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Los Angeles

Computation of Ordinal-Invariant Trajectory Solutions to Multiperson Bargaining Problems

A dissertation submitted in partial satisfaction
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by

Johann Yonghee Choi

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The dissertation of Johann Yonghee Choi is approved.

James McQueen

Don Blasius

Thomas Ferguson

Lloyd S. Shapley, Committee Chair

University of California, Los Angeles

2004

To my parents to whom

I owe everything,

And to Cecilia Maria who

is an inspiration.

"Between my eyes which do not see above,

there lies a footstep billions of years old.

Between your breasts which separte hate and love,

there lies a stream with stories yet untold."

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Vita

1989	B.S. (Mathematics), University of Pittsburgh, Pittsburgh, Pennsylvania.
1989-1990	Teaching Assistant, California Institute of Technology.
1994	M.A. (Mathematics), UCLA, Los Angeles, California.
1993-1997	Teaching Assistant, Mathematics Department, UCLA.
1997-1999	Instructor, Mathematics Department, Pasadena City College.

Abstract of the Dissertation

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Professor Lloyd S. Shapley, Chair

A solution to the bargaining problem among 3 or more players is explicitly computed and discussed from the point of view of its invariance under continuously differentiable order-preserving transformations of the individual bargainers' utility scales. It is of the trajectory type, reflecting that the problem narrows down as negotiations among the bargainers proceed. Assuming the continuous differentiability of the individuals' utility scales gives us an opportunity to use differential equations in explicitly constructing the trajectory leading to this solution and in theoretically justifying its existence. For the 3 person case, this trajectory solution is graphically illustrated and the singularities of the Pareto surface are discussed. Numerical calculation of this trajectory solution for the 4 person case is also included.

CHAPTER 1

Introduction

The basic n-person bargaining format that we shall study consists of a set $\aleph$ of *utility vectors* $u \in \mathbf{R}^n$, called the *agreement set* or sometimes the *Pareto set*, and a particular such vector $d = (d_1, \ldots, d_n)$, called the *disagreement point.* The set $\aleph$ takes the form of a "Pareto surface over d"; more precisely we have conditions:

(i) $\aleph$ is closed and bounded, and does not contain d.

(ii) $a \geq d$ holds (componentwise) for all $a \in \aleph$.

(iii) $a \leq b$ never holds for $a \neq b$ in $\aleph$.

(iv) $\aleph$ is maximal with respect to (i)-(iii).

The set of utility vectors that lie between d and $\aleph$ will be denoted by A; thus

$$A = \{\ c \in \mathbf{R}^n \ \mid \ d \leq c \leq a \ \text{ for at least one } \ a \in \aleph \ \}.$$

An element of A may be termed as *imputation.* It is evident that A is a closed and bounded set of dimension n, and $\aleph$ is its "upper" boundary. The points in the interior of A may be interpreted as partial agreements among the bargainers. Note that A above is sufficient to specify a basic bargaining problem, since given the set A, both d and $\aleph$ – and even n! – are determined.

Conditions (i)-(iv) above can be shown to be equivalent to the statement that every unbounded "trajectory"(curve) in $\mathbf{R}^n$, beginning at d and moving weakly monotonically (i.e. non-decreasing in each coordinate) meets $\aleph$ in precisely one point. A rather complicated characterization! But the definition, calculation, analysis, and interpretation of such trajectories will constitute the major portion of this paper.

Our trajectories will be driven by differential equations dependent on the Pareto surface $\aleph$, so some smoothness assumptions on $\aleph$ will be required. To accomplish this, we introduce a C^1 function $F : \mathbf{R}^n \to \mathbf{R}$ that is zero on $\aleph$, negative on d, and is monotonically increasing throughout $\mathbf{R}^n$ in the sense that its partial derivatives, denoted by

$$F_i(x) = \frac{\partial F}{\partial u_i} |_{u=x}, \quad i = 1, \ldots, n,$$

are positive for all $x \in \mathbf{R}^n$. It should be pointed out that the function F is not specific to the problem in the sense that if G is another such function – that is, G is continuously differentiable, monotonic increasing such that $G(u) = 0 \Leftrightarrow F(u) = 0$ – then the pairs (F, d) and (G, d) define the same problem $(\aleph, d)$. Continuous differentiablity of F can be weakened, as we can sometimes allow nonsmoothness of $\aleph$.

We shall be interested in the behavior of our trajectories under *order-preserving transformation* of the individual bargainers' utility scales. An order-preserving transformation is a bijective, strictly monotone increasing function of the real line $\mathbf{R}$ onto itself. The set ORD of all such transformations forms a group under composition. We denote by ORD^n the direct sum of n copies of ORD. Thus it is a group of bijective functions of $\mathbf{R}^n$ onto itself, each component of which is a member of ORD.

When an individual bargainer i changes her utility scale by means of an order-preserving transformation $g_i \in$ ORD, the set of imputations A defined above is transformed by $g = (g_1, \dots, g_n) \in \mathrm{ORD}^n$. Thus $g(A) = \{g(u) | u \in A\}$ is the transformed set of imputations, and $g(\aleph) = \{g(x) | x \in \aleph\}$ is the transformed Pareto surface. Note that $g(\aleph)$ is defined by zeros of $F \circ g^{-1}$, since $y \in g(\aleph)$ if and only if $g^{-1}(y) \in \aleph$. From the fact that $g \in \mathrm{ORD}^n$ is not necessarily a C^1 function, we see that the smoothness of Pareto surface in general is not preserved under order-preserving transformations on the individual utility scale. For this reason we introduce a subgroup of ORD that consists of C^1 transformations. In order for the set of such transformations to form a group, it is required that each member has its inverse. We require the condition that the derivative is strictly positive in addition to the continuous differentiability. We name this subgroup DIFF, and we define DIFF^n as the direct sum of n copies of DIFF. Thus,

$$\mathrm{DIFF}^n = \{ \; g = (g_1, \dots, g_n) \quad | \quad g_i \text{ is } C^1 \text{ and } g_i' > 0, \text{ for all } i = 1, \dots, n \; \}.$$

We note that a DIFF^n transformation on individual utility scales preserves continuous differentiability as well as any singularities; thus smoothness or nonsmoothness of $\aleph$ is preserved under DIFF^n transformations.

A *solution* of $(\aleph, d)$ requires *at least* the following properties [1] :

(a) *unicity*: It will consist of a single point ξ in the payoff space $\mathbf{R}^n$.

(b) *feasibility* : $F(\xi) \leq 0$.

(c) *individual rationality* : $\xi \geq d$.

(d) *Pareto optimality* : $F(\xi) = 0$.

(e) *anonymity* : If π is a permutation on $\{1, \dots, n\}$,

then the solution of $(\pi\aleph, \pi d)$ is $\pi\xi$.

(Here, $\pi\aleph = \{\pi x | x \in \aleph\}, \pi d = (d_{\pi_1}, \dots, d_{\pi_n})$, and $\pi\xi = (\xi_{\pi_1}, \dots, \xi_{\pi_n})$)

Below, we will present a solution of a n-person basic bargaining problem $(\aleph, d)$ for $n \geq 3$ with certain additional properties, which will require some further

[1] The condition (e) *anonymity* includes *symmetry*: If $(\pi\aleph, \pi d) = (\aleph, d)$, then $\pi\xi = \xi$.

definitions. For a payoff vector $u \in A \setminus \aleph \subset \mathbf{R}^n$, we will assign an n-dimensional vector w_u, each of whose components is positive. The direction w_u at u will depend only on the behavior of F on a certain subset of $\aleph_u \stackrel{def}{=} \aleph \cap (u + \mathbf{R}^n_+)$. The collection $\{w_u | u \in A \setminus \aleph\}$ thus forms a *direction field* aimed at the Pareto set. For any fixed $u_0 \in A \setminus \aleph$, a *p*ath from u_0 may be defined as a continuous function φ_{u_0} from an interval $I = [a, b]$ to the payoff space $\mathbf{R}^n$, that is (i) differentiable on (a, b), (ii) $\varphi_{u_0}(a) = u_0$, (iii) $\varphi_{u_0}(b) \in \aleph$, and (iv) if $\varphi_{u_0}(t) = u$ for $t \in (a, b)$, then $\varphi'_{u_0}(t)$ is a positive multiple of w_u. The image of φ_{u_0}, $\{\varphi_{u_0}(t) | t \in [a, b]\}$, is called a *t*rajectory defined by u_0. A payoff vector $u \in A$ is said to be on the trajectory defined by u_0 if $u = \varphi_{u_0}(t)$ for some $t \in [a, b]$, and the parameter t can be thought of as the time at which the trajectory defined by u_0 reaches u. The trajectory defined by u_0 can then be thought of as the trajectory of time-progress of partial agreements of the bargaining process which began at u_0. A *trajectory solution* of $(\aleph, d)$ is the endpoint of the trajectory defined by d with the important "reduced problem property": If $\overline{d} \in A \setminus \aleph$ is on the trajectory defined by d, then the reduced problem $(\overline{\aleph}, \overline{d})$, where $\overline{\aleph} = \aleph \cap (\overline{d} + \mathbf{R}^n_+)$, has the same solution as $(\aleph, d)$.

Bargaining problems of the format $(\aleph, d)$ were studied by Zeuthen [16, 1930] and Nash [7, 1950]. Each suggested solutions whose mathematical equivalence is proved by Harsanyi [1, 1956]. The Nash-Zeuthen bargaining problem is a 2-person problem, with the assumption of convexity of A. The Nash-Zeuthen solution is a *cardinal* solution, i.e. it is covariant under affine transformations on the individual bargainer's utility scale. Shapley [12, 1969] demonstrated non-existence of a solution that is covariant under ORD^2 transformations. Shapley and Shubik [14, 1982] constructed an ordinal solution for 3-person problem, i.e. a solution which is covariant under ORD^3 transformations. Safra and Samet [11, 2004] generalized the Shapley-Shubik solution to more than 3 person bargaining problem. A bargaining solution of the trajectory type was first suggested by Raiffa [9, 1953]. Kalai [3, 1977] and Perles and Maschler [8, 1981] suggested solutions of the trajectory type for the 2-person bargaining problem. The bargaining solution which this paper constructs is a n-person bargaining solution of the trajectory type, with $n \geq 3$. It is covariant under a fairly large subgroup of ORD^n transformations, whereas the above mentioned trajectory solutions permit only affine transformations. We will confine ourselves for $n = 3$ in Chapter 2, and construct the solution (sections 2.1 and 2.2), give mathematical justification of its existence and uniqueness (section 2.3), and discuss covariance of the solution and the trajectory under a certain subgroup of DIFF^3 transformations (section 2.4). Generalization of the solution to the case of more than three bargainers is discussed in Chapter 3.

CHAPTER 2

A trajectory solution to a 3-person Bargaining Problem

2.1 The direction field

When $n = 3$, the set of feasible payoff vectors A is a distorted tetrahedron, with three flat, mutually perpendicular faces meeting at $d = (d_1, d_2, d_3)$, and one smoothly curved (possibly flat) triangular face $\aleph$ defined by the condition $F(u) = 0$ and $u \geq d$. The three "corners" of $\aleph$, labelled

$$B_1 = (b_1, d_2, d_3)$$
$$B_2 = (d_1, b_2, d_3)$$
$$B_3 = (d_1, d_2, b_3)$$

are called the *b*liss points for bargainers 1,2,3, respectively. Because of the monotonicity, we obviously have $b_i > d_i, i = 1, 2, 3$, and $F(d) < 0$. We note that A is certainly 3-dimensional but it is not necessarily a convex set. [1]

[1]But it is "finitely convexifiable" by means of transformations $g = (g_1, g_2, g_3)$ available in ORD^3, or even in DIFF^3; see Shapley [13]. This result is in contrast to Bradley and Shubik [15, 1974], who demonstrate that it is in general *not* possible to *flatten* a Pareto set in $\mathbf{R}^n$ by means of order-preserving transformations of the n individual coordinates, if $n > 2$.

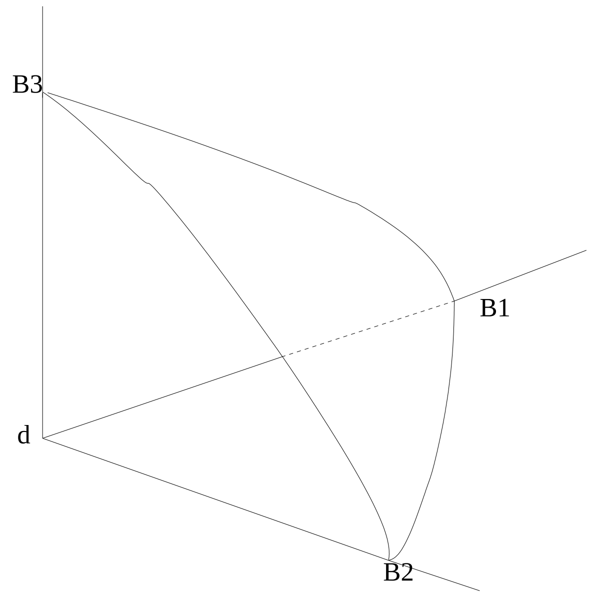


Figure 2.1: The set of feasible vectors A

On the tangent plane at player 1's bliss point B_1, we will have the equation

$$F_1(B_1)du_1 + F_2(B_1)du_2 + F_3(B_1)du_3 = 0,$$

where $F_i(B_1) = \frac{\partial F}{\partial u_i} \mid_{u=B_1}$, etc.. If we look at the family of parallel level lines on the planes defined by $u_1 =$ constant, we see that on each line the above equation reduces to

$$F_2(B_1)du_2 + F_3(B_1)du_3 = 0,$$

which gives us

$$-\frac{du_2}{du_3} = \frac{F_3(B_1)}{F_2(B_1)}.$$

This ratio, which represents the negative of the slope of the level lines on the tangent plane at B_1, defines the *exchange ratio from player 3 to 2 at player 1's bliss point*, and will be denoted by λ^1_{23}. It is worth writing this all out in full:

$$\lambda^1_{23} = -\frac{du_2}{du_3} \mid_{u=B_1} = \frac{F_3(B_1)}{F_2(B_1)}.$$

$$\lambda^2_{31} = -\frac{du_3}{du_1}\,|_{u=B_2} = \frac{F_1(B_2)}{F_3(B_2)},$$

$$\lambda^3_{12} = -\frac{du_1}{du_2}\,|_{u=B_3} = \frac{F_2(B_3)}{F_1(B_3)},$$

and

(2.1)

$$\lambda^1_{32} = -\frac{du_3}{du_2}\,|_{u=B_1} = \frac{F_2(B_1)}{F_3(B_1)} = \frac{1}{\lambda^1_{23}},$$

$$\lambda^2_{13} = -\frac{du_1}{du_3}\,|_{u=B_2} = \frac{F_3(B_2)}{F_1(B_2)} = \frac{1}{\lambda^2_{31}},$$

$$\lambda^3_{21} = -\frac{du_2}{du_1}\,|_{u=B_3} = \frac{F_1(B_3)}{F_2(B_3)} = \frac{1}{\lambda^3_{12}}.$$

We would like to point out that, although λ's are defined in terms of partial derivatives of F, they are essentially independent of F. As briefly discussed in the introduction, F was introduced to formulate the smoothness of the Pareto surface $\aleph$. If F and G are two different strictly increasing (i.e. all partials are positive) C^1 functions that are zeros on $\aleph$, then for any $x \in \aleph$, we will have ∇F and ∇G pointing in the same direction at x, namely, in the direction normal to the tangent plane to $\aleph$ at x. Therefore we have

$$\frac{F_1(x)}{G_1(x)} = \frac{F_2(x)}{G_2(x)} = \frac{F_3(x)}{G_3(x)},$$

and consequently,

$$\lambda^1_{23} = \frac{F_3(B_1)}{F_2(B_1)} = \frac{G_3(B_1)}{G_2(B_1)}.$$

Altogether we have three families of parallel level lines on tangent planes at each player's bliss point, the negative of whose slope represents the exchange ratio between the other two players. When the level lines on the tangent plane of player i's bliss point are projected down to player j, k's utility plane for all distinct $i, j, k \in \{1, 2, 3\}$, we see that these projected lines (still parallel) form an angular path going around the corner of the positive orthant at the disagreement point d, (see Figure 2.2).

This path may happen to form a closed loop on a single turn, but in general will form a "logarithmic spiral". That is, starting at any point on any player's positive axis near d, this path may or may not return to the starting point after one turn. In case it misses the starting point, the path will go around indefinitely forming a spiral in such a way that the ratio of coordinates of the ending point to the starting point of any single turn remains constant. When we choose the direction of turn as $1 \leftarrow 2 \leftarrow 3 \leftarrow 1$, this positive constant is given by

$$K = \lambda^1_{23} \lambda^2_{31} \lambda^3_{12},$$

which is a pure number. [2] The angular path will form a closed loop if and only if $K = 1$, and depending on whether $K > 1$ or $K < 1$, respectively, the logarithmic spiral will open up wider away from, or converge down to, the disagreement point as it turns $1 \leftarrow 2 \leftarrow 3 \leftarrow 1$ (see Figure 2.2).

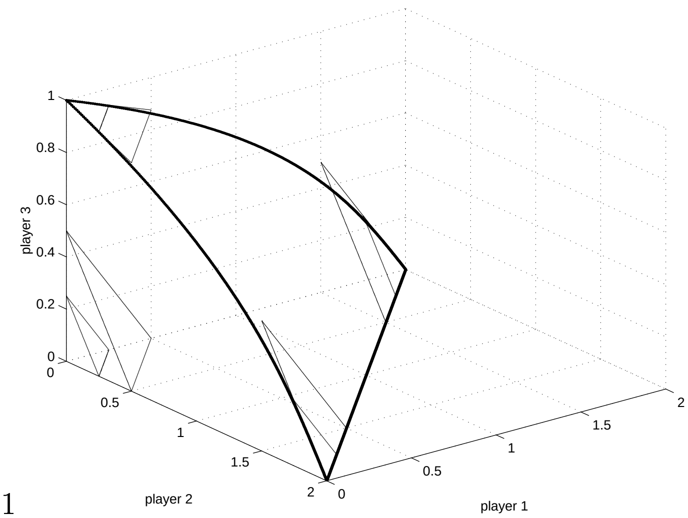
If it happens that $K = 1$, then the exchange ratios $\{\lambda^i_{jk}\}$ are said to be *c*onsistent. In a consistent system, we can define *w*eights for the three bargainers, (w_1, w_2, w_3), unique up to a scalar multiple, with the property that

$$\lambda^i_{jk} = \frac{w_j}{w_k} \qquad \text{for all distinct} \quad i, j, k.$$

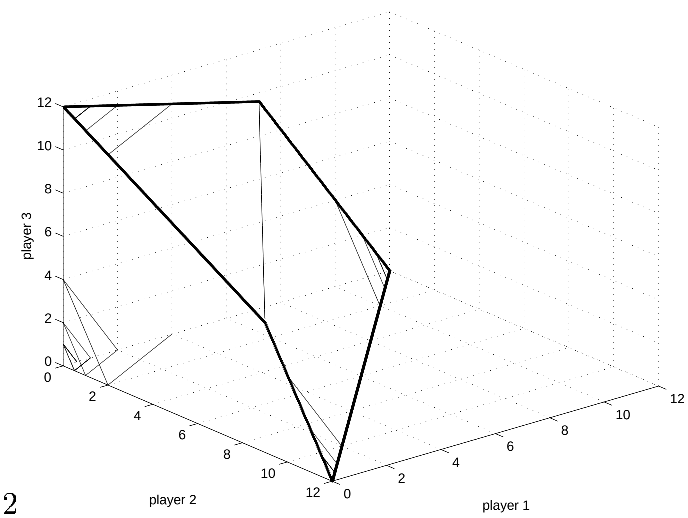
In fact, if we define w_j, $j = 1, 2, 3$, to be

$$w_1 = \sqrt[3]{\lambda^3_{12} \lambda^2_{13}}, \qquad w_2 = \sqrt[3]{\lambda^1_{23} \lambda^3_{21}}, \qquad w_3 = \sqrt[3]{\lambda^2_{31} \lambda^1_{32}}, \tag{2.2}$$

[2]We note that this pure number K is an absolute invariant under the action of the group DIFF^3. This K is denoted by K_{123} or K_{12} according to the notation which shall be introduced in Chapter 3.



$K = 1$



$K = 2$

Figure 2.2: Angular path forming a closed loop or a logarithmic spiral: Here we have taken $d = (0,0,0)$ in both figures and $B_1 = (2,0,0), B_2 = (0,2,0), B_3 = (0,0,1)$ in the first and $B_1 = (12,0,0), B_2 = (0,12,0), B_3 = (0,0,12)$ in the second. The first figure suggests a smooth (C^1) convex surface; the second suggests a piecewise linear Pareto surface which is not C^1. We shall work with both types of surfaces in the sequel. (See Section 2.5)

then we have

$$\frac{w_j}{w_k} = \sqrt[3]{\frac{\lambda^i_{jk}\lambda^k_{ji}}{\lambda^i_{kj}\lambda^j_{ki}}} = \sqrt[3]{\lambda^i_{jk}\lambda^i_{jk}\,\frac{\lambda^k_{ji}}{\lambda^j_{ki}}}$$

$$= \sqrt[3]{\lambda^i_{jk}\lambda^i_{jk}\,\frac{\lambda^k_{ji}\lambda^k_{ij}\lambda^i_{jk}}{\lambda^j_{ki}\lambda^k_{ij}\lambda^i_{jk}}} = \sqrt[3]{\frac{\lambda^i_{jk}\lambda^i_{jk}\lambda^i_{jk}}{\kappa}}$$

$$= \lambda^i_{jk},$$

where $\kappa = K$ if (i,j,k) is in the order of$(1,2,3)$ or $\kappa = 1/K$ if (i,j,k) is in the order of$(1,3,2)$. In any case $\kappa = 1$ since $K = 1$ when the exchange ratios are consistent.

Without consistency, however, we will have $K \neq 1$. [3] In order to restore consistency in a symmetric way, we shall apply the factor $K^{1/3}$ at each step of the logarithmic spiral described above: starting with player 1 define adjusted weights, w'_1, w'_2, w'_3, by

$$w'_1 = 1, \qquad w'_2 = \lambda^3_{21}K^{1/3}, \qquad w'_3 = \lambda^1_{32}\lambda^3_{21}K^{2/3}.$$

To symmetrize this definition, we temporarily define

$$w''_1 = \lambda^2_{13}\lambda^1_{32}K^{2/3}, \qquad w''_2 = 1, \qquad w''_3 = \lambda^1_{32}K^{1/3},$$

$$w'''_1 = \lambda^2_{13}K^{1/3}, \qquad w'''_2 = \lambda^3_{21}\lambda^2_{13}K^{2/3}, \qquad w'''_3 = 1,$$

and then take the *geometric mean* of the three adjustments. Thus,

$$w_1 = \sqrt[3]{1 \cdot \lambda^2_{13}\lambda^1_{32}K^{2/3} \cdot \lambda^2_{13}K^{1/3}} = \sqrt[3]{\lambda^2_{13}\lambda^1_{32}\lambda^2_{13} \cdot K}$$

$$= \sqrt[3]{\lambda^2_{13}\lambda^1_{32}\lambda^2_{13}\;\lambda^1_{23}\lambda^2_{31}\lambda^3_{12}} = \sqrt[3]{\lambda^3_{12}\lambda^2_{13}},$$

[3] $K \neq 1$ describes the situation known in economic markets as "arbitrage", where profits or losses can be incurred by trading among 3 or more commodities with inconsistent prices.

and symmetrically,

$$w_2 = \sqrt[3]{\lambda^3_{21}\lambda^1_{23}}, \quad \text{and} \quad w_3 = \sqrt[3]{\lambda^1_{32}\lambda^2_{31}},$$

which restores the definition (2.2). All four weight vectors w', w'', w''' and w are proportional; it will suffice to show that $\frac{w'_2}{w'_1} = \frac{w_2}{w_1}$. Indeed, we have

$$\frac{w'_2}{w'_1} = \frac{\lambda^3_{21} \cdot K^{1/3}}{1} = \sqrt[3]{\lambda^3_{21}\lambda^3_{21}\lambda^3_{21} \ \cdot \ \lambda^1_{23}\lambda^2_{31}\lambda^3_{12}} = \sqrt[3]{\lambda^3_{21}\lambda^3_{21} \ \cdot \ \lambda^1_{23}\lambda^2_{31}} = \frac{\sqrt[3]{\lambda^3_{21}\lambda^1_{23}}}{\sqrt[3]{\lambda^3_{12}\lambda^2_{13}}} = \frac{w_2}{w_1}.$$

Note that

$$\frac{w_2}{w_1} = \lambda^3_{21} \cdot K^{1/3}.$$

These weights describe a "box" at d (see Figure 2.3). The arrow from $d = (d_1, d_2, d_3)$ to $d+w = (d_1+w_1, d_2+w_2, d_3+w_3)$ is the direction that the trajectory will follow starting at d. Note that this box has fixed volume 1, since

$$w_1 w_2 w_3 = \sqrt[3]{\lambda^3_{12}\lambda^2_{13} \ \lambda^3_{21}\lambda^1_{23} \ \lambda^2_{31}\lambda^1_{32}} = 1 \qquad (2.3)$$

.

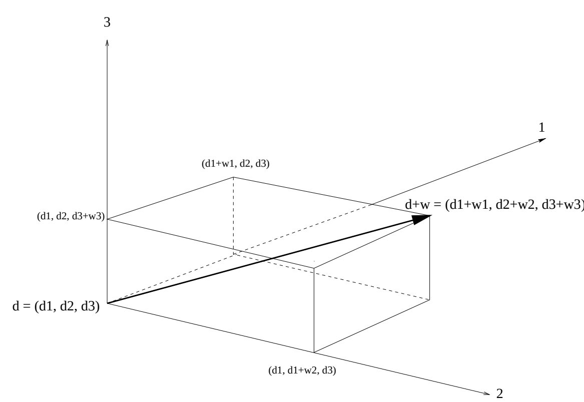


Figure 2.3: The "weight box"

The above-computed *weight vector*, (w_1, w_2, w_3), can be extended to all points u in $\mathbf{R}^3$. Since F is strictly monotonic in each coordinate direction, the lines which pass through the point u and are parallel to the coordinate axes will intersect with the level surface $\{F = 0\}$ precisely at one point. Let $B_i^u, i = 1, 2, 3$, denote the intersection point on the level surface $\{F = 0\}$, which includes the Pareto surface $\aleph$, with line passing through u parallel to each i's coordinate axis. The partial derivatives of F evaluated at B_i to compute the weight vector at d are now all evaluated at B_i^u to give the weight vector at u. Thus we have a mapping $w : \mathbf{R}^3 \to \mathbf{R}^3$ which assigns the weight vector

$$w(u) = (\ w_1(u), w_2(u), w_3(u)\ ) \tag{$*$}$$

to each point $u \in \mathbf{R}^3$. The weight vectors assigned at each point $u \in \mathbf{R}^3$ thus form the *direction field* that the trajectory will follow at u. We shall be interested the direction field confined to A, although it is defined through the whole space $\mathbf{R}^3$. From the continuous differentiability of F and the formulas used to compute w, we see that the mapping w is continuous on $\mathbf{R}^3$. Since A is compact, w is *uniformly continuous* on A. Since all the partial derivatives of F are assumed to be strictly positive, we see from the formulas defining weights that there shall exist a uniform lower bound δ_i, for $i = 1, 2, 3$, such that $w_i(u) \geq \delta_i > 0$, for all $u \in A$. It is straightforward to verify that the relation

$$w_1(u)w_2(u)w_3(u) = 1 \tag{2.3'}$$

holds for all $u \in \mathbf{R}^3$, and in view of this fact we also obtain a uniform upper bound of $w_i(u)$: $w_i(u) \leq \Delta_i$, for all $u \in A$.

We conclude this section by proving that the direction field constructed above is "covariant" under DIFF^3 transformations. Covariance of a direction field with respect to a transformation means that the process of obtaining the direction field commutes with the transformation. We could first transform a bargaining problem $(\aleph, d)$ by means of order preserving transformation g into a new bargaining problem $(g(\aleph), g(d))$ and get the direction field of $(g(\aleph), g(d))$. Or we could get the direction field of $(\aleph, d)$ and then transform it into a new direction field. Covariance of a direction field says that regardless of order of these two processes, we should get the same direction field. The direction field defined above by means of weights on the set A of feasible vectors of the bargaining problem $(\aleph, d)$ satisfies this property with respect to a class of transformations. We state this as in

Proposition 1 *The direction field defined by weights defined in (*) on the set A of feasible vectors is covariant with respect to $DIFF^3$ transformations.*
Moreover, the weight vector at $g(u)$,where $g = (g_1, g_2, g_3) \in DIFF^3$ and $u = (u_1, u_2, u_3) \in A$, of the trasformed problem $(g(\aleph), g(d))$ is given by

$$\frac{1}{m_g(u)}\ (g_1'(u_1)w_1(u),\ g_2'(u_2)w_2(u),\ g_3'(u_3)w_3(u)), \tag{2.4}$$

where

$$m_g(u) = \sqrt[3]{g_1'(u_1)g_2'(u_2), g_3'(u_3)}. \tag{2.5}$$

*R*emark 1. Note that the weight vector given in (2.4) has the property that each of its component is positive and the components multiply out to 1.

*R*emark 2. Later in the paper, we will generalize weights for bargaining problems among more than 3 bargainers. Proof of the generalized version of the present proposition, including the generalized formulas corresponding to (2.4) and (2.5), is given in Appendix A.

*R*emark 3. The continuity and uniform boundedness of the direction field plays an essential role in constructing a trajectory solution to be described below. Later we will discuss certain subclass of DIFF transformations which preserves a vital mathematical property of the direction field in forming the trajectory. Invariance (covariance) of the trajectory solution with respect to this subclass of DIFF transformation will be a direct consequence of the present proposition.

2.2 Constructing the solution

In actual numerical calculations we shall fix a positive multiplier s and let the vector from $d = (d_1, d_2, d_3)$ to $d + sw = (d_1 + sw_1, d_2 + sw_2, d_3 + sw_3)$ be the first leg of the piecewise linear trajectory, which is the diagonal of a box of volume s^3. The disagreement point then moves upward from d to $d + sw$ while the three bliss points move along the Pareto surface $\aleph$ to B_1', B_2', B_3', where B_i' for $i = 1, 2, 3$, is the intersection of $\aleph$ with the ray stemming from $d + sw$ parallel to the player i's coordinate axis (see Figure 2.4). This defines a "reduced problem" $(\aleph', d + sw)$, where $\aleph'$ is a subset of $\aleph$ having its "corners" at B_1', B_2', B_3'.

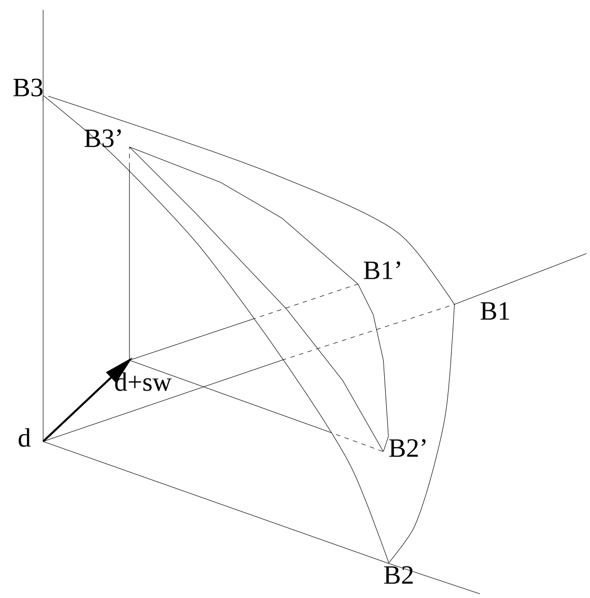


Figure 2.4: The first step, d to $d + sw$, of a finite trajectory

This procedure can then be applied to $(\aleph', d + sw)$ if $d + sw \in A \setminus \aleph$, and by successive iterations we obtain the piecewise linear trajectory of the disagreement points as well as the bliss points of each player. Therefore the trajectory of every disagreement point is driven by the weight vectors at the previous disagreement point, and the trajectories of the bliss points are "shadows" of the trajectory of the disagreement point onto the Pareto surface in the directions of the bargainers' utility scale axes.

Because the iteration multipler s is fixed and the weight w_i is uniformly bounded below by a positive constant δ_i, the sequence of new disagreement points $\{d_s^k\}_{k=0}^{\nu(s)}$, defined recursively by $d_s^{k+1} = d_s^k + s \cdot w(d_s^k)$, where $d_s^0 = d$, must eventually rise above the Pareto surface $\aleph$; that is, there exists a nonnegative integer $\nu(s)$ such that $F(d_s^{\nu(s)}) \geq 0$ and $F(d_s^k) < 0$ for $k = 0, \ldots, \nu(s) - 1$. We define the numerical bargaining solution ξ_s to $(\aleph,\ d)$ with iteration multipler s to be the intersection of the Pareto surface $\aleph$ with the line segment joining $d_s^{\nu(s)-1}$ and $d_s^{\nu(s)}$.

By a trajectory, we shall mean the image of a continuous map ϕ from a closed interval $I = [a, b]$ to the utility space $\mathbf{R}^3$, that is, $\{\phi(t) \in \mathbf{R}^3 \mid t \in I\}$. The class of trajectories we shall be interested in are those *rise from* d, *travel through* A, and *end in* $\aleph$. These are images of continuous function ϕ on $I = [a, b]$, with $\phi(t) \in A$ for all $t \in [a, b]$, $\phi(a) = d$, and $\phi(b) \in \aleph$. The interval I can be interpreted as time during which negotiation proceeds. Then, $\phi(t)$ represents the partial agreement reached at time $t \in I$, and the trajectory $\{\phi(t) \mid t \in I = [a, b]\}$ traces out time progress of the negotiation process which begins from the disagreement point d at time $t = a$, and reaches to a Pareto optimal solution $\phi(b) \in \aleph$ at time $t = b$.

By the *s-trajectory* we shall mean a trajectory which is a piecewise linear path of *steps* leading from d to ξ_s, each step of which is a linear segments joining d_s^k and d_s^{k+1}, for $k = 0, \ldots, \nu(s) - 2$, and finally $d_s^{\nu(s)-1}$ and ξ_s. It is convenient to describe the s-trajectory as a continuous function φ_s on a particular interval $\mathcal{I} \stackrel{\text{def}}{=} [F(d), 0]$, parametrized by the value of F in the following manner. Let us now replace $d_s^{\nu(s)}$ by ξ_s so that the sequence $\{d_s^k\}_{k=0}^{\nu(s)}$ constitutes the vertices of the s-trajectory. Define $t_s^k = F(d_s^k)$, for $k = 0, \ldots, \nu(s)$, so that $\{t_s^k\}_{k=0}^{\nu(s)}$ partitions the time interval $\mathcal{I}$, which takes nonpositive values. By construction, we have $d_s^{k+1} > d_s^k$ componentwise. Therefore, the s-trajectory strongly monotonically rises from d to ξ_s, and as it does, the values of F strictly increase from $t_s^0 = F(d)$ to $t_s^{\nu(s)} = 0$. For any $t \in \mathcal{I}$, the s-trajectory will intersect with the level surface $\{F = t\}$ at precisely one point. We define $\varphi_s(t)$ to be this unique intersection point. We then have

$$F(\varphi_s(t)) = t. \tag{2.6}$$

Therefore, we have $\varphi_s(t_s^k) = d_s^k$ for $k = 0, \ldots, \nu(s)$, and for $t \in (t_s^k, t_s^{k+1})$, $\varphi_s(t)$ is the point at which the line segment joining d_s^k and d_s^{k+1} intersects the level surface $\{F = t\}$. Evidently, φ_s is continuous on $\mathcal{I}$ and differentiable on every subinterval $(t_s^k, t_s^{k+1}), k = 0, \ldots \nu(s) - 1$, with the derivative $\varphi_s'(t)$ being a positive multiple of $w(d_s^k)$ on (t_s^k, t_s^{k+1}). Indeed, for $k = 0, \ldots \nu(s) - 1$ there exist differentiable functions ${}_k\zeta_s$ on $[t_s^k, t_s^{k+1}]$, which strictly increase from 0 at t_s^k to 1 at t_s^{k+1}, such that

$$\varphi_s(t) = \varphi_s(t_s^k) + s \cdot {}_k\zeta_s(t) \cdot w(d_s^k) \tag{2.7}$$

for $t \in [t_s^k, t_s^{k+1}]$. Roughly speaking, the function ${}_k\zeta_s$ measures the fraction of the s-trajectory contributed by the line segment from d_s^k to d_s^{k+1}. By differentiating (2.7) at $t \in (t_s^k, t_s^{k+1})$ we have

$$\varphi_s'(t) = s \cdot {}_k\zeta_s{}'(t) \cdot w(d_s^k). \tag{2.8}$$

Also, by (2.6) we have

$$\nabla F(\varphi_s(t)) \bullet \varphi_s'(t) = 1, \tag{2.9}$$

for all t in $\cup_{k=0}^{\nu(s)-1} (t_s^k, t_s^{k+1})$, where $\bullet$ is the dot product of two vectors. Combining

(2.8) and (2.9) gives

$$ {}_k{\zeta_s}'(t) = \frac{1}{s \cdot \nabla F(\varphi_s(t)) \bullet w(d_s^k)}, \tag{2.10} $$

which in turn enables us to rewrite (2.7) as

$$ \varphi_s(t) = \varphi_s(t_s^k) + \int_{t_s^k}^{t} \frac{w(d_s^k)}{\nabla F(\varphi_s(\theta)) \bullet w(d_s^k)}\, d\theta. \tag{2.11} $$

Discussion in the next section investigates conditions under which the s-trajectories will converge to a unique trajectory as s approaches 0. The end point ξ of the unique convergent trajectory, which is also the limit of the numerical solutions ξ_s as $s \to 0$, then is defined to be the *trajectory solution* of the bargaining problem $(\aleph, d)$.

2.3 Existence of the trajectory solution

Existence of ξ depends on the existence of a *unique* trajectory which rises from d and at every point of which is tangent to the direction field. More precisely, we require a unique, continuously differentiable function ϕ from an interval $I = [a, b]$ to A such that $\phi(a) = d, \phi(b) \in \aleph$, and for every $t \in I$, $\phi'(t)$ is a positive multiple of $w(\phi(t))$. In pursuit of such function, we will consult (see Henrici [2, 1962, 112-117]) on existence of unique solution to a differential equation in the following manner: Define a function Ω on $I \times \mathbf{R}^3$ by $\Omega(t, u) = w(u)$. Continuity of Ω assures the existence of a continuously differentiable function ϕ on I such that (i) $\phi(a) = d$ (initial condition), and (ii) $\phi'(t) = \Omega(t, \phi(t)) = w(\phi(t)), \forall t \in I$. These two conditions enable us to write

$$\phi(t) = d + \int_a^t w(\phi(\theta))d\theta.$$

In view of w being positive in each of its coordinate, we see that $\phi(t) > d$, for $t > a$. Thus, this trajectory defined by d travels through A, and in view of $w(u) > (\delta_1, \delta_2, \delta_3)$, for all $u \in A$, we can choose b such that (iii) $\phi(b) \in \aleph$.

The uniqueness of the function ϕ satisfying all three conditions (i), (ii), and (iii) requires a Lipschitz condition on Ω in u. That is, there should exist a constant L such that

$$||\Omega(t, u_1) - \Omega(t, u_2)|| \leq L||u_1 - u_2||, \quad \forall t \in I, \forall u_1, u_2 \in A.$$

Existence of our solution to the bargaining problem, therefore, requires a (global) Lipschitz condition on the direction field.

We shall say that a function f from a metric space X to a metric space Y is (globally) *Lipschitzian* on a subset S of X if there exists a constant L such that $||f(x) - f(y)|| \leq L\,||x - y||$ for all x and y in S. We say that f is *locally uniformly Lipschitzian at a point* x in X if there exists a neighborhood N_x of x and a constant L_x such that $||f(y) - f(z)|| \leq L_x\,||y - z||$ for all y and z in N_x. We say that f is *locally uniformly Lipschitzian* on a subset S of X if it is locally uniformly Lipschitzian at every point of S.

Let B denote the compact "box" having for the opposite corners the disagreement point d and the "triple bliss point" $(d_1 + b_1, d_2 + b_2, d_3 + b_3)$. The following theorem provides a sufficient condition of the existence of a unique trajectory.

Theorem 1 *Suppose that there exists an open set, containing B, on which the partial derivatives F_i are Lipschitzian. Then, for any $a \in \mathbf{R}$, there exists $b > a$ and a unique continuously differentiable function ϕ on $[a, b]$ to A such that (i) $\phi(a) = d$, (ii) $\phi(b) \in \aleph$, (iii) $\phi'(t) = w(\phi(t))$, for all $t \in [a, b]$.*

Remark. Later in the paper we will generalize the trajectory solution of the bargaining problem to more than 3 bargainers. The present statement and proof of Theorem 1 generalizes naturally to $n > 3$.

Proof of Theorem 1. Without loss of generality we may assume that F_i are Lipschitzian on $\mathbf{R}^3$, since if not, we can find such function G such that $F \equiv G$ on a open set containing B, and apply the following on the function G, without altering the trajectory. By a standard theorem (see Henrici [2, 1962, 112-117]) of differential equations, the existence and the uniqueness of of such a trajectory is assured by the Lipschitz continuity of the direction field on B. Recall that $w(u) = (w_1(u), w_2(u), w_3(u))$ and $w_i(u) = \sqrt[3]{\lambda_{ij}^k(u)\lambda_{ik}^j(u)}$. It is straightforward to verify that a product or a composition of two functions that are Lipschitzian is also Lipschitzian. The required Lipschitz condition on w on B would follow as soon as we prove the following

Proposition 2 *Let O be an open subset of $\mathbf{R}^3$ which contains B, and on which the partial derivatives F_i are Lipschitzian. Then, the exchange ratios $\{\lambda_{ij}^k\}$ are Lipschitzian on B.*

Proof of Proposition 2. Without loss of generality we will show that λ_{12}^3 is Lipschitzian on B. Here and below, we will use the Euclidean metric: for $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$, $||x - y|| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^3}$ Also, we will use notation $|| \cdot ||_{\overline{12}}$ for 2-dimensional norm of 3-dimensional vector which projects onto the 1,2-plane. So $||x - y||_{\overline{12}} = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$.

Given $u = (u_1, u_2, u_3) \in O$, let $B_3^u = (u_1, u_2, b_3^u)$ denote the unique intersection point of the level surface $\{F = 0\}$ with the line given by $\{(x, y, z) \in \mathbf{R}^3 | x = u_1, y = u_2, z \in \mathbf{R}\}$. If u is A, then B_3^u is player 3's bliss point on the Pareto set $\aleph$ based on the current disagreement point u. Then, by definition, $\lambda_{12}^3(u) = F_2(B_3^u)/F_1(B_3^u)$. From the assumption that F_i is Lipschitzian and is strictly positive on B, it follows easily that F_i/F_j (in particular, F_2/F_1) is Lipschitzian on B, and thus there is a constant L_1 such that

$$|\lambda_{12}^3(u) - \lambda_{12}^3(v)| = |\frac{F_2}{F_1}(B_3^u) - \frac{F_2}{F_1}(B_3^v)| \leq L_1 ||B_3^u - B_3^v||, \quad \text{for all } u, v \in B.$$

It remains to show that the mapping $u \mapsto B_3^u$ is (globally) Lipschitzian on B. Since F is defined to be continuously differentiable with all its partial derivatives strictly positive on $\mathbf{R}^3$, the implicit function theorem tells us that the mapping $(u_1, u_2) \mapsto b_3^u$ such that $F(B_3^u) = F(u_1, u_2, b_3^u) = 0$ is defined and is continuously differentiable in the neighborhood of (u_1, u_2). Via the mean value theorem we see that this mapping $(u_1, u_2) \mapsto b_3^u$ is locally uniformly Lipschitzian on $\mathbf{R}^2$. So there is a constant L_u such that

$$|b_3^u - b_3^v| \leq L_u ||u - v||_{\overline{12}}$$

for all v sufficiently close to u. Then,

$$\begin{aligned} ||B_3^u - B_3^v|| &\leq ||u - v||_{\overline{12}} + L_u ||u - v||_{\overline{12}} \\ &\leq (1 + L_u) ||u - v||_{\overline{12}} \\ &\leq (1 + L_u) ||u - v||, \end{aligned}$$

for all v sufficiently close to u, showing that the mapping $u \mapsto B_3^u$ is locally uniformly Lipschitzian on O. Consequently, it is (globally) Lipschitzian on B, by the Proposition 3 stated below. $\diamondsuit$

The last claim of the above proof is justified in the following proposition, which is proved in Appendix B.

Proposition 3 *A locally uniformly Lipschitzian function f from a metric space X to a metric space Y is (globally) Lipschitzian on any compact, connected, convex subset of X.*

The following two corollaries are immediate results from the Theorem 1 and the Propsition 3.

Corollary 1 *The conclusion of the Theorem 1 will follow if we assume that F is locally uniformly Lipschitzian (LUL) on $\mathbf{R}^3$.*

Corollary 2 *The conclusion of the Theorem 1 will follow if we assume that F is twice continuously differentiable on an open set containing B.*

The above discussion justifies an unique smooth trajectory and the existence and uniqueness of the solution ξ as the endpoint of the trajectory. In the previous section, we have constructed "s-trajectories", having the ξ_s as its endpoint. We have claimed that the s-trajectories converge to a unique continously differentiable trajectory. Now we prove that this limit trajectory is indeed the trajectory that the Theorem 1 provides, hence justifying our claim that the solution ξ of the Theorem 1 is the limit of ξ_s, as s tends to 0.

We begin by questioning ourselves the uniqueness of trajectoy of the Theorem 1. Note that the left end point a of the interval I is arbitrarily chosen. Then there exists a unique number $b > a$ and a unique continously differentiable function ϕ on $[a, b]$, such that $\phi(a) = d, \phi(b) \in \aleph$, and $\phi'(t) = \Omega(t, \phi(t)) = w(\phi(t))$, for all $t \in (a, b)$. If we had chosen $\bar{a} \neq a$, as the left endpoint of an interval, then we would have gotten a unique number $\bar{b} > \bar{a}$ and a unique continously differentiable function ψ on $[\bar{a}, \bar{b}]$, such that $\psi(\bar{a}) = d, \psi(\bar{b}) \in \aleph$, and $\psi'(\bar{t}) = \Omega(\bar{t}, \psi(\bar{t})) = w(\psi(\bar{t}))$, for all $\bar{t} \in (\bar{a}, \bar{b})$. In what sense then the trajectory unique? We shall say that the trajectories defined by ϕ and ψ are *identical* if the sets $\{\phi(t) | t \in [a, b]\}$ and $\{\psi(\bar{t}) | \bar{t} \in [\bar{a}, \bar{b}]\}$ are identical subsets of $\mathbf{R}^3$. [4] We shall also say that the trajectory ϕ is a *reparametrization* of the trajectory ψ, and vice versa. If we interpret the intervals represent the time during which negotiations take place, the partial agreement $\phi(t) \in A$ achieved at time t is also achieved by $\psi(\bar{t})$ for some $\bar{t} \in [\bar{a}, \bar{b}]$, although it may be achieved at a different time $\bar{t}$. More precisely, we shall say that ϕ on $[a, b]$ and ψ on $[\bar{a}, \bar{b}]$ defines the identical trajectory if there exist a continuously differentiable bijection $\beta : [a, b] \longrightarrow [\bar{a}, \bar{b}]$ such that $\psi(\beta(t)) = \phi(t)$, for all $t \in [a, b]$.

We presented s-trajectories in the previous section as a continuous, piece-

[4] In fact, this says not only that $\phi(b) \in \aleph$ and $\psi(\bar{b}) \in \aleph$, but also it says that $\phi(b) = \psi(\bar{b}) = \xi$.

wise linear functions φ_s on a particular interval $\mathcal{I} = [F(d), 0]$. In order to prove that s-trajectories converges to the smooth trajectory of the Theorem 1, it is convenient to reparametrize the limit trajectory as a continously differentiable function on the same interval $\mathcal{I}$.

Theorem 2 *There is a reparametrization φ on $\mathcal{I}$ of the trajectory ϕ on $[a, b]$ given by the Theorem 1. Furthermore, we have*

$$F(\varphi(\tau)) = \tau, \tag{2.12}$$

and

$$\varphi'(\tau) = \frac{w(\varphi(\tau))}{\nabla F(\varphi(\tau)) \bullet w(\varphi(\tau))}, \tag{2.13}$$

for all $\tau \in \mathcal{I}$.

Proof. Choose a real number a. By Theorem 1, we will have $b > a$ and a unique continuously differentiable funcition ϕ on $I = [a, b]$, such that $\phi(a) = d, \phi(b) \in \aleph$, and $\phi'(t) = w(\phi(t))$, for all $t \in (a, b)$. Define $\beta : I \to \mathcal{I}$ by $\beta(t) = (F \circ \phi)(t)$. Then, $\beta(a) = F(\phi(a)) = F(d)$, $\beta(b) = F(\phi(b)) = 0$, since $\phi(b) \in \aleph$, and β is continuously differentiable for $t \in (a, b)$ with

$$\beta'(t) = \nabla F(\phi(t)) \bullet \phi'(t) > 0.$$

So β is a bijection of $[a, b]$ onto $\mathcal{I}$, hence invertible. Now let $\varphi = \phi \circ \beta^{-1}$. Then, φ is a continuously differentiable funcion of $\mathcal{I}$ into A, and

$$\begin{aligned} F(\varphi(\tau)) &= F((\phi \circ \beta^{-1})(\tau)) \\ &= (F \circ \phi) \circ \beta^{-1}(\tau) \\ &= \beta(\beta^{-1}(\tau)) = \tau \end{aligned}$$

for all $\tau \in \mathcal{I}$, since $\beta = F \circ \phi$ by definition. In order to show (2.13), let $\gamma = \beta^{-1}$ so that $\varphi = \phi \circ \beta^{-1} = \phi \circ \gamma$. Fix $\tau \in \mathcal{I}$ and let $t = \gamma(\tau)$. Then $\gamma'(\tau) = \frac{1}{\beta'(t)}$, and

$\phi(t) = \phi(\gamma(\tau)) = \varphi(\tau)$. From $\varphi(\tau) = \phi(\gamma(\tau))$, we see that

$$\begin{aligned}
\varphi'(\tau) &= \phi'(\gamma(\tau)) \cdot \gamma'(\tau) \\
&= w(\phi(\gamma(\tau))) \cdot \frac{1}{\beta'(t)} \\
&= \frac{w(\varphi(\tau))}{\beta'(t)}.
\end{aligned}$$

It remains to show that $\beta'(t) = \nabla F(\varphi(\tau)) \bullet w(\varphi(\tau))$. But, indeed we have

$$\begin{aligned}
\beta'(t) &= \nabla F(\phi(t)) \bullet \phi'(t) \\
&= \nabla F(\phi(t)) \bullet w(\phi(t)) \\
&= \nabla F(\varphi(\tau)) \bullet w(\varphi(\tau)).
\end{aligned}$$

$\diamondsuit$

We are ready to prove the following

Theorem 3 *For all $t \in \mathcal{I}$, we have $\varphi_s(t) \longrightarrow \varphi(t)$ pointwise as $s \longrightarrow 0$. In particular, $\xi_s \longrightarrow \xi$ as $s \longrightarrow 0$.*

Proof. For a given $s > 0$, let us define a continuous function p_s on $\mathcal{I}$ as

$$p_s(t) = \varphi_s(t) - \varphi(t),$$

which represents the difference between partial disagreement points represented by the s-trajectory and the limit trajectory at the time $t \in \mathcal{I}$. We will show that for every $t \in \mathcal{I}$, $p_s(t) \to 0$ as s tends to 0. It is trivial that $p_s(t_0)$ tends to 0 as s tends to 0, for $p_s(t_0) = \varphi_s(t_0) - \varphi(t_0) = d - d = 0$, for all $s > 0$. So let us now fix $t \in (t_0, 0]$. Then for any given $s > 0$, there exists a unique index $k = k(s)$ such that $t_s^k < t \leq t_s^{k+1}$. We shall write $[s]$ for this unique $k(s)$. With this notation, the equation (2.7), which describes the s-trajectory, should be written

$$\varphi_s(t) = \varphi_s(t_s^{[s]}) + s \cdot {}_{[s]}\zeta_s(t) \cdot w(d_s^{[s]}), \tag{2.7$'$}$$

where the derivative of ${}_{[s]}\zeta_s$ is given by

$$ {}_{[s]}\zeta_s{}'(t) = \frac{1}{s \cdot \nabla F(\varphi_s(t)) \bullet w(d_s^{[s]})}. \tag{2.10'} $$

In this proof, we shall write $\zeta_{[s]}$ for ${}_{[s]}\zeta_s$, $t_{[s]}$ for $t_s^{[s]}$, and $d_{[s]}$ for $d_s^{[s]}$.

For a fixed $s > 0$, we shall write φ, which describes the limit trajectory, as

$$ \varphi(t) = \varphi(t_{[s]}) + s \cdot \eta_s(t), \tag{2.14} $$

where

$$ \eta_s(t) = \frac{1}{s} \int_{t_{[s]}}^{t} \varphi'(\theta)\, d\theta = \frac{1}{s} \int_{t_{[s]}}^{t} w(\varphi(\theta))\, d\theta. \tag{2.15} $$

From the uniform boundedness of the direction field w on the compact box B, there exists a positive constant M_w, independent of s, such that $||w(u)|| \leq M_w$, for all $u \in B$. Then we have

$$ ||\eta_s(t)|| \leq \frac{M_w}{s}\,(t - t_{[s]}). $$

From the relation $F(\varphi_s(t)) = t$, for all t in $\mathcal{I}$, we have

$$ \begin{aligned} t - t_{[s]} &\leq t_{[s]+1} - t_{[s]} \\ &= F(\varphi_s(t_{[s]+1})) - F(\varphi_s(t_{[s]})) \\ &= \nabla F(\gamma_{[s]}) \bullet (\varphi_s(t_{[s]+1}) - \varphi_s(t_{[s]})) \\ &\leq ||\nabla F(\gamma_{[s]})|| \cdot ||\varphi_s(t_{[s]+1}) - \varphi_s(t_{[s]})|| \\ &= ||\nabla F(\gamma_{[s]})|| \cdot s \cdot ||w(d_{[s]})|| \\ &\leq M_P \cdot s, \end{aligned} \tag{2.16} $$

where $\gamma_{[s]}$ is a point in the line segment joining $d_{[s]} = \varphi_s(t_{[s]})$ and $d_{[s]+1} = \varphi_s(t_{[s]+1})$, and M_P is a constant independent of s. [5] Now, consider

$$ p_s(t) - p_s(t_{[s]}) \;=\; \varphi_s(t) - \varphi(t) - \{\, \varphi_s(t_{[s]}) - \varphi(t_{[s]}) \,\} $$

[5]Combining with the above inequality, we could obtain a positive constant $\overline{M}$ such that $||\eta_s(t)|| \leq \overline{M}$, for all t in $\mathcal{I}$, and for all $s > 0$.

$$= \; \{\, \varphi_s(t) - \varphi_s(t_{[s]}) \,\} - \{\, \varphi(t) - \varphi(t_{[s]}) \,\}$$
$$= \; s \cdot \zeta_{[s]}(t) \cdot w(d_{[s]}) - s \cdot \eta_s(t).$$

With our current notation the equations (2.7) and (2.11) should be written as

$$\varphi_s(t) = \varphi_s(t_{[s]}) + s \cdot \zeta_{[s]}(t) \cdot w(d_{[s]}), \tag{2.7''}$$

$$\varphi_s(t) = \varphi_s(t_{[s]}) + \int_{t_{[s]}}^{t} \frac{w(d_{[s]})}{\nabla F(\varphi_s(\theta)) \bullet w(d_{[s]})}\, d\theta. \tag{2.11'}$$

so that

$$s \cdot \zeta_{[s]}(t) \cdot w(d_{[s]}) = \int_{t_{[s]}}^{t} \frac{w(d_{[s]})}{\nabla F(\varphi_s(\theta)) \bullet w(d_{[s]})}\, d\theta,$$

whereas we know from (2.15) that

$$s \cdot \eta_s(t) = \int_{t_{[s]}}^{t} w(\varphi(\theta))\, d\theta.$$

Then,

$$\begin{aligned} \|\, p_s(t) - p_s(t_{[s]}) \,\| \;&=\; \|\, s \cdot \zeta_{[s]}(t) \cdot w(d_{[s]}) - s \cdot \eta_s(t) \,\| \\ &=\; \left\| \int_{t_{[s]}}^{t} \frac{w(d_{[s]})}{\nabla F(\varphi_s(\theta)) \bullet w(d_{[s]})}\, d\theta \;-\; \int_{t_{[s]}}^{t} w(\varphi(\theta))\, d\theta \right\| \\ &\leq\; \int_{t_{[s]}}^{t} \left\| \frac{w(d_{[s]})}{\nabla F(\varphi_s(\theta)) \bullet w(d_{[s]})} - w(\varphi(\theta)) \right\| d\theta \\ &\leq\; \frac{1}{m} \int_{t_{[s]}}^{t} \left\|\; w(d_{[s]}) - \{\nabla F(\varphi_s(\theta)) \bullet w(d_{[s]})\} \cdot w(\varphi(\theta)) \;\right\|\; d\theta, \end{aligned} \tag{2.17}$$

where $m = \min_{u,v \in B} \{\nabla F(u) \bullet w(v)\}$, which is a positive constant independent of s. By using the triangular inequality, the integrand of the last integal above can be bounded by the sum of three terms $\mathbf{A}, \mathbf{B}$, and $\mathbf{C}$, where

$$\begin{aligned}
\mathbf{A} &= ||\, w(d_{[s]}) - w(\varphi_s(\theta))\, || \\
\mathbf{B} &= ||\, w(\varphi_s(\theta)) - \{\nabla F(u) \bullet w(d_{[s]})\} \cdot w(\varphi_s(\theta))\, || \\
\mathbf{C} &= \{\nabla F(u) \bullet w(d_{[s]})\} \cdot ||\, w(\varphi_s(\theta)) - w(\varphi(\theta))\, ||,
\end{aligned}$$

of which

$$\begin{aligned}
\mathbf{A} &\leq L_w \cdot ||d_{[s]} - \varphi_s(\theta)|| = L_w \cdot ||\varphi_s(t_{[s]}) - \varphi_s(\theta)|| \\
&\leq L_w \cdot ||\varphi_s(t_{[s]}) - \varphi_s(t_{[s]+1})|| \\
&= L_w \cdot s \cdot ||w(d_{[s]})|| \\
&\leq s \cdot K_A,
\end{aligned}$$

where L_w is a Lipschitz constant for the direction field w on the box B, and $K_A = L_w \cdot M_w$ is a constant independent of s. In order to find a bound for the term $\mathbf{B}$, note first that for all t in $\mathcal{I}$

$$\nabla F(\varphi(t)) \bullet w(\varphi(t)) = 1,$$

since $F(\varphi(t)) = t$, and $\varphi'(t) = w(\varphi(t))$. Then,

$$\begin{aligned}
\mathbf{B} &= ||\, \{\nabla F(\varphi(\theta)) \bullet w(\varphi(\theta))\} \cdot w(\varphi_s(\theta)) - \{\nabla F(\varphi_s(\theta)) \bullet w(d_{[s]})\} \cdot w(\varphi_s(\theta))\, || \\
&= |\, \nabla F(\varphi(\theta)) \bullet w(\varphi(\theta)) - \nabla F(\varphi_s(\theta)) \bullet w(d_{[s]})\, | \cdot ||w(\varphi_s(\theta))|| \\
&\leq M_w \cdot |\, \nabla F(\varphi(\theta)) \bullet w(\varphi(\theta)) - \nabla F(\varphi_s(\theta)) \bullet w(d_{[s]})\, |,
\end{aligned}$$

where $M_w = \max_{u \in B} ||w(u)||$. By the triangular inequality,

$$|\, \nabla F(\varphi(\theta)) \bullet w(\varphi(\theta)) - \nabla F(\varphi_s(\theta)) \bullet w(d_{[s]})\, | \quad \leq$$

$$|\, \nabla F(\varphi(\theta)) \bullet w(\varphi(\theta)) - \nabla F(\varphi_s(\theta)) \bullet w(\varphi(\theta))\, | \quad +$$

$$
\begin{aligned}
| \nabla F(\varphi_s(\theta)) \bullet w(\varphi(\theta)) - \nabla F(\varphi_s(\theta)) \bullet w(\varphi_s(\theta)) \,| &\quad + \\
| \nabla F(\varphi_s(\theta)) \bullet w(\varphi_s(\theta)) - \nabla F(\varphi_s(\theta)) \bullet w(d_{[s]}) \,| &\quad \leq
\end{aligned}
$$

$$
\begin{aligned}
M_w \cdot || \nabla F(\varphi(\theta)) - \nabla F(\varphi_s(\theta)) \,|| + M_F \cdot || \, w(\varphi(\theta)) - w(\varphi_s(\theta)) \,|| &\quad + \\
M_F \cdot || \, w(\varphi_s(\theta)) - w(d_{[s]}) \,|| &\quad \leq
\end{aligned}
$$

$$
\begin{aligned}
M_w \cdot L_F \cdot ||p_s(\theta)|| + M_F \cdot L_w \cdot ||p_s(\theta)|| + M_F \cdot s \cdot M_w &\quad = \\
M_I \cdot ||p_s(\theta)|| + s \cdot M_{II},
\end{aligned}
$$

where L_F is a Lipschtz constant for ∇F on B, $M_F = \max_{u \in B} ||\nabla F(u)||$, $M_I = M_w \cdot L_F + M_F \cdot L_w$, and $M_{II} = M_F \cdot M_w$. So we have

$$\mathbf{B} \leq K_B \cdot ||p_s(\theta)|| + s \cdot \overline{K}_B,$$

for some constants K_B and $\overline{K}_B$ independent of s. And, finally

$$
\begin{aligned}
\mathbf{C} &= \{\nabla F(u) \bullet w(d_{[s]})\} \cdot || \, w(\varphi_s(\theta)) - w(\varphi(\theta)) \,|| \\
&\leq M_P \cdot || \, w(\varphi_s(\theta)) - w(\varphi(\theta)) \,|| \\
&\leq M_P \cdot L_w \cdot || \, \varphi_s(\theta) - \varphi(\theta) \,|| \\
&= K_C \cdot ||p_s(\theta)||,
\end{aligned}
$$

where $M_P = \max_{u,v \in B} \nabla F(u) \bullet w(v)$, and $K_C = M \cdot L_w$.

Getting back to (2.17) we have

$$|| \, p_s(t) - p_s(t_{[s]}) \,|| \leq \int_{t_{[s]}}^{t} (K_I \cdot ||p_s(\theta)|| + s \cdot K_{II}) \, d\theta,$$

for some constants K_I and K_{II}, both of which are independent of s. Applying the mean value theorem we have

$$|| \, p_s(t) - p_s(t_{[s]}) \,|| \leq (t - t_{[s]}) \cdot (K_I \cdot ||p_s(c)|| + s \cdot K_{II}),$$

for some $c \in (t_{[s]}, t)$, which in turn becomes

$$|| \, p_s(t) - p_s(t_{[s]}) \, || \leq s \cdot \overline{K}_I \cdot ||p_s(c)|| + s^2 \cdot \overline{K}_{II}, \tag{2.18}$$

by means of (2.16).

Now, given any $\epsilon > 0$, by the uniform continuity of the limit trajectory φ choose $\delta > 0$ such that $|t_1 - t_2| < \delta$ imples $||\varphi(t_1) - \varphi(t_2)|| < \epsilon$. From (2.16) we know that

$$\mu_s \stackrel{def}{=} \max_{k=0,\ldots,\nu(s)} \left(\, t_s^{k+1} - t_s^k \, \right)$$

tends to 0 as s does. Therefore, we could choose s so small that $\mu_s < \delta$. Then we have

$$\begin{aligned} || \, p_s(c) - p_s(t_{[s]}) \, || &\leq || \, \varphi_s(c) - \varphi_s(t_{[s]}) \, || + || \, \varphi(c) - \varphi(t_{[s]}) \, || \\ &\leq || \, \varphi_s(t_{[s]+1}) - \varphi_s(t_{[s]}) \, || + || \, \varphi(c) - \varphi(t_{[s]}) \, || \\ &= s \cdot || \, w(\varphi(t_{[s]})) \, || + || \, \varphi(c) - \varphi(t_{[s]}) \, || \\ &\leq s \cdot M_w + \epsilon, \end{aligned}$$

and consequently,

$$|| \, p_s(c) \, || \leq || \, p_s(t_{[s]}) \, || + s \cdot M_w + \epsilon.$$

The inequality (2.18) now becomes

$$\begin{aligned} || \, p_s(t) - p_s(t_{[s]}) \, || &\leq s \cdot \overline{K}_I \cdot (\, || \, p_s(t_{[s]}) \, || + s \cdot M_w + \epsilon \,) + s^2 \cdot \overline{K}_{II} \\ &= s \cdot \overline{K}_I \cdot || \, p_s(t_{[s]}) \, || + s \cdot \overline{K}_I \cdot \epsilon + s^2 \cdot \overline{K}_{III}, \end{aligned}$$

where $\overline{K}_{III} = M_w + \overline{K}_{II}$ is a constant independent of s. Finally, we arrive at

$$||p_s(t)|| \leq A_s + B_s \, ||p_s(t_{[s]})||, \tag{2.19}$$

where

$$A_s = 1 + s \cdot \overline{K}_I \tag{2.20a}$$

$$B_s = s \cdot (\, \overline{K}_I \cdot \epsilon + s \cdot \overline{K}_{III} \,). \tag{2.20b}$$

Now we use the following lemma, which is stated as Auxiliary Lemma 1.2. on p.18 of Henrici([2, 1962]).

Lemma 1 *If numbers* $\{x_n\}_{n=0}$ *satisfy*

$$|x_{n+1}| \leq A|x_n| + B,$$

where A *and* B *are nonnegative constants independent of* n*, then*

$$|x_n| \leq A^n \, |x_0| + \begin{cases} \frac{A^n-1}{A-1} \, B, & \textit{if } A \neq 1 \\ n \, B, & \textit{if } A = 1 \end{cases}$$

Henrici remarked on the same page (p.18) that in many application of Lemma 1, the number A takes the form of

$$A = 1 + \rho,$$

for a small positive number ρ, in which case the conclusion of Lemma 1 takes the form

$$|x_n| \leq e^{n\rho} \, |x_0| + \frac{e^{n\rho} - 1}{\rho} \, B. \tag{2.21}$$

In applying Lemma 1 in our present discussion, we first note that, although we have started the argument by fixing $t \in (t_0, 0]$, the inequality (2.19) stands for any $t \in (t_0, 0]$, with the understanding that $[s]$ is a function of t. Note further that the terms A_s and B_s are independent of t. Now going back to the argument with a fixed $t \in (t_0, 0]$, we can choose s so small that the numbers $\{\, x_n \,\}_{n=0}^{[s]+1}$ defined by

$$\begin{aligned} x_n &= ||\, p_s(t_s^n) \,||, \quad \text{for } n = 0, \ldots, [s] \\ x_{[s]+1} &= ||\, p_s(t) \,|| \end{aligned}$$

will have the property of

$$|x_{n+1}| \leq A_s |x_n| + B_s,$$

with A_s and B_s given by (2.20). Note also that A_s and B_s may depend on s, but are independent of n, which depends only on s and t. Now since $x_0 = ||\ p_s(t_0)\ || = 0$, we have by means of (2.21) that

$$||\ p_s(t)\ || = x_{[s]+1} \leq \frac{e^{\rho_s \cdot ([s]+1)} - 1}{\rho_s} \cdot B_s, \tag{2.22}$$

where $\rho_s = s \cdot \overline{K}_I$ as in $A_s = 1 + \rho_s$. Now,

$$\begin{aligned} t_s^{k+1} - t_s^k &= F(\varphi_s(t_s^{k+1})) - F(\varphi_s(t_s^k)) \\ &= \nabla F(\gamma_s^k)) \bullet \{\ \varphi_s(t_s^{k+1}) - \varphi_s(t_s^k)\ \} \\ &= \nabla F(\gamma_s^k)) \bullet \{\ s \cdot w(d_s^k)\ \} \\ &\geq s \cdot m_p, \end{aligned}$$

where $m_p = \min\limits_{u,v \in B} \{\ \nabla F(u) \bullet w(v)\ \} > 0$. Then,

$$|F(d)| = \sum_{k=0}^{\nu(s)-1} (\ t_s^{k+1} -\ t_s^k\ ) \geq s \cdot \nu(s) \cdot m_p,$$

so that

$$\nu(s) \leq \frac{K_{IV}}{s}, \tag{2.23}$$

where $K_{IV} = \frac{|F(d)|}{m_p}$ is independent of s. Now, the right hand side of (2.22) is bounded by

$$\frac{e^{\rho_s([s]+1)} - 1}{s \cdot \overline{K}_I} \cdot s \cdot (\ \overline{K}_I \cdot \epsilon + s \cdot \overline{K}_{III}\ ) = (\ e^{\rho_s([s]+1)} - 1\ ) \cdot \frac{\overline{K}_I \cdot \epsilon + s \cdot \overline{K}_{III}}{\overline{K}_I}. \tag{2.24}$$

Finally,

$$\rho_s\ (\ [s] + 1\ ) \leq \rho_s \cdot \nu(s) \leq \frac{K_{IV}}{s} \cdot s \cdot \overline{K}_I = K_V,$$

for a positive constant K_V. Then the right hand side of (2.24) is bounded by

$$K_U \cdot \epsilon + s \cdot K_V,$$

for some constants K_U and K_V, which can be chosen arbitrarily small. $\diamondsuit$

Note that the trajectory φ of the Theorem 2 has the following property given by the equation (2.13). There we see that the derivative φ' at time t is a positive multiple of direction $w(\varphi(t))$ at $\varphi(t)$. This araises the following question: Given a vector field A in $\mathbf{R}^3$ and a initial point $d \in \mathbf{R}^3$, create the vector field B by multiplying a positive scalar to each vector in A. Would the trajectory defined by d under A be identical to the trajectory defined by d under B? We give the condition on positive scalar multiplication under which the answer to above question is "yes" in the following theorem [6].

Theorem 4 *Let $I = [a, b]$ be an interval and let f be defined on $I \times \mathbf{R}^n$ to $\mathbf{R}^n$ by $f = f(t, u) = w(u)$ is Lipschitzian with respect u. For any $h \in ORD$, put $J = h(I) = [h(a), h(b)]$ and define g on $J \times \mathbf{R}^n$ to $\mathbf{R}^n$ such that $g(s, u) = \alpha(u)f(h^{-1}(s), u)$, for some real-valued function α on $\mathbf{R}^n$. If α is positive and Lipschitzian with respect to u, Then for any $d \in \mathbf{R}^n$, the trajectories defined by d under f and g are identical.*

Proof. Let ϕ be the unique function on $I \times \mathbf{R}^n$ to $\mathbf{R}^n$ such that (i) $\phi(a) = d$ (ii) $\phi'(t) = f(t, \phi(t)) = w(\phi(t)), \forall t \in I$. By definition

$$g(s, u) = \alpha(u)f(h^{-1}(s), u) = \alpha(u)w(u)$$

is Lipschitzian w.r.t. u since both α and f are Lipschitzian w.r.t. u. Let ψ be the unique trajectory such that (i) $\psi(a) = d$ (ii) $\psi'(s) = g(s, \psi(s)) = \alpha(\psi(s))f(h^{-1}(s), \psi(s)) = \alpha(\psi(s))w(\phi(s)), \forall s \in J$.

[6] The key assumption of this theorem is that f is a function of a point in space and independent of time.

Now define k on $I \times J$ such that

$$k(t,s) = \frac{1}{\alpha(\psi(s))}.$$

Since α is positive, the above equation is well-defined. Moreover since both α and ψ is Lipschitzian, by appling Henrici's theorem to k we will have a continuously differentiable function $\beta : I \to J$ such that

$$\beta(a) = \bar{a} = h(a),$$

and

$$\beta'(t) = k(t, \beta(t)),$$

for all $t \in I$. Now let $\Psi = \psi \circ \beta$. We claim that $\Psi = \phi$. Indeed, Ψ is defined on I to have values in $\mathbf{R}^b$ and (i)$\Psi(a) = \psi(\beta(a)) = \psi(\bar{a}) = d$. and (ii)

$$\begin{aligned}
\Psi'(t) &= \psi'(\beta(t)) \;\; \beta'(t) \\
&= g(\beta(t), \psi(\beta(t))) \;\; k(t, \beta(t)) \\
&= \frac{\alpha(\psi(\beta(t)) \;\; f(h^{-1}(\beta(t)))}{\alpha(\psi(\beta(t)))} \\
&= f(\, h^{-1}(\beta(t)), \Psi(t)\,) \\
&= f(t, \Psi(t)),
\end{aligned}$$

where the last equality follows since f is function of u only. However ϕ is the unique function which satisfies conditions (i) and (ii). By uniqueness then we have $\Psi = \phi$. $\diamondsuit$

2.4 Covariance of the trajectory solution

As we have discussed in section 2.1, the C^1 assumption on our Pareto-surface-determining-function F is enough to produce the direction field. It was also shown that the direction field thus formed is covariant under DIFF^3 transformations. As we have discovered in section 2.2, however, C^1 smoothness of the Pareto surface may not be enough to determine a trajectory tangential to the direction field. The discussion in section 2.2 provides a sufficient condition that F, in addition to being C^1, has all its partial derivatives locally uniformly Lipschitzian (LUL). This is stronger than C^1 but certainly weaker than a C^2 assumption on F. We may conveniently call it $C^{1.5}$ smoothness.

Therefore, it is not that our trajectory solution is covariant under DIFF^3 transformations, even though the direction field is. When we start with a $C^{1.5}$ smooth Pareto surface and apply a DIFF^3 transformation, then the transformed Pareto surface is only guaranteed to be C^1-smooth, and hence may fail to produce a unique trajectory tangent to the transformed direction field. Therefore, we should limit our transformations to those which preserve the Lipschitzian property.

For this reason we shall refine our definition of DIFF to the following list of subgroups. We shall consider the previously defined DIFF as "1-DIFF", which is a group of order preserving transformations of class C^1. For any positive integer m, we shall define "m-DIFF" to be a subgroup of 1-DIFF, composed of order-preserving transformations of class C^m. When we say DIFF, we shall mean 1-DIFF. Thus

$$m\text{-DIFF} = \{ \ g \in C^m \ \mid \ g' > 0 \ \}, \qquad \text{for any positive integer } m.$$

For any positive integer n, we shall denote by m-DIFF^n a direct product of n copies of m-DIFF:

$$m\text{-DIFF}^n = \{\ g = (g_1, \ldots, g_n) \ \mid\ g_i \in m\text{-DIFF},\ \ i = 1, \ldots, n\ \}.$$

We shall be particularly interested in a subset of 1-DIFF, which is composed only of those transformations whose derivatives are locally uniformly Lipschitzian (LUL). We will name this subset 1-DIFFLIP, (or merely DIFFLIP). That is,

$$\text{DIFFLIP} = \text{1-DIFFLIP} = \{\ g \in \text{DIFF} \ \mid\ g' \text{ is LUL }\}.$$

That this subset of DIFF forms a group, and hence a subgroup of DIFF, is proved in Appendix C. The same argument can be extended to define a similar subgroup m-DIFFLIP of m-DIFF, for any positive integer m, as

$$m\text{-DIFFLIP} = \{\ g \in m\text{-DIFF} \ \mid\ g^{(m)} \text{ is LUL }\},$$

where $g^{(m)}$ is the mth derivative of g. Thus we have the chain of subgroups:

$$\text{1-DIFF} \supset \text{1-DIFFLIP} \supset \text{2-DIFF} \supset \text{2-DIFFLIP} \supset \text{3-DIFF} \supset \text{3-DIFFLIP} \cdots$$

One can easily construct examples to illustrate that every inclusion above is strict. For instance,

$$g(x) = \begin{cases} x & x \le 0 \\ x^2 + x & x > 0 \end{cases}$$

is a member of 1-DIFFLIP, but not a member of 2-DIFF. As usual, we will denote the direct product of n copies of m-DIFFLIP as m-DIFFLIP^n. Also, when we say 1-DIFFLIP, we will often drop "1" to say merely DIFFLIP.

Now we have the terminolgy to state the main result of the present section.

Theorem 5 *Under the same hypothesis of Theorem 1, both the solution* ξ *and its trajectory of the bargaining problem* $(\aleph,\ d)$ *are covariant under DIFFLIP* ^{3} *transformations.*

Proof. Let $g = (g_1, g_2, g_3)$ be in DIFFLIP ^{3}. In considering the transformed bargaining problem $(g(\aleph), g(d))$, note first that the Pareto surface $g(\aleph)$ is defined by the relations $g(v) \geq g(d)$, and $G(v) = 0$, where $G = F \circ g^{-1}$. In particular $G(g(d)) = F(g^{-1}(g(d))) = F(d) = t_0$. Let $\tilde{w}$ denote the direction field of the transformed problem $(g(\aleph), g(d))$. By Proposition 1 of the section 2.1, $\tilde{w}$ on any $v \in g(A)$ is given by

$$\tilde{w}(v) = \frac{1}{m_g(u)}\,(g_1'(u_1)w_1(u),\ g_2'(u_2)w_2(u),\ g_3'(u_3)w_3(u)),$$

where $u = g^{-1}(v)$ and $m_g(u) = \sqrt[3]{g_1'(u_1)g_2'(u_2)g_3'(u_3)}$. By applying Theorem 1 and Theorem 2, we have a unique continuously differentiable function ψ on $\mathcal{I}$ such that $\psi(t_0) = g(d)$, and

$$\psi'(t) = \frac{\tilde{w}(\psi(t))}{\nabla G(\psi(t)) \bullet \tilde{w}(\psi(t))}.$$

By applying the same two theorems to the original untransformed problem $(\aleph, d)$, we had a unique continuously differentiable function φ on $\mathcal{I}$ such that $\varphi(t_0) = d$, and

$$\varphi'(t) = \frac{w(\varphi(t))}{\nabla F(\varphi(t)) \bullet w(\varphi(t))}.$$

We claim that $\psi = g \circ \varphi$.

Let us define a function f on $\mathcal{I} \times \mathbf{R}^3$ to $\mathbf{R}^3$ by

$$f(t, x) = \frac{\tilde{w}(x)}{(\nabla G \bullet \tilde{w})(x)}.$$

Then by the Theorem 1 and Theorem 2 we know that ψ is the unique continuously differentiable function on $\mathcal{I}$ satisfying $(i)\psi(t_0) = g(d)$, and $(ii)\psi'(t) = f(t, \psi(t))$.

Now we consider $\phi = g \circ \varphi$, which is a continously differentiable funtion on $\mathcal{I}$. We verify $(i)\phi(t_0) = g(\varphi(t_0)) = g(d)$, and $(ii)\phi'(t) = f(t, \phi(t))$, so that we will have proved our claim using the uniqueness of ψ. It remains to verify that $(ii)\phi'(t) = f(t, \phi(t))$. On one hand we have

$$
\begin{aligned}
\phi'(t) &= g'(\varphi(t))\ \varphi'(t) \\
&= \begin{pmatrix} g_1'(u_1) & 0 & 0 \\ 0 & g_2'(u_2) & 0 \\ 0 & 0 & g_3'(u_3) \end{pmatrix} \frac{(w_1(u), w_2(u), w_3(u))^T}{\alpha_F(u)},
\end{aligned}
\tag{2.25}
$$

where

$$u = (u_1, u_2, u_3) = \varphi(t),$$

and

$$\alpha_F(u) = (\nabla F \bullet w)(u).$$

On the other hand, if we let $v = g(u) = g(\varphi(t))$ and $h = g^{-1}$, we have

$$
\begin{aligned}
\nabla G(v) &= \nabla(F \circ h)(v) \\
&= \nabla F(h(v))\ h'(v) \\
&= (F_1(u), F_2(u), F_3(u)) \begin{pmatrix} h_1'(v_1) & 0 & 0 \\ 0 & h_2'(v_2) & 0 \\ 0 & 0 & h_3'(v_3) \end{pmatrix},
\end{aligned}
$$

and

$$\tilde{w}(v) = \frac{1}{m_g(u)} \begin{pmatrix} g_1'(u_1) & 0 & 0 \\ 0 & g_2'(u_2) & 0 \\ 0 & 0 & g_3'(u_3) \end{pmatrix} (w_1(u), w_2(u), w_3))^T,$$

so that

$$(\nabla G \bullet \tilde{w})(v)$$

$$= \frac{(F_1(u), F_2(u), F_3(u))}{m_g(u)} \begin{pmatrix} h_1'(v_1) & 0 & 0 \\ 0 & h_2'(v_2) & 0 \\ 0 & 0 & h_3'(v_3) \end{pmatrix} \begin{pmatrix} g_1'(u_1) & 0 & 0 \\ 0 & g_2'(u_2) & 0 \\ 0 & 0 & g_3'(u_3) \end{pmatrix} \begin{pmatrix} w_1(u) \\ w_2(u) \\ w_3(u) \end{pmatrix}$$

$$= \frac{\nabla F(u) \bullet w(u)}{m_g(u)} = \frac{\alpha_F(u)}{m_g(u)},$$

since $h_1'(v_1)\ g_1'(u_1) = 1$, and so on.

Consequently,

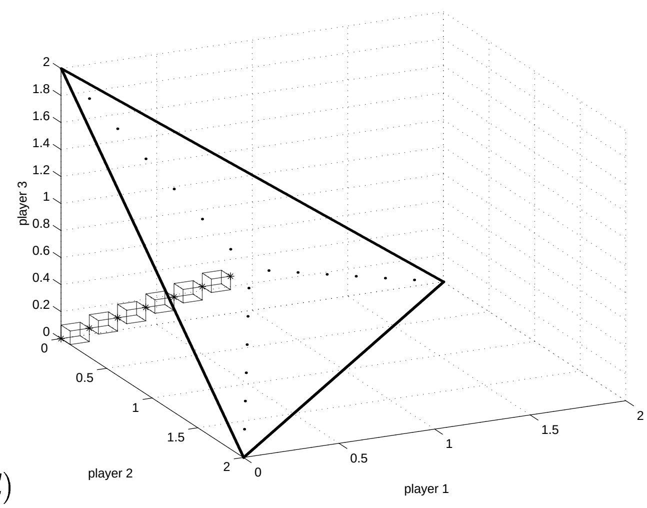
$$\begin{aligned} f(t, \phi(t)) &= f(t, g \circ \varphi(t)) \\ &= f(t, v) \\ &= \frac{\tilde{w}(u)}{(\nabla G \bullet \tilde{w})(u)} \\ &= \frac{\frac{1}{m_g(u)} \begin{pmatrix} g_1'(u_1) & 0 & 0 \\ 0 & g_2'(u_2) & 0 \\ 0 & 0 & g_3'(u_3) \end{pmatrix} (w_1(u), w_2(u), w_3(u))^T}{\frac{\alpha_F(u)}{m_g(u)}} \\ &= \begin{pmatrix} g_1'(u_1) & 0 & 0 \\ 0 & g_2'(u_2) & 0 \\ 0 & 0 & g_3'(u_3) \end{pmatrix} \frac{(w_1(u), w_2(u), w_3(u))^T}{\alpha_F(u)} = \phi'(t), \end{aligned}$$

by the equation (2.25). $\diamondsuit$

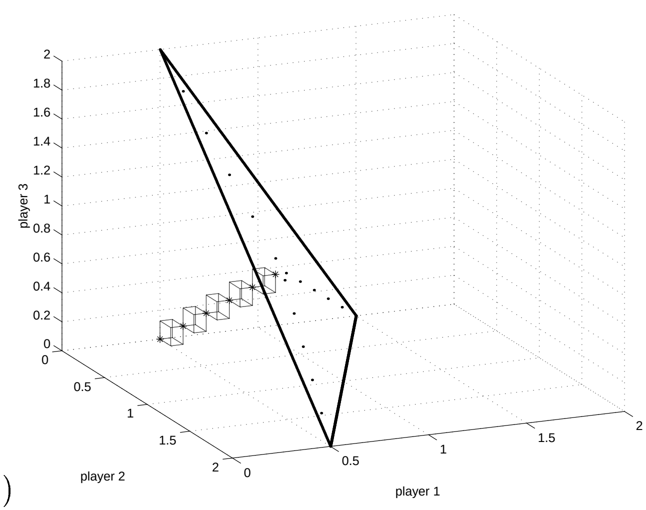
We conclude this section with a few numeical examples illustrating the covariance under DIFF^3 transformations.

*E*xample 1. Let $F(u_1, u_2, u_3) = u_1 + u_2 + u_3 - 2$ and $d = (0, 0, 0)$ define the bargaining problem ($\aleph$, d). Thus $B_1 = (2, 0, 0)$, $B_2 = (0, 2, 0)$, $B_3 = (0, 0, 2)$. By symmetry, $\xi = (2/3, 2/3, 2/3)$ is the solution of $(\aleph, d)$. Suppose player 1 changes her utility scale by means of $g_1(u_1) = \frac{u_1+1}{2}$. The covariance property says $\xi' = (g_1(2/3), 2/3, 2/3) = (5/6, 2/3, 2/3)$ should be the solution of the transformed problem $(\aleph', d')$, where $d' = (g_1(0), 0, 0) = (1/2, 0, 0)$, and $\aleph'$ is the Pareto surface reprented by zeros of $F'(u_1, u_2, u_3) = F(g_1^{-1}(u_1), u_2, u_3) = (2u_1 - 1) + u_2 + u_3 - 2 = 2u_1 + u_2 + u_3 - 3$. Then the three corners of $\aleph'$ are $B_1' = (g_1(2), 0, 0) = (3/2, 0, 0)$, $B_2' = (g_1(0), 2, 0) = (1/2, 2, 0)$, and $B_3' = (g_1(0), 0, 2) = (1/2, 0, 2)$. Our trajectory method applies to $(\aleph', d')$ as follows: $\lambda^1_{23} = F_3'(B_1')/F_2'(B_1') = 1$, $\lambda^2_{31} = F_1'(B_2')/F_3'(B_2') = 2$, $\lambda^3_{12} = F_2'(B_3')/F_1'(B_3') = 1/2$, which gives $w_1 = \sqrt[3]{\lambda^3_{12}\lambda^2_{13}} = \sqrt[3]{1/4}$, $w_2 = \sqrt[3]{\lambda^1_{23}\lambda^3_{21}} = \sqrt[3]{2}$, $w_3 = \sqrt[3]{\lambda^2_{31}\lambda^1_{32}} = \sqrt[3]{2}$. Since all partial derivatives are constant, $(w_1, w_2, w_3) = \sqrt[3]{1/4}(1, 2, 2)$ gives the trajectory direction of a straight line starting at $d' = (1/2, 0, 0)$. The expected solution (5/6, 2/3, 2/3) indeed lies on this trajectory. Shown in the Figure 4 are trajectories and weight boxes of the bargaining problems ($\aleph$, d) and ($\aleph'$, d').

*E*xample 2. Consider a bargaining problem ($\aleph''$, d'') where $d'' = (0, 0, 0)$, and $\aleph''$ is represented by $F''(u_1, u_2, u_3) = u_1 + u_2 + u_3^3 + u_3 - 2 = 0$. Thus, $B_1'' = (2, 0, 0)$, $B_2'' = (0, 2, 0)$, $B_3'' = (0, 0, 1)$. This is a transformed problem of $(\aleph, d)$ given in example 1 by means of $g = (i, i, g_3)$, where i is the identity transformation and $g_3^{-1}(u_3) = u_3^3 + u_3$. By the covariance property, the solution



(a) $(\aleph,\, d)$



(b) $(\aleph',\, d')$

Figure 2.5: Example 1 - Notice that the edges of the boxes for $(\aleph,\, d)$ are in the proportion of 1:1:1, whereas it is 1/2:1:1 for $(\aleph',\, d')$ due to the DIFF transformation $g_1(u_1) = \frac{u_1+1}{2}$ on player 1's utility scale.

of $(\aleph'', d'')$ is $(2/3, 2/3, g_3(2/3))$. i.e. $(2/3, 2/3, \zeta)$, where $\zeta^3 + \zeta = 2/3$. The numerical solution verifies this fact.

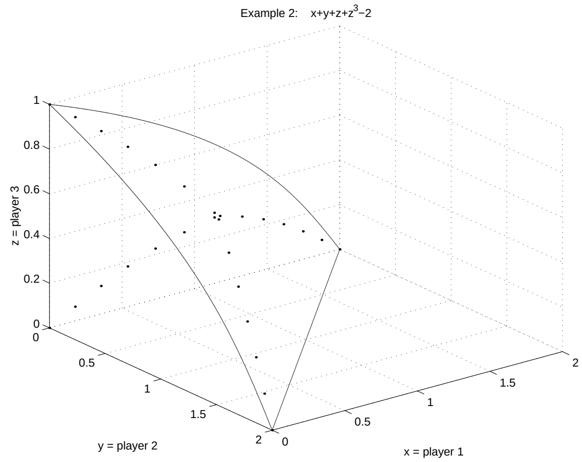


Figure 2.6: Example 2: $s = .1$ is used for this plot.

s	solution
.1	(0.655666, 0.655666, 0.535289)
.01	(0.665563, 0.665563, 0.524546)
.001	(0.666556, 0.666556, 0.523457)

Table 2.1: Numerical solutions to Example 2 when various s are used.

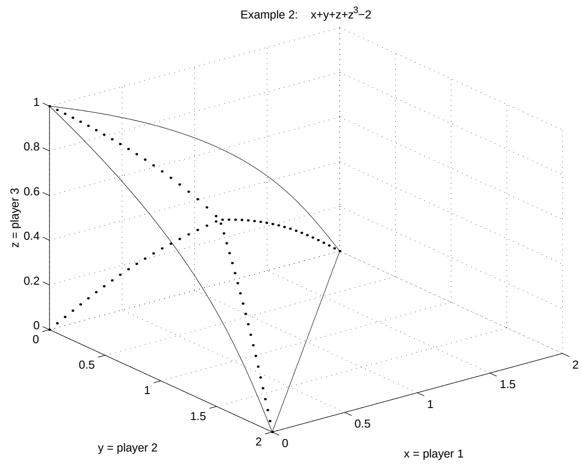


Figure 2.7: Example 2: $s = .03$ is used for this plot.

2.5 Singularities on the Pareto surface ℵ

As mentioned earlier, "nonsmoothness" is allowed in the Pareto surface. Next few examples examine how much this "nonsmoothness" is allowed for our trajectory method. First we take two planes intersecting at a single line in such a way that the resulting surface forms a "ridge" at the intersection. When this surface is projected onto the simplex triangle, the "ridge" will divide the simplex triangle into two regions; let us call them upper and lower regions corresponding to the upper and lower parts of the surface. If the "ridge" is steep enough, the direction field in the upper region will be directed toward the lower region and vice versa. Of course the partial derivatives do not exist along the ridge and hence our method would fail at a point on the ridge. However, since our trajectory is generated by means of finite iteration, it will "hit the ridge" with probability 0. The trajectory near the ridge will constantly go across the ridge. (see Figure 2.8) As the iteration multiplier decreases, the trajectory moving along the surface would be "trapped" along the ridge, and therefore we expect the solution would lie on the ridge as well. Shown in the figure is the example of such surface by taking F to be $F(u_1, u_2, u_3) = max(u_1 + 4u_2 + 5u_3 - 30, 5u_1 + 5u_2 + u_3 - 30)$. Plots on the triangles are trajectories of the three bliss points projected onto the simplex triangle $u_1 + u_2 + u_3 = 1$, along the rays from $d = (d_1, d_2, d_3)$.

If two planes intersect forming a "valley" instead of a "ridge", the effect is exactly the opposite; the direction field diverges away from the "boundary line". In the example presented the disagreement point is at $(0, 0, 0)$ and the Pareto surface is given by two planes intersecting to form a valley. Moreover, one player (player 1) has her bliss point at the intersection of two planes. Her bliss point is a singularity where partial derivatives fail to exist. Clearly our method cannot

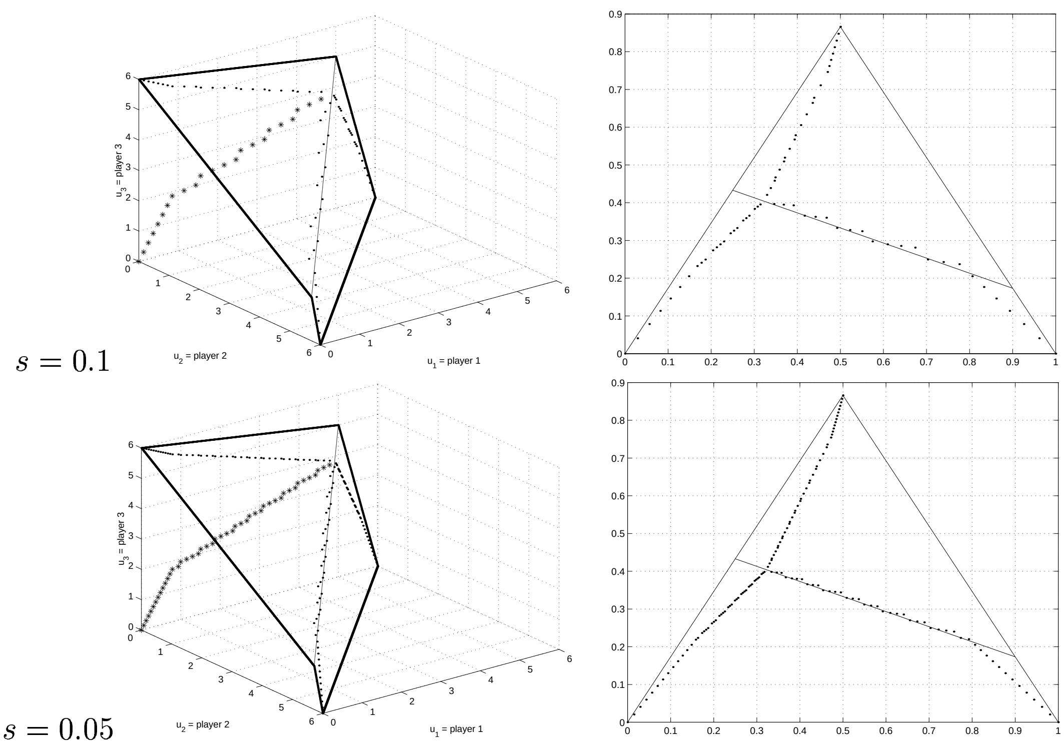
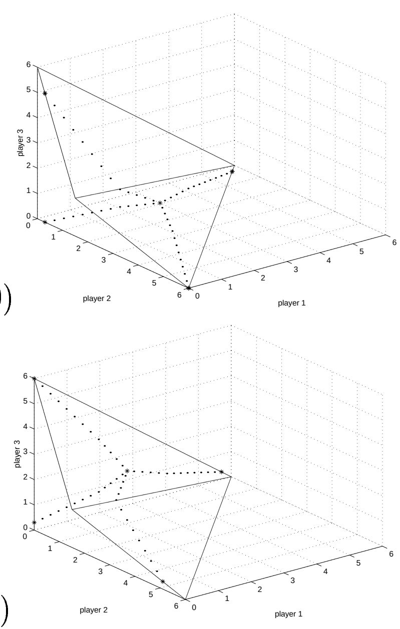


Figure 2.8: Two planes meeting in a line to illustrate a "ridge" type singularity: Plots on the triangles at right are projections of trajectories of three bliss points onto the simplex triangle viewed from below at d.

be applied in this case. However, let us consider a small pertubation of the disagreement point. If we move the disagreement point from $(0,0,0)$ to $(0,\delta,0)$ for arbitrarily small positive number δ (see Figure 2.9(a)), bliss points of player 1 and 3 need to move accordingly, i.e. to a point on the Pareto surface whose 2nd coordinate is δ. Player 1's resulting bliss point is no longer singular and our method is well applied. Note also that her new bliss point lies on the "lower" side of the valley. Due to the "diverging" effect of the valley, the trajectory of her bliss points will stay on the "lower" side and move further away from the intersection of the two planes. As a result, the solution is in the "lower" side of the valley.



(a) $d = (0, \delta, 0)$

(b) $d = (0, 0, \delta)$

Figure 2.9: Singularity of the "valley" type

If we instead perturb the 3rd coordinate of the disagreement from $(0,0,0)$ to $(0,0,\delta)$ (see Figure 2.9(b)), player 1's bliss point now moves up to the "upper" side of the valley, and by following similar reasoning we see that the solution converges to a point on the "upper" side. Hence from this example we learn that having a singlarity of "valley" type on the Pareto surface makes the solution extremely sensitive to the precise location of the disagreement point. As a result, continuity of the solution as a function of the disagreement point is disrupted by having the "valley" type of singularity on the Pareto surface.

The "valley" type singlarity causes further trouble as the next example illustrates. In it, the Pareto surface has a "ridge" and at one end of the ridge begins a "valley". As illustrated in an earlier example (Figure 2.10), the trajectory can move along the ridge, but when it reaches the point where the ridge meets the valley, it won't know which direction to go. Although the finite iteration would make the trajectory hit the singularity with probability 0, trajectories having different iteration multipliers may enter the different side of the valley when they get off the ridge.

s	solution
.1	(6.23424, 2.41569, 7.7478)
.05	(7.08262, 2.3672, 6.71091)

Table 2.2: Numerical solutions to Example 2 when different s are used.

As a result, a radically different solution would be reached using different iteration multiplier, as illustrated in the figure. When the iteration multiplier 0.1 is used the trajectory ends at the left side of the valley, whereas the iteration multiplier 0.05 puts it on the right side of the valley. Here, the Pareto surface is defined as zeros of $F(u_1, u_2, u_3) = max(u_1 + 2.7u_2 + 3u_3 - 36, 3u_1 + 3u_2 + u_3 - 36, G(u_1, u_2, u_3))$, where $G(u_1, u_2, u_3) = min(u_1/10 + u_2/15 + u_3/36 - 1, u_1/36 + y/\eta + u_3/10 - 1)$, and $\eta = 17.9104$.

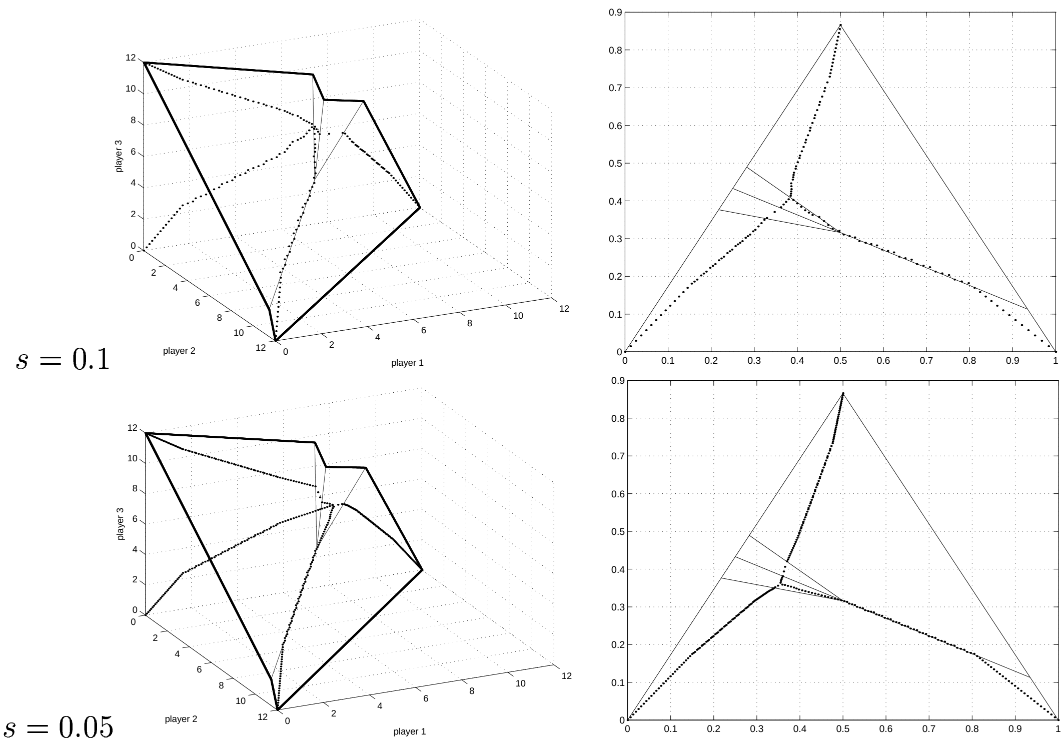


Figure 2.10: Failure of the method: Note that different iteration multipliers s put the solution on the different sides of the "valley".

2.6 Approximation of a piecewise linear Pareto surface by $\mathbf{C}^{1.5}$ surfaces

By a piecewise linear Pareto surface we will mean a surface formed by intersection of two or more, but a finite number of planes. We have discussed a few examples of such surfaces in the previous section. Clearly, these surfaces pose problems in applying our method: at any point $u \in A$, one or more of whose bliss points happen to be on the intersection of the planes, the direction at u cannot be defined. (We have been using the term "singularity" to describe such situations.) As a result the direction field is not continous, and hence Theorem 1, which justifies the existence of our trajectory solution, cannot be applied. Nevertheless, when we discuss the singularity of a "ridge" type in the previous section (see Figure 2.8), we came up with an intuition that the solution should exist on the ridge. The present section intends to give mathematical justification of such intuition.

The method employed here is to place a cylinder at the intersection of two planes in such a way that it is tangent to both planes, and to "round-off" the sharp corner by replacing it by the cylinder. The resulting surface would be C^1, in the sense that the function whose zero describes the surface is C^1. Finding the numerical formula of such function is not too difficult in many cases. Although it is not C^2 along the line of tangency of the cylinder to planes, the "left" and "right" second partial derivatives exist and are finite. Therefore this function has locally uniformly Lipschitzian first partial derivative (i.e. it is of the class $C^{1.5}$.), and therefore existence of the trajectory solution is well justified by Theorem 1. By choosing a different radius of cylinders to be placed, we can have a sequence of $C^{1.5}$ surfaces, each with well defined trajectory solution. We then approximate the given piecewise linear surface by sequence of $C^{1.5}$ surfaces by letting the radii of cylinders approach 0.

To illustrate this idea, we have taken the "ridge" example and round off the sharp edge with cylinders with various radii. Shown in the table are the results when $s = 0.001$ is used with decreasing radii. Figure 2.11 and 2.12 show the plots with $s = 0.1$ with two different radii, $r = 1$ and $r = 0.5$. Note that the "zig-zag" effect given in previous section disappears, precisely because the new direction field is continuous. (Compare with Figure 2.8.)

radius	solution
1	(3.93885, 0.964815, 4.38131)
0.1	(4.17283, 0.920197, 4.4235)
0.01	(4.1965, 0.915755, 4.42753)
0.001	(4.22255, 0.887748, 4.44529)
0.0001	(4.25353, 0.851613, 4.468)
0	(4.25447, 0.849859, 4.46922)

Table 2.3: Numerical solutions when $s = 0.001$ is used

This is an example in which the solution concept as a limit of solutions of approximating surfaces is rather well perceived by intuition. The rest of the present section will attempt to find conditions on the manner that the approximating surfaces converges to the limit surface, so that this limit concept of solution is mathematically justified.

We will begin the argument by letting us be given a disagreement point d in $\mathbf{R}^3$ and a sequence $\{\aleph^n\}_{n=1}^{\infty}$ of Pareto surfaces over d, each of which is characterized by zeros of a real-valued $C^{1.5}$ class function F^n of $\mathbf{R}^3$, all of which take negative value t_0^n at d. Note that we may assume $t_0^n = t_0 < 0$ for all n, since, if not, we can take $\aleph_n$ to be defined by zeros of the functions G^n, where $G^n(u) = \frac{t_0}{F^n(d)} F^n(u)$, then $G^n(d) = t_0$ and $G^n(u) = 0$ if and only if $F^n(u) = 0$, so that the pairs (F^n, d) and (G^n, d) define the same bargaining problem $(\aleph^n, d)$. By applying the Theorem 1 to each bargaining problem $(\aleph^n, d)$, we will have a sequence φ_n of continuously differentiable trajectories on the interval $[t_0, 0]$ with properties of (i) $\varphi_n(t_0) = d$, (ii) $\varphi_n(0) \in \aleph^n, (iii) F^n(\varphi_n(t)) = t$, for all t in $[t_0, 0]$, and (iv) $\varphi_n'(t) = \frac{w^n(\varphi_n(t))}{(\nabla F^n \bullet w^n)((\varphi_n(t))}$, for all t in $(t_0, 0)$. We then search for the conditions under which $\varphi(t)$ converges pointwise on $[t_0, 0]$, including the convergence of $\varphi_n(0)$ which concerns us the most. Evidently, each φ_n is uniformly continous on $[t_0, 0]$. We will do a similar computation to that was involved in the proof of the Theorem 3. For $l = 1, \ldots \infty$ let $\Gamma_l = \{t_0^l = t_0, \ldots, t_l^l = 0\}$ be the partition of $[t_0, 0]$ into l subintervals of equal length. Then for each fixed $t \in (t_0, 0]$ there is an index $[l]$ such that $t_{[l]} < t \leq t_{[l]+1}$ and we can write

$$\varphi_n(t) = \varphi_n(t_{[l]}) + \int_{t_{[l]}}^{t} \frac{w^n(\varphi_n(\theta))}{(\nabla F^n \bullet w^n)(\varphi^n(\theta))} \, d\theta.$$

For $m \neq n$, define

$$p_{m,n}(t) = \varphi_m(t) - \varphi_n(t)$$

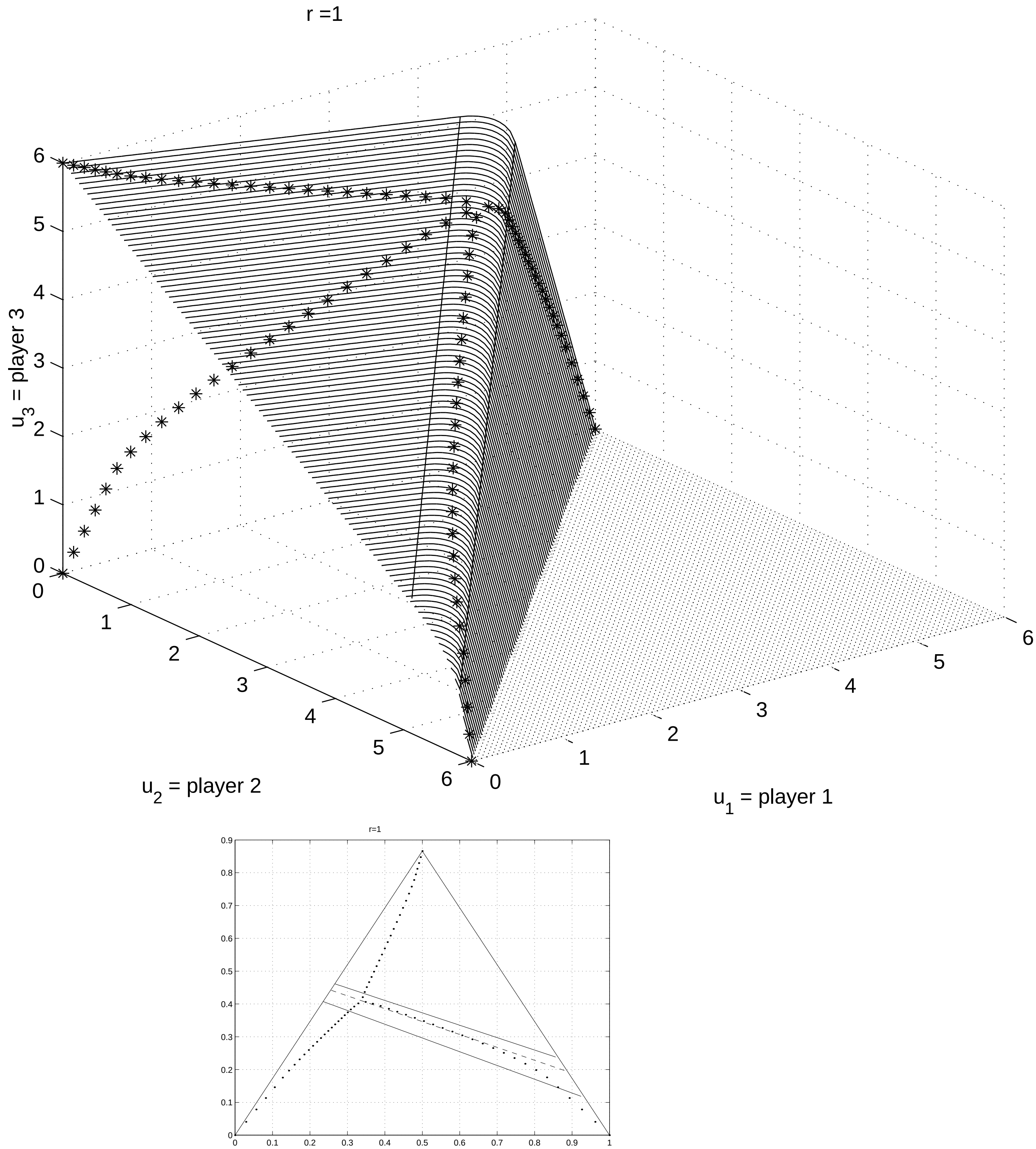


Figure 2.11: Rounding off the ridge with a cylinder of radius 1

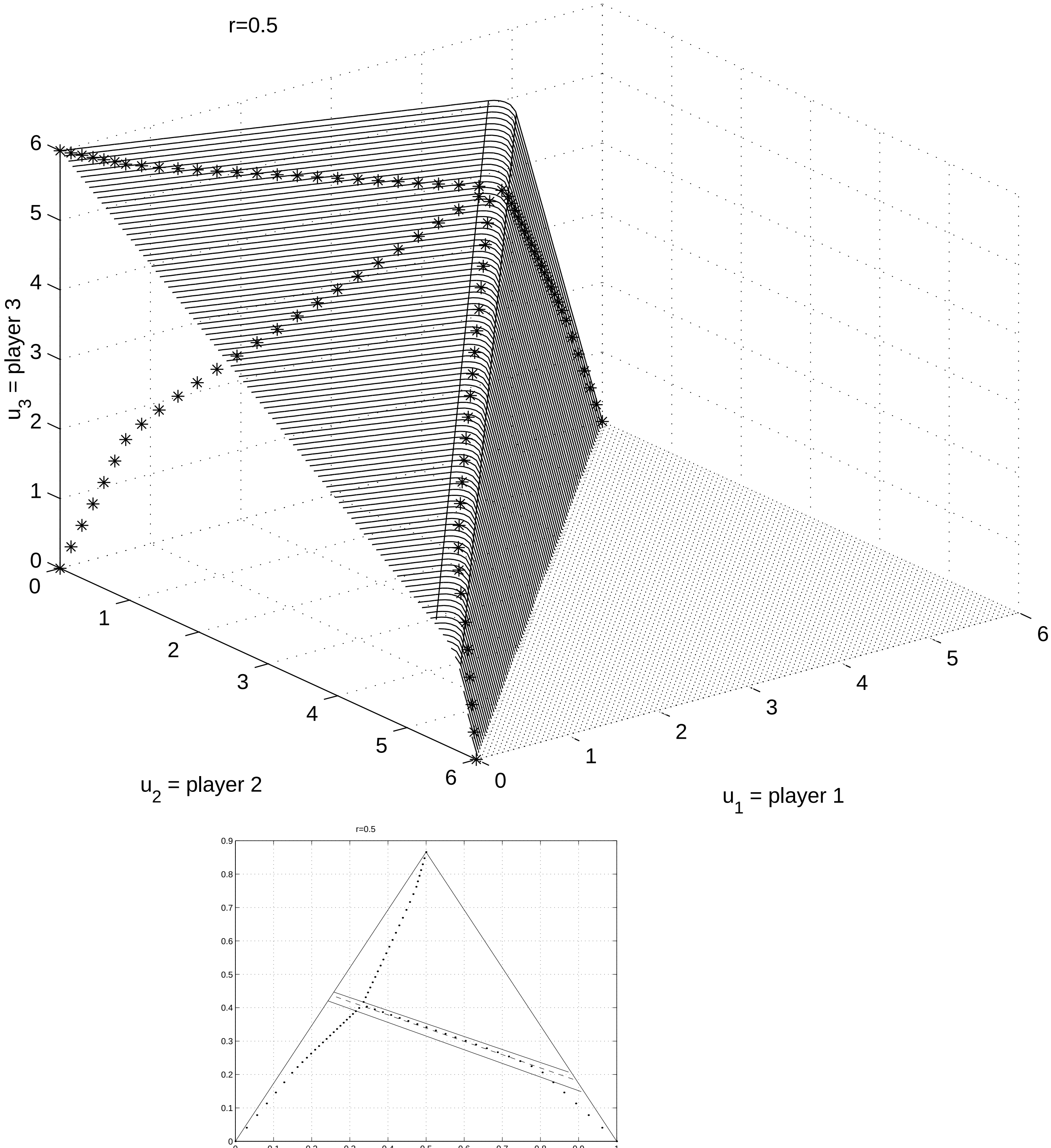


Figure 2.12: Rounding off the ridge with a cylinder of radius 0.5

for $t \in [t_0, 0]$. Then

$$p_{m,n}(t) = p_{m,n}(t_{[l]}) + \int_{t_{[l]}}^{t} \frac{w^m(\varphi_m(\theta))}{(\nabla F^m \bullet w^m)(\varphi^m(\theta))} - \frac{w^n(\varphi_n(\theta))}{(\nabla F^n \bullet w^n)(\varphi^n(\theta))}\, d\theta.$$

We will find a positive bound $M = M(m,n)$ for the integrand, so that

$$||\, p_{m,n}(t) - p_{m,n}(t_{[l]})\, || \leq M\frac{D}{l}, \tag{2.26}$$

where $D = |t_0|$. The left hand side can be made arbitraily small as $l \to \infty$.

Let us write the integrand as

$$\frac{\mathbf{x}_m}{\alpha_m} - \frac{\mathbf{x}_n}{\alpha_n},$$

where

$$\mathbf{x}_m = w^m(\varphi_m(\theta)),$$

$$\alpha_m = (\nabla F^m \bullet w^m)\,(\varphi_m(\theta)),$$

and similar expressions for the index n. We would be able to say that

$$||\, \frac{\mathbf{x}_m}{\alpha_m} - \frac{\mathbf{x}_n}{\alpha_n}\, || \leq C_1 \;\; ||\, \alpha_n \mathbf{x}_m - \alpha_m \mathbf{x}_n\, ||,$$

for some constant C_1 if we have the following

Assumption I There is a compact set B containing all the set of imputations

$$A^n = \{u \in \mathbf{R}^3 \; : \; u \geq d \text{ and } F^n(u) \leq 0\},$$

and there exist positive constants δ_F and Δ_F such that

$$\delta_F \leq F_i^n(u) \leq \Delta_F,$$

for all the partial derivatives $F_i^n(u)$ for all $u \in B$.

Consequently, there exist postive constants $\delta_D, \Delta_D, \delta_W, \Delta_W$ such that

$$\delta_D \leq |\ (F^m \bullet w^m)(\varphi_m(t))\ | \leq \Delta_D,$$

$$\delta_W \leq\ ||w^m(\varphi_m(t))||\ \leq \Delta_W,$$

for all $m = 1, \ldots$ and for all $t \in [t_0, 0]$. So it remains to find a bound for

$$||\ \alpha_n \mathbf{x}_m - \alpha_m \mathbf{x}_n\ ||.$$

By the triangular inequality, the above is bounded by

$$|\alpha_m| \cdot ||\mathbf{x}_m - \mathbf{x}_n|| + ||\mathbf{x}_m|| \cdot |\alpha_m - \alpha_n|. \qquad (*)$$

Now $|\alpha_m| \leq \Delta_D$, and

$$\begin{aligned} ||\mathbf{x}_m - \mathbf{x}_n|| &= ||\ w^m(\varphi_m(\theta)) - w^n(\varphi_n(\theta))\ || \\ &\leq ||\ w^m(\varphi_m(\theta)) - w^m(\varphi_n(\theta))\ || + ||\ w^m(\varphi_n(\theta)) - w^n(\varphi_n(\theta))\ || \\ &\leq L_m\ ||\ \varphi_m(\theta) - \varphi_n(\theta)\ || + ||\ w^m(\varphi_n(\theta)) - w^n(\varphi_n(\theta))\ || \\ &\leq L_m\ ||\ p_{m,n}(\theta)\ || + ||\ w^m(\varphi_n(\theta)) - w^n(\varphi_n(\theta))\ ||, \end{aligned}$$

$$(**)$$

where L_m is a Lipschitz constant for the vector-valued function w_m. We will give a few definitions in order to obtain a bound for the terms of (**).

Definition 1 A sequence $\{f_n\}_{n=1}^{\infty}$ of functions from a metric space X to a metric space Y is said to be *uniformly Lipschitzian* if there exists a constant L, independent of n, such that

$$||f_n(x) - f_n(y)|| \leq L\ ||x - y||,$$

for all x and y in X, and for all $n = 1, \ldots, \infty$.

Assumption II The family $\{\aleph^n\}$ of the Pareto surfaces is uniformly Lipschitzian in the sense that sequence of the partial derivatives $\{F_i^n\}_{i=1,2,3}^{n=1,\ldots,\infty}$ is uniformly Lipschitzian on the compact set B.

With the assumption II we can replace the constant L_m in (**) by another constant L, which does not depend on m. In order to give a bound for the second term in (**), we would like to have the family $\{w^n\}$ of the direction fields be *uniformly Cauchy* on B: i.e., for any given $\epsilon > 0$, there exists M such that $m, n \geq M$ implies that $||w^m(u) - w^n(u)|| < \epsilon$, for all u in B. Then the right hand side of (**) is bounded by

$$L\,||p_{m,n}(\theta)|| + \epsilon.$$

Recall that the direction field at a point $u = (u_1, u_2, u_3)$ is defined in terms of the partial derivatives at the corresponding bliss points. Therefore, given a sequence $\{F^n\}_{n=1,\ldots,\infty}$ of C^1 functions, we would have the corresponding sequences $\{(\lambda_{12}^3)^n\}_{n=1,\ldots,\infty}$, and so on. Write λ^n for $(\lambda_{12}^3)^n$, and consider

$$|\,\lambda^m(u) - \lambda^n(u)\,| = |\,\frac{F_2^m}{F_1^m}(B_3^m) - \frac{F_2^n}{F_1^n}(B_3^n)\,|,$$

where

$$B_3^n = (u_1, u_2, b_3^n),$$

and b_3^n is the image of the point $u_{12} = (u_1, u_2) \in \mathbf{R}^2$ under the C^1 function defined on a neighborhood $U \in \mathbf{R}^2$ of u_{12}, provided by the Implicit Function Theorem, such that

$$F^n(u_1, u_2, b_3^n(u_1, u_2)) = 0.$$

From this we see that in order to have the sequence $\{w^n\}$ uniformly Cauchy on B, we need to have the sequence $\{(\lambda_{jk}^i)^n\}$ uniformly Cauchy on B, which in turn requires not only

Assumption III The sequence $\{F_i^n\}$ of partial derivatives is uniformly Cauchy on B,

but also

Assumption IV The sequence of the local functions $\{b_3^n\}$ of neighborhood of $u_{12} = (u_1, u_2) \in \mathbf{R}^2$ is uniformly Cauchy on $B_{12} \stackrel{def}{=} \{u_{12} | u \in B\}$, and similar conditions for the other coordinates as well.

By the assumptions I through IV, together with the uniform continuity of $p_{m,n}$, it is easy to see that the terms of (*) can be made to be bounded by

$$M(m,n) = A \, ||p_{m,n}(t)|| + B \, \epsilon,$$

where A, B are constants independent of m and n. Then the inequality (2.26) becomes

$$|| \, p_{m,n}(t) - p_{m,n}(t_{[l]}) \, || \leq A_l \, ||p_{m,n}(t)|| + B_l \, \epsilon,$$

where $A_l, B_l \to 0$ as $l \to \infty$. From the above inequlity, we have

$$||p_{m,n}(t)|| \leq \frac{1}{1 - A_l} \, ||p_{m,n}(t_{[l]})|| \; + \; \frac{B_l \, \epsilon}{1 - A_l},$$

and by applying Lemma 1 of the section 2.3, we see that $||p_{m,n}(t)||$ can be made to be arbitrarily small for all t in $[t_0, 0]$. Thus, for any t in $[t_0, 0]$, the sequence $\{\varphi_m(t)\}$ is a Cauchy sequence, and hence converges.

In particular, ξ^n converges to a point ξ. In order to take ξ as a solution of a piecewise linear surface $\aleph$, we need to have ξ lie in $\aleph$. For this purpose we will have further

Assumption V (Monotonicity) For a given piecewise linear surface $\aleph$, we assume that there exist a sequence of $C^{1.5}$ class surfaces $\aleph^n$, which converges to $\aleph$ *weakly monotonically.* increasing manner.

i.e. If $(u_1, u_2, u_3^m) \in \aleph^m$, $(u_1, u_2, u_3^m) \in \aleph^n$, and $m > n$, then $u_3^m \geq u_3^n$.

Assumption VI (Convexity) For each n, $(\aleph^n, d)$ is a *convex* bargaining game.

i.e. $A^n = \{ u \in \mathbf{R}^3 \mid u \geq d, F^n(u) \leq 0 \}$ is a convex set.

We sum up the present section's discussion into the following

Proposition 4 *(Monotone Convergence) Let a piecewise linear surface $\aleph$, be defined by the set of zeros of a* continuous, monotonic *function F. Suppose that there exists a sequence $\{\aleph^n\}$, of class $C^{1.5}$ surfaces each with a well defined solution ξ^n, converges from below in the sense of Assumptions I through VI. Suppose also that ξ^n converges to ξ monotonically, then $\xi \in \aleph$.*

Proof. It only remains to show that $\xi \in \aleph$, or $F(\xi) = 0$. Let us write

$$\xi^n = (\xi_1^n, \xi_2^n, \xi_3^n),$$

$$\xi = (\xi_1, \xi_2, \xi_3).$$

Choose a sequence $\{b_3^n\}$ so that

$$F^n(\xi_1, \xi_2, b_3^n) = 0.$$

By Assupmtion IV, we have

$$F(\xi_1, \xi_2, b_3) = 0,$$

where $b_3 = \lim_{n \to \infty} b_3^n$. So in order to prove $F(\xi_1, \xi_2, \xi_3) = 0$, it is enough to show that $\xi_3 = b_3$. Let us define

$$\zeta^n = (\zeta_1^n, \zeta_2^n, \zeta_3^n) = \frac{1}{2}\,(\xi_1^n, \xi_2^n, \xi_3^n) \;+\; \frac{1}{2}\,(\xi_1, \xi_2, b_3^n).$$

Now choose the sequence $\{p_3^n\}$ such that

$$F(\xi_1^n, \xi_2^n, p_3^n) = 0.$$

We claim that $p_3^n \to b_3$ as $n \to \infty$ and that

$$p_3^n > \xi_3^n \geq \zeta_3^n.$$

Consequently, $b_3 \geq \xi_3 \geq \frac{\xi_3 + b_3}{2}$, which results in our goal of $b_3 = \xi_3$.

From $F^n(\xi_1^n, \xi_2^n, \xi_3^n) = 0$ and the monotonicity (Assumption V), we have

$$F(\xi_1^n, \xi_2^n, \xi_3^n) < 0.$$

But we have

$$F(\xi_1^n, \xi_2^n, p_3^n) = 0.$$

By the monotonicity of F, we have $p_3^n > \xi_3^n$.

Since ξ^n converges to ξ monotonically, we have

$$\xi_1 \geq \xi_1^n \quad \text{and} \quad \xi_2 \geq \xi_2^n.$$

However, we have

$$F^n(\xi_1^n, \xi_2^n, \xi_3^n) = 0,$$

and

$$F^n(\xi_1, \xi_2, b_3^n) = 0.$$

By the fact that F^n defines the Pareto surface $\aleph^n$, we must have $\xi_3^n \geq b_3^n$.

Finally, from $F(\xi_1^n, \xi_2^n, p_3^n) = 0$, and from the continuity of F, we have

$$\lim_{n\to\infty} F(\xi_1^n, \xi_2^n, p_3^n) = F(\xi_1, \xi_2, \lim_{n\to\infty} p_3^n) = 0.$$

But $F(\xi_1, \xi_2, b_3) = 0$, thus $b_3 = \lim_{n\to\infty} p_3^n$, which complete the proof. $\diamondsuit$

In the previous section, we have noted that the *concavity* (opposite of the convexity defined above) of the Pareto surface leads to a direction field which diverges out (see Figure 2.9) We conclude this section by having Proposition 4 provide a sufficient condintion under which smooth surfaces converges to a piecewise linear surface by means of the above propositon. The example introduced at the opening of the present section certainly falls into the categories which Proposition 4 requires. It is not clear whether or not it is possible to approximate a piecewise linear surface which is not convex by means of sequence of $C^{1.5}$ smooth surfaces. The proof or a counterexample is open to further research.

CHAPTER 3

The case $n \geq 4$

When there are 4 bargainers, exchange ratio between two players should be considered at each of the two remaining players' bliss. λ_{12}^3 and λ_{12}^4 denote the exchange ratio from 2 to 1 with respect to 3's and 4's bliss, respectively. The exchange ratio from 2 to 1, denoted by λ_{12}, then takes the geometric mean of these two ratios.

$$\lambda_{12} = \sqrt{\lambda_{12}^3 \lambda_{12}^4}$$

Note that the exchange ratio from 1 to 2 is the reciprocal of that from 2 to 1:

$$\lambda_{21} = \sqrt{\lambda_{21}^3 \lambda_{21}^4} = \frac{1}{\lambda_{12}}.$$

Similarly, when there are $n \geq 4$ bargainers, exchange ratio from player j to i, denoted λ_{ij} will be the geometrical mean of the exchange ratios from j to i at the remaining $n-2$ players' bliss. Thus,

$$\lambda_{ij} = \sqrt[n-2]{\prod_{k \neq i,j} \lambda_{ij}^k}, \qquad \text{for all distinct } i, j. \tag{3.1}$$

When we speak of consistency, we could speak of consistency among 3 players at a time. For $n \geq 4$ and for any distinct players $i, j, k \in \{1, \ldots, n\}$, we may define a positive number,

$$K_{ijk} = \lambda_{jk}^i \lambda_{ki}^j \lambda_{ij}^k,$$

which is analogous to K in the 3-person case. The exchange ratio among the 3 players i, j, k is said to be consistent if $K_{ijk} = 1$. By definition, we have $K_{ijk} = K_{jki} = K_{kij}$ and $K_{ikj} = 1/K_{ijk}$. So, there are total of $2 \times \binom{n}{3}$ such numbers ($\binom{n}{3}$ pairs of reciprocals), which are absolute invariants under the action of the group DIFF^n.

When all of these numbers are equal to 1, we have consistency among all n players and we can define the weight vector $(w_1, \ldots, w_n)$ such that $\frac{w_i}{w_j} = \lambda_{ij}$. This can be achieved by defining

$$w_i = \sqrt[n]{\prod_{j \neq i} \lambda_{ij}}, \qquad i = 1, \ldots, n. \tag{3.2}$$

In order to verify $\frac{w_i}{w_j} = \lambda_{ij}$, it will be convenient to introduce the following two Lemmas, which are proved at the end of the present chapter.

*L*emma 4.1. For $n \geq 4$,

$$\frac{\lambda^l_{ij}}{\lambda^l_{kj}} = \frac{\lambda^l_{jk}}{\lambda^l_{ji}} = \lambda^l_{ij}\lambda^l_{jk} = \lambda^l_{ik}, \qquad \text{for all distinct } i, j, k, l.$$

*L*emma 4.2. For $n \geq 4$,

$$\frac{\lambda_{ij}}{\lambda_{kj}} = \frac{\lambda_{jk}}{\lambda_{ji}} = \lambda_{ij}\lambda_{jk} = \lambda_{ik} \sqrt[n-2]{K_{ijk}}, \qquad \text{for all distinct } i, j, k.$$

Assuming consistency among all n players, Lemma 4.2 reduces to $\frac{\lambda_{ij}}{\lambda_{kj}} = \lambda_{ik}$. Then, we have

$$\frac{w_i}{w_j} = \sqrt[n]{\frac{\prod_{k \neq i} \lambda_{ik}}{\prod_{k \neq j} \lambda_{jk}}} = \sqrt[n]{\frac{\lambda_{ij}}{\lambda_{ji}} \prod_{k \neq i,j} \frac{\lambda_{ik}}{\lambda_{jk}}} = \sqrt[n]{\lambda_{ij}\lambda_{ij}\left(\lambda_{ij}\right)^{n-2}} = \lambda_{ij}.$$

Without consistency, however, the weights must be adjusted, but it is too much to expect that the all 3-person spirals would be made to close down simultaneously. Instead, we are forced to "adjust the adjustments" in a consistent manner; i.e. in a way that is both symmetric among the players and invariant with respect to DIFF^n. For this purpose we introduce the following notation and a few Lemmas, which are also proved in appendix A;

$$K_{ij} \stackrel{\mathrm{def}}{=} \sqrt[n-2]{\prod_{k \neq i,j} K_{ijk}}.$$

*L*emma 4.3. For $n \geq 4$,

$$K_{ijl}K_{jkl}K_{kil} = K_{ijk} \qquad \text{for all distinct } i, j, k, l.$$

*L*emma 4.4. For $n \geq 4$,

$$\frac{\lambda_{ij}}{\sqrt[n]{K_{ij}}} \frac{\lambda_{jk}}{\sqrt[n]{K_{jk}}} = \frac{\lambda_{ik}}{\sqrt[n]{K_{ik}}} \qquad \text{for all distinct } i, j, k.$$

*L*emma 4.5. For $n \geq 4$,

$$K_{ij}K_{ji} = 1.$$

*L*emma 4.6. For $n \geq 4$,

$$\prod_{j \neq i} K_{ij} = 1.$$

Without consistency we see from the definition (2) and Lemma 4.2 above that

$$
\begin{aligned}
\frac{w_i}{w_j} &= \sqrt[n]{\frac{\prod\limits_{k\neq i}\lambda_{ik}}{\prod\limits_{k\neq j}\lambda_{jk}}} = \sqrt[n]{\frac{\lambda_{ij}}{\lambda_{ji}}\prod_{k\neq i,j}\frac{\lambda_{ik}}{\lambda_{jk}}} \\
&= \sqrt[n]{\lambda_{ij}\lambda_{ij}\prod_{k\neq i,j}\lambda_{ij}\sqrt[n-2]{K_{ikj}}} \\
&= \lambda_{ij}\sqrt[n]{K_{ji}} = \frac{\lambda_{ij}}{\sqrt[n]{K_{ij}}}
\end{aligned}
$$

In order to adjust the weights in a consistent manner, therefore, we shall multiply the factor $\sqrt[n]{K_{ij}}$ (or equivalently divide by $\sqrt[n]{K_{ji}}$, see Lemma 4.5) whenever a trade is made from i to j. To illustrate this for $n = 4$, note first that there are $\frac{(4-1)!}{2}$ =3 distinct cycles among 4 players: they are 12341, 12431, 13241. We call them cycle A, B, C, in that order. For cycle A, we define the temporary weights, based on player 1, as

$$
w_1' = 1 \qquad w_2' = \frac{\lambda_{21}}{\sqrt[4]{K_{21}}}
$$

$$
w_3' = \frac{\lambda_{21}}{\sqrt[4]{K_{21}}}\frac{\lambda_{32}}{\sqrt[4]{K_{32}}} = \frac{\lambda_{31}}{\sqrt[4]{K_{31}}} \qquad w_4' = \frac{\lambda_{21}}{\sqrt[4]{K_{21}}}\frac{\lambda_{32}}{\sqrt[4]{K_{32}}}\frac{\lambda_{43}}{\sqrt[4]{K_{43}}} = \frac{\lambda_{41}}{\sqrt[4]{K_{41}}}
$$

where the simpifications have been made according to Lemma 4.4. For symmetrization we define temporary weights based on 2,3,4 in a similar manner, and after simpification we would have

$$
\begin{array}{llll}
w_1'' = \frac{\lambda_{12}}{\sqrt[4]{K_{12}}} & w_2'' = 1 & w_3'' = \frac{\lambda_{32}}{\sqrt[4]{K_{32}}} & w_4'' = \frac{\lambda_{42}}{\sqrt[4]{K_{42}}} \\
w_1''' = \frac{\lambda_{13}}{\sqrt[4]{K_{13}}} & w_3''' = \frac{\lambda_{23}}{\sqrt[4]{K_{23}}} & w_3''' = 1 & w_4''' = \frac{\lambda_{43}}{\sqrt[4]{K_{43}}} \\
w_1^{iv} = \frac{\lambda_{14}}{\sqrt[4]{K_{14}}} & w_2^{iv} = \frac{\lambda_{24}}{\sqrt[4]{K_{24}}} & w_3^{iv} = \frac{\lambda_{34}}{\sqrt[4]{K_{34}}} & w_4^{iv} = 1
\end{array}
$$

This would complete the definitions for the cycle A. For the cycle B=12431, we would define w^v based on player 1 as

$$w_1^v = 1 \qquad w_2^v = \frac{\lambda_{21}}{\sqrt[4]{K_{21}}}$$

$$w_3^v = \frac{\lambda_{21}}{\sqrt[4]{K_{21}}}\frac{\lambda_{42}}{\sqrt[4]{K_{42}}}\frac{\lambda_{34}}{\sqrt[4]{K_{34}}} = \frac{\lambda_{31}}{\sqrt[4]{K_{31}}} \qquad w_4^v = \frac{\lambda_{21}}{\sqrt[4]{K_{21}}}\frac{\lambda_{42}}{\sqrt[4]{K_{42}}} = \frac{\lambda_{41}}{\sqrt[4]{K_{41}}}$$

However, due to the identity from Lemma 4.4, these weights are exactly the same as w' for the cycle A. From this we see that the procedure of multiplying the factor $\sqrt[n]{K_{ij}}$ when going from i to j would produce the same sets of temporary weights independent of the specific cycle. Therefore we may define the "permanent" weights w as the geometric mean of the temporary weights w', w'', w''', w^{iv}. Then

$$w_1 = \sqrt[4]{\frac{\lambda_{12}\lambda_{13}\lambda_{14}}{\sqrt[4]{K_{12}}\sqrt[4]{K_{13}}\sqrt[4]{K_{14}}}} = \sqrt[4]{\lambda_{12}\lambda_{13}\lambda_{14}},$$

since $K_{12}K_{13}K_{14} = 1$ by Lemma 4.6. Similarly, we have

$$w_2 = \sqrt[4]{\lambda_{21}\lambda_{23}\lambda_{24}}, \qquad w_3 = \sqrt[4]{\lambda_{31}\lambda_{32}\lambda_{34}}, \qquad w_4 = \sqrt[4]{\lambda_{41}\lambda_{42}\lambda_{43}},$$

which restores the definition (3.2) above.

We conclude this chapter with a numerical example for the case $n = 4$. Let a bargaining problem $(\aleph, d)$ be given by $d = (0,0,0,0)$, and $F(x,y,z,w) = (x+1)(y+1)(z+1) + (x+1)(y+2)(w+3) - 14$ charcterizing $\aleph$. The four blisspoints are

$$B_1 = (1,0,0,0),$$

$$B_2 = (0, \frac{7}{4}, 0, 0),$$

$$B_3 = (0,0,7,0),$$

$$B_4 = (0, 0, 0, \frac{7}{2}).$$

Shown in the table 4 and 5 are trajectories of disagreement points leading up to the solution when $s = .1$ and $s = .05$ are used.

d_1	d_2	d_3	d_4	$F(d)$
0	0	0	0	-7
0.0350155	0.0539711	0.295612	0.179002	-5.82843
0.0702176	0.108628	0.590293	0.355374	-4.54112
0.105671	0.164094	0.883784	0.528642	-3.13212
0.141426	0.220479	1.17593	0.698428	-1.59503
0.175921	0.27534	1.45377	0.857079	0

Table 3.1: Numerical solutions when $s = .1$ is used

d_1	d_2	d_3	d_4	$F(d)$
0	0	0	0	-7
0.0175077	0.0269855	0.147806	0.0895009	-6.42857
0.0350583	0.0541364	0.295388	0.178374	-5.82891
0.0526612	0.0814693	0.442705	0.266552	-5.20031
0.0703243	0.109	0.589723	0.353975	-4.54204
0.088055	0.136742	0.736417	0.440591	-3.85333
0.10586	0.16471	0.882767	0.526353	-3.13337
0.123743	0.192917	1.02876	0.611219	-2.38134
0.141712	0.221376	1.17438	0.695152	-1.59638
0.159768	0.2501	1.31963	0.778119	-0.77759
0.176332	0.276567	1.45185	0.852933	0

Table 3.2: Numerical solutions when $s = .05$ is used

Proof of Lemmas

*P*roof of Lemma 4.1.

$$\lambda_{ij}^{l}\lambda_{jk}^{l} = \frac{F_j(B_l)}{F_i(B_l)}\,\frac{F_k(B_l)}{F_j(B_l)} = \frac{F_k(B_l)}{F_i(B_l)} = \lambda_{ik}^{l}.$$

*P*roof of Lemma 4.2.

$$\begin{aligned}
\lambda_{ij}\lambda_{jk} &= \sqrt[n-2]{\prod_{l\neq i,j} \lambda_{ij}^{l} \prod_{l\neq j,k} \lambda_{jk}^{l}} \\
&= \sqrt[n-2]{\lambda_{ij}^{k}\lambda_{jk}^{i} \prod_{l\neq i,j,k} \lambda_{ij}^{l}\lambda_{jk}^{l}} \\
&= \sqrt[n-2]{(\lambda_{ij}^{k}\lambda_{jk}^{i})\, \lambda_{ki}^{j}\, \lambda_{ik}^{j} \prod_{l\neq i,j,k} \lambda_{ik}^{l}} \\
&= \sqrt[n-2]{\lambda_{ij}^{k}\lambda_{jk}^{i}\lambda_{ki}^{j} \prod_{l\neq i,k} \lambda_{ik}^{l}} \\
&= \lambda_{ik} \sqrt[n-2]{K_{ijk}},
\end{aligned}$$

where the third equality follows from Lemma 4.1.

*P*roof of Lemma 4.3.

$$\begin{aligned}
K_{ijl}K_{jkl}K_{kil} &= (\lambda_{jl}^{i}\lambda_{li}^{j}\lambda_{ij}^{l})\,(\lambda_{kl}^{j}\lambda_{lj}^{k}\lambda_{jk}^{l})\,(\lambda_{il}^{k}\lambda_{lk}^{i}\lambda_{ki}^{l}) \\
&= (\lambda_{jl}^{i}\lambda_{lk}^{i})\,(\lambda_{kl}^{j}\lambda_{li}^{j})\,(\lambda_{il}^{k}\lambda_{lj}^{k})\,(\lambda_{ij}^{l}\lambda_{jk}^{l}\lambda_{ki}^{l}) \\
&= (\lambda_{jk}^{i}\lambda_{ki}^{j}\lambda_{ij}^{k})(\lambda_{ik}^{l}\lambda_{ki}^{l}) \\
&= \lambda_{jk}^{i}\lambda_{ki}^{j}\lambda_{ij}^{k} = K_{ijk}.
\end{aligned}$$

*P*roof of Lemma 4.4.

By Lemma 4.2, it suffices to show that

$$\frac{\sqrt[n-2]{K_{ijk}}}{\sqrt[n]{K_{ij}}\ \sqrt[n]{K_{jk}}} = \frac{1}{\sqrt[n]{K_{ik}}},$$

or equivalently,

$$\sqrt[n]{K_{ij}K_{jk}K_{ki}} = \sqrt[n-2]{K_{ijk}}.$$

But, by definition,

$$K_{ij}K_{jk}K_{ki} = \sqrt[n-2]{\prod_{l\neq i,j} K_{ijl} \prod_{l\neq j,k} K_{jkl} \prod_{l\neq i,k} K_{kil}}.$$

So it will suffice to verify the term inside the $n-2$ nd root is actually K_{ijk}^n. And since

$$\begin{aligned}
\prod_{l\neq i,j} K_{ijl} \prod_{l\neq j,k} K_{jkl} \prod_{l\neq i,k} K_{kil} &= K_{ijk}K_{jki}K_{kij} \prod_{l\neq i,j,k} K_{ijl}K_{jkl}K_{kil} \\
&= K_{ijk}^3 \prod_{l\neq i,j,k} K_{ijl}K_{jkl}K_{kil} \\
&= K_{ijk}^n,
\end{aligned}$$

where the last equality follows form the Lemma 4.3.

Proof of Lemma 4.5.

Clear from the definition of K_{ij}.

Proof of Lemma 4.6.

$$\begin{aligned}
\prod_{j\neq i} K_{ij} &= \prod_{j\neq i} \sqrt[n-2]{\prod_{k\neq i,j} K_{ijk}} \\
&= \sqrt[n-2]{\prod_{j\neq i} \prod_{k\neq i,j} K_{ijk}} \\
&= \sqrt[n-2]{\prod_{\substack{j\neq i,k \\ k\neq i,j}} K_{ijk}} \\
&= \sqrt[n-2]{\prod_{\substack{j\neq i,k\neq i \\ j<k}} K_{ijk}K_{ikj}} \\
&= 1.
\end{aligned}$$

Appendix A

*P*roof of Proposition 1. Fix $u = (u_1, \dots, u_n) \in A$. Let $w = w(u) = (w_1, \dots, w_n)$ denote the weight vector at u of the bargaining problem $(\aleph, d)$. For $g = (g_1, \dots, g_n) \in \mathrm{DIFF}^n$, let w_g denote the transformation of w under g and let $\tilde{w}$ denote the weight vector of the transformed problem $(g(\aleph), g(d))$ at $\tilde{u} = g(u)$. We need to show that w_g and $\tilde{w}$ are in the same direction.

First, note that w_g is the derivative of g in the direction of w. Therefore,

$$\begin{aligned}
w_g &= \lim_{s\to 0} \frac{g(u+sw) - g(u)}{s} \\
&= (\lim_{s\to 0} \frac{g_1(u_1+sw_1) - g_1(u_1)}{s}, \dots, \lim_{s\to 0} \frac{g_n(u_n+sw_n) - g_n(u_n)}{s}) \\
&= (\lim_{s\to 0} \frac{g_1(u_1+sw_1) - g_1(u_1)}{sw_1}\, w_1, \dots, \lim_{s\to 0} \frac{g_1(u_n+sw_n) - g_n(u_n)}{sw_n}\, w_n) \\
&= (g_1'(u_1)w_1, \dots, g_1'(u_n)w_n).
\end{aligned}$$

To compute $\tilde{w}$, first note that $x \in g(\aleph)$ iff $g^{-1}(x) \in \aleph$, which implies that $F \circ g^{-1}(x) = 0$. For convenience we introduce the notations $\tilde{F} = F \circ g^{-1}$, and $f = g^{-1}$. The second means of course that $f = (f_1, \dots, f_n)$, where $f_i = g_i^{-1}$. According to this notation we have $\tilde{F}(x_1, \dots, x_n) = F \circ g^{-1}(x_1, \dots, x_n) = F(f_1(x_1), \dots, f_n(x_n))$.

Next, in order to use the formulas to compute the weight vector at $\tilde{u} = g(u) = (\tilde{u}_1, \dots, \tilde{u}_n)$, (where $\tilde{u}_i = g_i(u_i)$), we need to evaluate partial derivatives of $\tilde{F}$ at n bliss points of each player "based" on $\tilde{u}$. For this purpose let, for any $k \in \{1, \dots, n\}$, $A_k = A_k(u) = (u_1, \dots, a_k, \dots, u_n)$ denote the bliss point of the kth player "based" on u of the original, untransformed problem $(\aleph, d)$, so that $F(A_k) = 0$. Put $\tilde{A}_k = g(A_k) = (\tilde{u}_1, \dots, \tilde{a}_k, \dots, \tilde{u}_n)$, where $\tilde{a}_k = g_k(u_k)$. Then

$\tilde{F}(\tilde{A}_k) = F \circ g^{-1}(g(A_k)) = F(A_k) = 0$, i.e., $\tilde{A}_k$ is the bliss point of the kth player of the transformed problem $(g(\aleph), g(d))$.

Then, for any $i, k \in \{1, \ldots, n\}, k \neq i$,

$$\begin{aligned}
\tilde{F}_i(\tilde{A}_k) &= \frac{\partial}{\partial x_i} \tilde{F}(x_1, \ldots, x_n) \mid_{x=\tilde{A}_k} \\
&= \frac{\partial}{\partial x_i} F(f_1(x_1), \ldots, f_n(x_n)) \mid_{x=\tilde{A}_k} \\
&= F_i(f_1(\tilde{u}_1), \ldots, f_k(\tilde{a}_k), \ldots, f_n(\tilde{u}_n)) \cdot f_i'(\tilde{u}_i) \\
&= F_i(u_1, \ldots, a_k, \ldots, u_n) \cdot \frac{1}{g'(u_i)} \\
&= \frac{1}{g'(u_i)} F_i(A_k).
\end{aligned}$$

We are ready to compute λ's. We will use the notation $\tilde{\lambda}$'s for the transformed problem $(g(\aleph), g(d))$. So, below all the λ's are evauated at u before transformation, and all the corresponding $\tilde{\lambda}$'s are evaluated at $\tilde{u} = g(u)$.

For any distinct $i, j, k \in \{1, \ldots, n\}$, we have

$$\tilde{\lambda}_{ij}^k = \frac{\tilde{F}_j(\tilde{A}_k)}{\tilde{F}_i(\tilde{A}_k)} = \frac{\frac{1}{g_j'(u_j)} F_j(A_k)}{\frac{1}{g_i'(u_i)} F_i(A_k)} = \frac{g_i'(u_i)}{g_j'(u_j)} \cdot \frac{F_j(A_k)}{F_i(A_k)} = \frac{g_i'(u_i)}{g_j'(u_j)} \lambda_{ij}^k.$$

Then,

$$\tilde{\lambda}_{ij} = \sqrt[n-2]{\prod_{k \neq i,j} \tilde{\lambda}_{ij}^k} = \sqrt[n-2]{\prod_{k \neq i,j} \frac{g_i'(u_i)}{g_j'(u_j)} \lambda_{ij}^k} = \frac{g_i'(u_i)}{g_j'(u_j)} \sqrt[n-2]{\lambda_{ij}^k} = \frac{g_i'(u_i)}{g_j'(u_j)} \lambda_{ij},$$

and finally,

$$\tilde{w}_i = \sqrt[n]{\prod_{j \neq i} \tilde{\lambda}_{ij}} = \sqrt[n]{\prod_{j \neq i} \frac{g_i'(u_i)}{g_j'(u_j)} \lambda_{ij}} = \frac{g_i'(u_i)}{\sqrt[n]{\prod_{j=1}^n g_j'(u_j)}} \sqrt[n]{\prod_{j \neq i} \lambda_{ij}} = \frac{g_i'(u_i)}{m_g(u)} w_i,$$

$$\text{where } m_g(u) = \sqrt[n]{\prod_{j=1}^n g_j'(u_j)}.$$

This shows that

$$\tilde{w} = \frac{1}{m_g(u)} (g_1'(u_1)\, w_1, \ldots, g_n'(u_n)\, w_n),$$

and

$$w_g = (g_1'(u_1)\, w_1, \ldots, g_n'(u_n)\, w_n)$$

are in the same direction. ◇

Appendix B

Proposition 3 A locally uniformly Lipschitzian function f from a metric space X to a metric Y is (globally) Lipschitzian on any compact, connected, convex subset of X.

Proof. Let A be a compact, convex subset of X. For each $x \in A$ we can choose an open ball B_x around x in which there exists a constant L_x such that

$$||f(y) - f(z)|| \leq L_x||y - z||, \quad \forall y, z \in B_x.$$

Then, $\bigcup_{x \in A} B_x$ forms an open cover of A, and therefore there is a finite subcover $B_1, \ldots, B_N$, for some N, and constants $L_1, \ldots, L_N$ such that

$$x, y \in B_i \Longrightarrow ||f(x) - f(y)|| \leq L_i||x - y||, \quad i = 1, \ldots, N.$$

Since A is convex, for any arbitrary two points $\xi, \eta \in A$, we could find a finite sequence of points $\{\zeta_k\}_{k=0}^n$ on the line segments joining ξ and η such that $\zeta_0 = \xi, \zeta_n = \eta$, and that each consecutive ζ_k, ζ_{k+1} , $k = 0, \ldots n-1$, belongs to the same open ball B_i in the finite cover of A. Consequently, $||f(\zeta_k) - f(\zeta_{k+1})|| \leq L||\zeta_k - \zeta_{k+1}||$, where $L = max\{L_i\}_{i=1}^N$. Since there are only N such balls, we have $n \leq N$. Then,

$$\begin{aligned} ||f(\xi) - f(\eta)|| &\leq \sum_{k=0}^{n-1} ||f(\zeta_k) - f(\zeta_{k+1})|| \\ &\leq \sum_{k=0}^{n-1} L||\zeta_k - \zeta_{k+1}|| \\ &\leq L\sum_{k=0}^{n-1} ||\xi - \eta|| = nL||\xi - \eta|| \leq NL||\xi - \eta||. \end{aligned}$$

$\diamondsuit$

Proposition 3.a below sharpens Theorem 1 (See Section 2.3.) We only require locally uniformly Lipschizian property on the set of imputations A. A point y of a set $S \subset \mathbf{R}^n$ can see a point $x \in S$ via S if the line segment $\overline{x}y$ is contained in S. S is said to be starshaped [10] if there exists at least one point $u \in S$ such that each point of S can see u via S. Note that A is a starshaped set with d a point which can see any other point in A.

Proposition 3.a A locally uniformly Lipschitzian function f from a metric space X to a metric space Y is (globally) Lipschitzian on any compact, starshaped subset of X.

proof. Let A be a compact, starshaped subset of X. First we claim that the compactness of A forces a local Lipschitzian property to be an *uniform* Lipschitzian property. More precisely, we claim that there exists $\delta > 0$ and $L > 0$ such that for all ξ and η in A,

$$||\xi - \eta|| < \delta \Longrightarrow ||f(\xi) - f(\eta)|| \leq L||\xi - \eta||.$$

To see this, we apply locally uniformly Lipschitzian (LUL) property to every point $x \in A$ to choose δ_x and L_x such that for any y and z in the open ball $B(x, \delta_x)$ centered at x with radius δ_x, we have $||f(y) - f(z)|| \leq L_x||y - z||$. Then, $\bigcup_{x \in A} B(x, \frac{\delta_x}{3})$ is an open cover of A, with each ball having one third of radius chosen by LUL property. By compactness of A, choose a finite subcover $\bigcup_{i=1}^{n} B(x_i, \frac{\delta_i}{3})$, and let L_i be associatied with x_i. Put $\delta = \min_i \frac{\delta_i}{3}$ and $L = \max_i L_i$. Now choose any ξ and η in A such that $||\xi - \eta|| < \delta$. Then $\xi \in B(x_i, \frac{\delta_i}{3})$ for some i. But since

$$\begin{aligned} ||x_i - \eta|| &\leq ||x_i - \xi|| + ||\xi - \eta|| \\ &< \frac{\delta_i}{3} + \delta \\ &\leq \frac{\delta_i}{3} + \frac{\delta_i}{3} < \delta_i, \end{aligned}$$

both ξ and η belong to the same ball $B(x_i, \delta_i)$. Therefore, by choice of L_i we have

$$||f(\xi) - f(\eta)|| \leq L_i||\xi - \eta|| \leq L||\xi - \eta||,$$

which proves the claim.

Now since A is starshaped, there exists a point $d \in A$ such that the line segment joining d to any point in A is contained in A. We let $\lambda = \sup_{x \in A} ||x - d||$ and $\mathcal{L} = max(\frac{2\lambda}{\delta}L, L)$, and claim the proof of this lemma with 3 times this constant $\mathcal{L}$. That is,

$$||f(x) - f(y)|| \leq 3\,\mathcal{L}||x - y||,$$

for any x and y in A.

So choose x and y arbirtrarily in A. If $||x - y|| < \delta$, we are done by the above claim. So assume $||x - y|| \geq \delta$. Then we can find a point ξ on the line segment joinining x and d, and another point η on the line segment joinining y and d, such that $||\xi - \eta|| < \delta$. Note here that the line segment joining ξ and η needs not to be contained in A. All we require for ξ and η is the inequality $||f(\xi) - f(\eta)|| \leq L||\xi - \eta||$. Now we have,

$$||f(x) - f(y)|| \leq ||f(x) - f(\xi)|| + ||f(\xi) - f(\eta)|| + ||f(\eta) - f(y)||.$$

We are done if we show each of three terms on the right hand side is less than or equal to $\mathcal{L}||x - y||$. Of these, the middle term satisfies the inequality by the choice of ξ and η being less than δ away from each other. Thus, $||f(\xi) - f(\eta)|| \leq L||\xi - \eta|| \leq \mathcal{L}||x - y||$.

To see $||f(x) - f(\xi)|| \leq \mathcal{L}||x - y||$, we can choose sequence $\{x_i\}_{i=0}^N$, with $x_0 = \xi, x_N = x$, such that any two consecuitive points are away from each other by a distance between $\frac{\delta}{2}$ and δ. Then the number of sequence points are less than or equal to $\frac{2\lambda}{\delta}$, and thus

$$\begin{aligned} ||f(x) - f(\xi)|| &\leq \sum_{i=0}^{N-1} ||f(x_i) - f(x_{i+1})|| \\ &\leq \sum_{i=0}^{N-1} L||x_i - x_{i+1}|| \\ &\leq \frac{2\lambda}{\delta} L \, \delta \\ &\leq \mathcal{L}||x - y||. \end{aligned}$$

The last inequlity follows since we are assuming $||x - y|| \geq \delta$. Similarly, $||f(y) - f(\eta)|| \leq \mathcal{L}||x - y||$, and we are done. $\diamondsuit$

Appendix C

Lemma C DIFFLIP is a group.

Proof. Let f and g be two arbitrary member of DIFFLIP. Fix $x \in \mathbf{R}$. Since $\mathbf{R}$ is a locally compact space, and since g' is LUL, we can choose a compact neighborhood N_x of x, and a positive constant L_x such that

$$|g'(x_1) - g'(x_2)| \leq L_x|x_1 - x_2| \quad \text{whenever} \quad x_1, x_2 \in N_x.$$

Put $y = g(x)$. Choose compact neighborhood N_y and a positive constant L_y such that

$$|f'(y_1) - f'(y_2)| \leq L_y|y_1 - y_2| \quad \text{whenever} \quad y_1, y_2 \in N_y.$$

Let $\tilde{N}_x = N_x \cap g^{-1}(N_y)$. Then, whenever $x_1, x_2 \in \tilde{N}_x$, we have $g(x_1), g(x_2) \in N_y$, and so

$$\begin{aligned}
|(f \circ g)'(x_1) - (f \circ g)'(x_2)| &= |f'(g(x_1))g'(x_1) - f'(g(x_2))g'(x_2)| \\
&\leq |f'(g(x_1))g'(x_1) - f'(g(x_2))g'(x_1)| + |f'(g(x_2))g'(x_1) - f'(g(x_2))g'(x_2)| \\
&\leq M_g|f'(g(x_1)) - f'(g(x_2))| + M_f|g'(x_1) - g'(x_2)| \\
&\qquad \text{where } M_g = \max_{N_x} g', \text{ and } M_f = \max_{N_y} f', \\
&\leq M_g L_y|x_1 - x_2| + M_f L_x|x_1 - x_2| \\
&= (M_g L_y + M_f L_x)|x_1 - x_2|.
\end{aligned}$$

This show that DIFFLIP is closed under composition.

To show that DIFFLIP is closed under taking inverse, take $g \in$ DIFFLIP and fix $y \in \mathbf{R}$. Let us write h for g^{-1}, and put $x = h(y)$. Choose a compact neighborhood N_x of x and a positive constant L_x such that

$$|g'(x_1) - g'(x_2)| \leq L_x |x_1 - x_2| \quad \text{whenever} \quad x_1, x_2 \in N_x.$$

Put $N_y = g(N_x)$, which is also compact. Then for any $y_1, y_2 \in N_y$, we have

$$\begin{aligned}
|h'(y_1) - h'(y_2)| &= \left|\frac{1}{g'(x_1)} - \frac{1}{g'(x_2)}\right| \quad \text{where } x_i = h(y_i), \quad i = 1, 2, \\
&= \frac{1}{g'(x_1)g'(x_2)}|g'(x_1) - g'(x_2)| \\
&\leq \frac{1}{m_g^2} L_x |x_1 - x_2| \quad \text{where } m_g = \min_{N_x} g', \\
&= \frac{L_x}{m_g^2}|h(y_1) - h(y_2)| \\
&= \frac{L_x}{m_g^2} h'(c)|y_1 - y_2|, \quad \text{where } c \text{ is a point between } y_1 \text{ and } y_2 \\
&\leq \frac{L_x M_h}{m_g^2}|y_1 - y_2|,
\end{aligned}$$

where $M_h = \max_{N_y} h'$, which completes the proof.

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