumulation one is that the players have always behaved in the same way. Then, they experienced only one course (maximal feasible sequence) in the objective protocol. In this case, the accumulated memory set $T^*(\mathsf{m}_i^o)$ is very simple, and the constructed view has only one line such as in Figure 3.3 when he can recall every thing at the ending position. This is called the *cane* case. There are many intermediate cases between the cane one and the perfect accumulation one. We develop a theory dealing with such possibly restricted experiences.

In such an intermediate case, an inductively derived personal protocol must be much smaller than the objective one. Players may still behave, but may not be objectively optimal at all. A directly related problem is how much experiences are needed for a Nash equilibrium to emerge.

(3) Nash Equilibrium and Sufficient Experiences: We note that even the Nash equilibrium itself may be defined up to the experienced domain. To compare this with the Nash equilibrium in the objective sense, we need a quite specific set of experiences. It was partially discussed in Kaneko-Matsui [8]. We need now a more general treatment, which will be discussed in a separate paper.

(4) Unexperienced Options and Rationalization: Sometimes, a player notices unexperienced options for him. One possibility is to act and experience such options. The other possibility is to avoid them, but when he is already in the mode of optimization, he would rationalize his avoidance by putting low payoffs to the unexperienced options. Such rationalization complements some lack of experiences.

These remarks suggest that we need to develop a lot of new concepts and to modify what we have done in Part II as well as some extant concepts. These will be discussed in a separate paper.

Finally, we give one comment on our avoidance of probability.

(5) Probability in our Context: The probability has been used quite freely as a convenient tool in the traditional game theory. In our context, an objective use of a probability in the sense of the frequentist may be relevant. But the individual evaluation of a probability should be based inductively upon the experiences. This may be the problem of the emergence of a subjective probability (as an individual probabilistic belief) from individual experiences. To a great extent, this is related to the frequentist probability such as von Mises's *ex post* consideration of a random sequence.[4] However, this needs much more repetitions of the same situation than ours for $T^*(\mathtt{m}_i^o)$. Therefore, we do not touch the problem of the inductive derivation of an *ex post* probability in our project.

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