The Role of Time-Inconsistent Preferences in Intertemporal Investment Decisions and Bargaining

Zafer Akin [*†]
Pennsylvania State University
April 30, 2004

1 Introduction

1.1 The Problem

There is a growing literature documenting the tendency of people to procrastinate when they face intertemporal decision problems. This kind of behavior creates some *time-inconsistency* that cannot be completely explained by conventional economic theory. These agents are *"time-inconsistent"* in the sense that they are very keen on their near-future selves' gratification such that their distant-future selves probably will be regretful about this pursuit of immediate gratification. Due to this tendency, they behave differently than what they planned to behave in the future.

The most obvious symptom of *time inconsistent* preferences is procrastination of unpleasant and costly tasks with the hope of completing them in the future. This procrastination tendency sometimes results in inefficient behavior such that even if taking an action is optimal, agent may procrastinate doing it. Writing a proposal for your thesis, finishing your paper, filing your taxes, starting a diet, attending gym regularly are some of the examples of costly tasks that we always have the tendency to delay up to the deadline (handing in your proposal at the very last moment, long lines in front of post office on April 15, etc.) or to procrastinate in a nonreversible way (not quitting smoking and die early, not going gym regularly and get fat, not filing your taxes and lose the tax return, etc.).

Although inefficient procrastinative behavior is observed, there is still controversy in explaining it, Mukherji et all [2002], Rubinstein [2003], Dasgupta and Maskin [2002] all give alternative explanations why such behavior might be observed. A different point of view is taken by Laibson [1997] and by O'Donoghue and Rabin [1999, 2003]. We adopt their formulation of time-inconsistent preferences, which they call quasi-hyperbolic discounting. We do not, in this paper, express any opinion about the foundational questions involved; the Laibson-O'Donoghue-Rabin formulation poses some interesting questions that we seek to answer.

The problem we are going to consider consists of two parts. In the first stage, an individual agent, whose time preferences exhibit quasi-hyperbolic discounting, has to

*Department of Economics, Pennsylvania State University, 608 Kern Graduate Building, University Park, PA, 16802. E-mail: zafer@psu.edu
†I am indebted to Prof. Kalyan Chatterjee for his valuable guidance and advice throughout this project. I also thank Prof. Thomas Sjostrom for his helpful comments. All remaining errors are mine.

choose when to complete a sequence of costly, inalienable investments in his human capital. After completing all the investments, a surplus is generated for the employer, if the agent uses his newly acquired skills. The employer and the agent bargain over the division of this surplus at the time the sequence of investments is completed. Thus, this time is endogenous in our model. Also, the employer knows the nature of the agent's time preferences. Bargaining takes place through a Rubinstein alternating offers procedure with the discounting being standard exponential for the employer and quasi-hyperbolic for the agent. The introduction of two players in this setting that includes bargaining is, as far as we know, new in this paper. We therefore have to discuss the equilibrium concept, which we shall do later. (It is what we consider a natural extension of subgame perfectness to this setting.) We determine the behavior of the quasi-hyperbolic agent who, as in O'Donoghue and Rabin, can be either naive or sophisticated. We do not, in this paper, consider partially naive agents as they do [2001].

1.2 Examples

Example 1 *Think about a doctor or a technician who wants to work in a hospital. There are some requirements of the hospital that the doctor must fulfill. She must pass some eligibility exams, have some kind of special training and have some certificates before she goes to interview with the hospital. These activities that she must complete are costly tasks including self investments. If the agent (doctor) has time-inconsistent preferences, then how is she going to make these investments? Is she going to finish all tasks in an efficient way or is she going to have some procrastination motive that makes her postpone the investments?*

Example 2 *Think about a junior employee in an insurance firm who has to complete several actuarial examinations before he or she can undertake some assignments for the company. In our model, the employee has time inconsistency problems but the employer is time-consistent. Preparing for the exams is a costly task that needs some work such as reading and research. In this situation, how does she allocate her time on these costly tasks? Does she procrastinate taking the exams or does she invest herself quickly and take the exams as soon as possible?*

Example 3 *Think about a university student. She has the opportunity to take extra classes such as computer programming, leadership, management besides her major. By taking these courses, she increases her chance to get paying-internships in summers. Moreover, if she works in summers then, she can get a job more easily in those companies she worked or in other companies after graduation. However, taking those courses is costly in terms of time, effort and money. If she has time-inconsistent preferences, does she take those classes at each semester regularly? does she postpone taking them? Or doesn't she take them at all?*

People's lives are full of the problems like above ones and more in which they always face with the trade-off between finishing long-term paying projects and their immediate costs. In this paper, we will try to explain the people's, sometimes inefficient, behavior in these kinds of situations.

1.3 Related Literature

First, Strotz [1956] suggested that people are more impatient when they make short-run trade-offs than when they make long-run trade-offs. When two payoffs are both far away in time, decision-maker tends to be more patient, on the other hand, when two payoffs are relatively close in time, then decision-maker is likely to behave more impatient (e.g., choosing between "15 minutes break now, today or 1 hour break tomorrow" and "15 minutes break in 30 days or 1 hour break in 31 days", in this example people -of course, generally- prefer the first choice in the first offer and second in the second offer which reflects the idea of discounting future in a different way). Briefly, we wish to act patiently in the long run but the desire for instant satisfaction frequently overwhelms our good intentions.

Traditionally, it is assumed that discount factors are exponential that means a util delayed τ periods is worth δ^{τ}as much a util enjoyed immediately ($\tau = 0$). Some examples of the hyperbolic discount functions that are used in the literature, Chung (1961), are like $1/\tau$ and $1/(1 + \alpha\tau)$ with $\alpha > 0$. Laibson [1997a] used called "quasi-hyperbolic discount" function, $\{1, \beta\delta, \beta\delta^2, \beta\delta^3, ...\}$ where $\beta \in [0, 1]$ and $\delta \in [0, 1]$.

Since this is a relatively new approach, there is controversy about whether we really observe hyperbolic discounting in the consumer behavior, Dasgupta and Maskin [2002]. Since this is not a foundation, this can only be argued via experiments on people and by some real-life facts. There is a growing literature in this aspect of the issue both in favour of and against it, Mukherji et all [2002], Rubinstein [2003].

Our second stage game is bargaining game between time-consistent principal and time-inconsistent hyperbolic agent. We apply the alternating-offer bargaining framework proposed first by Rubinstein [1982]. Rubinstein assumes stationary preferences overtime. Coles and Muthoo [2003], examined bargaining situations in a non-stationary environment. In their paper, they study Rubinstein's Bargaining game in which the set of possible utility pairs evolves through time in a non-stationary but smooth manner. They found that when the time interval between offers goes to zero, there exists a unique subgame perfect equilibrium.

Behavioral characterization of economic agents in the context of costly investment under the assumption of time-inconsistent preferences was examined by O'Donoughue and Rabin in a series of papers, [1999a], [1999b], [2001], [2003]. In their paper called "Doing It Now or Later" [1999a], they assume two different characteristics of time-inconsistent behavior, naive and sophisticated, and two different cost and reward structures, immediate rewards and immediate costs. Naive hyperbolic agent, NHA, is not aware of her future preference reversals or self-control problems, however, sophisticated hyperbolic agent, SHA, is fully aware of her self-control problems so that she predicts how her future selves will behave in the future correctly. O'Donoghue and Rabin found that naive agents procrastinate immediate-cost activities and do immediate-reward activities too soon. Whereas, Sophistication relieves procrastination and exacerbates preproperation. In addition, When there are multiple tasks, there is no general result saying that sophisticates always finish the tasks before naives as in the case of only one task.

In the next paper [1999b], they introduce a model in which how principals can design incentives to induce time-inconsistent procrastinators to complete tasks in an efficient way. Risk neutral agents faces with a task having a stochastic cost structure. They found that if task-cost distribution is common knowledge, the efficient outcome, which minimizes the sum of the task cost for agent and waiting cost for principal, can be achieved. If task-cost distribution is only known by the agent, efficiency often cannot be achieved for procrastinators. Also, they showed that optimal incentives for procrastinators involve an increasing punishment for delay as time passes.

Partial naivete (neither completely naive nor completely sophisticated) and menu of tasks for economic agents were introduced in their "Choice and Procrastination" paper, [2001]. They basically show that providing a nonprocrastinator additional options can induce procrastination and a person may procrastinate worse pursuing important goals than unimportant ones.

When time-inconsistent agents face with long-term projects rather than projects that are completed once begun, they can choose when and whether to complete each stage of the projects. In their recent paper, [2003], O'Donoghue and Rabin showed that not only procrastination in starting the project but also never completing the project, even if some cost was already incurred, can be observed for naive agents. Cost distribution is the key in determining the behavior of agents in this environment. If the cost structure is endogenous then, people are more likely to choose cost structures that make them to start but not finish the project.

1.4 Contributions of This Paper

We will use Laibson's "quasi-hyperbolic discount" function in this paper. Since Laibson's representation is widely used in modeling time-inconsistent behavior, we will take this as given in our paper and we will apply it to our framework without arguing whether it is the true approach or not. In O'Donoghue and Rabin's 2003 paper, there are two stages of investment. While we also have the sequence of investments (here we generalized it to k units) they have, in their case the wage after completion is exogenously given. However, the preferences and types also affect the payoff. In our model, the payoff is endogenized by introducing bargaining as the second stage game. We believe that this approach carries O'Donoghue and Rabin's framework to a broader context and it is more realistic and plausible in such situations that have not only interactions among the agents' selves but also strategic interaction of different agents. Introducing the second stage bargaining game is a new approach and introducing boundedly rational players in a bargaining game is also a new approach. So, one of the most interesting aspects of our problem is the interplay between alternating offers bargaining and time-inconsistent preferences. Our version of subgame perfect equilibrium is also therefore somewhat new. This is a twist from the classical rational expectations approach because the beliefs of time inconsistent agent turn out to be wrong eventually.

We do not give a formal definition of the equilibrium and will refer to it as "subgame perfect equilibrium" in the sequel, the definition of which is well-known. Rational expectations that is used in almost all strategic games assumes that people do not make systematic errors when predicting the future and forming beliefs about the future. Obviously, Naive hyperbolic agent (NHA) is systematically wrong about herselves' behavior in the future and also about the opponent's belief about herselves. So, she is boundedly rational. However, the crucial thing in this strategic environment, giving best responses mutually even if there exists this kind of inconsistent beliefs. Informally, a Nash equilibrium involves players playing best responses to their beliefs about the other player and the beliefs are correct and mutually consistent. Not surprisingly, with time-inconsistent behavior, the last requirement is difficult to satisfy. In our case, the beliefs about the other player's actions are correct but the hyperbolic discounter believes that the exponential player has wrong beliefs. (Since the time-inconsistent player is wrong about her own future actions, it is not surprising that she believes the exponential player is also wrong about these.) Given this caveat, subgame perfectness is defined in the usual way as being Nash after every history.

In this nonstationary environment, since beliefs about the actions and the actual actions may differ, it is crucial to look at the beliefs of each agent. In the bargaining game, the beliefs are like the following: Exponential agent, EA, believes that "NHA believes that I am EA" and "NHA will have this self-control problem not only in the very near future but also in the distant future" and she is right in these beliefs. On the other hand, NHA believes that "the opponent is actually EA", "Today, I follow my immediate gratification but this will not be the case in the future" and "EA believes that I am naive and I will behave time inconsistently in the future but she is wrong about this". However, NHA is wrong in all these beliefs except the opponent's type. She thinks that EA is wrong about herself but actually EA is right. She is wrong about herself too.

Again, the equilibrium concept includes the mutual best responses given the above beliefs. First, they do not want any delay in agreement. Second, since in equilibrium the game will end in the first period, the wrong beliefs of NHA will not be implemented. However, the equilibrium shares are determined based on these beliefs. If EA makes the first offer, then she will offer NHA a share that will be accepted given NHA's wrong beliefs about herself. NHA will accept this offer because she thinks that she will be time consistent in the future and EA takes this into account while making the offer. In other words, NHA expects a lower share because of her wrong beliefs from EA but when EA makes her offer, she takes into account that NHA is thinking to be time consistent in the future and she offers a share that is confirming the wrong beliefs of NHA about herself. By offering more, EA makes NHA believe that she (EA) believes that she (NHA) will be time consistent in the future as she (NHA) believes about herself. Since this high offer is kind of a signal confirming NHA's wrong beliefs then she accepts the offer.

In our paper, we have an exogenous time constraint for the completion of the project. As an extension, O'Donoghue and Rabin think a partial reward scheme (payoff is given not only when the investment stage is finished but also after each unit is invested) under the assumption of fixed total reward (partial rewards are basically transfers from the total payoff in the end) that causes more severe procrastination. However, we will consider a variable (not constant) total reward scheme as an incentive mechanism to make agents not to procrastinate by introducing some "bonus" scheme for each invested unit (the principal actually gives up from some of her surplus that she will earn in the end).

One other assumption is that type of each agent is known by the other agent in bargaining stage. The rationale behind this is that in the first stage, the investment pattern of each agent signals the type of the agent, so in the bargaining game that is played right after the investment stage is finished, there is complete information about the types of the agents. Incomplete information case will be examined in the future.

The other interesting point of introducing the bargaining stage is that since NHA is mistaken in predicting her wage (she predicts a higher wage than she will actually get, which shall be clear later) as the result of second stage game, this may create a motive of regret that she may finish a project that is *not* worth finishing since she will get a less payoff than she expects. So, this may be an interesting approach in understanding people's disappointments resulting from their great expectations about future.

Moreover, we assume an infinite and discrete time horizon but we assume an exogenous deadline for the first stage of the game rather than assuming a limitless investment phase. However, this assumption is plausible in some situations but it is not in some other situations, so, it can be relaxed according to the characteristic of the interested problem. In this long-term project framework, O'Donoghue and Rabin also examined the behavioral consequences of the partial naivete, which is defined and explained in [2001], and found that degree of naivete alters the investment patterns of the agents. We can easily extend our analysis to that case too.

1.5 Description of the Model and Preliminary Results

In order to make the comparison between different characteristics of preferences in our model, we will suppose that the principal is exponential discounter and the agent is either exponential agent (EA, time-consistent) or naive hyperbolic agent (NHA) or sophisticated hyperbolic agent (SHA). Exponential discounter has a sequence of discount factors $\{1, \delta, \delta^2, \delta^3, ...\}$. Naive and sophisticated hyperbolic agents have the same sequence of discount factors $\{1, \beta\delta, \beta\delta^2, \beta\delta^3, ...\}$. The parameter δ represents standard time-consistent impatience. The parameter β represents the self-control problem of the agent where smaller β means the agent has more significant self-control problems. In other words, β is called as the time-inconsistent preference for immediate gratification. For $\beta = 1$, agent has completely time-consistent preferences. The only difference between SHA and NHA is that NHA is not aware of her future preference reversals or self-control problems, however, SHA is fully aware of her self-control problems so that she predicts how her future selves will behave in the future correctly. Thus, NHA thinks that she will evaluate future payoffs with discount factor δ but, in fact, she will evaluate them with $\beta\delta$ whereas SHA correctly predicts that she will evaluate future payoffs with $\beta\delta$.

The model is as follows. Time is discrete and the time horizon is infinite. There is one principal and one agent, e.g. a hospital and a doctor to be hired as in example-1. Along the exogenously given $\hat{T}$ periods, the agent invests to reach to some amount of human capital (exogenously given also). The agent faces a 0 or 1 investment decision to make at each time t. This implies a fixed cost, C, for each unit of investment. In order to finish the investment phase, k units of investment have to be made where $k \leq \hat{T}$. T is the time at which the investment is finished and satisfies $k \leq T \leq \hat{T}$. Provided that the agent finishes the investment phase, the second stage game is played that is bargaining between the agent and the principal, which ends at the period that it is played in equilibrium. After completing all the investments, a surplus is generated for the employer, if the agent uses his newly acquired skills and this surplus will be shared in the bargaining game. Define w as the equilibrium wage determined in the bargaining stage of the game that will be earned at T and each period after T .

All agents will work backwards such that they will predict the wage -EA and SHA predicts correctly but NHA is mistaken in her prediction- that they will earn and depending on this wage earnings, they have to decide on the distribution of investments (k units of investment in $\hat{T}$ periods) and automatically on the time they finish the first phase.

By using the above specification of the model, we get the following results:

- In the bargaining game, under the assumption that both NHA and SHA has same β, preference for immediate gratification, and δ, standard time-consistent impatience, NHA always gets strictly more payoff than SHA.

If we have a homogeneous cost structure and a no-partial-reward system in the investment stage and assume optimality of finishing the investment stage then,

- Agents always invest consecutively regardless of their preferences.
- Naïve agent finishes investment stage without any delay whereas Sophisticated agent has a periodical investment schedule along the time path- SHA's belief about her investment behavior is cyclical. Given that once she starts investing, she will invest consecutively, she will expect to start project at every t_k^* periods where t_k^* is the maximum tolerable delay time and it will be defined formally later.
- Depending on the parametric values, investment schedule of SHA is found.
- The existence of a specific value for β -preference for immediate gratification-

that makes SHA finish investment stage without delay is shown. Analogously, the existence of a specific value for C -homogeneous cost of each unit of investment- that makes SHA finish investment stage without delay is shown.

If we have a bonus scheme in the investment stage and the project is not "worthwhile" for the SHA then:

- The bonus should increase in order for NHA to continue to invest and finish the investment stage. Otherwise, she will procrastinate.
- The agents with higher self-control problems - lower $\beta's$- should be given higher bonus, with a small caveat, by the principal in order to induce them to complete the same investment project.

The rest of the paper is organized as follows. Section 2 describes the formal model in detail. Section 3 considers the subgame perfect equilibrium of the considered game. Section 4 provides some extensions of the existing model. Section 5 concludes the paper with a brief discussion of the results.

2 The Formal Model

We suppose that the principal is exponential discounter and the agent is either exponential agent (EA, time-consistent) or naive hyperbolic agent (NHA) or sophisticated hyperbolic agent (SHA). The environment is as follows. Time is discrete and the time horizon is infinite. There is one principal and one agent. Along the exogenously given $\hat{T}$ periods, the agent invests to reach to some amount of human capital $\hat{V}$ (exogenously given also). For simplicity, the agent faces with a 0 or 1 investment decision, e_t, at each time t;

$$e_t = 0 \ or \ e_t = 1$$

Provided that the agent finishes the investment phase, the second stage game is played that is bargaining between the agent and the principal. In order to finish the investment phase, k units of investment have to be made. If we call the marginal value of investment as $f(e_t)$, where $f(0) = 0$, then, since the agent can only make 0 or 1 unit of investment in each period, we can write

$$\hat{V} = kf(1)$$

where $\hat{V}$ is the value of human capital that is required for the job.

In this framework, investments are costly tasks as reading journals, attending courses, training programs... etc. The cost can be interpreted as the opportunity cost of having spent the time on these activities or the disutility of these activities to the agent. Human capital investment accumulates over time.

We can suppose $\hat{V}$ is exogenously given because, e.g., the basic requirements or skills (or self-investment) needed to be acquired are announced by the principal. $\hat{V}$ is basically the size of the pie to be shared between the principal and the agent after the agent's $T \leq \hat{T}$ period investment phase. Since we know the pie size and the types of the players are known by each player in the bargaining stage, we can find the subgame perfect equilibrium (SPE) of it. Then, since they know the reward that they will earn, agents can decide on their investment distribution in the first stage, $\{e_t\}_{t=0}^{\hat{T}}$.

Along the paper, unless others are indicated, the following assumptions will be made:

- Whenever $\hat{V}$ is completed, bargaining game starts (and ends at that period, in equilibrium)
- After $\hat{T}$ periods, the agent does not have to make any investment.

- No depreciation on accumulated capital.
- Outside option for both players is zero.
- Accumulating capital does not give any utility other than the expected future wage income, which is determined by the bargaining game.

Definitions: Let the marginal cost of investment be $C(e_t)$ and it is time independent, strictly convex and satisfies $C(0) = 0$. Since we allow only zero or one unit investment choice, for notational convenience, let $C(1) = C$. The value of human capital accumulated up to time t is $\hat{V}_t$. $\hat{V}_t$ is a step function and it is weakly increasing between 0 and $\hat{T}$. After $\hat{T}$, it is constant. The relationship between $\hat{V}_t$ and e_t can be written as:

$$\hat{V}_{t+1} - \hat{V}_t = f(e_t) \textit{ or } \hat{V}_{t+1} = f(e_0) + f(e_1) + \ldots + f(e_t) = \sum_{j=0}^{t} f(e_j).$$

Indeed, if agent finishes the project, then:

$$\hat{V} = \hat{V}_{\hat{T}} = \sum_{j=0}^{\hat{T}-1} f(e_j) = f(e_0) + f(e_1) + \ldots + f(e_{\hat{T}-1})$$

Define k as the amount of investment required to be completed. The following means agent finishes the first stage within the time constraint:

$$\sum_{j=0}^{T-1} e_j = k.$$

For convenience, we suppose that

$$k \leq \hat{T}$$

which means, if the agent wishes, she can finish the investment phase. In other words, the required investment is doable in $\hat{T}$ periods.

Define T as the time at which the investment is finished:

$$\hat{V} = kf(1) = \sum_{j=0}^{T-1} f(e_j) \textit{ where } e_j = 0 \textit{ or } e_j = 1$$

T also satisfies the following:

$$k \leq T \leq \hat{T}$$

which means, she can finish the investment phase at least in k periods without any delay or at most in $\hat{T}$ period.

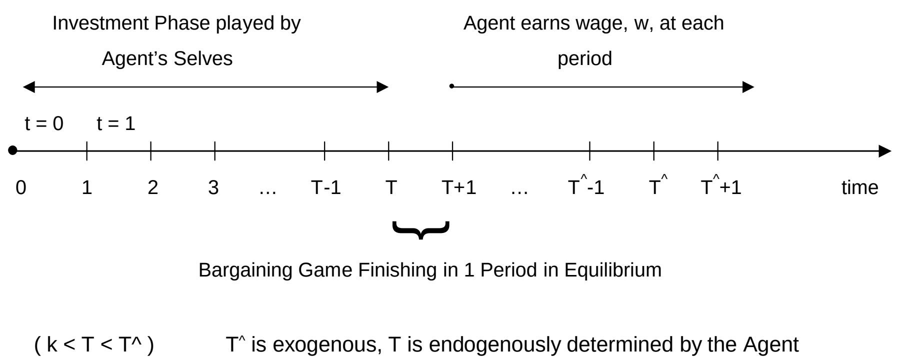
If the following is the case, then it means that agent completes the required investment amount in the maximum amount of time $\hat{T}$ and continues to bargaining game:

$$\hat{V} = \sum_{j=0}^{\hat{T}-1} f(e_j).$$

If the required investment amount is not completed:

Figure-1

Timing of the Game



$$\hat{V} < \sum_{j=0}^{\hat{T}-1} f(e_j)$$

the agent cannot continue to second stage and gets zero payoff.

Define $w(\hat{V})$ as the equilibrium wage determined in the bargaining stage of the game. Since the value of $\hat{V}$ is given and we can find the equilibrium partition of the bargaining game, we know the equilibrium wage $w(\hat{V})$.

Problem: The problem can be summed up like the following, at time 0, the agent maximizes her discounted utility subject to her time constraint:

$$\max_{\{e_t\}_{t=0}^{\hat{T}-1}} \{\beta\delta^{T+1}(\frac{w(\hat{V})}{1-\delta}) - [C(e_0) + \beta \sum_{j=0}^{\hat{T}-1} \delta^j C(e_j)]\}$$

subject to

$$\sum_{j=0}^{\hat{T}-1} f(e_j) = \hat{V}$$

$$k \leq T \leq \hat{T}$$

Timing of the Game: We now briefly mention how the game proceeds in time. First T periods, agent's his selves play the investment game. At period T, the bargaining game is played and it ends in period T with equilibrium wage for the agent. From T onwards (including T), the agent will earn the equilibrium wage at each period. Now the problem can be defined as the following. Given $f(e_t)$, $C(e_t)$ and $\hat{V}$, at each period, agent will decide whether to invest on himself to complete the $\hat{V}$ in order to maximize his expected payoff. In other words, problem is to choose $\{e_t\}_{t=0}^{\hat{T}-1}$ sequence to maximize the expected payoff such that $\hat{V} = \sum_{j=0}^{\hat{T}-1} f(e_j)$ supposing the agent being exponential discounter, naive hyperbolic discounter or sophisticated hyperbolic discounter. The first stage of this game is, in some sense, similar to the Admati-Perry's joint project investment framework but the difference is that the investment here is one-sided and made not by different players but by exponential and hyperbolic agent's selves.

3 Characterizing Equilibrium

3.1 Second Stage Game:

We think about the Rubinstein's alternating offers model [1982]. There are two players, at each time, one of the players makes an offer and the other player accepts or rejects it. In case of acceptance, the payoffs (the offered share of pie) are realized at that period, otherwise roles are reversed. As the purpose of the paper, we assume first that, first player, the employee in the insurance company, say, has hyperbolic discounting like in Laibson (1997a) $\{1, \beta\delta, \beta\delta^2, \beta\delta^3, ...\}$. So player 1, is relatively impatient for tomorrow -which is close- but more patient for distant future. In the hyperbolic discounting literature, there are two types of agents having this kind of discounting, naive and sophisticated, as explained above. we will think both here but first we begin with the sophisticated one.

Since it will be needed in the following results, it is useful to write down the equilibrium of the Rubinstein Bargaining game when both agents are exponential and have discount factors δ_1 and δ_2. We can write the result as either the limit case of the finite horizon game or the recursive way of solving it like in Shaked-Sutton, using stationarity of the game. The result is like the following:

Remark 1: In the infinite horizon alternating offers game with both players have exponential discounting with discount factors δ_1 and δ_2, the equilibrium payoffs are:

$$(x^*, 1 - x^*) \ where \ x^* = \frac{1 - \delta_2}{1 - \delta_1 \delta_2}$$

and x^* is the payoff of the agent 1, making first offer.

Claim 1: In the infinite horizon alternating offers game on a size 1 pie with only the first player (sophisticated) has hyperbolic discounting, SHA, the equilibrium payoffs are like the following:

if SHA makes the first offer, payoffs are:

$$(x^*, 1 - x^*) \ where \ x^* = \frac{1 - \delta}{1 - \beta\delta^2}.$$

if EA makes the first offer, payoffs are:

$$(y^*, 1 - y^*) \ where \ y^* = \beta\delta(\frac{1 - \delta}{1 - \beta\delta^2}).$$

Before looking at the equilibrium of the game, it is good to emphasize the behavior of the sophisticated hyperbolic player again. We know from the general framework of the Rubinstein model that in equilibrium of this game, since there is discounting, players reach to an immediate agreement. So in equilibrium, first player offers an x^* such that second player accepts and realized payoffs would be $(x^*, 1 - x^*)$. In this context with Hyperbolic Agent (HA), hyperbolic discounting will have significant role if the future time periods are reached and apparently, whether HA is naive or sophisticated is also important. The HA is relatively impatient for tomorrow and more patient for periods onward, but if future periods are reached then she makes his own maximization again and this makes the difference in general. So, since we think about alternating offers model ending in the first period with the equilibrium value offers x^* or y^*, the HA will be behaving consistently (at least she believes like that) because future will not be reached (of course, we think about the problem by applying backward induction in finite case and we solve it as we are at the last stage and find equilibrium offers, and work backwards). In SHA case, EA will anticipate that SHA is sophisticated and that if she delays the game by rejecting SHA's offer, she (SHA) will make the trade off between that day and the next day at rate $\beta\delta$ and she is aware of this. So, she simply thinks that SHA's discounting rate is effectively $\beta\delta$ and makes her offers according to this conjecture. Thus, since we look at the future from the perspective of the first period, the behavior of the SHA, who knows that she will reverse her preferences in the future, will seem as time consistent and her effective discount rate would be "$\beta\delta$". Simply, Claim 1 says that SHA will get the amount of payoff where she has an effective discount rate "$\beta\delta$" and EA has "δ".

Proof of Claim 1:

We can use the Shaked-Sutton's solution concept to find the equilibrium payoffs in this problem. In order to use this approach, we need to have stationarity. Stationarity (in the sense of Rubinstein, 1982) can be interpreted as, the preference of (x, t) over $(y, t+1)$ is independent of t. This means that the preference over getting x at time t and getting y at time $t + 1$ is independent of time where x and y are the shares of a size 1 pie. In other words, the agents do not have any preference reversals over time.

As it is explained above, the game, with an EA and a SHA, turns out to be the stationary Rubinstein's alternating offers Bargaining game with $\delta_1 = \beta\delta$ and $\delta_2 = \delta$. Then, the following would be true since we have stationarity:

$$1 - x^* = \delta(1 - y^*)$$

$$y^* = \beta\delta x^*$$

which gives the following results:

$$y^* = \beta\delta(\frac{1-\delta}{1-\beta\delta^2})\ and\ x^* = \frac{1-\delta}{1-\beta\delta^2}$$

So, when SHA offers first then she (SHA) gets x^* and when EA offers first, she (SHA) will get y^*. ∎

The Naive HA case is a little more tricky. Since the HA player is naive, she would change his preferences because of the hyperbolic discounting characteristic and also she is not aware of this kind of characteristic of herself. She would think that, for example, at $t = 1$, she trades off payoffs today and tomorrow at $\beta\delta$, but again at $t = 1$, she thinks that she will trade off future payoffs between time t and $t + 1$

, like EA, at the rate δ. However, once she gets to the time t, she uses $\beta\delta$ again for tomorrow. So, The Naive HA (when she responds) in bargaining would assume that she will accept δx, where x is his payoff from the offer she makes, which is accepted, three periods from now. However now, she is ready to accept $\beta\delta$ times his continuation payoff in which she assumes δ discounting in all future periods.

Claim 2: In the infinite horizon alternating offers game with only the first player (naive) has hyperbolic discounting, NHA, the equilibrium payoffs are like the following:

1.if NHA makes the first offer, payoffs are:

$$(x^*, 1-x^*) \ where \ x^* = \frac{1}{1+\delta}.$$

2.if EA makes the first offer, payoffs are:

$$(x^*, 1-x^*) \ where \ x^* = 1-\beta\delta(\frac{1}{1+\delta}).$$

Proof of Claim 2: First offer is always represented by x^* and second offer is always represented by y^*.

1. As it can be noticed the equilibrium payoffs when NHA makes the first offer is same as the one with both agents are EA. This is because of the following: by considering the situation of NHA, we can say that NHA should convince EA to accept the offer x^* at the very first period. She thinks that EA evaluates each t and $t+1$ with discount factor δ for all t and also since she is naive, she thinks that she will evaluate each t and $t+1$ with discount factor δ for $t \geq 2$. In order to make EA indifferent between accepting and rejecting, she offers EA the payoff that she can, at most, get by rejecting the offer that is

$$1-x^* = 1-(1+(-\delta)+(-\delta)^2+(-\delta)^3+(-\delta)^4+...)$$

$$1-x^* = \frac{\delta}{1+\delta}$$

2. EA makes the first offer and there is a major change now like the following. EA, by offering x^*, she should make NHA indifferent between accepting and rejecting. She knows that HA has discount factor $\beta\delta$ between $t=1$ and $t=2$, so she should make an offer y^* satisfying

$$1-x^* = \beta\delta(1-y^*).$$

But, since NHA is naive, she would think that the value of $(1-y^*)$ can, at most, be $\frac{1}{1+\delta}$, which is the payoff of the NHA when both have discount factor δ for $t \geq 2$, and NHA makes the offer at $t=2$. If we plug this into the above equality then,

$$1-x^* = \beta\delta(\frac{1}{1+\delta}).$$

we get the result in the claim. ■

We now check the payoffs of the agents when they have different characteristics:

We assume complete information, which means, each player knows the other player's characteristic, e.g., in a NHA-NHA game, a Naive HA knows that the other player is naive but she does not know she, herself, behaves naively.

Row player makes the first offer to the column player in the alternating offers bargaining game.

Payoffs are in the P matrix such that first entry is the payoff of the row player and second entry is the payoff of the column player.

For example, P_{23} represents the game where SHA makes the first offer to the NHA and gets the first entry of the P_{23}.

P_{11} is the case of classical alternating offer bargaining game.

P_{12} and P_{21} are the results of the *claim*1.

P_{13} and P_{31} are the results of the *claim*2.

P_{22} is the same case where two EA with effective discount factors "$\beta\delta$".

P_{23} is a similar case to the P_{13} with EA has an effective discount factor "$\beta\delta$".

P_{32} is a similar case to the P_{31} with EA has an effective discount factor "$\beta\delta$".

P_{33} is a similar case to the P_{13} since both would think that they will have discount factor "δ" for $t \geq 2$.

$$P = \begin{array}{cccc} & EA & SHA & NHA \\ EA & (\frac{1}{1+\delta}, \frac{\delta}{1+\delta}) & (\frac{1-\beta\delta}{1-\beta\delta^2}, \frac{\beta\delta(1-\delta)}{1-\beta\delta^2}) & (1 - \frac{\beta\delta}{1+\delta}, \frac{\beta\delta}{1+\delta}) \\ SHA & (\frac{1-\delta}{1-\beta\delta^2}, 1 - \frac{1-\delta}{1-\beta\delta^2}) & (\frac{1}{1+\beta\delta}, \frac{\beta\delta}{1+\beta\delta}) & (1 - \frac{\beta\delta-\beta^2\delta^2}{1-\beta\delta^2}, \frac{\beta\delta-\beta^2\delta^2}{1-\beta\delta^2}) \\ NHA & (\frac{1}{1+\delta}, \frac{\delta}{1+\delta}) & (\frac{1-\beta\delta}{1-\beta\delta^2}, \frac{\beta\delta(1-\delta)}{1-\beta\delta^2}) & (1 - \frac{\beta\delta}{1+\delta}, \frac{\beta\delta}{1+\delta}) \end{array}$$

We can compare the payoffs of each player by checking the payoff matrix P.

Lets state a result according to above payoff structure:

Result: Assume each agent has the same time-consistent impatience, δ. Further, assume that the self-control problem of the agents, if they have, are also same, β. Then,

1. If a player makes the first offer, then regardless of the opponents type, his payoff will be $P_{EA} = P_{NHA} > P_{SHA}$ according to his type.
2. If a player makes the second offer, then regardless of the opponents type, his payoff will be $P_{EA} > P_{NHA} > P_{SHA}$ according to his type, where P_i represents the payoff of i.

Proof: The types here are being either time consistent or naive hyperbolic or sophisticated hyperbolic agent. We can infer the result from the above payoff matrix:

1. If a player makes the first offer, then in terms of payoffs, if she is EA or NHA, then she gets the same payoff, which is better payoff than the case where she is SHA, in other words $P_{EA} = P_{NHA} > P_{SHA}$, since
$$\frac{1}{1+\delta} > \frac{1-\delta}{1-\beta\delta^2}; \ \frac{1-\beta\delta}{1-\beta\delta^2} > \frac{1}{1+\beta\delta} \ ; \ 1 - \frac{\beta\delta}{1+\delta} > 1 - \frac{\beta\delta - \beta^2\delta^2}{1-\beta\delta^2}$$
2. If a player makes the first offer, then in terms of payoffs, if she is EA, then she does better than the case where she is NHA who does better than the case where she is SHA, in other words, $P_{EA} > P_{NHA} > P_{SHA}$, since
$$\frac{\delta}{1+\delta} > \frac{\beta\delta}{1+\delta} > \frac{\beta\delta(1-\delta)}{1-\beta\delta^2} \ ; \ 1 - \frac{1-\delta}{1-\beta\delta^2} > \frac{\beta\delta - \beta^2\delta^2}{1-\beta\delta^2} > \frac{\beta\delta}{1+\beta\delta}$$

Briefly, this result states that given the assumptions on preferences; 1. If someone is the first mover and if she is EA or NHA, then she gets more payoff than the case if she were SHA. However, being EA or NHA makes her earn the same payoff. 2. If someone is second mover, then she gets the largest payoff if she is EA and the least payoff if she is SHA. $\blacksquare$

3.2 First Stage Game

Lets try to find the equilibrium path sequence of investment levels for the agents. Let the equilibrium sequence investment levels be $\{\hat{e}_i\}_{i=0}^{\hat{T}-1}$ and $\hat{e}_i \geq 0$, $i = 1, 2, ..., \hat{T}-1$. Also, define the following;

$$(t_1; t_2; ...; t_k) \text{ means invest at } t_1,\ t_2, ..., t_{k-1} \text{ and finish at } t_k.$$

We assume that in second stage bargaining game the principal makes the first offer. Since the bargaining game is ahead at least k periods from now on, NHA will think that she will behave consistently at the bargaining game and she predicts the outcome of it according to the perception that she and the principal have same preferences, δ. So, from the second stage game, the expected wage of NHA would be

$$w_{NHA}(\hat{V}) = \hat{V}\frac{\delta}{1+\delta}$$

However, when she actually finishes the first stage (if she does so) and go on bargaining game, she will get $\beta w_{NHA}(\hat{V})$. On the other hand, SHA predicts the true wage as it will be in the bargaining game. Expected wage (and actually realized) of SHA would be

$$w_{SHA}(\hat{V}) = \hat{V}\frac{\beta\delta(1-\delta)}{1-\beta\delta^2}$$

Similarly, expected wage of EA would be

$$w_{EA}(\hat{V}) = \hat{V}\frac{\delta}{1+\delta}$$

$$w_{EA}(\hat{V}) = w_{NHA}(\hat{V}) > w_{SHA}(\hat{V})$$

We also make an assumption making our life easier that finishing the first stage game is optimal for all agents, that is, SHA finds it optimal to finish the game (This implies it is optimal for NHA and EA too).

Assumption: Completion of the investment phase of the game is optimal for SHA:

$$(0, 1, 2..., k) = \frac{\beta\delta^k}{1-\delta}w_{SHA}(\hat{V}) - \beta\sum_{j=1}^{k-1}\delta^j C - C > 0$$

This implies optimality for NHA and implies the following too, for EA:

$$(0, 1, 2..., k) = \frac{\delta^k}{1-\delta}w_{EA}(\hat{V}) - \sum_{j=0}^{k-1}\delta^j C > 0$$

Remark 2: One implication of the above assumption is the following: For NHA, finishing investment by starting immediately and investing consecutively is always better than postponing this one period for any amount of investment left. In other words, the following assumption

$$\frac{\beta\delta^k}{1-\delta}w_{SHA}(\hat{V}) - \beta\delta^{k-1}C - \beta\delta^{k-2}C - ... - \beta\delta C - C > 0$$

or

$$(t_1; t_1+1; ...; t_1+k-1) > 0$$

implies that for NHA:

$$(t_1; t_1+1; ...; t_1+j-1) > (t_1+1; t_1+2...; t_1+j) \; \forall j \leq k$$

This can be shown like the following:
Assume the following:

$$\frac{\beta\delta^k}{1-\delta}w_{SHA}(\hat{V}) - \beta\delta^{k-1}C - \beta\delta^{k-2}C - ... - \beta\delta C - C > 0 \Longrightarrow$$

$$\frac{\beta\delta^k}{(1-\delta)}\frac{1}{(1+\beta\delta+...+\beta\delta^{k-2}+\beta\delta^{k-1})}\hat{V}\frac{\beta\delta(1-\delta)}{1-\beta\delta^2} > C \Longrightarrow$$

$$\frac{\beta^2\delta^{k+1}\hat{V}(1-\delta)}{(1+\beta\delta-\delta-\beta\delta^k)(1-\beta\delta^2)} > C \tag{1}$$

Now, check whether the following is true:

$$(t_1; t_1+1; ...; t_1+j-1) > (t_1+1; t_1+2...; t_1+j) \; \forall j \leq k$$

check first for $j = k$;

$$\frac{\beta\delta^k}{1-\delta}w_{NHA}(\hat{V}) - \beta\delta^{k-1}C - ... - \beta\delta C - C > \frac{\beta\delta^{k+1}}{1-\delta}w_{NHA}(\hat{V}) - \beta\delta^k C - ... - \beta\delta^2 C - \beta\delta C$$

$$\frac{\beta\delta^{k+1}\hat{V}}{(1-\beta\delta^k)(1+\delta)} > C \tag{2}$$

If Inequality (2) is satisfied for $j = k$, then it is satisfied for all $j < k$, since it is decreasing in k. Lets show that the assumption (1) implies (2). Inequality (1) can be written as

$$\underbrace{\frac{\beta\delta^{k+1}\hat{V}}{(1-\beta\delta^k)(1+\delta)}}_{(a)}\underbrace{\frac{(1-\beta\delta^k)(1+\delta)\beta(1-\delta)}{(1+\beta\delta-\delta-\beta\delta^k)(1-\beta\delta^2)}}_{(b)} > C \tag{3}$$

In (3), (a) is same with expression (2). The condition $(b) < 1$ is sufficient for $(a) > C$ to be satisfied, which is the desired result. Fortunately, it is easy to show that $(b) < 1 \; \forall k > 1$. For $k = 1$, it is satisfied too since for NHA,

$$(t_2; t_2+1) \geq (t_2+1; t_2+2) \Rightarrow (t_2+1) > (t_2+2)$$

■

In order to solve this perfect information game, we will use backward induction by just starting from the period that one unit of investment is left.

Step 1: Suppose we have at time t_1 ($k-1 \leq t_1 \leq \hat{T}-2$), which is the first time that only one unit of investment is needed to finish the first stage -for $t_1 = \hat{T}-1$, every agent finishes the investment right away by assumption. She can either finish by investing one unit or she postpones investing one period and finishes it next period. We now examine what the agent does when she has different preferences:

1a. Exponential Agent (EA): If the agent has exponential discounting, then she has two different choices. Either she invests now, t_1, and finishes it or she postpones investing and finishes it next period. If she invests now, she gets the payoff:

$$-C + \delta w_{EA}(\hat{V}) + \delta^2 w_{EA}(\hat{V}) + ... = \frac{\delta}{1-\delta} w_{EA}(\hat{V}) - C = \frac{\delta^2}{1-\delta^2}\hat{V} - C \tag{1a-1}$$

If she postpones investment one period and then finishes, she gets:

$$\frac{\delta^3}{1-\delta^2}\hat{V} - \delta C \tag{1a-2}$$

As easily seen, $\delta(1a-1) = (1a-2) > 0$ implies $(1a-1) > (1a-2) > 0$. It is also obvious that postponing more than one period is not optimal either. Thus, EA chooses to finish the investment phase right away when only one unit of investment left.

1b. Naive Hyperbolic Agent (NHA): *:* NHA compares two different options like EA such that

$$\underbrace{0}_{\text{do nothing}} < \underbrace{\frac{\beta\delta^2}{1-\delta^2}\hat{V} - C}_{\text{finish right away}} \tag{1b-1}$$

and

$$\underbrace{0}_{\text{do nothing}} < \underbrace{\frac{\beta\delta^3}{1-\delta^2}\hat{V} - \beta\delta C}_{\text{postpone one period}}$$

By *Remark* 2, finishing right away is better than postponing one period or $(t_1) > (t_1+1)$. Thus, NHA finishes investment whenever one unit of investment left.

1c. Sophisticated Hyperbolic Agent (SHA): What SHA does is a little different from NHA. We will work backwards. We know that if SHA is at $t_1 = \hat{T}-1$, then She will invest for sure. Lets define a critical value for delaying time like the following:

$$t_1^* = \min\{s \in \{1,2,...\} | \frac{\beta\delta w_{SHA}(\hat{V})}{1-\delta} - C \geq \frac{\beta\delta^{s+1}}{1-\delta} w_{SHA}(\hat{V}) - \beta\delta^s C\}$$

This means that SHA can tolerate $t_1^* - 1$ periods of delay. In other words,

$$\frac{\beta\delta^2}{1-\delta}w_{SHA}(\hat{V}) - \beta\delta C > \frac{\beta\delta^3}{1-\delta}w_{SHA}(\hat{V}) - \beta\delta^2 C > ...$$

$$... > \frac{\beta\delta^{t_1^*}}{1-\delta}w_{SHA}(\hat{V}) - \beta\delta^{t_1^*-1}C > \frac{\beta\delta}{1-\delta}w_{SHA}(\hat{V}) - C > \frac{\beta\delta^{t_1^*+1}}{1-\delta}w_{SHA}(\hat{V}) - \beta\delta^{t_1^*}C$$

or

$$(t_1+1) > (t_1+2) > ... > (t_1+t_1^*-1) > (t_1) > (t_1+t_1^*)$$

Note that we are at period t_1 and above expression means that delaying $t_1^* - 1$ period is acceptable but t_1^* is not. We can now continue our analysis. Suppose that SHA is at $t_1 = \hat{T} - 2$. Since she knows that if she postpones to $\hat{T} - 1$ then she will finish it for sure. Since one period delay is acceptable, then SHA will postpone it one period. Suppose that SHA is at $t_1 = \hat{T} - 3$. Since she knows that if she postpones to $\hat{T} - 2$ then she will finish it at $\hat{T} - 1$. Since two period delay is acceptable, then SHA will postpone it to $\hat{T} - 2$ and then to $\hat{T} - 1$ and finish at $\hat{T} - 1$. If we make this $t_1^* - 1$ times, SHA again will postpone because $t_1^* - 1$ period delay is acceptable. Now, suppose SHA is at period $t_1 = \hat{T} - 1 - t_1^* = \hat{T} - t_1^* - 1$. Then, she knows that if she postpones investing now then she will invest at $\hat{T} - 1$ and it is not optimal to postpone t_1^* periods which means $(\hat{T} - t_1^* - 1) > (\hat{T} - 1)$. Thus, she invests at $t_1 = \hat{T} - t_1^* - 1$ and finish the investment phase. Suppose $t_1 < \hat{T} - t_1^* - 1$. The above argument can be repeated that she will postpone if $\hat{T} - 1 - 2t_1^* < t_1 < \hat{T} - t_1^* - 1$. However, if $t_1 = \hat{T} - 1 - 2t_1^*$, then she invests.

Notice that we can go on this iteration and we can say that SHA has a periodic investment plan. The behavior characterizations of SHA and other agents are established in the following result.

RESULT 1: Suppose Agents are at time t_1 and there is only one unit of investment left. Also suppose that completion of the investment phase of the game is optimal. Then, For all parametric values, $\beta, \delta, \hat{T}, \hat{V}$ and for all functional forms $w(\hat{V}), f(.), C(.)$;

$\hookrightarrow$EA: She always finishes the first stage immediately,

$\hookrightarrow$NHA: She always finishes the first stage immediately,

$\hookrightarrow$SHA: She invests and finish the first stage at t_1 if and only if $t_1 \in \{\hat{T} - 1 - it_1^*\}$ where $i \in \{0, 1, 2, ...\}$ such that $0 \leq \hat{T} - 1 - it_1^*$. If $t_1 \notin \{\hat{T} - 1 - it_1^*\}$, then she will postpone investing up to the closest time such that $t_1 \in \{\hat{T} - 1 - it_1^*\}$. If $\hat{T} - 1 - t_1^* < 0$, then SHA follows the $(\hat{T} - 1)$ strategy.

Proof:

The argument is in the above substeps, $1a, 1b, 1c$. The explanation for the last part is that If $\hat{T} - 1 - t_1^* < 0$ or $\hat{T} - 1 < t_1^*$ then the available time for investment is, $\hat{T} - 1$, less than or equal to the maximum tolerance time, $t_1^* - 1$, which means she can tolerate to postpone the investment up to the period $\hat{T} - 1$. Thus, she follows $(\hat{T} - 1)$ strategy. ■

Step 2: Suppose we are at time t_2, which is the first time that only two units of investment are needed to finish the first stage. She can either invest one unit and leaves one unit to finishing or she postpones investing. We now again examine what the agent does when she has different preferences:

2a. Exponential Agent (EA): Suppose we are at t_2 such that $k-2 \leq t_2 \leq \hat{T}-2$. What the EA can do is that she can either invest now and finish it next period ($from\ step1$) or she can postpone investing $\tilde{t}$ period at t_2 to $t_2+\tilde{t}$ and then finish it at $t_2+\tilde{t}+1$ where $0 < \tilde{t} \leq \hat{T}-t_2-1$. Comparison of the payoffs is as follows:

$$\underbrace{\frac{\delta^2}{1-\delta}w_{EA}(\hat{V})-\delta C-C}_{\text{invest at } t_2 \text{ and } t_2+1} > \underbrace{\frac{\delta^{\tilde{t}+2}}{1-\delta}w_{EA}(\hat{V})-\delta^{\tilde{t}+1}C-\delta^{\tilde{t}}C}_{\text{invest at } t_2+\tilde{t} \text{ and } t_2+\tilde{t}+1} \tag{2a-1}$$

So, for any time, whenever two units of investment needed to finish, the EA finishes it in two periods by investing consecutively.

2b. Naive Hyperbolic Agent (NHA): : Suppose we are at t_2 such that $k-2 \leq t_2 \leq \hat{T}-2$. NHA compares different options like the following:

$$\begin{array}{lllll}
(t_2;t_2+1) & (t_2+1;t_2+2) & (t_2+2;t_2+3) & .. \quad . & (\hat{T}-2;\hat{T}-1) \\
(t_2;t_2+2) & (t_2+1;t_2+3) & (t_2+2;t_2+4) & ..\ (\hat{T}-3;\hat{T}-1) & \\
(t_2;t_2+3) & . & . & .. & \\
. & . & . & . & \\
. & . & (t_2+2;\hat{T}-1) & & \\
. & (t_2+1;\hat{T}-1) & & & \\
(t_2;\hat{T}-1) & & & &
\end{array} \tag{2b-1}$$

The number of options that she has to consider is a lot ($(T-t_2-1)(T-t_2)\backslash 2$) but indeed, when we compare the payoffs of them, we recognize the fact that choosing the investment periods close to each other is better than having more periods between investments -e.g., $(t_2+2;t_2+3) > (t_2+1;t_2+3) > (t_2;t_2+3)-$ and that making the investments as close as possible is better than postponing more and more, e.g., $(t_2+1;t_2+2) = \frac{1}{\delta}(t_2+2;t_2+3) = \frac{1}{\delta^2}(t_2+3;t_2+4)$. So, only options that should be compared are $(t_2;t_2+1)$ and $(t_2+1;t_2+2)$.We know from *remark* 2 that

$$(t_2;t_2+1) \geq (t_2+1;t_2+2)$$

She invests at t_2. When tomorrow, t_2+1, comes, she will solve the problem in *Step*1. But

$$(t_2;t_2+1) \geq (t_2+1;t_2+2) \Rightarrow (t_2+1) > (t_2+2)$$

Thus, she will follow $(t_2;t_2+1)$ strategy.

2c. Sophisticated Hyperbolic Agent (SHA): There are two cases:

$\hookrightarrow$If $(t_2;t_2+1) \geq (t_2+1;t_2+2)$ or

$$\underbrace{\frac{\beta\delta^2}{1-\delta}w_{SHA}(\hat{V})-\beta\delta C-C}_{\text{invest at } t_2 \text{ and } t_2+1} \geq \underbrace{\frac{\beta\delta^3}{1-\delta}w_{SHA}(\hat{V})-\beta\delta^2 C-\beta\delta C}_{\text{invest at } t_2+1 \text{ and finish at } t_2+2}$$

Then, since $(t_2;t_2+1) \geq (t_2+1;t_2+2) \Rightarrow (t_2+1) > (t_2+2)$, she will finish it by investing consecutively, $(t_2;t_2+1)$.

$\hookrightarrow$If $(t_2; t_2+1) < (t_2+1; t_2+2)$, she knows that if she postpones one period, she will keep postponing up to $\hat{T}-2$. To solve SHA's problem, we will work backwards again. We know that if SHA is at $t_2 = \hat{T}-2$, then She will invest for sure. Suppose $t_2 = \hat{T}-3$. Then SHA has two options $(\hat{T}-3; \hat{T}-1)$ and $(\hat{T}-2; \hat{T}-1)$, because she knows that if she invest at $\hat{T}-3$, she will invest the last unit at $\hat{T}-1$ from 1c. Also, if she postpones investment to next period, she will finish it for sure by assumption. Thus, SHA postpones investment. Suppose $t_2 = \hat{T}-4$. Then, by the same argument, she will compare the following options: $(\hat{T}-4; \hat{T}-1)$, $(\hat{T}-3; \hat{T}-1)$ and $(\hat{T}-2; \hat{T}-1)$. The last option is optimal for him, so SHA postpones investment two periods. This iteration goes on up to the period where $t_2 = \hat{T}-2-t_1^*$. At $t_2 = \hat{T}-2-t_1^*$, if SHA invests then she will invest the last unit at $\hat{T}-1-t_1^*$ by Result 1. Then SHA will compare the following two options $(\hat{T}-2-t_1^*; \hat{T}-1-t_1^*)$ and $(\hat{T}-2; \hat{T}-1)$.

$\hookrightarrow$If $(\hat{T}-2-t_1^*; \hat{T}-1-t_1^*) \geq (\hat{T}-2; \hat{T}-1)$, or

$$\frac{\beta\delta^2}{1-\delta} w_{SHA}(\hat{V}) - \beta\delta C - C \geq \frac{\beta\delta^{t_1^*+2}}{1-\delta} w_{SHA}(\hat{V}) - \beta\delta^{t_1^*+1}C - \beta\delta^{t_1^*}C \tag{2c-1}$$

Then, SHA will finish it immediately.

We can repeat exactly the same argument above. Now, SHA knows that she will follow $(\hat{T}-2-t_1^*; \hat{T}-1-t_1^*)$ strategy. For $t_2 \in \{\hat{T}-3-t_1^*, \hat{T}-4-t_1^*, ..., \hat{T}-1-2t_1^*\}$, she keeps postponing up to the period $t_2 = \hat{T}-2-t_1^*$. At $t_2 = \hat{T}-2-2t_1^*$, she will compare $(\hat{T}-2-2t_1^*; \hat{T}-1-2t_1^*)$ and $(\hat{T}-2-t_1^*; \hat{T}-1-t_1^*)$. Notice that this comparison is same with the one in expression $(2c-1)$. So, she decides to finish it right away, $(\hat{T}-2-2t_1^*); \hat{T}-1-2t_1^*)$. Like in Step1, we can go on this iteration and we can say that SHA has a periodical investment plan.

$\hookrightarrow$If $(\hat{T}-2-t_1^*; \hat{T}-1-t_1^*) < (\hat{T}-2; \hat{T}-1)$, Then, she will postpone. Remember we are at time $t_2 \leq \hat{T}-2-t_1^*$. Lets define a critical value for second level delaying time like the following:

$$t_2^* = \min\{s \in \{j(t_1^*)\} | \frac{\beta\delta^2 w_{SHA}(\hat{V})}{1-\delta} - \beta\delta C - C \geq \frac{\beta\delta^{s+2}}{1-\delta} w_{SHA}(\hat{V}) - \beta\delta^{s+1}C - \beta\delta^{s}C\}$$

$$j \in \{1, 2, ...\} \text{ such that } 0 \leq \hat{T}-1-(j-1)t_1^*.$$

(notice that the above case, $2c-1$, is just $s = t_2^* = t_1^*$). This definition implies that t_2^*-1 is tolerable amount of time if there are two units of investment but t_2^* is not tolerable.

Then SHA will follow $(\hat{T}-2-t_2^*; \hat{T}-1-t_2^*)$. Since She knows this, she will have again a periodically structured strategy. She will postpone the investment to $\hat{T}-2-t_2^*$ if $\hat{T}-2-2t_2^* < t_2 < \hat{T}-2-t_2^*$. Moreover, if $t_2 = \hat{T}-2-2t_2^*$, she will invest immediately. Similarly, we can continue iteration like this. The behavior characterizations of SHA and other agents are established in the following result.

RESULT 2: Suppose Agents are at time t_2 and there are two units of investment left. Also suppose that completion of the investment phase of the game is optimal. Then, For all parametric values, $\beta, \delta, \hat{T}, \hat{V}$ and for all functional forms $w(\hat{V}), f(.), C(.)$;

$\hookrightarrow$EA: She always finishes the first stage immediately,
$\hookrightarrow$NHA: She always finishes the first stage immediately,

$\hookrightarrow$SHA: She finishes immediately, e.g. $(t_2; t_2+1)$, if and only if $t_2 \in \{\hat{T}-2-it_2^*\}$ where $i \in \{0,1,2,...\}$ such that $0 \leq \hat{T}-2-it_2^*$. If $t_2 \notin \{\hat{T}-2-it_2^*\}$, then she will postpone investing up to the closest time such that $t_2 \in \{\hat{T}-2-it_2^*\}$. If $\hat{T}-2-t_2^* < 0$, then SHA follows the $(\hat{T}-2; \hat{T}-1)$ strategy.

Proof:

The argument is in the above substeps, $2a, 2b, 2c$. The explanation for the last part is that If $\hat{T}-2-t_2^* < 0$ or $\hat{T}-2 < t_2^*$ then the available time for investment is, $\hat{T}-2$, less than or equal to the maximum tolerance time, t_2^*-1, which means she can tolerate to postpone the investment up to the period $\hat{T}-2$. Thus, she follows $(\hat{T}-2; \hat{T}-1)$ strategy. ∎

We can continue these steps and at each time t_i that i units of investment left, we can find a critical value t_i^* for t_i. We are going to use the method of induction in order to show that this is true.

Theorem 1: Suppose Agents are at time $t_k \geq 0$ and there are k units of investment needed to finish the first stage. Also suppose that completion of the investment phase of the game is optimal. Define t_k^* as follows:

$$t_k^* = \min\{s \in \{j(t_{k-1}^*)\} | \frac{\beta\delta^k w_{SHA}(\hat{V})}{1-\delta} - \beta\sum_{j=1}^{k-1}\delta^j C - C \geq \frac{\beta\delta^{s+k} w_{SHA}(\hat{V})}{1-\delta} - \beta\sum_{j=0}^{k-1}\delta^{j+s}C\}$$

$$j \in \{1,2,...\} \textit{ such that } 0 \leq \hat{T}-k-(j-1)(t_{k-1}^*).$$

Then, For all parametric values, $\beta, \delta, \hat{T}, \hat{V}$ and for all functional forms $w(\hat{V}), f(.), C(.)$;

$\hookrightarrow$EA: She, always, finishes the first stage without any delay,

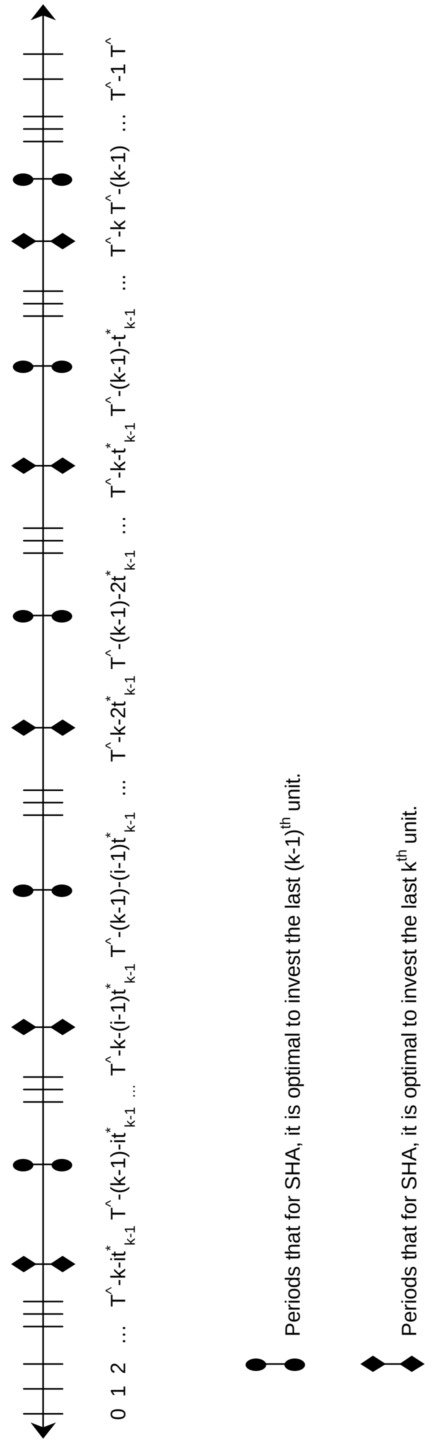
$\hookrightarrow$NHA: She, always, finishes the first stage without any delay,

$\hookrightarrow$SHA: She finishes immediately, e.g. $(t_k; t_k+1; ...; t_k+k-1)$, if and only if $t_k \in \{\hat{T}-k-it_k^*\}$ where $i \in \{0,1,2,...\}$ such that $\hat{T}-k-it_k^* \geq 0$. If $\{\hat{T}-k-it_k^*\} \notin t_k$, then she will postpone investing up to the closest time such that $t_k \in \{\hat{T}-k-it_k^*\}$. If $\hat{T}-k-t_k^* < 0$, then SHA follows the $(\hat{T}-k; \hat{T}-k+1; ...; \hat{T}-2; \hat{T}-1)$ strategy.

Proof:

First of all, we will mention some important points about the investment schedule of the agents. As it is explained above, no agent want to leave any time gaps between investment periods. The first reason is the homogeneous cost structure in the investment game and there is no uncertainty about the cost that the agent should pay at each period if she invests. This makes agents certain about their contingent plans for the future. The second reason is that the reward system is constructed in such a way that there is no partial or gained utility unless the investment is completely finished. Leaving time gap between investment periods always makes them worse off. Given the completion time, all agents want to make the costly investments as close as possible to the completion period in order to minimize the cost since there is discounting.

For EA, the argument is same as above steps. For her, postponing is never optimal because of her exponential discounting type. Thus, she finishes the first stage without any delay.


0 1 2 … $\hat{T}-k-it^*_{k-1}$ $\hat{T}-(k-1)-it^*_{k-1}$ … $\hat{T}-k-(i-1)t^*_{k-1}$ $\hat{T}-(k-1)-(i-1)t^*_{k-1}$ … $\hat{T}-k-2t^*_{k-1}$ $\hat{T}-(k-1)-2t^*_{k-1}$ … $\hat{T}-k-t^*_{k-1}$ $\hat{T}-(k-1)-t^*_{k-1}$ … $\hat{T}-k$ $\hat{T}-(k-1)$ … $\hat{T}-1$ $\hat{T}$
Periods that for SHA, it is optimal to invest the last $(k-1)^{th}$ unit.
Periods that for SHA, it is optimal to invest the last k^{th} unit.

For NHA, she will again compare only the options of finishing immediately and postponing one period like the following:

$$(t_k; t_k+1; ...; t_k+k-1) \ \& \ (t_k+1; t_k+2; ...; t_k+k)$$

From the *remark* 2 again, finishing immediately is always optimal for her:

$$(t_k; t_k+1; ...; t_k+k-1) \ > \ (t_k+1; t_k+2; ...; t_k+k)$$

Thus, she finishes the first stage without any delay.

For SHA, we use the induction method. The first two steps were explained above. So, we assume that we have a value for t^*_{k-1} and show that we have a t^*_k such that above condition for SHA is satisfied.

First, we clarify the point that t^*_i is chosen from the multiples of t^*_{i-1}. The reason for this is that, by definition of t^*_{i-1}, SHA knows for sure that she will only invest at periods that are in the following time schedule $\hat{T}-(i-1)-jt^*_{i-1}$, if $i-1$ units of investment left. Other than those periods, she will postpone investing up to the closest time that is in that time schedule because by investing at $\hat{T}-i-jt^*_{i-1}$, she minimizes the cost that she will incur. So, she invests the i^{th} left unit according to the time schedule $\hat{T}-i-jt^*_{i-1}$. Then, she compares the payoffs of investing for different j values and she does this according to the maximum tolerable time of postponing that is basically the definition of t^*_k.

As it is explained above SHA invests the k^{th} unit at $(\hat{T}-k-it^*_{k-1})$, which is one period before than the periods that she will surely invest the $(k-1)^{th}$ unit (and the rest too), which are $\hat{T}-(k-1)-it^*_{k-1}$ (above figure). In other periods, investing the k^{th} unit is not optimal because next period she will not invest the rest and she will wait the closest period that is in the time schedule $\hat{T}-(k-1)-it^*_{k-1}$. So, she invests at periods $(\hat{T}-k-it^*_{k-1})$. Then, she will calculate the maximum tolerable time for postponing but the definition of t^*_k gives this. "t^*_k-1" is the maximum tolerable time for SHA to invest the last k^{th} unit. Thus, she invests at period $\hat{T}-k-t^*_k$. With the same logic used in earlier steps, since she is sophisticated, she knows this and at earlier periods than $\hat{T}-k-t^*_k$, she will take this into account and again will have a periodic investment scheme like $\hat{T}-k-it^*_k$. Thus, she will invest consecutively whenever she is at a period that is in the time schedule $\hat{T}-k-it^*_k$.

The explanation for the last part is that If $\hat{T}-k-t^*_k<0$ or $\hat{T}-k<t^*_k$, then the available time for investment is, $\hat{T}-k$, less than or equal to the maximum tolerance time, t^*_k-1, which means she can tolerate to postpone the investment up to the period $\hat{T}-k$. Thus, she follows the $(\hat{T}-k; \hat{T}-k+1; ...; \hat{T}-2; \hat{T}-1)$ strategy. ■

Corollary: The following is always satisfied:

$$t^*_k \geq t^*_{k-1} \geq ... \geq t^*_2 \geq t^*_1 \geq 1$$

Moreover, if maximum tolerable time is zero for the first investment or $t^*_k=1$, then $t^*_k=t^*_{k-1}=...=t^*_2=t^*_1=1$. Thus, SHA finishes the first stage without any delay.

Proof:

The reason for the first condition is the following: $t^*_1 \geq 1$ by definition. Moreover, for all $i=k, k-1, k-2, ..., 2$, the definition of the t^*_i entails that any t^*_i is chosen from the multiples of t^*_{i-1} or $t^*_i \in j(t^*_{i-1})$ where $j = 1, 2, 3...$ Thus the condition should be satisfied. If $t^*_k = 1$, then from the first condition, all other $t^{*\prime}s$ should be one, too. This can be shown in a different way like the following:

$t^*_k = 1$ implies that

$$\frac{\beta\delta^k w_{SHA}(\hat{V})}{1-\delta} - \beta\sum_{j=1}^{k-1}\delta^j C - C \geq \frac{\beta\delta^{1+k} w_{SHA}(\hat{V})}{1-\delta} - \beta\sum_{j=0}^{k-1}\delta^{j+1} C \Longrightarrow$$

$$\frac{\beta\delta^k w_{SHA}(\hat{V})}{1-\beta\delta^k} \geq C$$

The left side of the above inequality is decreasing in k implies that the inequality is valid for values smaller than k.Thus, if $t^*_k = 1$, then all other critical values should be one too. ■

Theorem 2: For given values of $\delta, \hat{T}, \hat{V}, k$ and C, $\exists\ \beta^*$ such that for all $\beta \geq \beta^*$, SHA finishes the first stage without any delay.

Proof: Assume $\delta, \hat{T}, \hat{V}, k$ and C values are given. In order SHA to finish the first stage without any delay, the following condition should be satisfied:

$$(t_k; t_k + 1; ...; t_k + k - 1) \ > \ (t_k + 1; t_k + 2; ...; t_k + k)$$

or

$$\frac{\beta\delta^k w_{SHA}(\hat{V})}{1-\beta\delta^k} \geq C$$

Now, plug the expression for $w_{SHA}(\hat{V})$ into the above inequality and get:

$$\frac{\beta\delta^k}{1-\beta\delta^k}\hat{V}\frac{\beta\delta(1-\delta)}{(1-\beta\delta^2)} \geq C \Longrightarrow$$

$$\underbrace{\frac{\beta^2}{(1-\beta\delta^2-\beta\delta^k-\beta^2\delta^{k+2})}}_{h(\beta)}\delta^{k+1}(1-\delta)\hat{V} \geq C$$

It is not difficult to show that $dh(\beta)/d\beta > 0$. In the above equation, if the inequality is actually satisfied with equality then that specific β value is β^*. The reason is that when we increase β a little bit, then the left hand-side (LHS) will be larger than the right hand-side (RHS), which means SHA still wants to finish first stage without any delay or $t^*_k = 1$. If we decrease β, then the RHS will be larger than the LHS, which means SHA finds it optimal to postpone investment or $t^*_k > 1$. Moreover since the following is satisfied:

$$\frac{\beta\delta^k w_{SHA}(\hat{V})}{1-\beta\delta^k} \geq C \Rightarrow \frac{\beta\delta^j w_{SHA}(\hat{V})}{1-\beta\delta^j} \geq C,\ \forall j \leq k$$

The planned investment schedule is implemented by SHA.

So, there exist a β^* such that

$$\frac{(\beta^*)^2}{(1-(\beta^*)\delta^2-(\beta^*)\delta^k-(\beta^*)^2\delta^{k+2})}\delta^{k+1}(1-\delta)\hat{V} = C$$

and for all $\beta \geq \beta^*$, the LHS will be higher than the RHS and vice versa. This completes the proof. ■

We can write the *Theorem* 2 for the value of cost as well since C is also a specific characteristic of the agent. It would be like the following: For given values of $\beta, \delta, \hat{T}, \hat{V}$ and k , $\exists$ C^* such that for all $C \leq C^*$, SHA finishes the first stage without any delay. The rationale of the *Theorem* 2 is to point out the role of immediate gratification preference of the agent in the decision making process. It says there exists an immediate gratification preference level, β^*, such that at and above that level, she just starts investing immediately.

In *Theorem* 2, we assume $\delta, \hat{T}, \hat{V}, k$ and C are all given. Actually, values for $\hat{T}, \hat{V}$ and k are already exogenous in the problem. Agents take these as given by definition of the problem. For δ, it is a common discount factor of both agent and the principal and taking it as given is not a very strong assumption. For the cost value, C, it may differ from agent to agent but in this formulation we assume that C is constant. In the future, we will assume a different cost scheme that allows more flexibility in the context.

4 Some Extensions

4.1 Equilibrium With The Bonus Motive

We now turn to the examples again. The insurance company worker can get a bonus in her wage by incurring the cost of taking and passing the exam. The doctor can be rewarded by the hospital prior to the actual wage agreement for each of her costly investments. The student can take the courses in each semester and the firm can reward her by giving her the opportunity to be an intern each summer. These can be the examples for having a bonus scheme in this framework. The rationale behind doing this in terms of the company or the hospital is to make the agent not procrastinate or finish the investment earlier than the deadline. Because for the principal, waiting is costly in the sense that it cannot earn the profit that it would get in the periods where agents can finish the investment but instead they procrastinate. Another approach may be that these bonuses may make a job desirable that is actually not worth to finish without bonuses. Thus, as an incentive mechanism, the principal can take advantage of this bonus motive to make the agents finish the investment earlier or to make the investment worth finishing. The bonus structure that will be presented here is different from the one that is mentioned in O'Donoghue and Rabin [2003] as an extension. They assume a fixed total reward scheme causing more severe procrastination but here we will assume that this new reward scheme is, in fact, a "bonus" in the sense that it is paid extra to the agent by principal without any reduction in the agent's expected future wage income. In other words, the firm gives up some part of its payoff that will be earned after the agent finishes the first stage.

One of the simplest modelling ways of the bonus motive for the firm is to offer a fixed amount of benefit to the agent after each unit of investment is made. We will call the bonus amount as "x". It is earned with a one period lag, e.g., if the agent invests at time t then, she gets x at time $t+1$. We can also interpret the one-time bonus as the present value of a continuous benefit initiated by the completion of each unit of investment.

In this context, we do not have to assume the optimality of finishing the investment phase of the game because bonus scheme may make an unworthy investment project worthwhile. In the previous section, the optimality of finishing by SHA was implying optimality for others. The interesting case was optimal finishing because if it is not, then they will not even start to project and the game ends. Here again the interesting case is the following: for NHA, it is optimal to finish the project without any delay *without the bonus scheme* and for SHA, it is not optimal to finish the project without the bonus scheme but optimal to finish without any delay *with the bonus scheme.* Thus, now we can get an interesting case where SHA has a similar investment structure but NHA may have a procrastinative behavior. Actually, since there is this strategic interaction between the principal and the agent it is not the case that NHA procrastinate inefficiently. In other words, the principal will offer a bonus scheme (if it is optimal for herself) that just makes NHA invest consecutively. Otherwise, if there is no optimal bonus scheme for the principal, she will not introduce it.

So, we assume that For NHA and EA, finishing investment phase in k periods by investing consecutively is optimal;

$$\frac{\beta\delta^k w_{NHA}(\hat{V})}{1-\delta} - \beta\sum_{j=1}^{k-1}\delta^j C - C \geq 0$$

or

$$\frac{\beta\delta^k w_{NHA}(\hat{V})}{1-\beta\delta^k+\beta\delta-\delta} \geq C \tag{4.1.1}$$

However, for SHA it is not;

$$\frac{\beta\delta^k w_{SHA}(\hat{V})}{1-\delta} - \beta\sum_{j=1}^{k-1}\delta^j C - C < 0$$

or

$$\frac{\beta\delta^k w_{SHA}(\hat{V})}{1-\beta\delta^k+\beta\delta-\delta} < 0 \tag{4.1.2}$$

Lets now add the bonus scheme to the model: Equation 4.1.1 implies the following;

$$\beta\delta^k[\frac{w_{NHA}(\hat{V})}{1-\delta}+x] - \beta\sum_{j=1}^{k-1}[\delta^j C - x] - C > 0 \tag{4.1.3}$$

We also assume that bonus scheme makes finishing worthwhile for SHA;

$$\beta\delta^k[\frac{w_{SHA}(\hat{V})}{1-\delta}+x] - \beta\sum_{j=1}^{k-1}[\delta^j C - x] - C \geq 0 \tag{4.1.4}$$

So 4.1.1, 4.1.2 and 4.1.4 are assumed and 4.1.3 is indicated by 4.1.1.

The principal is going to give a bonus to the agents to make them finish the investment phase without delay. So, the Naive agent, e.g., will calculate whether postponing one period is optimal for her or not:

$$\underbrace{\beta\delta^k[\frac{w_{NHA}(\hat{V})}{1-\delta}+x] - \beta\sum_{j=1}^{k-1}[\delta^j C - x] - C}_{(0;1;2;\ldots;k-1)} \; and \; \underbrace{\beta\delta^{k+1}[\frac{w_{NHA}(\hat{V})}{1-\delta}+x] - \beta\sum_{j=1}^{k}[\delta^j C - x]}_{(1;2;\ldots;k)} \tag{4.1.5}$$

The comparison of these two payoffs is the same with the following comparison:

$$\frac{\beta\delta^k w_{NHA}(\hat{V}) + \beta\delta x(1-\delta^k)}{1-\beta\delta^k} \ and \ C$$

Now, it is time to mention a sufficient condition for NHA to finish the investment project right away. By using 4.1.1;

$$\frac{\beta\delta^k w_{NHA}(\hat{V}) + \beta\delta x(1-\delta^k)}{1-\beta\delta^k} \ \geq \ \underbrace{\frac{\beta\delta^k w_{NHA}(\hat{V})}{1-\beta\delta^k+\beta\delta-\delta} \geq C}_{Assumption\ 4.1.1} \tag{4.1.6}$$

If the first inequality is satisfied then, This implies that NHA will start investment at time 0 and invest consecutively and finish it in $k-1$ periods, $(0;1;2;...;k-1)$. Now, by using inequality in 4.1.6, we can find a condition on the bonus, x, like the following;

$$x \geq \frac{(1-\beta)\delta^k w_{NHA}(\hat{V})}{(1-\delta^k)(1-\beta\delta^k+\beta\delta-\delta)} \tag{4.1.7}$$

Notice that, without bonus scheme, NHA will postpone if the following is true:

$$\frac{\beta\delta^k w_{NHA}(\hat{V})}{(1-\beta\delta^k+\beta\delta-\delta)} \geq C \geq \frac{\beta\delta^k w_{NHA}(\hat{V})}{1-\beta\delta^k}$$

So, bonus scheme will certainly make NHA follow $(0;1;2;...;k-1)$ strategy if and only if 4.1.7 is satisfied for all k. NHA thinks that if bonus satisfies 4.1.7 then, she will follow $(0;1;2;...;k-1)$ strategy. However, when the next period comes, NHA again makes the same calculation. She will have the same expression with 4.1.7 except that instead of k now she has $k-1$ since she invested the k^{th} unit. Since the right-hand-side in expression 4.1.7 is decreasing in k, the new bonus amount that makes NHA decide to invest is higher now. This may make NHA decide not to invest (it is a weak result because 4.1.7 is sufficient but not necessary condition.) Thus, as long as 4.1.7 is satisfied for all $k \geq 1$, then NHA follow $(0;1;2;...;k-1)$ strategy. Otherwise, NHA may start to invest but after some point she may decide not to continue because of the insufficient bonus amount (but since this kind of behavior is obviously not optimal for the principal, she will not allow this to happen by arranging the bonus scheme).

Now we think about the Principal's problem. She is going to compare the payoff from the agent and the bonus that she will give to her. She will give the bonus if and only if

$$\underbrace{\sum_{j=0}^{k-1} \delta^j x_j}_{(3)} \ \leq \ \underbrace{\hat{V}(1-\frac{\beta\delta}{1+\delta})}_{(1)}\underbrace{(\frac{\delta^k}{1-\delta})}_{(2)} \tag{4.1.8}$$

where (1) is the payoff of principal from hiring the agent per period, $(1)*(2)$ gives the total discounted payoff of principal and (3) is the discounted value of the bonus that she gives to the agent ($x_j's$ will be specified later and it will be shown that they are actually different for all periods for NHA case). If 4.1.8 is satisfied then, the principal decides to give the bonus.

The problem of NHA is to compare the immediate cost and the short run benefit of investing. That is, she only chooses between investing today and tomorrow by assuming, once started, she will continue investing consecutively. This is a problem because when future investment periods arrive, her immediate gratification may overwhelm her optimistic beliefs of consecutive investment at previous periods and she may give up investing after some point where the bonus is no longer enough to make her continue investing. From the perspective of NHA, whenever condition 4.1.7 is satisfied, she continues investing and vice versa. Thus, we may well have a situation where NHA starts to invest by having an optimistic beliefs about her future-selves but after sometime she is defeated to her self-control problem and stop investing and this deprives her from getting the wage after investment phase.

However, we have interactions here between agents and bonus scheme will be implemented by the principal if and only if she believes that NHA will finish the investment stage as planned (assume a complete information framework for now, e.g., principal knows the type and the self-control problem of the agent). So, in this kind of environment, either Principal implements the bonus scheme and NHA finishes "efficiently" or she does not implement it and NHA may either not even start or finish it depending on the payoff-cost comparison. In fact, there will not be a case where NHA starts but not finish because of again the homogeneous cost structure and not having immediate rewards (in case of bonus scheme is not implemented).

The question is which bonus scheme, $x_j's$, will the principal choose -if there exists a bonus scheme that is optimal for both NHA and the principal? What the principal does is the following: at $t = 0$, she is going to promise the agent to give her at least the bonus amount that is:

$$x_0 = \frac{\hat{V}\beta\delta(1-\beta)\delta^k}{(1-\delta^k)(1+\delta)(1-\beta\delta^k+\beta\delta-\delta)} \tag{4.1.9}$$

The agent chooses between investing today and postponing one period. This bonus scheme guarantees her a higher payoff if she invests today than postponing-by assuming that she will get at least this much bonus in the future for each of her investment unit. In the next period, the bonus amount, which is required to convince NHA to continue investing, will be higher because the expression in 4.1.9 is decreasing in k. So when $k - j$ units left, the principal will give the agent:

$$x_j = \frac{\hat{V}\beta\delta(1-\beta)\delta^{k-j}}{(1-\delta^{k-j})(1+\delta)(1-\beta\delta^{k-j}+\beta\delta-\delta)} \tag{4.1.10}$$

and the agent will accept this bonus and invest, again by assuming that she will get at least this much bonus in the future for each of her unit of investment. Thus the optimal bonus scheme is like in expression 4.1.10, $\forall\ 0 \leq j \leq k-1$. This bonus scheme is a sufficient bonus scheme that the principal is willing to give to the agent and the agent is also willing to take it and invest consecutively as she planned at $t = 0$.

Now, the existence of the bonus scheme boils down to the following condition:

$$\hat{V}(1-\frac{\beta\delta}{1+\delta})(\frac{\delta^k}{1-\delta}) \geq \sum_{j=0}^{k-1} \delta^j x_j;\ \forall k$$

or

$$\hat{V}(1-\frac{\beta\delta}{1+\delta})(\frac{\delta^k}{1-\delta}) \geq \sum_{j=0}^{k-1} \delta^j \frac{\hat{V}\beta\delta(1-\beta)\delta^{k-j}}{(1-\delta^{k-j})(1+\delta)(1-\beta\delta^{k-j}+\beta\delta-\delta)} \tag{4.1.11}$$

or

$$(\frac{1+\delta-\beta\delta}{\beta\delta(1-\beta)})(\frac{1}{1-\delta}) \geq \sum_{j=0}^{k-1} \frac{1}{(1-\delta^{k-j})(1-\beta\delta^{k-j}+\beta\delta-\delta)}$$

Proposition 1: Assume, for given values of δ, β and k, 4.1.11 is satisfied (existence of a bonus scheme). Then, in the investment game with bonus scheme, the bonus amount for NHA is a monotonically increasing function of k. In other words, the bonus should increase in order for the NHA to continue to invest and finish the investment stage.

Proof: The argument is above.

The proposition generates a very similar result with the O'Donoghue and Rabin obtained in their paper [1999b]. In that paper, they find that the optimal incentives for procrastinators typically involve an increasing punishment for delay as time passes. We actually find the complement of this result saying that whenever the agent continues to invest, she should get a higher bonus. These are very similar results because increasing punishment when the agent did not invest is almost the same thing with increasing reward or bonus when she invests. So, we generate the same result in our framework that is more general in the sense that we have more than two units of investment but it is more restricted in the sense that we have a homogeneous cost structure along the investment path. There is this trade-off between having longer projects and having a simpler cost structure. Including both longer projects and more complex cost structure remains to be done.

Proposition 2: For the above game, the bonus scheme is an increasing function of self-control problem of the agent, β (with a small caveat). In other words, the agents with higher self-control problems - lower $\beta' s$- should be given higher bonus by the principal in order to induce the agents to complete the same investment project.

Proof: In the expression 4.1.11, if we take derivative with respect to β, we get the following:

$$(1-\delta)(1-2\beta) \leq \beta^2\delta(1-\delta^{k-1}) \tag{4.1.12}$$

in order to get the desired result. $\forall\beta \leq 0.5$, the result is trivial. Otherwise, 4.1.12 should be satisfied for the desired result. The rationale behind this is that when the agents with more severe self-control problems face with the same project, since their immediate gratification tendency is higher, in order to induce them to complete the investments, the principal should give more immediate rewards to them -which is the bonus here.

5 Discussion and Conclusion

Our main purpose in this paper is to investigate the role of different preference structures -other than classical time consistent preferences- in bargaining and investment games. We use the well-known quasi-hyperbolic discounting function to incorporate the time inconsistency into our framework. By keeping the environment as simple and general as possible, we explore the behavioral characteristics of different economic agents when they face with intertemporal decisions and with bargaining situations.

We believe that introducing time-inconsistency in a strategic environment is the most important aspect of this paper. Incorporating boundedly rational agents into this kind of a game provides us observational differences among them. In case of no bonus, while exponential and navie hyperbolic agent finishes the investment game without any delay, sophisticated agent has a periodic investment plan -explained in section 1.5 and in *Theorem* 1. When there is bonus scheme, an increasing reward or bonus is necessary for NHA to continue investing. On the other hand, Since exponential agent is time consistent, a fixed amount of bonus makes her to finish the project. The behavior of SHA will be examined later.

The degree of time inconsistency factor is important in determining the behavioral differences. Depending on the severity of the self-control problem, optimal incentives schemes and behavior of the agents change. In the bargaining game, we apply a different kind of equilibrium concept because of bounded rationality. Beliefs of naive hyperbolic agents turn out to be wrong but again by using best response argument, we find the subgame perfect equilibrium.

An interesting observational difference about the naive hyperbolic agent is that she is mistaken in predicting her wage. She overestimates it and because of this, she is disappointed about her realized wage. This misperception leads to a regret motive that she may pursue a goal that is not worthwhile to pursue, since she will get a less payoff than she expects. Thus, this observation may be helpful in understanding people's disappointments resulting from their great expectations about the future.

The puzzling question in this context is that why does not NHA learn from her experiences? She always behaves consistenly in being time inconsistent. It is not that she is not learning but she is always defeated by her tendency to pursue her immediate gratification. She knows and remembers what happened in the past but since she is highly overoptimistic about herself in being time consistent in the future, she basically ignores her past actions and does not take lessons from her previous experiences. Introducing partial naivete may be a more realistic approach in order to incorporate the learning or bounded memory motive.

There are several possible extensions that can be made in our framework. Some of them are the following: Incomplete information about types can be introduced into the bargaining stage. Different cost structures (endogenous, stochastic...etc) could be examined. Underestimation or overestimation of the costs depending on the agents' types can be examined. Partial naivete can also be introduced into the model.

6 References

1. Chung Shin-Ho, and Richard J. Herrnstein, Journal of the Experimental Analysis of Animal Behavior, *"Relative and absolute Strenghts of Response as a function of Frequency of reinforcement."* 1961, 4, Pages 267-272.

2. Coles Melvyn G. and Muthoo A., Journal of Economic Theory, *"Bargaining in a non-stationary environment"* March 2003, Volume 109, Issue 1, Pages 70-89.

3. Dasgupta P., Maskin E., "Uncertainty, Waiting Costs and Hyperbolic Discounting" April 2002.

4. Laibson David I., QJE, *"Golden eggs and Hyperbolic Discounting."*, 1997a, 62: 2, Pages 443-478.

5. Mukherji A., Villaverde, J.F., PIER Working Paper,"*Can We Really Observe Hyperbolic Discounting?"*, March 2002, N0.02-008

6. O'Donoghue T., Rabin M., AER, *"Doing It Now or Later"* , March 1999a, 89:1, Pages 103-124

7. O'Donoghue T., Rabin M., QJE, *"Incentives for Procrastinators"* , August 1999b, Pages 769-816

8. O'Donoghue T., Rabin M., QJE, *"Choice and Procrastination"* , February 2001, 116:1, Pages 121-160

9. O'Donoghue T., Rabin M., *"Procrastination on Long-Term Projects"* , February 24 2003

10. Rubinstein A., Econometrica, *"Perfect Equilibrium in a Bargaining Model"*, Vol. 50, No. 1. (Jan., 1982), Pages 97-110.

11. Rubinstein A., forthcoming in International Economic Review, *"Economics and Psychology? The Case of Hyperbolic Discounting"*, 2003.

12. Perry M., Admati A. R., Review of Economic Studies, *"Joint Projects without Commitment"*, Volume 58:2, 1991, Pages 259-276.

13. Shaked A., Sutton J., Econometrica, *"Involuntary unemployment as a perfect equilibrium in a Bargaining Model",* Vol. 52, No.6, (Nov. 1984), Pages 1351-1364.

14. Strotz Robert H., Review of Economic Studies, *"Myopia and inconsistency in Dynamic Utility Maximization."*, 1956, 23, Pages 165-180.