Feasible Beliefs in Noncooperative Games

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Abstract

This paper refines the "equilibrium under uncertainty" introduced in Eichberger and Kelsey (2000) and modified in Albers (2000). We assume that a player's uncertainty prevents him from choosing certain beliefs. In particular, frightened players cannot choose (most) additive beliefs. Therefore, for each player we use a feasible set that specifies all beliefs that are consistent with his uncertainty. It is possible to impose such a restriction in a very general way and still guarantee the existence of an equilibrium in feasible beliefs.

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1 Introduction

In this paper we talk about noncooperative games in normal form. This part of game theory analyzes situations, in which the payoff of a player depends on his own actions as well as on the actions of his opponents—which are unknown to him. Thus a player has to make a decision, without being completely informed about the consequences.

So while choosing his strategy a player is going to make assumptions *(beliefs)* about his opponents' behavior. Since a player will never exactly know his opponents' behavior in advance, this situation is similar to a random experiment—except the probabilities are unknown.

In traditional game theory there are attempts to model such beliefs in various ways. All of them have one aspect in common: at one point a player's assumptions about his opponents' behavior are modeled via a probability distribution. The player then optimizes just as if he *knew* his opponents to act according to this distribution.

However there are reasons to doubt, whether random type situations where the true probabilities are unknown can be modeled by classical probability distributions at all. Ellsberg (1961) presented a simple experiment in which subjects had to evaluate lotteries with unknown probability distributions. He pointed out that preference structures on those lotteries cannot be supported by classical probability distributions in a coherent way. He concluded that *risk* has to be distinguished from *uncertainty*. Here risk refers to a classical random experiment with known probabilities, while uncertainty denotes a situation, in which there is no known probability distribution.

In the late eighties Gilboa (1987) and Schmeidler (1989) described a concept that used capacities (nonadditive probabilities) to model this uncertainty. Later Dow and Werlang (1994) applied this model in noncooperative 2-person games and formulated an equilibrium they called "Nash Equilibrium under Uncertainty". Eichberger and Kelsey (2000) generalized the model to the n-player case and showed some nice properties of the equilibrium in beliefs. In Albers (2000) we modified the model presented by Dow and Werlang (1994) and Eichberger and Kelsey (2000).

In this paper we mainly follow the model of Eichberger and Kelsey. However, we will always talk about an "Equilibrium in Beliefs" while we use the term "Nash Equilibrium" for the classical equilibrium concept only.

It is the central idea of these equilibrium models that a player might not be certain enough about his opponents' behavior to model his beliefs via (additive) probabilities. In this context Eichberger and Kelsey introduced the "degree of confidence", and showed that even if for every player only beliefs of a certain (low) degree of confidence are feasible there exists an equilibrium in feasible beliefs. (See Definition 2.2 and Proposition 3.1 in Eichberger and Kelsey (2000).) Unfortunately the model presented in Albers (2000) does not fulfill this important property.

In this paper we present a similar theorem: if for every player only beliefs out of a set with certain properties are feasible, there exists an equilibrium in feasible beliefs. (See Theorem 4.1.) It was our intention to keep the theoretical constraints for feasibility low in order to make the theorem applicable to a broad class of sets.

In Section 5.1 we give an example how sets of feasible beliefs could look like. There we talk about a certain class of capacities: *distorted probabilities.* They have been used by Tversky and Kahneman (1992) to explain subjects' behavior in the evaluation of lotteries. We briefly discuss the relationship between distorted probabilities and feasible sets. In particular we show that the set of distorted probabilities related to a subject in Tversky and Kahneman (1992) form a feasible set of capacities in the sense of our theorem.

Finally in Section 6 we give an example of a game and feasible sets where the only equilibrium in feasible beliefs is not related to the unique Nash equilibrium. We thereby establish a counterexample to the objection that equilibria in feasible beliefs do not really differ from Nash equilibria.

2 Capacities

In this paper we will always use finite probability spaces Ω. Thus probability measures will be defined on the powerset 2^Ω.

First we will define a nonadditive probability measure in a standard way.

Definition 2.1 *A* ***capacity*** *(nonadditive probability measure) c on a finite set Ω is a function*

$$c : 2^\Omega \longrightarrow [0,1]$$

for which following axioms hold true:

$$\begin{array}{ll} \textit{C1.} & c(\Omega) = 1 \\ \textit{C2.} & c(\emptyset) = 0 \\ \textit{C3.} & A \supseteq B \;\Rightarrow\; c(A) \geq c(B) \qquad (A, B \subseteq \Omega)\ . \end{array}$$

The set of capacities on Ω is denoted as $\mathscr{C}(\Omega)$.

Since we deal with finite probability spaces we may interpret a capacity as a $2^{|\Omega|}$-dimensional vector with elements in the unit interval $[0,1]$. With this interpretation the axioms of Definition 2.1 can be interpreted as restrictions to some closed half spaces. So $\mathscr{C}(\Omega)$ is a convex, compact polytope in $\mathbb{R}^{2^{|\Omega|}}$. If later we talk about topological properties of capacities (like convergence or compactness) we always think about capacities as elements of $\mathbb{R}^{2^{|\Omega|}}$.

Verbally we interpret a capacity in the following way:

> *A player is willing to rely on the event $A \subseteq \Omega$ to occur with probability $c(A)$.*

This interpretation of capacities cannot be found in Definition 2.1! Just using the axioms it is impossible to interpret statements like $c(A) = 0.04$. *E. g.*, if we take a given probability measure p, the function $c(A) = p(A)^2$ is a capacity. In this case the meaning of $c(A) = 0.04$ is that the true probability of event A is $\sqrt{0.04} = 2\%$.

From a mathematicians point of view, the interpretation of a capacity is implemented by the way we compute the expected value. We use the Choquet integral:

$$E_c(Z) = \int_{\mathbb{R}_+} c(Z \geq x)\, dx - \int_{\mathbb{R}_-} c(\Omega) - c(Z \geq x)\, dx \ .$$

For probability measures, the Choquet integral coincides with the classical expected value. This reflects the close relation between probabilities and capacities used here.

In this paper we will restrict the scope to nonnegative random variables. This way we may drop the second summand which greatly simplifies writing. Since we deal with finite probability spaces only, this restriction has no influence in the validity of propositions made in this paper.

Definition 2.2 *The* ***expected value*** *of a nonnegative random variable $Z : \Omega \to \mathbb{R}_+$ according to a capacity c is defined by the Choquet integral:*

$$E_c(Z) := \int_0^\infty c(Z \geq x)\, dx \ .$$

Since we deal with finite probability spaces only, it is possible to write down the Choquet integral as a sum.

Lemma 2.3 *Let c be a capacity on a finite probability space Ω and Z be a nonnegative random variable. Further denote the values of Z by $z_1, \ldots, z_n$ in increasing order,* i. e.*:*

$$\begin{aligned} Z(\Omega) &= \{z_1, \ldots, z_n\} \\ z_i < z_{i+1} & \qquad (1 \leq i < n) \ . \end{aligned}$$

Then we can write the Choquet integral as

$$\int_{\mathbb{R}_+} c(Z \geq x)\, dx = z_1 + \sum_{i=2}^{n} (z_i - z_{i-1})\, c(Z \geq z_i)\ .$$

Proof:

$$\begin{aligned} \int_{\mathbb{R}_+} c(Z \geq x)\, dx &= \int_0^{z_1} c(\Omega)\, dx + \sum_{i=2}^{n} \int_{z_{i-1}}^{z_i} c(Z \geq x)\, dx \\ &= z_1 + \sum_{i=2}^{n} (z_i - z_{i-1})\, c(Z \geq z_i) \end{aligned}$$

□

From this lemma we can directly conclude:

Lemma 2.4 *If $(c_k)_{k\in\mathbb{N}}$ is a series of capacities on a finite Ω, and $c_k \to \bar{c}$, then $E_{c_k}(Z) \to E_{\bar{c}}(Z)$ for any nonnegative random variable Z on Ω.*

In addition to the expected value we will need another, more crude interpretation of a capacity—supplied by the support (carrier). Verbally, the support of a capacity contains all results $\omega \in \Omega$ that are considered possible according to c.

Other authors use various definitions of the support and thus use a model with two concurring interpretations of a capacity—the Choquet expected value as well as the support. In our opinion, if two random variables Z, Y coincide on the support of a capacity c, then their expected values ought to be the same. For the support used in Dow and Werlang (1994) and Eichberger and Kelsey (2000) this is not the case.

In contrast, we will show that the support (as defined here) and the Choquet expected value have a close connection.

Definition 2.5 *The* ***support*** *of a capacity c is defined by:*

$$\operatorname{supp} c := \big\{\omega \in \Omega \,\big|\, \exists A \subseteq \Omega : c(A) \neq c(A \cup \{\omega\})\big\}\ .$$

The connection of the support and the Choquet expected value is described by following theorem:

Theorem 2.6 *Let c be a capacity on some finite Ω. Then ω is in the support of c if and only if there exist random variables Y and Z that coincide on $\Omega \setminus \{\omega\}$ but $E_c(Y) \neq E_c(Z)$.*

Proof: "$\Longrightarrow$": Let $\bar{\omega} \in \operatorname{supp} c$. Then there is $A \subseteq \Omega$ such that $c(A) < c(A \cup \{\bar{\omega}\})$. Then let:

$$Z(\omega) := \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{otherwise} \end{cases} \quad , \quad Y(\omega) := \begin{cases} 1 & \text{if } \omega \in A \cup \{\bar{\omega}\} \\ 0 & \text{otherwise} \end{cases} \quad .$$

Then $E_c(Z) = c(A)$ and $E_c(Y) = c(A \cup \{\bar{\omega}\})$, so $E_c(Z) \neq E_c(Y)$.

"$\Longleftarrow$": Let Z, Y coincide on $\operatorname{supp} c$. Then

$$\begin{aligned} E_c(Z) &= \int_{\mathbb{R}_+} c\big(\{\omega \in \Omega \mid Z(\omega) \geq x\}\big)\, dx \\ &= \int_{\mathbb{R}_+} c\big(\{\omega \in \Omega \mid Z(\omega) \geq x\} \cap \operatorname{supp} c\big)\, dx \\ &= \int_{\mathbb{R}_+} c\big(\{\omega \in \Omega \mid Y(\omega) \geq x\} \cap \operatorname{supp} c\big)\, dx \\ &= \int_{\mathbb{R}_+} c\big(\{\omega \in \Omega \mid Y(\omega) \geq x\}\big)\, dx \\ &= E_c(Y) \end{aligned}$$

□

3 Equilibrium in Beliefs

By "game" we mean the standard definition of a game in normal form:

Definition 3.1 *A **game** Γ is a tuple $(N, \mathscr{S}, a)$ where*

$N = \{1, \ldots, n\}$	*is the set of players*
$\mathscr{S} = \mathscr{S}^1 \times \cdots \times \mathscr{S}^n$	*are the finite strategy sets of the players*
$a = (a^1, \ldots, a^n)$,	
$a^i : \mathscr{S} \to \mathbb{R}_+ \quad (i \in N)$	*are the payoff functions of the players*

Now we want to model players' beliefs about their opponents' behavior as capacities. Of course later we want to use these capacities to compute expected payoffs. But here we stumble across a serious problem. When we take some player's beliefs b and his own mixed strategy a we need to evaluate expressions like $E_{b\otimes a}\cdots$.

While this is no problem in classical probability theory, for capacities this does not work. This is because Fubini's Theorem:

$$\int\int \cdots d\mu\, d\nu = \int\int \cdots d\nu\, d\mu$$

does generally *not* hold for capacities.

This forces us to two important design decisions

First: beliefs are defined as capacities on the strategies of *all* other players, *i. e.*, player i's belief b^i is a capacity on

$$\mathscr{S}^{-i} := \mathscr{S}^1 \times \cdots \times \mathscr{S}^{i-1} \times \mathscr{S}^{i+1} \times \cdots \times \mathscr{S}^n\ .$$

Second: we use the von Neumann-Morgenstern interpretation of an equilibrium. *I. e.*, something like: a tuple of beliefs $(b^1, \dots, b^n)$ is an equilibrium when the following holds: if player i considers action $\bar{s}^j$ of player j as possible, then $\bar{s}^j$ must be best reply to player j's belief:

$$\bar{s}^j \in \operatorname*{argmax}_{s^j \in \mathscr{S}^j} E_{b^j}\left[a^j(s^j, \bullet)\right]\ .$$

To find out which actions player i considers possible, we use the support (Definition 2.5).

Definition 3.2 *Given a game Γ the* ***belief*** *b^i of a player i is a capacity on the strategy sets of all his opponents:*

$$\mathscr{S}^{-i} = \bigtimes_{k\in N\setminus\{i\}} \mathscr{S}^k\ .$$

The space of all possible beliefs of player i is denoted as $\mathscr{B}^i$.

The ***best replies*** *of player i to his belief b^i are given by:*

$$\operatorname{br} b^i := \operatorname*{argmax}_{s^i \in \mathscr{S}^i} E_{b^i}\left[a^i(s^i, \bullet)\right]\ .$$

A tuple of beliefs $(\bar{b}^1, \dots, \bar{b}^n)$ is called an ***equilibrium in beliefs*** *if and only if*

$$\operatorname{supp} \bar{b}^i \subseteq \bigtimes_{k\in N\setminus\{i\}} \operatorname{br} \bar{b}^k \qquad (i \in N)\ .$$

The definition of an equilibrium can be interpreted in the following way: $\operatorname{supp} \bar{b}^i$ is a subset of $\mathscr{S}^{-i}$, *i. e.*, it contains *tuples* of pure strategies—one for each opponent. These are all tuples that player i considers possible. Every pure strategy s^k in every tuple $s^{-i} \in \operatorname{supp} \bar{b}^i$ has to be a best reply for opponent k.

4 Feasible Beliefs

Our motivation to model beliefs by capacities was to analyze situations, in which players' assumptions about their opponents cannot be reflected by additive probabilities.

It is easy to see that every game has an equilibrium in beliefs: if m is a (mixed) Nash equilibrium and every player assumes his opponents are playing their equilibrium strategies, *i. e.* $b^i = \bigotimes_{k \neq i} m^i$, then b is an equilibrium in beliefs.

However, this result is not satisfying, since it does not take us any further than the Nash equilibrium did.

Instead, we think about a player being strictly uncertain about what his opponents are doing. This means, his uncertainty prevents him from choosing certain beliefs—among them especially additive capacities. Therefore we assume that for each player i there is an a-priory *feasible* set of beliefs $\mathscr{F}^i \subseteq \mathscr{B}^i$ that are consistent with his individual uncertainty. (Typically we will have $\mathscr{F}^i \neq \mathscr{B}^i$.)

The following theorem shows conditions that sets of feasible beliefs have to fulfill while we still can guarantee the existence of an equilibrium.

Let $\triangle^A$ denote the set of all probability distributions on the set A. Since we deal with finite worlds, $\triangle^A$ coincides with the $|A|$-dimensional unit simplex.

Theorem 4.1 *Let Γ be a game and $\mathscr{F}^i \subseteq \mathscr{B}^i$ $(i \in N)$ be sets of feasible beliefs. If for every player i we have a subset $\bar{\mathscr{F}}^i \subseteq \mathscr{F}^i$ and a continuous function*

$$f^i : \triangle^{\mathscr{S}^{-i}} \longrightarrow \bar{\mathscr{F}}^i$$

such that

$$\operatorname{supp} f(m^{-i}) \subseteq \operatorname{supp} m^{-i} \qquad \forall m^{-i} \in \triangle^{\mathscr{S}^{-i}} . \tag{1}$$

Then Γ has an equilibrium in beliefs $\bar{b}$ where all $\bar{b}^i$ are feasible.

A function f^i generates a membrane shaped subset $\bar{\mathscr{F}}^i$. Indeed, sets like these have been used to model real peoples behavior—as we will see in the next section. Still, for the sake of generality we only demand a feasible set $\mathscr{F}^i$ to contain $\bar{\mathscr{F}}^i$ as a subset.

Furthermore the support-condition (1) ensures that a player i always has the option to narrow his view to some subset $S^{-i} \subsetneq \mathscr{S}^{-i}$. If the player thinks, some opponent k would be a complete fool to play some strategy $s^k \in \mathscr{S}^k$, player i should have the option to exclude s^k from his further considerations.

Proof (Theorem 4.1): Let $\mathscr{M}^i = \triangle^{\mathscr{S}^i}$ denote the set of mixed strategies of player i.

To simplify notation we identify strategies in $\mathscr{S}^i$ with strategies in $\mathscr{M}^i$ via $s^i \mapsto \delta_{s^i}$. In a similar way we identify tuples of pure strategies $s^{-i} \in \mathscr{S}^{-i}$ with beliefs $b_{s^{-i}} \in \mathscr{B}^i$ via

$$b_{s^{-i}}(S) = \begin{cases} 1 & \text{if } S \ni s^{-i} \\ 0 & \text{otherwise} \end{cases} .$$

Furthermore we write $m^i_{s^i}$ instead of $m^i(\{s^i\})$. Finally let

$$\bar{m}^{-i} := \bigotimes_{k \in N \setminus i} \bar{m}^k .$$

This way we find natural representations for a pure strategy tuple $s^{-i} \in \mathscr{S}^{-i}$ in the space of mixed strategies $\mathscr{M}^{-i}$ (all players $j \neq i$ play s^j) as well as in the space of beliefs $\mathscr{B}^i$ (player i thinks that his opponents play s^{-i}).

We perform the proof in two steps. First we define a new game $\tilde{\Gamma}$ and show that it always has an equilibrium. Then we will show that from any equilibrium in $\tilde{\Gamma}$ we can derive an equilibrium in beliefs b in Γ such that all $b^i \in \bar{\mathscr{F}}^i$.

Step 1 (The new game):

Let $\Gamma = (N, \mathscr{S}, a)$ be a game and f^i as demanded in the theorem. Then define the game $\tilde{\Gamma} = (N, \mathscr{M}, \tilde{a})$ as follows:

- N is the set of players of Γ
- $\mathscr{M} = \mathscr{M}^1 \times \cdots \times \mathscr{M}^n$ is the set of mixed strategies of Γ.
- The payoff is defined by:

$$\tilde{a}^i(m^i, m^{-i}) := E_{m^i}\big(E_{f^i \circ m^{-i}} a^i(*, \bullet)\big) .$$

i. e., $\tilde{\Gamma}$ is a game with continuous strategy sets and it is played in pure strategies.

In the last formula we see two expected values. The inner one $E_{f^i \circ m^{-i}}$ is a Choquet integral while the outer one E_{m^i} is a classical expected value. The function f^i is only applied to the strategies of the opponents—not to the strategy of player i himself.

Obviously $\tilde{a}^i : \mathscr{M}^i \times \mathscr{M}^{-i} \to \mathbb{R}_+$ is linear (and thus quasi-concave) on $\mathscr{M}^i$. Furthermore the Choquet integral is continuous in the space of capacities. (see Lemma 2.4.) Also f^i is continuous by definition. Therefore $\tilde{a}^i$ is continuous on $\mathscr{M}^{-i}$.

Therefore we may apply the standard existence theorem. (See *e. g.* Proposition 20.3 in Osborne and Rubinstein (1994) or Proposition 8.D.3. in Mas-Colell, Whinston and Green (1995).)

Step 2 (coincidence):

We want to show: if $\bar{m}$ is an equilibrium in $\tilde{\Gamma}$, then $\bar{b} = (\bar{b}^1, \ldots, \bar{b}^n)$ defined by

$$\bar{b}^i := f^i \circ \bar{m}^{-i} \qquad (i \in N)$$

is an equilibrium in feasible beliefs in Γ.

As we mentioned above $\tilde{a}^i$ is linear on $\mathscr{M}^i$. So $\operatorname{argmax}_{m^i \in \mathscr{M}^i} \tilde{a}(m^i, m^{-i})$ is the convex hull of all *pure* strategies that maximize $\tilde{a}^i(\bullet, m^{-i})$. Therefore

$$\bar{m}^i_{s^i} > 0 \quad \Longrightarrow \quad s^i \in \operatorname*{argmax}_{r^i \in \mathscr{S}^i} \tilde{a}(r^i, \bar{m}^{-i}) \qquad (i \in N, s^i \in \mathscr{S}^i) \ .$$

By definition argmax $\cdots$ in $\tilde{\Gamma}$ coincides with the best replies to $f^i \circ \bar{m}^{-i}$ in Γ:

$$\begin{aligned} \operatorname*{argmax}_{r^i \in \mathscr{S}^i} \tilde{a}(r^i, \bar{m}^{-i}) &= \operatorname*{argmax}_{r^i \in \mathscr{S}^i} E_{f^i \circ \bar{m}^{-i}} \big(a^i(r^i, \bullet)\big) \\ &= \operatorname{br} f^i \circ \bar{m}^{-i} \qquad (i \in N) \ . \end{aligned}$$

Thus we have

$$\bar{m}^i_{s^i} > 0 \quad \Longrightarrow \quad s^i \in \operatorname{br} \bar{b}^i \qquad (i \in N, s^i \in \mathscr{S}^i) \ .$$

Suppose player i considers s^{-i} possible, *i. e.*, $s^{-i} \in \operatorname{supp} f^i \circ \bar{m}^{-i}$. Adding the support-condition (1) we can conclude:

$$s^{-i} \in \operatorname{supp} \bar{m}^{-i}$$

which means that for every player i we have:

$$\begin{aligned} s^{-i} \in \operatorname{supp} \bar{b}^i &\implies s^{-i} \in \operatorname{supp} \bar{m}^{-i} \\ &\implies m^k_{s^k} > 0 \qquad (k \in N \setminus \{i\}) \\ &\implies s^k \in \operatorname{br} \bar{b}^i \qquad (k \in N \setminus \{i\}) \ . \end{aligned}$$

Thus $\bar{b}$ is an equilibrium in Γ. □

5 Distorted Probabilities

One example for such restricted sets are distorted probabilities. They have been used by Tversky and Kahneman (1992) to explain subjects behavior in the evaluation of lotteries. Originally they were used to model subjects risk aversion rather than their uncertainty. So using them here means ripping them out of their context. Still I think distorted probabilities are a nice example of how feasible beliefs *could* look like.

For each subject Tversky and Kahneman determined an individual distortion function w. Then, for different probability measures μ they used capacities $w \circ \mu$ to compute the subjects' expected utility.

Definition 5.1 *A* ***distortion function*** *w is a continuous, monotone function $w : [0,1] \to [0,1]$ with $w(0) = 0$, $w(1) = 1$.*

Given an additive probability measure μ on a finite space Ω and a distortion function w, the function

$$c = w \circ \mu \ .$$

is a capacity. It is called a ***distorted probability measure****.*

Given a game Γ and distortion functions $w^1, \ldots, w^n$ we use

$$\mathscr{F}^i_{w^i} := \left\{ w^i \circ \mu^i \,\middle|\, \mu^i \in \triangle^{\mathscr{S}^{-i}} \right\} \qquad (i \in N)$$

as the sets of feasible beliefs. Obviously, since w^i is continuous so is $f^i : \mu^i \longmapsto w^i \circ \mu^i$. Furthermore from the monotony of w^i we can conclude

$$s^{-i} \in \operatorname{supp} w^i \circ \mu^i \quad \implies \quad \mu^i(s^{-i}) > 0 \quad \implies \quad s^{-i} \in \operatorname{supp} \mu \ .$$

Therefore, every set $\mathscr{F}^i_{w^i}$ is a feasible set of beliefs in the sense of Theorem 4.1. Thus we know that for every tuple of distortions functions $(w^i)_{i \in N}$ there exists an equilibrium in beliefs $(\bar{b}^i)_{i \in N}$ such that $b^i \in \mathscr{F}^i_{w^i} (i \in N)$.

6 The Example

This example shows a game and sets of feasible beliefs with exactly one equilibrium in feasible beliefs. The equilibrium does not relate to an equilibrium in mixed strategies.

Definition 6.1 *We say, a tuple b of beliefs* relates *to a tuple m of mixed strategies if the best replies in pure strategies coincide,* i. e., *if*

$$\operatorname{br} m^{-i} = \operatorname{br} b^i \qquad (i \in N) \ .$$

where $m^{-i} = \otimes_{k \in N \setminus \{i\}} m^k$.

Let

$$A^1 = \begin{pmatrix} 0 & 4 & 9 \\ 4 & 0 & 10 \end{pmatrix} \quad , \quad A^2 = \begin{pmatrix} 4 & 0 & 1 \\ 0 & 4 & 1 \end{pmatrix}$$

be the payoff matrices in a bimatrix game. Strategies are $\mathscr{S}^1 = (u, d)$ (up/down) and $\mathscr{S}^2 = (\ell, c, r)$ (left, center, right). It is easy to see that the game has exactly one equilibrium in mixed strategies: $\bar{m}^1 = (\frac{1}{2}, \frac{1}{2})$ and $\bar{m}^2 = (\frac{1}{2}, \frac{1}{2}, 0)$.

Figure 1 shows the best replies for player 2. Note that at the borders between

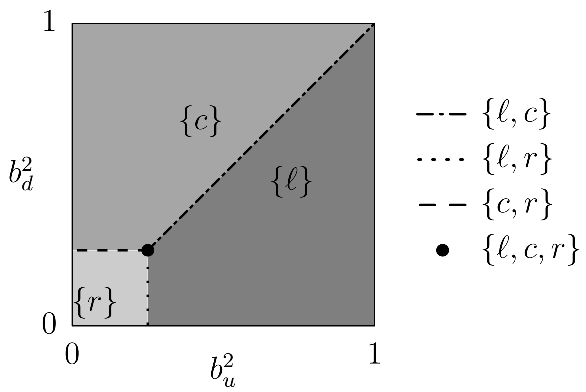

Figure 1: Best replies for player 2

the shaded areas the best replies take special values—as sketched on the right of the figure.

Further, let

$$w^2(x) = x^3 \quad , \quad w^1(x) = x$$

be distortion functions. So for player 1, the set feasible beliefs $\mathscr{F}^1_{w^1}$ consists of all additive probability measures on $\mathscr{S}^2$.

The set of player 2's feasible beliefs consists of the points

$$\mathscr{F}^2_{w^2} = \left\{ \left(x^3, (1-x)^3\right) \,\middle|\, x \in [0,1] \right\} .$$

I. e., $\mathscr{F}^2_{w^2}$ is the graph of the function $x \mapsto \sqrt[3]{(1-x)^3}$. It is displayed as a fat curve in Figure 2. For a tuple of beliefs b to relate to the equilibrium in

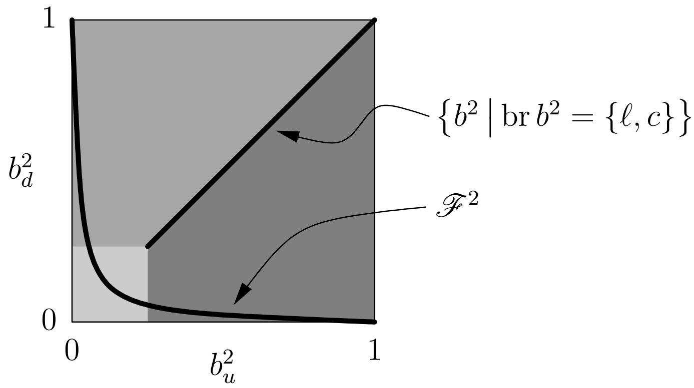

Figure 2: Feasible beliefs of player 2

mixed strategies, the equation

$$\mathrm{br}\, b^2 = \{\ell, c\} \tag{2}$$

has to hold. In Figure 2 the set of all beliefs b^2 with $\mathrm{br}\, b^2 = \{\ell, c\}$ is displayed as a fat straight line. We see with a glance that there is no distorted belief b^2 for which (2) holds, since the two fat lines do not intersect.

But if there is no tuple of feasible beliefs that relates to the mixed equilibrium, certainly also an *equilibrium* in feasible beliefs cannot relate to the equilibrium in mixed strategies!

Since the existence of an equilibrium in feasible beliefs has been proved, we are done … but what is the equilibrium in feasible beliefs?

The expected payoffs for player 1 are:

$$E_{b^1}\left(a^1(u, \bullet)\right) = 4b^1_{cr} + 5b^1_r \quad , \quad E_{b^1}\left(a^1(d, \bullet)\right) = 4b^1_{\ell r} + 6b^1_r \ .$$

Obviously there is no equilibrium in beliefs, in which player 2 only considers one action of player 1 possible. (This is due to the same reason as there is no equilibrium in pure strategies.) Therefore $\operatorname{supp} b^2$ will be $\{u, d\}$. In case of an equilibrium we demand $E_{b^1}\big(a^1(u,\bullet)\big) = E_{b^1}\big(a^1(d,\bullet)\big)$, *i. e.*:

$$\begin{aligned} 4b^1_{cr} + 5b^1_r &= 4b^1_{\ell r} + 6b^1_r \\ \Longrightarrow \quad 4b^1_{cr} &= 4b^1_{\ell r} + b^1_r \ . \end{aligned} \tag{3}$$

Now for an equilibrium belief b^2 it must either hold:

$$\operatorname{br} b^2 = \{\ell, r\} \quad \text{or} \quad \operatorname{br} b^2 = \{c, r\} \ ,$$

because beliefs with $\operatorname{br} b^2$ being $\{\ell, c\}$ or $\{\ell, c, r\}$ are not in $\mathscr{F}^2$ (see Figure 2) and single valued best replies would imply an equilibrium in pure strategies.

Suppose $\operatorname{br} b^2 = \{\ell, r\}$. Since we assume b to be an equilibrium we have $\operatorname{supp} b^1 \subseteq \{\ell, r\}$, or

$$c \notin \operatorname{supp} b^1 \ .$$

Then obviously $b^1_{cr} = b^1_r$ and because of equation (3):

$$4b^1_r = 4b^1_{cr} = 4b^1_{\ell r} + b^1_r \geq 5b^1_r$$

—*i. e.*,

$$b^1_r = b^1_{cr} = b^1_{\ell r} = 0 \ . \tag{4}$$

Lets look at all values of b^1:

S^2	$b^1(S^2)$	explanation
$\emptyset$	0	by definition
$\{\ell\}$	0	$b^1_\ell \leq b^1_{\ell r} \overset{(4)}{=} 0$
$\{c\}$	0	$c \notin \operatorname{supp} b^1$
$\{r\}$	0	(4)
$\{\ell, c\}$	0	$c \notin \operatorname{supp} b^1 \quad \Longrightarrow \quad b^1_{\ell c} = b^1_\ell$
$\{\ell, r\}$	0	(4)
$\{c, r\}$	0	(4)
$\{\ell, c, r\}$	1	by definition

But this means $\operatorname{supp} b^1 = \{\ell, c, r\}$! (Because $b^1(S) = 0 < 1 = b^1(\Omega)$ for any $S \subseteq \Omega$ with $|S| = 2$.) This is a contradiction.

Now we know the supports for the equilibrium beliefs:

$$\operatorname{supp} b^1 = \{c, r\} \quad , \quad \operatorname{supp} b^2 = \{u, d\} \ .$$

Now we calculate the precise beliefs.

From the definition of the equilibrium we know that $\operatorname{br} b^2 \supseteq \operatorname{supp} b^1$. The only feasible belief of player 2 satisfying this criterion is marked with a circle in Figure 3. It is the point

$$b^2 = \left[\left(1 - \sqrt[3]{\tfrac{1}{4}}\right)^3, \tfrac{1}{4}\right] . \tag{5}$$

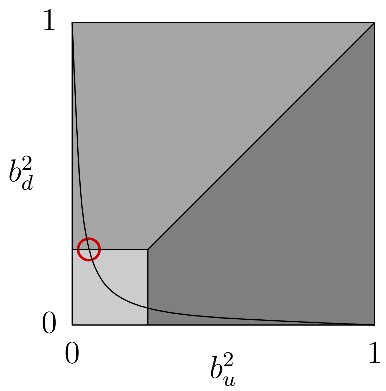

Figure 3: Equilibrium belief for player 2

Concerning b^1, we fixed $\mathscr{F}^1$ to all additive beliefs. Therefore we can calculate b^1 easily from (3):

$$\begin{aligned} 4b^1_c + 4b^1_r &= \underbrace{4b^1_\ell}_{0} + 4b^1_r + b^1_r \\ \Longrightarrow \qquad 4b^1_c &= b^1_r \end{aligned}$$

Therefore

$$b^1 = \left(0, \tfrac{1}{5}, \tfrac{4}{5}\right) . \tag{6}$$

The point defined by (5) and (6) is the unique equilibrium in feasible beliefs.

The properties of this example are not restricted to a null set of games. *I. e.*, in a neighborhood of the game and the feasible sets it holds, that the equilibrium in mixed strategies does not relate to the equilibrium in beliefs. (When we talk about a neighborhood of a feasible set we mean a neighborhood in the sense of the Hausdorff metric.)

Proof: We get an intuition of the proof from Figure 2. Basically in the case of an equilibrium the two fat lines have to intersect. As all involved functions are continuous, if you change the game and the feasible sets just slightly, also these fat lines will only move by an arbitrary small amount. This way we can easily prevent them from intersecting.

Take some small ε (say 0.1).

Suppose the game is given by

$$A^1 = \begin{pmatrix} 0+\Delta_1 & 4+\Delta_3 & 9+\Delta_5 \\ 4+\Delta_2 & 0+\Delta_4 & 10+\Delta_6 \end{pmatrix} \quad , \quad A^2 = \begin{pmatrix} 4+\Delta_7 & 0+\Delta_9 & 1+\Delta_{11} \\ 0+\Delta_8 & 4+\Delta_{10} & 1+\Delta_{12} \end{pmatrix}$$

with $|\Delta_i| < \varepsilon, (i = 1, \ldots, 12)$. Concerning the set of feasible beliefs we are very tolerant and set the new set of feasible beliefs for player 2 (see Figure 4):

$$\hat{\mathscr{F}}^2 = \left\{ \left(b_u^2, b_d^2\right) \in [0,1]^2 \,\middle|\, \min\left\{b_u^2, b_d^2\right\} \le \tfrac{1}{8} + \varepsilon \right\} .$$

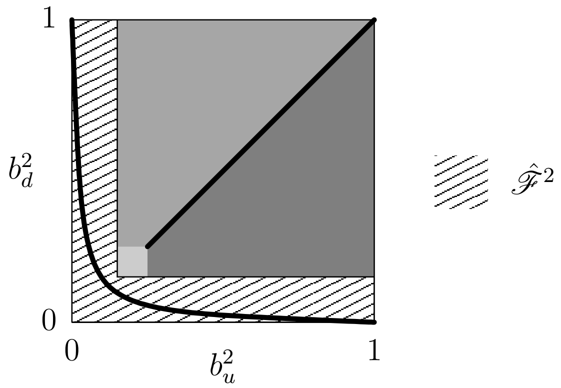

Figure 4: New feasible beliefs for player 2

Obviously, every set $A \subseteq \mathscr{B}^2$ with Hausdorff distance $d(A, \mathscr{F}^2_{w^2}) < \varepsilon$ is covered by $\hat{\mathscr{F}}^2$.

It is easy to see that still there is only one equilibrium in mixed strategies $\bar{m}$, and

$$\bar{m}^1(u) > 0, \quad \bar{m}^1(d) > 0, \quad \bar{m}^2(\ell) > 0, \quad \bar{m}^2(c) > 0, \quad \bar{m}^2(r) = 0$$

hold.

Now assume we *do* have an equilibrium in feasible beliefs b that is related to $\bar{m}$.

This means:

$$\operatorname{br} b^2 \supseteq \{\ell, c\} \; .$$

What is the expected outcome of player 2 if he played one of the (assumed optimal) strategies ℓ or c?

The expected payoffs for player 2 for his actions ℓ and c have to coincide:

$$E_{b^2}\big(a^2(\ell, \bullet)\big) = E_{b^2}\big(a^2(c, \bullet)\big) \; .$$

I. e.:

$$(4 + \Delta_7) \cdot b_u^2 = (4 + \Delta_{10}) \cdot b_d^2 \; .$$

Since the payoff must be feasible we know that either b_u^2 or b_d^2 must be less than $\frac{1}{8} + \varepsilon$. Therefore

$$(4 + \Delta_7) \cdot b_u^2 = (4 + \Delta_{10}) \cdot b_d^2 \leq (4 + \varepsilon)\big(\tfrac{1}{8} + \varepsilon\big) < 1 - \varepsilon \; .$$

But player 2 can get an expected outcome of at least $1 - \varepsilon$ by playing r! This contradicts the assumption.

Thus there is still no equilibrium in feasible beliefs which is related to the equilibrium in mixed strategies. □

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