Construction of Equilibrium Components with Arbitrary Index and Degree

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Arndt von Schemde

London School of Economics
Department of Mathematics

Abstract

In this talk we give a brief introduction into the theory of index and degree of Nash-equilibria. After explaining the general concepts in a rather informal way, we will, by means of certain examples, show how Nash-equilibria components of arbitrary index (or degree) can be constructed. We will then discuss certain properties of these components and the question, whether the index (or degree) of a component can be used to capture certain refinement criteria.

Overview

1) Introduction and Motivation
2) Degree and Index for Components of NE
3) Properties of Index and Degree
4) Construction of Components with arbitrary Index
5) Some Results and Open questions

1) Introduction and Motivation

- The number of Nash-equilibria in a non-degenerated game is odd (Lemke-Howson algorithm)

  → index as “orientation”

- Kohlberg-Mertens **Structure theorem**:
  The space of games is homotopic to the graph of the NE-correspondence

  → “degree” of the projection map

- The NE of a game are the fixed points of certain mappings, mapping the strategy space into itself

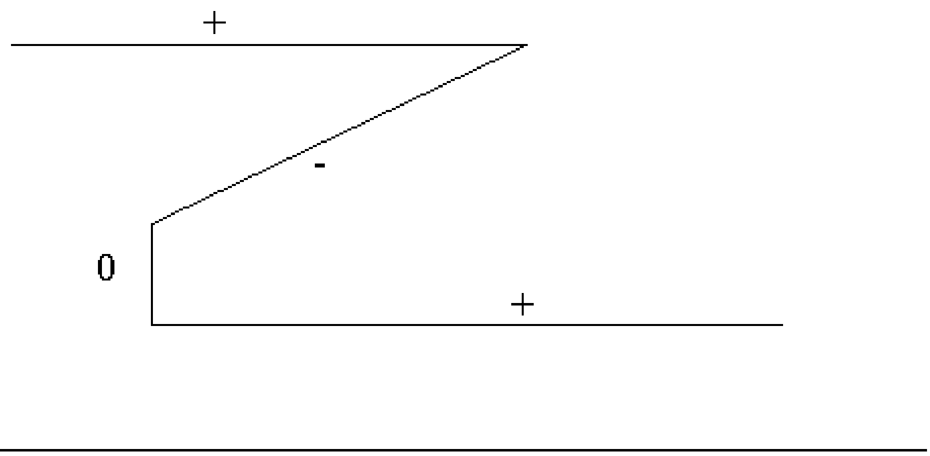
  → index as the “local degree” of the *displacement map*

- Can index and degree capture certain aspects of NE?

  → stability, refinement

2) Degree and Index for componets of NE

- KM-structure Theorem:



Definition: deg (C) = local degree of projection map at the component

→ number of cycles around the original game traversed by the image of a cycle in the graph around the component

→ sum of the degrees of the Nash-equilibria of some non-degenerated game close to the original game that are close to the component.

- Index of a component

Fix a game G. Let Σ be the strategy space. Consider some mapping *F*: Σ → Σ whose fixed points coincide with the set of Nash-equilibria.

Definition: Ind(C) = local degree of the displacement map *F-id*

The definition of index by Shapley (1974)

- Motivated by the Lemke-Howson algorithm for non-degenerate games

  → "orientation" of Nash-equlibria

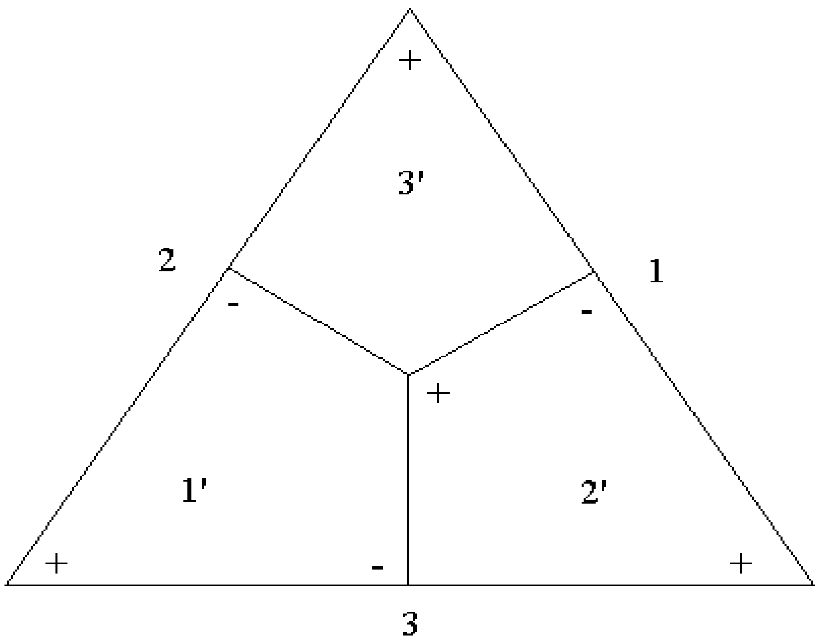
- Example:

A=

10	0	0
0	10	0
0	0	10

B=

10	0	0
0	10	0
0	0	10



Definition: I(σ)= - sign $\begin{vmatrix} 0 & B' \\ A' & 0 \end{vmatrix}$ det

A' and B' are the payoff matrixes consisting of those rows and columns of A and B that are played with positive probability.

3) Properties of Index and Degree

- Index and degree are the same (also for Shapley in the non-degenerate case) (Govindan and Wilson, 1997; DeMichelis and Germano, 1998)

- The sum of indices of NE-components of a game is +1.

- Pure strategy equilibria have index +1

- The index of an equlibirum component is invariant under adding redundant strategies as new strategies (Govindan and Wilson, 1997).

  => non-zero index components are essential and hyperessential

  Example:

0,0

0,0	0,0	0,0
0,0	0,0	0,0
0,0	0,0	0,0

a_{11},b_{11}	a_{12},b_{12}	a_{13},b_{13}
a_{21},b_{21}	a_{22},b_{22}	a_{23},b_{23}
a_{31},b_{31}	a_{32},b_{32}	a_{33},b_{33}

4) Construction of components with arbitrary index

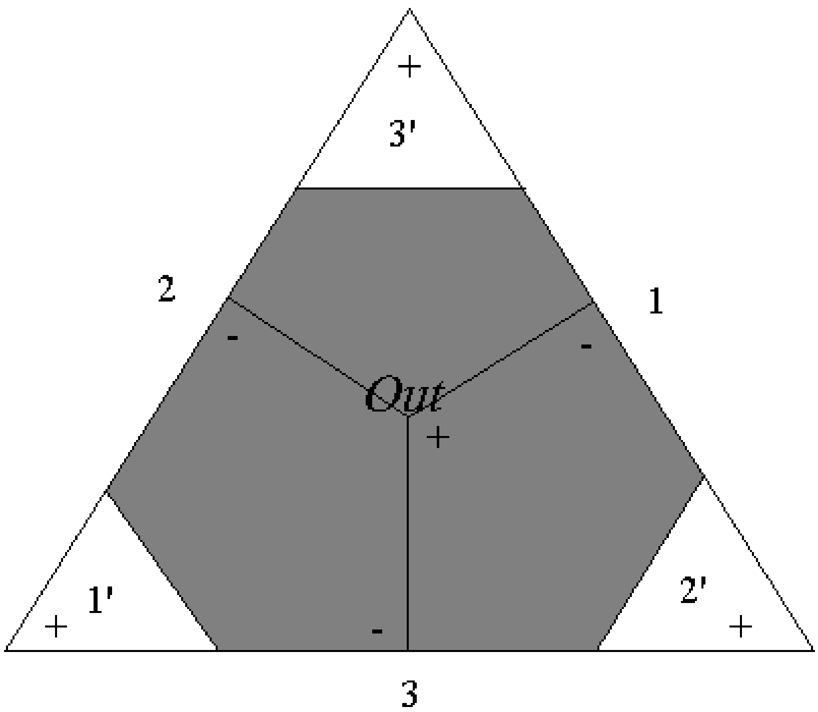
- Construction of Components with arbitrary high positive and negative index via outside option games.

  Idea:

  - Overall index is +1
  - Pure strategy equilibria have index +1
  - Cutting off equlibria with outside options creates indizes of desired size

I)

10,10	0,0	0,0
0,0	10,10	0,0
0,0	0,0	10,10
8,0	8,0	8,0



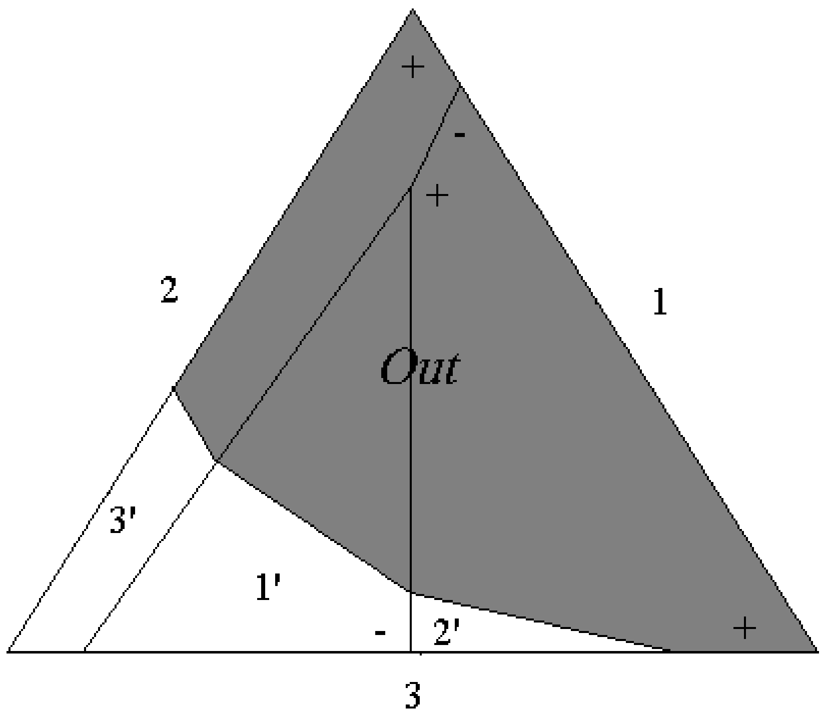
The Game has 4 equlibrium componets: The 3 index +1 pure strategy equilibria and the outside option equilibrium component, in which player I plays *Out*.

=> The component has index –2.

This method allows us to construct arbitrarily high negative index components.

II)

13,13	7,12	0,14
12,7	8,8	0,0
14,0	0,0	1,1
8,0	8,0	8,0



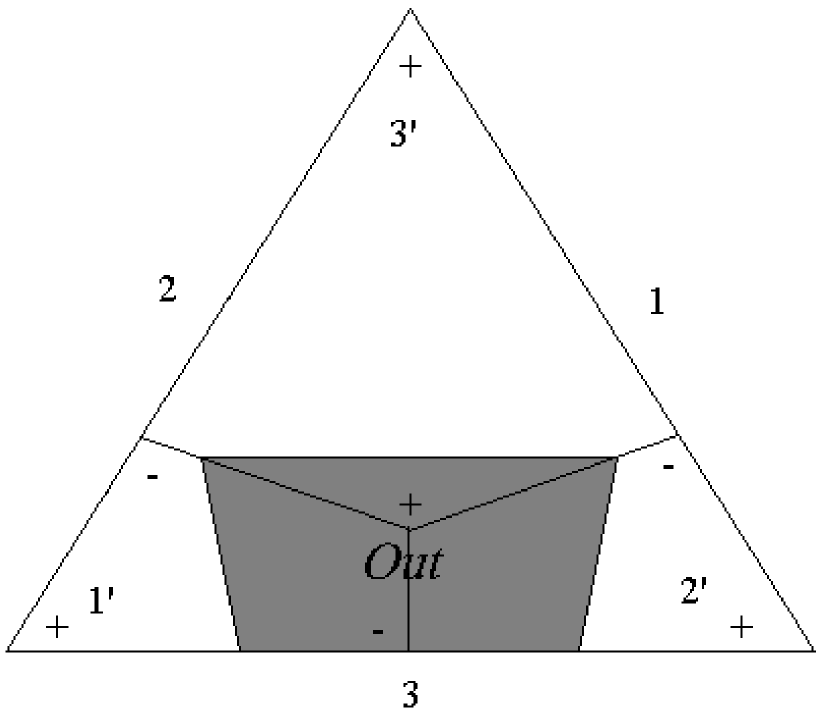
The game has two equlibrium components: [(0.5, 0.5, 0, 0); (0.5, 0.5, 0)] and the outside option equlibrium component. The first equilibrium has index –1.

=> The component has index +2.

This method allows us to construct arbitrarily high positive index equilibrium components.

III)

10,10	0,0	0,0
0,0	10,10	0,0
0,0	0,0	20,20
8,0	8,0	8,0



The outside option equilibrium component has index 0. The component is not essential

Remark: It was conjectured by Govindan and Wilson (1997) that index 0 equilibrium components cannot be essential. This conjecture turned out to be false (Hauk and Hurkens, 1999).

5) Some Results and Open Questions

- The construction methods from above can be used to show that q-stable sets violate the weak symmetry axiom as defined by Govindan (2001) (Govindan, von Stengel, von Schemde, 2002; von Schemde, 2002).

- Index 0 components can be essential (Hauk and Hurkens, 1999).

  Question: Can index 0 components be hyperstable

  Conjecture: No!

- What other properties of NE might be captured by the index?