A Minimax Procedure for Electing Committees

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Abstract

A new voting procedure for electing committees, called the *minimax procedure*, is described. Based on approval voting (AV), it chooses the committee that minimizes the maximum "Hamming distance" to all voters (*minimax outcome*). Such an outcome may be diametrically opposed to the outcome obtained from aggregating votes in the usual manner, which minimizes the sum of the Hamming distances to all voters (*minisum outcome*). Computer simulation is used to assess how much minimax and minisum outcomes tend to diverge. The manipulability of the minimax procedure is also investigated.

The minimax procedure is applied to the 2003 Game Theory Society (GTS) election of a council of 12 new members from a list of 24 candidates. The 9th and 10th biggest vote-getters would have been displaced by the 16th and 17th biggest vote-getters if the minimax procedure had been used; there would have been more substantial differences if the size of the council had been made endogenous rather than being fixed at 12. It is argued that when few if any voters cast identical AV ballots, as was true in the GTS election (there were $2^{24} \approx 16.8$ million possible ballots), a minimax committee will better represent the interests of all voters than a minisum committee.

A Minimax Procedure for Electing Committees[1]

1. Introduction

In this paper we propose a new voting procedure, called the *minimax procedure*, for electing committees. This procedure is based on approval voting (AV)—whereby voters can approve of as many candidates as they like (Brams and Fishburn, 1978, 1983)—but votes are not aggregated in the usual manner.[2]

Instead of selecting the candidates that get the most votes, the minimax procedure selects the set of candidates that minimizes the maximum "Hamming distance" to all voters. We define and illustrate this metric in section 2, showing how a process called fallback bargaining, with a decision rule of unanimity, yields such a *minimax outcome*.

By contrast, the set of candidates that minimizes the sum of the Hamming distances to all voters, which in fact is just the set of majority winners, yields a *minisum outcome*. We show that the minimax and the minisum outcomes may be unique and diametrically opposed when there are as few as three candidates, given that there may be more than one voter who casts the same AV ballot. If no two voters cast identical ballots, this diametric opposition can occur if and only if there are four or more candidates when ties are permitted, and five candidates when not.[3]

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[1] We thank Eric van Damme, secretary-treasurer of the Game Theory Society (GTS), for providing ballot data for the 2003 council election of the GTS, and Peter C. Fishburn for providing his notes analyzing these data (Fishburn, 2004). We are especially grateful to Shawn Ling Ramirez for research assistance; she is responsible for the computer analysis of the ballot data and simulating all 3-candidate and 4-candidate election outcomes analyzed in section 4.

[2] Merrill and Nagel (1987) distinguish between a balloting method and a procedure for aggregating voter choices on the ballot. Throughout we assume the balloting method is AV; what we will compare are two different ways of aggregating approval votes.

[3] In showing these and other results, we put the more technical proofs in the Appendix, using the text for examples and more informal reasoning.

How often does such a dramatic difference in outcomes occur, and under what conditions? Which outcome is preferable when the minimax and minisum outcomes conflict?

We argue that in elections in which few if any voters cast the same AV ballot, there are good reasons for preferring the minimax outcome. It ensures that no voter is "too far away" from the committee that is elected, whereas the minisum outcome may be one in which some voter (1) *disapproves* of every candidate elected and (2) *approves* of every candidate not elected.

In section 3, we show that the minimax procedure is manipulable in theory, whereas the minisum procedure is not. In practice, however, the minimax procedure is almost certainly not vulnerable because of (i) voters' lack of information about each others' preferences and (ii) the computational complexity of processing such information, even if it were available.

In section 4 we analyze the 2003 Game Theory Society (GTS) election of a council of 12 new members from a list of 24 candidates. There were $2^{24} \approx 16.8$ million possible ballots under AV, because each voter could either approve or not approve of each of the 24 candidates. Given this huge number, it is hardly surprising that all but two of the 161 GTS members who voted in this election cast different ballots.[4]

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[4] If all ballots are assumed equiprobable, the probability that no two (of the 161) voters cast identical ballots is $[(s)(s - 1)(s - 2) \ldots (s - 159)(s - 160)]/s^{161}$, where s is the number of possible ballots (16,774,216 in this case). This follows from the fact that the first voter can cast one of s different ballots; for each of these, if the second voter is to cast a different ballot, there are $(s - 1)$ different ways that he or she can do so; and so on to the 161st voter. This number, divided by the number of possible ballots, s^{161}, gives the probability that no two voters cast identical ballots; the complement of this probability is the probability that at least two voters cast the same ballot. The latter probability in the GTS election was only 0.000768, or less than 1 in 1,000, indicating that it was highly improbable that two or more voters would cast the same ballot, given all ballots are equiprobable (also highly unlikely). In section 4 we will define a more "empirical" probability, based on the numbers of voters voting for different numbers of candidates, and argue that the probability of some identical ballots is much higher. These calculations are similar to those

We show that the minimax procedure would have elected a different council from the minisum procedure when it is mandated that 12 candidates must be elected. If there had not been this requirement but, instead, the size of the council had been endogenous, then the minimax procedure would have elected a council of 10 members. (This figure would have been 9 if the one member who voted for all candidates, and presumably liked everybody, had not voted, illustrating how "outliers" may affect minimax outcomes.) By comparison, the minisum procedure would have elected a council of only 5 members—specifically, the 5 candidates who were approved of by a majority of at least 81 of the 161 voters.[5]

In the GTS election, a committee elected by the minimax procedure may have better represented the interests of *all* voters by ensuring that no voter was more distant from the election outcome than necessary. On the other hand, when there are relatively few candidates and there are, therefore, likely to be identical ballots, the minisum procedure seems preferable, because it takes into account the numbers casting identical ballots (the minimax procedure reflects only the distinct ballots cast, not the numbers casting them).

We conclude in section 5 that the minimax procedure should be used to elect committees when there is a relatively large number of candidates to choose from and, therefore, duplicate ballots are unlikely. Also, instead of fixing the size of a committee,

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used to solve the "birthday problem" in probability theory, which asks how many people must be in a room to make the probability greater than 1/2 that at least two people have the same birthday (the answer is 23 or more).

[5] These projections need to be qualified by the fact that GTS voters knew they were electing a council of 12 new members. If they thought they were electing significantly fewer candidates (say, 5), the average voter would probably have cast fewer votes. As it was, these voters approved an average of 9.8 candidates.

we suggest that its size, within limits, be made endogenous so as better to ensure that the committee reflects the interests of the entire electorate.

Besides professional societies like the GTS, we commend the minimax procedure to colleges, universities, and other organizations that rely substantially on representative committees to make recommendations and decisions. In such organizations, there will often be many members eligible to serve (e.g., the tenured faculty in colleges and universities). In addition, the choice of representative councils, especially in faction-ridden societies, would seem another important application. The minimax procedure should facilitate finding a committee or council that mirrors the diversity of views in the organization or society.

2. Minimax and Minisum Outcomes

Assume there n voters and k candidates. Under AV, a *ballot* is a binary k-vector, $(p_1, p_2, \ldots, p_k)$, where p_i equals 0 or 1. These binary vectors indicate the approval or disapproval of each candidate by a voter. To simplify notation, we write ballots such as (1, 1, 0) as 110, which indicates that the voter approves of candidates 1 and 2 but disapproves of candidate 3. (We also use vectors like 110 to represent election outcomes—that is, the committees that can be chosen by the voters.) Note that the number of possible ballots or outcomes is 2^k.

The *top preference* of a voter is the committee it most prefers. We assume that voters rank all outcomes according to their "Hamming distance" from their top preferences. The *Hamming distance*, $d(p, q)$, between two binary k-vectors, p and q, is the number of components on which they differ.[6] For example, if $k = 3$ and a voter's top preference is 110, the distances, d, between it and the eight binary 3-vectors (including itself) are shown below:

Top Preference	$d = 0$	$d = 1$	$d = 2$	$d = 3$
110	110	100 010 111	000 101 011	001

Observe that three vectors are tied for second place at distance $d = 1$, and three more are tied for third place at $d = 2$. In general, a preference ordering is not strict because of such ties.

To find a minimax outcome, we use a descent process called *fallback bargaining with unanimity* (FB_n):[7]

1. Assume that the voters approve only of committees at distance $d = 0$ from their top preferences—that is, only their top preference is acceptable to them. If one or more committees are approved by all voters, then those with the most approvals are winning, and the process stops.

2. If no committee is approved of by all voters at distance $d = 0$, then consider all committees at distance $d \leq 1$ from the voters' top preferences. If one or more ballots are

[6] Other metrics, such as "root-mean-square," which is essentially Euclidean distance, could be used. We think the Hamming metric is particularly well suited for measuring the distance of a voter from an outcome, because it reflects equally the voter's disagreements with the candidates elected and with those not elected.

[7] Technically, Brams and Kilgour (2001) define fallback bargaining only when preferences form a linear order, which assumes that players have strict preferences. But it is easy to extend this procedure to nonstrict preferences, as we do here. In an electoral context, wherein the decision rule is assumed to be majority rule, this procedure is called the "majoritarian compromise"; see Hurwicz and Sertel (1999), Sertel and Sanver (1999), and Sertel and Yilmaz (1999).

approved by all voters, then those with the most approvals are winning, and the process stops.

3. If no committee is approved by all voters at distance $d \leq 1$, continue the descent until, *for the first time*, one or more committees is approved of by all voters at distance $d \geq 2$. The committees with the most approvals are winning, and the process stops.

Exactly when the process stops depends on, among other things, the number of voters n and the number of candidates k (Brams and Kilgour, 2001). Our first result characterizes winning outcomes under FB_n.

Proposition 1. *The FB_n winners are the* <u>*minimax outcomes*</u>*—they minimize the maximum Hamming distance to the top preferences of all voters*.

Proof. See Brams and Kilgour (2001, p. 292, Theorem 3).

The idea behind the proof is the following. Suppose an FB_n outcome is not a minimax outcome. Then there is some other outcome for which the maximum distance to the top preferences of all voters is less. But the descent under FB_n, which stops at the first point at which all voters approve of some ballot, must stop at this other ballot. Therefore, this other ballot must be a minimax outcome.

It is useful to compare minimax outcomes with outcomes of *majority voting* (MV)—that is, outcomes in which only those candidates who are approved of by a majority of voters are elected.[8]

[8] If there is a tie between the yes (1) and no (0) votes for a candidate, minisum outcomes may either include or exclude this candidate.

Proposition 2. *The MV winners are the minisum outcomes—they minimize the sum of the Hamming distances to the top preferences of all the voters.*

Proof. See Brams, Kilgour, and Sanver (2004, Proposition 4).

The idea underlying the proof is that the MV outcomes minimize the number of disagreements of the voters with the outcome, because at least as many voters agree as disagree with the MV outcome on each candidate. Because the sum of disagreements across the k candidates equals their sum across the voters, the latter—which is the total distance of all voters from their top preferences—is also minimized.

To illustrate the difference between minimax and minisum outcomes, consider Example A, in which there are $n = 5$ voters and $k = 4$ candidates, which yields $2^4 = 16$

Example A about here

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possible AV ballots. Note that the top preferences of the 5 voters (#2, #5, #9, #11, and #15) are all different. They are shown in boldface and also are distinguished by having a "1" in the $d = 0$ column, indicating that one voter makes each of these ballots his or her top preference.

Notice that the first committee approved of by all 5 voters under FB_n is ballot #4 (0010) at $d = 2$, so this is the minimax outcome, though it is not the top preference of any one of the 5 voters.[9] By contrast, the minisum outcome is committee #11 (0011), because a majority of at least 3 voters disapproves of candidates 1 and 2 and approves of candidates 3 and 4. Example A, in fact, establishes the following proposition:

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[9] Minisum as well as minimax outcomes need not be the top preference of any voter, as discussed in Brams, Kilgour, and Zwicker (1997, 1998).

Proposition 3. *A <u>unique</u> minimax outcome may differ from a <u>unique</u> minisum outcome*.

That minimax outcome #4 (0010) and minisum outcome #11 (0011) differ on one candidate (candidate 4), who would win under MV but not under FB_n, is not surprising. More surprising is the fact that the minimax and minisum outcomes may be diametrically opposed. For example, if the minimax outcome is 1100, the minisum outcome may be 0011; such a pair of ballots is called *antipodal*; each ballot is the *antipode* of the other.

Proposition 4. *<u>Tied</u> minimax and <u>tied</u> minisum outcomes may include antipodes*.

Proof. Assume there are 2 voters with the following top preferences for 2 candidates: (1) 00; (2) 11. The minimax outcomes, 01 and 10, are antipodes ($d = 1$ from each voter); these outcomes, as well as outcomes 00 and 11, which are also antipodes, are minisum outcomes, too (total $d = 2$ from the 2 voters).[10] Q.E.D.

Outcomes 01 and 10 lead to the election of just one person. This is not a committee as this term is usually used, but it is easy to show that Proposition 4 holds for the larger tied minimax/minisum committees.

In some of the examples that follow, we will, for reasons of exposition, use antipodes like 0000 (no candidate elected) and 1111 (all candidates elected). These outcomes can readily be converted into antipodes, like 1100 and 0011, that more plausibly reflect the election of committees comprising some but not all candidates.

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[10] Consider a slightly more complicated example comprising 4 voters and 3 candidates: (1) 110; (2) 101; (3) 010; (4) 001. There are four minimax outcomes, 000, 100, 011 and 111, which include two antipodal pairs; all eight outcomes are minisum. Thus, when a minimax/minisum committee increases in size (e.g., from 1 to 2 members in the case of antipodes 100 and 011), the original winner may not be included on the larger committee. This is a failure of *monotonicity*—large committees may not include smaller committees

The necessary and sufficient conditions under which antipodes can be elected depend on (i) the number of candidates and (ii) whether some voters have the same top preference.

Proposition 5. *A* <u>*unique*</u> *minimax and a* <u>*unique*</u> *minisum outcome may be antipodes if and only if there are at least* <u>*3 candidates*</u>.

Proof of "if." Consider Example B, with n = 13 voters, who cast 7 distinct

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Example B about here

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ballots, and k = 3 candidates. The minimax outcome is #1 (000), which occurs at d = 2, whereas the minisum outcome is the antipode, #8 (111), because a majority of 7 of the 13 voters approve of each of the 3 candidates. This 3-candidate example of antipodal outcomes can easily be extended to any larger number of candidates. For the proof of "only if," see the Appendix. Q.E.D.

When, as in Example B, the minimax and minisum outcomes are antipodes, is one more desirable than the other? Because the minisum outcome, #8 (111), is d = 1 from 9 of the 13 voters (69.2%), whereas the minimax outcome, #1 (000), is d = 2 from these voters, 111 will be preferred by more than 2/3 of the voters.

Note, however, that voter #1 is d = 3 from 111. Is it fair that this voter can swing the outcome, in his or her favor, against the wishes of more than 2/3 of the voters?

A judgment on this question depends on how much importance one attaches to improving the lot of the most disadvantaged voter(s), whom minimax helps out. Because

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as subsets—that also characterizes other voting procedures that have been proposed to elect committees in this example voter #1 is the only one of the 13 voters who disapproves of all candidates, perhaps his or her opinion should not be weighed so heavily.

We hasten to add that it is multiple voters with the same top preference that make the difference between the minimax and minisum outcomes in Example B. If there were no multiple voters—if there were one instead of three voters each with top preferences #5 (110), #6 (101), and #7 (011)—then the minisum outcome would be #1 (000) and coincide with the minimax outcome.

This leads us to ask the following: How great can the difference between minimax and minisum outcomes be when all voters have different top preferences? Example A illustrated that these two outcomes may differ on one component when there are 4 candidates. We next show that 4 candidates are necessary and sufficient to give diametric opposition, but the minimax outcome will not be unique in such a situation.

Proposition 6. *When all voters cast different ballots, a* <u>*nonunique*</u> *minimax and a* <u>*unique*</u> *minisum outcome may be antipodes if and only if there are at least* <u>*4 candidates*</u>.

Proof of "if." Consider Example C with $n = 5$ voters and $k = 4$ candidates.

Example C about here

One of the 11 minimax winners is #1 (0000), which occurs at $d = 3$, whereas the unique minisum winner is the antipode, #16 (1111), because a majority of 3 of the 5 voters approve of each candidate. This 4-candidate example of diametric opposition can easily be extended to any larger number of candidates. For the proof of "only if," see the Appendix. Q.E.D.

(Ratliff, 2003).

As in Example B, it is one voter (#1) who prevents the minisum outcome (#16) from being a minimax outcome. But unlike Example B, there are 10 other minimax outcomes whose *total d*'s (i.e., the sums of their Hamming distances from all voters) are 9 or 10, which is less than that of minimax outcome #1 (0000), whose total *d* is 12. Presumably, one of these other ten minimax outcomes would be preferable if, ignoring voter #1, one desires to make all voters as satisfied as possible.

The most stark clash of minimax and minisum occurs when they are not only diametrically opposed but also unique.

Proposition 7. *When all voters cast different ballots, a <u>unique</u> minimax and a <u>unique</u> minisum outcome may be antipodes if and only if there are at least <u>5 candidates</u>.*

Proof of "if." We give an example with $k = 5$ candidates and $n = 11$ voters. Instead of showing each of the 32 possible outcomes in a table and calculating how many voters approve of each outcome at each value of *d*, it is simpler to list the top preferences of the 11 voters, 10 of whom approve of exactly 3 candidates in the $\binom{5}{3} = 10$ different ways that this is possible; voter #11 approves of no candidates:

Example D

1. 11100
2. 11010
3. 11001
4. 10110
5. 10101
6. 10011
7. 01110
8. 01101
9. 01011
10. 00111
11. 00000

That each of the 5 candidates is approved by 6 of the 11 voters proves that outcome 11111 is the unique minisum outcome. The total and maximum Hamming distances of this outcome from the voters are

11111: total $d = 10(2) + 1(5) = 25$; max $d = 5$,

because the first 10 voters are $d = 2$ from 11111, and the 11th voter is $d = 5$ from this outcome.

Similarly, these values for the other distinct outcomes are given below. For each outcome representing a committee of from 0 to 4 candidates, we carry out only one calculation, because the other outcomes yield the same total d and max d by symmetry (e.g., the calculation for 10000, given below, is identical to the calculation for 00001). Note that the summands are shown in the order of the above listing of voters, from 1 to 11:

00000: total $d = 10(3) + 1(0) = 30$; max $d = 3$;

10000: total $d = 6(2) + 4(4) + 1(1) = 29$; max $d = 4$;

11000: total $d = 3(1) + 6(3) + 1(5) + 1(2) = 28$; max $d = 5$;

11100: total $d = 1(0) + 4(2) + 1(4) + 2(2) + 1(4) + 1(3) = 27$; max $d = 4$;

11110: total $d = 2(1) + 1(3) + 1(1) + 2(3) + 1(1) + 3(3) + 1(4) = 26$; max $d = 4$.

Clearly, the unique minimax outcome is 00000 (max $d = 3$), the antipode of the unique minisum outcome, 11111. This 5-candidate example of diametric opposition can easily be extended to any larger number of candidates. For the proof of "only if," see the Appendix. Q.E.D.

We think the minimax outcome in Example D, #1 (00000), where max $d = 3$ and total $d = 30$, is at least as defensible as the minisum outcome, #16 (11111), where max $d = 5$ and total $d = 25$. The minimax outcome brings voter #11 40% closer than the minisum outcome (from 5 to 3), whereas the minisum outcome decreases the total, or average, distance of voters by 17% (from 30 to 25).

Of course, the 0's and 1's in Example D can be permuted to give different opposites from 00000 (elect nobody) and 11111 (elect everybody). Thus, we might get opposites like 11000 and 00111. If, however, the results were those in Example D, and we wanted a committee of either 2 or 3 members, minimax would help us choose one: 11100 would be preferable to 11000 not only because its max d is lower (4 versus 5) but also because its total d is lower (27 versus 8). In fact, because of the symmetry in this example, *any* 3-member committee would be preferable to any 2-member committee, based on both the minimax and minisum criteria.

Our examples so far raise the following question: When is it better to minimize (i) the sum of the distances from an outcome (MV winner) or (ii) the maximum distance of voters from an outcome (FB_n winner), given there is a conflict? If the goal is to avoid antagonizing any player "too much," there are good grounds for choosing the FB_n (minimax) outcome, especially if there are few or no instances of voters having the same top preference, as in Example B.

If there are such voters, however, their numbers can largely determine the MV (minisum) outcome. For example, assume there is 1 voter with top preference 10, and 99 voters with top preference 01. Then the two minimax outcomes are 00 and 11 (total $d = 100$; max $d = 1$), whereas the minisum outcome is 01 (total $d = 2$; max $d = 2$).

We think the minisum outcome, which is sensitive to the numbers of voters with different preferences, is certainly more compelling in this example than the two minimax outcomes, which are insensitive to the preponderance of 01 voters. On the other hand, when all or almost all voters cast different ballots, as was the case in the Game Theory Society election we will analyze in section 4, minimax outcomes make more sense—they give outcomes closer to the most distant, and least represented, voters.

3. Manipulability and Endogeneity

A voting procedure is *manipulable* if a voter, by misrepresenting his or her preferences, can obtain a preferred outcome. To determine what is "preferred," we require only the following plausible assumption about preferences over sets: If a is preferred to b_1 and a is preferred to b_2, then $\{a\}$ is preferred to $\{a, b_1, b_2\}$.

Proposition 8. *The minimax procedure is manipulable, whereas the minisum procedure is not*.

Proof. We start with the minisum procedure. Because a voter's choices are binary on each candidate, it is always in his or her interest to support only those candidates of whom he or she approves. Moreover, because a voter's decision on each candidate does not affect which other candidates are elected, each voter has a dominant strategy of voting sincerely on *all* k candidates.

Now consider the minimax procedure. In Example E, there are $n = 4$ voters and

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Example E about here

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and $k = 4$ candidates, which yields 16 possible outcomes, but not all are shown. For the "true top preferences" of the 4 voters in Example Ea, there are three minimax winners {#1 (1000) = a, #5 (0000) = b_1, and #6 (1001) = b_2}, for each of which max $d = 2$.

If the voter whose top preference is #1 (1000) in Example Ea misrepresents his or her top preference as #5 (1110) in Example Eb, then the *unique* minimax winner is outcome #1 (1000) = a, which is in fact this voter's true top preference.[11] Thus, by indicating an insincere top preference, this voter can induce {a}, which he or she prefers to the sincere outcome, {a, b_1, b_2}. Hence, the minimax procedure is manipulable. Q.E.D.

In theory, therefore, the minimax procedure is vulnerable to manipulation. In practice, however, this procedure is probably almost as resilient to manipulation as the minisum procedure, because its possible exploitation would require that a manipulative voter have virtually complete information about the voting intentions of other voters, which is unlikely in most real-world situations.

We next consider situations in which the number of candidates to be elected is fixed rather than left open. For example, suppose that a committee of exactly 2 of the 4 candidates in Example A is to be chosen. Given this stipulation, 4 of the 6 possible 2-winner outcomes are minimax (#7, #9, #10, and #11, all of which have "5" in the $d = 2$ column of Example A), but only outcome #11 is minisum (total $d = 7$).

Now suppose there is more flexibility, and a committee of either 2 or 3 members is acceptable. The fact that every committee of size 2 has a total $d = 10$, and every committee of size 3 has a total $d = 9$, indicates that a committee of size 3 is preferable,

[11] Outcome 1000 also happens to be the unique minisum winner.

based on the minisum criterion. In effect, this criterion can be used to break ties among the minimax outcomes.[12]

When the committee size is *not* restricted to 2 or 3 members, a phantom committee of size 0 in Examples C and D—with total d's of 8 and 27, respectively—is the minisum outcome. We think a restriction is appropriate if there are good reasons for a committee to be of a certain size, or within some range of sizes.

The election of 1-person committees raises the question of whether it is possible for minimax and minisum outcomes to be different in single-winner elections. Our previous examples of different minimax and minisum outcomes, including antipodal ones, do not provide an answer, because they all involved different numbers of winners or tied winners.

Proposition 9. *If the size of a committee is fixed, a <u>unique</u> minisum outcome may be different from all <u>nonunique</u> minimax outcomes.*

Proof. Consider Example F, in which there are $n = 4$ voters and $k = 4$ candidates and a committee of size 1 is to be elected.

Example F

1. 1100
2. 1010
3. 1001
4. 0111

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[12] Recall in Example D (section 2) that it was a committee of size 3, rather than 2, that also better satisfied both the minimax and the minisum criteria. Lest one think that only larger committees better satisfy these criteria, however, assume that there are 3 voters who vote as follows: (1) 100; (2) 010; (3) 011. Suppose it is stipulated that either a (degenerate) committee of size 1, or a committee of size 2, will be chosen. While both 010 and 110 are minimax outcomes (max $d = 2$), only 010 is minisum (total $d = 3$ versus total $d = 4$ for 110). Hence, it is the smaller committee in this example that better satisfies both criteria.

It is easy to see that 1000 is the unique minisum outcome (total $d = 7$; max $d = 4$), whereas outcomes 0100, 0010, and 0001 are the (tied) minimax outcomes (total $d = 9$; max $d = 3$).[13] By adding candidates approved of by all voters, this example can be extended to any fixed committee size. Q.E.D.

Because candidate 1 is the only candidate approved of by a majority of 3 of the 4 voters in Example F, he or she would seem to be the obvious social choice in a single-winner election. However, not only does voter #4 not approve of this candidate, but he or she approves of the other 3 candidates, putting #4 at a distance of $d = 4$ from outcome 1000. On the other hand, no voter is more than a distance of $d = 3$ from the three other single-winner outcomes (0100, 0010, 0001), making each of these more of a "compromise" choice.

It may seem bizarre not to elect the most approved candidate, especially one approved of by a majority, in a single-winner election. We will revisit this issue in the concluding section, asking whether the minimax criterion is reasonable to apply to single-winner elections.

We next turn to a real-world election that suggests how the minimax procedure might be used in practice. This election renders concrete some of the theoretical and practical issues we have discussed and raises some new questions as well.

4. The Game Theory Society Election and Computer Simulations

In 2003, the Game Theory Society (GTS) used AV to elect 12 new council members from a list of 24 candidates. (The council comprises 36 members, with 12

[13] If there were no single-winner restriction, the election of candidate 1 and any one, or any two, of the other three candidates (i.e., outcomes 1100, 1010, 1001, 1110, 1101, and 1011) are tied minimax outcomes

elected each year to serve 3-year terms.) The method of aggregation was the standard one of electing the 12 biggest vote-getters, which yields the minisum outcome when the outcome is restricted to 12 candidates. This is shown by the following proposition, which is related to Proposition 2.

Proposition 10. *When the size of a committee is restricted to c members, a minisum outcome is every set of c candidates that receives the most votes.*

Proof. See Appendix.

The idea behind the proof is the following. We know from Proposition 2 that when there is no restriction, the minisum outcome is the set of candidates that win a majority of votes. Assume that the number of majority winners is less than the desired committee size, *c*. Then add to the majority winners those candidates with the largest, next-largest, etc., number of approvals—but not a majority—until the committee size is exactly *c*. The difference between the number of approvals of each added new member and a majority is the number of disagreements of this new member with the outcome.

The sum of these disagreements for the new members is the total Hamming distance added to the original minisum outcome. Because of the way in which new members are added, the committee comprising the original majority winners, plus the new members, minimizes the sum of the Hamming distances among all committees of size *c*. If the number of the original majority winners is greater than the desired committee size, *c*, subtracting the candidates with the fewest approvals to reduce the committee size to *c* gives an analogous minimization result.

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that are, like outcome 1000, also minisum (total $d = 7$; max $d = 3$).

If there had been no restriction in the GTS election, the 5 candidates that were approved by a majority—at least 81 of the 161 members who voted (45% of the membership of the GTS)—would have been elected. Adding the next 7 biggest vote-getters gives the minisum outcome under the restriction that 12 candidates must be elected.

But was this the most representative set of new members, based on the minimax criterion? Examining the $2^{24} \approx 16.8$ million possible ballots that can be cast under AV, we found that a different council would have been elected under the minimax procedure.

In fact, there were 199 different 12-member councils for which the maximum Hamming distance,14 in this instance, was minimal. Applying the minisum criterion to these councils yielded a minimax council in which the 9th and 10th biggest vote-getters would have been replaced by the16th and 17th biggest vote-getters.[14]

We give next the different minisum and minimax outcomes for a council restricted to 12 members. The candidates are ordered from those who get the most votes (on the left) to those that get the fewest votes (on the right); differences between those elected to each council are underscored.

Council Restricted to 12 Members

[14] It is worth pointing out that minisum outcomes like this are always Pareto-optimal; if this were not the case, then there would be some other outcome such that some voter is less distant and no voter is more distant, contradicting the supposition that the original outcome is minisum. Minimax outcomes, however, need not be Pareto-optimal. To illustrate, assume the top preferences of 4 voters for 3 candidates are as follows: (1) 100; (2) 010; (3) 110; (4) 011. There are four minimax outcomes, each a maximum distance of 2 from each voter: (a) 000; (b) 110; (c) 010; (d) 111. Because outcome (b) is at least as good as outcome (a) for all voters, and better for voter (3), and outcome (c) is at least as good as outcome (d) for all voters, and better for voter (2), only outcomes (b) and (c) are Pareto-optimal. Although these two outcomes happen to be the minisum outcomes (total $d = 4$), Pareto-optimal outcomes need not be minisum. This is illustrated by Example Ea, in which the three minimax outcomes (1000, 0000, 1001) are all Pareto-optimal, but only outcome 1000 is minisum (total $d = 4$, as opposed to total $d = 5$ for the other two minimax outcomes).

Minisum: 111111111111000000000000 (total *d* = 1516; max *d* = 17).

Minimax: 111111110011000110000000 (total *d* = 1584; max *d* = 14).

Observe that the percentage decrease in max *d* from 17 to 14 (17.6%) by the aforementioned replacement is four times greater than the percentage increase in the minisum value from 1516 to 1584 (4.5%), suggesting a trade-off that favors a minimax over a minisum council.

If the size of the council had not been restricted to 12 winners but instead had been endogenous, the minisum and minimax councils would have been quite different:

Unrestricted Council (Minisum, 5 Members; Minimax, 10 Members)

Minisum: 111110000000000000000000 (total *d* = 1381; max *d* = 19).

Minimax: 111111110010000100000000 (total *d* = 1506; max *d* = 14).

As noted earlier, the minisum council would have comprised only the 5 majority winners. By contrast, councils of 10, 11, 12, and 13 members all minimize the maximum Hamming distance of 14. Applying the minisum criterion to these yields a 10-member council on which, in addition to the 5 majority winners, the next 3 biggest vote-getters (6th, 7th, and 8th), and the 11th and 16th biggest vote-getters, are added. Observe that the percentage decrease in max *d* from 19 to 14 (26.3%) by these additions is almost three times greater than the percentage increase in the minisum value from 1381 to 1506 (9.1%), again suggesting a trade-off that would favor a minimax over a minisum council.

To shed further light on the GTS election, we give below the numbers of voters who voted for from 1 to all 24 candidates (no voters voted for between 19 and 23 candidates):

Votes cast	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	24
# of voters	3	2	3	10	8	6	13	12	21	14	9	25	10	7	6	5	3	3	1

Casting a total of 1574 votes, the 161 voters approved, on average, $1574/161 \approx 9.8$ candidates; the median number of candidates approved of, 10, is almost the same. It is not surprising, therefore, that the (unrestricted) minimax outcome is also 10 candidates.

The modal number of candidates approved of is 12 (by 25 voters), perhaps due to the fact that the instructions to members on the ballot stipulated that 12 of the 24 candidates were to be elected. The approval of candidates ranged from a high of 110 votes (68.3% approval) to a low of 31 votes (19.3% approval). The average approval received by a candidate was 40.7%, though only candidates who received at least 69 votes (42.9 % approval) were elected.

Fishburn (2004) shows that the 12 winners tended to be supported somewhat more strongly by voters who voted for few candidates, whereas the reverse was true for the losers. In effect, voters who approved of few candidates were more discriminating, helping to put the winners over the top.

The ballot data shed light on the legitimacy of comparing the minisum and minimax outcomes (with and without the restriction to 12 candidates). As we argued earlier, unless there are few if any identical ballots, the minisum procedure, which takes into account the numbers of voters casting identical ballots, seems preferable.

It turns out that only 2 of the 161 voters cast identical ballots. As one might expect, the identical ballot, 111100011001101000000111, was cast by 2 of the 25 modal voters who voted for 12 candidates. If all ballots approving of 12 candidates are assumed equiprobable, the probability that no two of 25 ballots are identical is

$$[t(t-1)(t-2)\ldots(t-24)]/t^{25} \approx 0.999889,$$

where $t = \binom{24}{12} = 2{,}704{,}156$, based on reasoning similar to that given in note 4. The complement of this probability, 0.000111, is the probability that at least two voters cast identical ballots.[15]

We showed in section 3 how it was possible, in theory, for a voter successfully to manipulate a minimax outcome, but we contended that this would be well-nigh impossible in most elections. As a case in point, consider the single voter who voted for all 24 candidates in the GTS election and who, we presume, was indifferent among all candidates.[16]

But might this voter have influenced the outcome if the size of the council had been endogenous? In fact, if minisum had been the criterion, the 5 biggest vote-getters would still have been elected. But the minimax outcome would have changed

From: $1111\underline{1}11100\underline{01}00000\underline{1}0\underline{0}0000$ (with voter who approved of all candidates)

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[15] We have made this calculation for each category of voter—from those who cast 1 vote to those who cast 18 votes—excluding only the category containing the one voter who voted for all 24 candidates (there is only one such ballot). The most likely voters to cast an identical ballot are the 3 voters who vote for one candidate; the probability that at least 2 of them cast the same ballot is 0.117905. To generalize for all voters, let p_i be the probability that no two voters who cast i votes chose an identical ballot. It follows that the probability that no two voters in any category cast an identical ballot is $p_1p_2\ldots p_{17}p_{18}$, so the complement of this probability is the probability that at least two voters in one or more categories cast identical ballots. The latter probability in the GTS election is 0.131006; it is far bigger than the probability that we calculated in note 4 (0.000768), which did not take into account the 18 categories into which voters sorted themselves. But even this bigger probability is likely an underestimate, because it does not reflect the fact that some candidates were far more approved of than others, rendering dubious the assumption that all ballots in each category are equiprobable.

[16] In fact, this voter might simply have relished the role of being an outlier by approving of everybody, even though he or she had no effect on the actual (minisum) outcome under the GTS rules.

To: $1111\underline{0}11100\underline{10}00000\underline{0}0\underline{1}0000$ (without this voter).

Although 7 of the original 10 winners would have been elected, 2 would have been replaced by other candidates and 1 would not have won (only 9 candidates would have been elected without this voter). Thus, this voter’s support of all candidates would have elected one more candidate if the size of the council had been endogenous, as well as replacing some of the original winners.

When the number of candidates to be elected is made endogenous and the minisum procedure is used, this is tantamount to electing only candidates approved of by a majority (Proposition 2). In fact, this rule is used to determine who is admitted to certain societies, though the threshold for entry is not always a simple majority. While the minimax procedure has no such straightforward consequence, our preliminary analysis suggests that the size of a minimax committee will be roughly equal to the average number of candidates approved of by the voters.

To get some idea of the degree of divergence between minisum and minimax committees, we analyzed all ballots in which voters have different top preferences for k = 2, 3, and 4 candidates. Thus, when $k = 2$, voters can cast 4 distinct ballots: (00, 01, 10, 11) = (A, B, C, D). There will be no duplicate ballots when there are 0 voters (casting no ballot: 1 case); 1 voter (casting A, B, C, or D: 4 cases); 2 voters (casting AB, AC, AD, BC, BD, or CD: 6 cases); 3 voters (casting ABC, ABD, ACD, or BCD: 4 cases); or 4 voters (casting ABCD: 1 case). Altogether, there are $2^4 = 16$ cases in which voters cast no duplicate ballots; when $k = 3$, there are $2^8 = 256$ cases, and when $k = 4$ there are $2^{16} = 65{,}536$ cases.[17]

To illustrate the calculation of divergence when $k = 2$, if there are two voters and their ballot pair is (10, 01), the minimax outcomes are {00, 11}; the minisum outcomes are {00, 01, 10, 11}. The maximum distance between outcomes in these two sets is $d = 2$ (between 00 and 11). For the 16 cases we have the following:

- max $d = 0$ in 9 cases [e.g., for ballot single 10], or 56.25%
- max $d = 1$ in 4 cases [e.g., for ballot pair (10, 11)], or 25.00%
- max $d = 2$ in 3 cases [e.g., for ballot pair (10, 01)], or 18.75%.

Hence, in the majority of cases (56%), the minimax and minisum outcomes agree.

This is decidedly not true when there are $k = 3$ or 4 candidates. In a large majority of cases (78% when $k = 3$, and 91% when $k = 4$), these outcomes disagree, and by relatively large amounts, as shown in the table below:

max *d*:	**0**	**1**	**2**	**3**	**4**
***k* = 2**	<u>9 (56%)</u>	4 (25%)	3 (19%)		
***k* = 3**	57 (22%)	56 (22%)	<u>102 (40%)</u>	41 (16%)	
***k* = 4**	5,905 (9%)	8,400 (13%)	12,768 (19%)	<u>26,696 (41%)</u>	11,767 (18%)

Thus, for $k = 3$ and 4, maximum divergences of 2 and 3 are the most common (all modal cases are underscored). Past these, there is a fall-off, with the number of possibly antipodal minimax and minisum outcomes (max $d = 2$, 3, or 4) ranging between 16% and

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[17] If there are k candidates, there are $m = 2^k$ distinct ballots, because a voter may or may not vote for each of the k candidates. Consequently, there are 2^m cases in which voters cast no duplicate ballots.

19% for $k = 2$, 3, and 4. All in all, we would expect that minimax and minisum outcomes will diverge, given that the different top preferences of voters are equally likely, especially as the number of candidates increases and the number of distinct ballots increases exponentially.

That the divergence between minimax and minisum outcomes is not purely theoretical is demonstrated by the GTS election. In the concluding section, we will summarize our results and comment on the feasibility of the minimax procedure in different kinds of elections.

5. Conclusions

We proposed a procedure, based on AV and fallback bargaining with unanimity, that we argued may elect committees more representative of all voters than the usual alternative. Specifically, we compared outcomes under the minimax procedure, which minimizes the maximum Hamming distance of voters from the outcome, with those given by minisum procedure, which minimizes the sum of the Hamming distances from the outcome.

Both procedures begin with a set of candidates, whom voters may either approve or not, as under AV. Once voters indicate a top preference by approving of some subset of candidates, the minimax and minisum procedures differ in how they aggregate these top preferences to yield an outcome.

We showed that the two procedures may yield diametrically opposed outcomes (antipodes) that are unique when there are as few as three candidates and some voters have the same top preference. If all voters have different top preferences, however, it takes a minimum of four candidates with ties, and five candidates without ties, for the minimax and minisum procedures to give antipodal outcomes.

We analyzed the 2003 election by the Game Theory Society (GTS) of 12 new members to its council. We showed that the minimax procedure would have given a different outcome from the minisum procedure, which was the procedure actually used by the GTS.

Although only 2 of the 12 winners would have different, the difference would have been considerably greater if the number of candidates elected had been endogenous (i.e., not restricted to 12 winners). The minisum procedure would have elected only the 5 candidates who were approved of by a majority, whereas the minimax procedure would have elected 10 candidates, including two candidates not among the top 10 vote-getters.

In the actual election of 12 candidates, the minimax procedure would have reduced the maximum distance of voters from the outcome by four times as much as the minisum procedure would have reduced the average distance of voters from the outcome. Without the restriction to 12 candidates, the reduction factor would have been three times as great for minimax. Insofar as reducing the maximum distance and the average distance are deemed equally desirable, the minimax procedure would have done, quantitatively speaking, much better.

The computer simulation we performed showed that minimax and minisum only rarely give highly discrepant outcomes. When they do, however, minimax seems preferable, because it elects committees that represent, insofar as possible, the interests of all voters, not just those of a majority.

Does the same principle hold in the election of single winners? In such elections, it is hard to contend that the winner should not be the most approved candidate. But as we showed in Example F, a different candidate may less antagonize a minority (1 of the 4 voters in this example), so it is not apparent, even in single-winner elections, that the usual AV winner should always triumph.

Whereas the minisum procedure is not manipulable, the minimax procedure is. In practice, however, it would be virtually impossible for a voter to induce a preferred outcome. As a case in point, the outlier in the GTS election who voted for all candidates succeeded in increasing the (endogenous) minimax outcome from 9 to 10 winners, and replacing two of the original winners, but we see no way that he or she could have anticipated these results.

If there are likely to be several voters who cast identical ballots, we do not recommend the minimax procedure, because it fails to reflect the numbers of voters who cast these ballots. This was not the case in the GTS election, and it is not generally the case in elections in which there are many eligible candidates (e.g., all tenured members of a department or school).

We think the unanimity rule we assumed in fallback bargaining—that the descent continues until all voters approve of an outcome—is plausible in the election of most committees, though less stringent rules are possible (Brams and Kilgour, 2001; Brams, Kilgour, and Sanver, 2004). Whether the size, or range of sizes, of a committee should be fixed or endogenous will depend, we think, on the importance of electing a committee whose size significantly affects its ability to function.

Even if size is made endogenous, voters should probably be given some guidance as to what would be an appropriate size of a committee. Without this information, it may be hard for them to gauge how many candidates to approve of in an election.

These practical considerations aside, we believe that more theoretical research on the possible effects of the minimax procedure needs to be conducted. For example, if minimax is used, is it appropriate to break ties among the minimax winners using minisum? What effect do the correlated preferences of voters, or perceived similarities in candidates, have on the divergence between minimax and minisum outcomes, especially antipodal ones? How might information (e.g., from polls) affect the manipulability of minimax?

In addition to these questions, other procedures, especially those that allow for proportional representation (Potthoff and Brams, 1998; Brams and Fishburn, 2002; Ratliff, 2003), should be considered. Just as AV in single-winner elections stimulated considerable theoretical and empirical research beginning a generation ago (Brams and Fishburn, 2002, 2004; Brams and Sanver, 2004; Weber 1995), we hope that the minimax procedure generates new research on using AV to elect committees under a different aggregation method.

Appendix

Proposition 5. *A* <u>*unique*</u> *minimax and a* <u>*unique*</u> *minisum outcome may be antipodes if and only if there are at least* <u>*3 candidates*</u>.

Proof of "only if." We show that 2 candidates are not sufficient. Let the non-negative integer, n_i, denote the number of voters who cast the following ballots for 2 candidates:

n_1 voters: 00 n_2 voters: 10 n_3 voters: 01 n_4 voters: 11.

Without loss of generality, let 11 be the unique minisum outcome. Suppose, to obtain a contradiction, that the antipode, 00, is the unique minimax outcome. Because 11 is the unique MV outcome, by Proposition 2 it must be the case that

$$n_2 + n_4 > n_1 + n_3$$

$$n_3 + n_4 > n_1 + n_2.$$

Adding the left and right sides of these inequalities gives $n_4 > n_1$, so $n_4 > 0$. But $d(00, 11) = 2$, so if 00 is a minimax outcome, then so are 10, 01, and 11. This contradicts the assumption that 00 is the unique minimax outcome. Q.E.D.

Proposition 6. *When all voters cast different ballots, a* <u>*nonunique*</u> *minimax and a* <u>*unique*</u> *minisum outcome may be antipodes if and only if there are at least* <u>*4 candidates*</u>.

Proof of "only if." We show that 3 candidates are not sufficient. Let $n_i = 0$ or 1 denote the number of voters who cast the following ballots for 3 candidates:

n_1: 000 n_2: 100 n_3: 010 n_4: 001 n_5: 110 n_6: 101 n_7: 011 n_8: 111.

Without loss of generality, let 111 be the unique minisum outcome. Suppose, to obtain a contradiction, that the antipode, 000, is a minimax outcome. Because 111 is the unique MV outcome, by Proposition 2 it must be the case that

$$n_2 + n_5 + n_6 + n_8 > n_1 + n_3 + n_4 + n_7 \tag{1}$$

$$n_3 + n_5 + n_7 + n_8 > n_1 + n_2 + n_4 + n_6 \tag{2}$$

$$n_4 + n_6 + n_7 + n_8 > n_1 + n_2 + n_3 + n_5. \tag{3}$$

Adding the left and right sides of these inequalities gives

$$n_5 + n_6 + n_7 + 3n_8 > 3n_1 + n_2 + n_3 + n_4. \tag{4}$$

Suppose $n_8 = 1$. Because 000 is a minimax outcome and $d(000, 111) = 3$, every outcome must be a minimax outcome since it is at most $d = 3$ from all other outcomes. Moreover, none of n_1, n_2, …, or n_7 can equal 0, because otherwise the antipodal outcomes would be at most $d = 2$ from every voter's top preference, and 000 would not be a minimax outcome. But we have now shown that $n_1 = n_2 = \ldots = n_8 = 1$, which contradicts inequality (4). Therefore, $n_8 = 0$ that, by (4), implies $n_1 = 0$ also.

Having shown that $n_1 = n_8 = 0$, we now add the left and right sides of (1) and (2), giving $n_5 > n_4$. Similarly, adding (1) and (3) gives $n_6 > n_3$. Finally, adding (2) and (3) gives $n_7 > n_2$. Hence, $n_2 = n_3 = n_4 = 0$. Therefore, the only top preferences are those with two approvals. But 111 is $d = 1$ from them, whereas 000 is $d = 2$ from them, contradicting the assumption that 000 is a minimax outcome.[18] Q.E.D.

[18] It is worth noting that there are 3-candidate examples in which the minimax outcomes are all different from a unique minisum outcome *in one component.* Assume the top preferences of 5 voters for 3 candidates are as follows: (1) 000; (2) 110; (3) 101; (4) 011; (5) 111. The three minimax outcomes are 110, 101, and 011, whereas the unique minisum outcome is 111.

Proposition 7. *When all voters cast different ballots, a unique minimax and a unique minisum outcome may be antipodes if and only if there are at least 5 candidates*.

Proof of "only if.". We show that 4 candidates are not sufficient. Let $n_i = 0$ or 1 denote the number of voters who cast the following ballots for 4 candidates:

n_1: 0000 n_2: 1000 n_3: 0100 n_4: 0010 n_5: 0001 n_6:1100 n_7: 1010 n_8: 1001
n_9: 0110 n_{10}: 0101 n_{11}: 0011 n_{12}: 1110 n_{13}: 1101 n_{14}:1011 n_{15}: 0111 n_{16}: 1111.

Without loss of generality, let 1111 be the unique minisum outcome. Suppose, to obtain a contradiction, that the antipode, 0000, is the unique minimax outcome. Then we can give four inequalities, analogous to the three inequalities in the proof of Proposition 6, that ensure that the total number of voters not approving of each candidate exceeds the total approving of each. Adding these inequalities gives

$$2n_{12} + 2n_{13} + 2n_{14} + 2n_{15} + 4n_{16} > 4n_1 + 2n_2 + 2n_3 + 2n_4 + 2n_5. \quad (5)$$

Now for 0000 to be the unique minimax outcome, we must have $n_{16} = 0$. Therefore, at least one of n_{12}, n_{13}, n_{14}, or n_{15} must equal 1 by inequality (5). The corresponding outcome is $d = 3$ from 0000. It follows that the minimax outcomes are exactly those with an antipode not approved by any voter.

But if 0000 is to be the unique minimax outcome, then $n_{16} = 0$ and $n_1 = n_2 = ... = n_{15} = 1$. This contradicts inequality (4), completing the proof. Q.E.D.

Proposition 10. *When the size of a committee is restricted to c members, a minisum outcome is every set of c candidates that receives the most votes.*

Proof. To establish notation, assume that there are n voters and k candidates, and that voter i's top preference is the binary k-vector $p^i = (p_1^i, p_2^i, \ldots, p_k^i)$. For an arbitrary binary k-vector $x = (x_1, x_2, \ldots, x_k)$, define

$$d_j(x, p^i) = \begin{cases} 0 & \text{if } x_j = p_j^i \\ 1 & \text{if } x_j \neq p_j^i \end{cases}$$

for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, k$. Then it is clear that the Hamming distance from x to p^i is given by $d(x, p^i) = \sum_{j=1}^{k} d_j(x, p^i)$. We consider how to select x so as to minimize $D(x) = \sum_{i=1}^{n} d(x, p^i)$, subject to the constraint $\sum_{j=1}^{k} x_j = c$, which guarantees that x represents a committee with exactly c members.

For any x and j, define

$$S_j(x) = \sum_{i=1}^{n} d_j(x, p^i), \tag{5}$$

which represents the number of voters who disagree with k-vector x on candidate j. Note that $D(x) = \sum_{j=1}^{k} S_j(x)$, and that $S_j(x)$ depends only on x_j and not on the other $k - 1$ components of x.

We wish to identify the committees, x, with exactly c members that minimize

$$D(x) = \sum_{i=1}^{n} d(x, p^i) = \sum_{j=1}^{k} S_j(x).$$

Suppose the top preferences of all voters, $p^1, \ldots, p^n$, are given. Define $K_j = |\{i : p_i^j = 1\}|$, so K_j is the number of voters who vote for candidate j; clearly, $n - K_j$ is the number of voters who vote against j. Now

$$S_j(x) = \begin{cases} n - K_j & \text{if } x_j = 1 \\ K_j & \text{if } x_j = 0 \end{cases}$$

so a choice of $x = (x_1, \ldots, x_k)$ minimizes $D(x)$ and satisfies $\sum_{j=1}^{k} x_j = c$ if and only if $x_j = 1$ for the c values of j for which K_j is largest. Q.E.D.

Example A

Number of Voters Approving of Each Outcome at Different Distances *d*

(Top Preferences of 5 Voters in Boldface)

Comb.	$d = 0$	$d = 1$	$d = 2$	$d = 3$	$d = 4$
1. 0000	0	2	4	5	5
2. 1000	**1**	**1**	**2**	**4**	**5**
3. 0100	0	1	4	5	5
4. 0010	0	2	5*	5	5
5. 0001	**1**	**2**	**4**	**5**	**5**
6. 1100	0	1	2	4	5
7. 1010	0	1	3	5	5
8. 1001	0	2	3	4	5
9. 0110	**1**	**2**	**3**	**5**	**5**
10. 0101	0	2	4	5	5
11. 0011	**1**	**3**	**4**	**5**	**5**
12. 1110	0	1	3	4	5
13. 1101	0	1	4	5	5
14. 1011	0	0	3	5	5
15. 0111	**1**	**3**	**4**	**4**	**5**
16. 1111	0	1	3	5	5
Total	**5**	**25**	**55**	**75**	**80**

*FB_n (minimax) winner; the MV (minisum) winner is #11 (0011).

Example B

Number of Voters Approving of Each Outcome at Different Distances *d*

(Top Preferences of 13 Voters in Boldface)

Combination	***d* = 0**	***d* = 1**	***d* = 2**	***d* = 3**
1. 000	**1**	**4**	**13***	**13**
2. 100	**1**	**7**	**10**	**13**
3. 010	**1**	**7**	**10**	**13**
4. 001	**1**	**7**	**10**	**13**
5. 110	**3**	**5**	**12**	**13**
6. 101	**3**	**5**	**12**	**13**
7. 011	**3**	**5**	**12**	**13**
8. 111	0	9	12	13
Total	**13**	**49**	**91**	**104**

*FB_n (minimax) winner; the MV (minisum) winner is #8 (111).

Example C

Number of Voters Approving of Each Outcome at Different Distances *d*

(Top Preferences of 5 Voters in Boldface)

Comb.	*d* = 0	*d* = 1	*d* = 2	*d* = 3	*d* = 4
1. 0000	**1**	**1**	**1**	**5***	**5**
2. 1000	0	0	3	4	5
3. 0100	0	0	3	4	5
4. 0010	0	0	3	4	5
5. 0001	0	0	3	4	5
6. 1100	0	2	2	5*	5
7. 1010	0	2	2	5*	5
8. 1001	0	2	2	5*	5
9. 0110	0	2	2	5*	5
10. 0101	0	2	2	5*	5
11. 0011	0	2	2	5*	5
12. 1110	**1**	**1**	**4**	**5***	**5**
13. 1101	**1**	**1**	**4**	**5***	**5**
14. 1011	**1**	**1**	**4**	**5***	**5**
15. 0111	**1**	**1**	**4**	**5***	**5**
16. 1111	0	4	4	4	5
Total	**5**	**23**	**45**	**70**	**80**

*FB_n (minimax) winner; the MV (minisum) winner is #16 (1111).

Example E

Number of Voters Approving of Each Combination at Different Distances

(Top Preferences of 4 Voters in Boldface)

Ea. True Top Preferences

Combination	***d* = 0**	***d* = 1**	***d* = 2**
1. 1000	**1**	**3****	**4***
2. 0001	**1**	**1**	**2**
3. 1100	**1**	**2**	**3**
4. 1010	**1**	**2**	**3**
5. 0000	0	2	4*
6. 1001	0	2	4*
10 other comb.'s	0	At most 2	At most 3

Eb. Voter 1 Above Misrepresents as Voter 5 Below

Combination	***d* = 0**	***d* = 1**	***d* = 2**
1. 1000	0	2	4*
2. 0001	**1**	**1**	**1**
3. 1100	**1**	**2**	**3**
4. 1010	**1**	**2**	**3**
5. 1110	**1**	**3**	**3**
11 other comb.'s	0	At most 1	At most 3

*FB_n (minimax) winner.

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