Forming societies and the Shapley NTU value*

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Abstract

We design a simple non-cooperative mechanism in the classs of NTU-games. We study it in the context of a particular class of pure exchange economies. When the corresponding NTU game (N, V) satisfies that $V(N)$ is flat, the only payoff which arises in equilibrium is the Shapley NTU value.

1 Introduction

The Shapley value (Shapley, 1953) is considered as one of the most important solution concepts in the class of transferable utility games (TU games). However, its generalization to nontransferable utility games (NTU games) is not clear. For NTU games, there are three main solution concepts which generalize the Shapley value: The Harsanyi value (Harsanyi, 1963), the Shapley NTU value (Shapley, 1969), and the consistent value (Maschler and Owen, 1989, 1992). Some characterizations for these values are given by Harsanyi (1963), Aumann (1985) and Hart and Mas-Colell (1996), respectively.

Moreover, Hart and Mas-Colell (1996) design a non-cooperative mechanism[1] such that the consistent value arises in subgame perfect Nash equilibria. As far as we know, no similar result has been obtained for the Harsanyi value nor the Shapley NTU value.

In this paper, we describe a simple mechanism of negotiation. The main idea of the mechanism is the creation and further ampliation of a union or society of players. The members of this society agree on a rule to share their resources. Players outside the society can apply to enter the society by agreeing on the

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[1] We use the term non-cooperative *mechanism* instead of non-cooperative *game* in order to avoid confusion with cooperative games.

established internal rule. However, in the admission negotiation, candidates may also propose to change the internal rule on entrance. Furthermore, unanimity is required among every member of the society to change the rules.

Notice that a similar mechanism is used by supranational institutions such like the European Union or NATO when new countries apply to join. In the particular case of the European Union, refusal from any current member may abort this ampliation. Thus, unanimity is required in changing the rules.

Surprisingly, in a particular class of games, the mechanism described above implements the Shapley NTU value. This provides further support to this value. The particular class of games we should restrict ourselves to are games (N, V) such that $V(N)$ is delimited by a hyperplane.

An example of such a game may be find in the following particular environment. Consider a pure exchange economy where big *Factories* acquire products from *farmers*, who have limited liability. Suppose that the government would like to favor the productivity of the farmers[2], avoiding the factories to take advantage of farmers' lack of liability. Our analysis shows that this handicap can be avoided by forcing the proposed mechanism. The Shapley NTU value, as opposed to other values, such like Harsanyi's and consistent, provide all agents (both farmers and factories) with the Shapley value of the game which arises from the economy when a common utility is freely transferable.

Next example is an adaptation of the game presented by Owen (1972). It has also been used by Hart and Kurz (1983) and Hart and Mas-Colell (1996):

Example 1 *Consider a pure exchange economy with three players* $\{1, 2, 3\}$ *and three commodities* $\{x, y_1, y_2\}$. *Initial endowments are given by:*

$$\begin{aligned} z_1^0 &= (0, 1, 0) \\ z_2^0 &= (0, 0, 1) \\ z_3^0 &= (1, 0, 0) \end{aligned}$$

and utility functions are given by

$$\begin{aligned} u_1(x, y_1, y_2) &= x + \min\{y_1, y_2\} \\ u_1(x, y_1, y_2) &= x + \frac{1}{4}\min\{y_1, y_2\} \\ u_1(x, y_1, y_2) &= x + \min\{y_1, y_2\} - 1. \end{aligned}$$

Thus, commodity x (money) is additive and linear in every player's utility function. Commodities y_1 and y_2 may be considered as 'left gloves' and 'right gloves', respectively. Players only get utility from pairs of gloves. However, player 2 does not have as much production (or selling) ability as the rest of the players. If players had unlimited liability, players 1 and 2 could agree on the consumptions $z_1 = \left(-\frac{1}{2}, 1, 1\right)$ *and* $z_2 = \left(\frac{1}{2}, 0, 0\right)$, *so that the final payoff would be* $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$.

[2] A similar idea may be found in Dam and Pérez-Castrillo (2001), were they present a model with tenants and landowners, tenants with limited liability.

However, if we consider only nonnegative commodities, the above consumptions are not feasible. We are, thus, in the context of the non-transferable utility (NTU) game given by

$$
\begin{aligned}
V(\{i\}) &= \left\{t \in \mathbb{R}^{\{i\}} : t \leq 0\right\} \text{ for all } i \in \{1,2,3\} \\
V(\{1,2\}) &= \left\{(t_1,t_2) \in \mathbb{R}^{\{1,2\}} : t_1 + 4t_2 \leq 1, t_1 \geq 1, t_2 \geq \frac{1}{4}\right\} \\
V(\{i,3\}) &= \left\{(t_i,t_3) \in \mathbb{R}^{\{i,2\}} : t_i + t_2 \leq 0, t_i \leq 1, t_3 \leq 0\right\} \text{ for all } i \in \{1,2\} \\
V(\{1,2,3\}) &= \left\{(t_1,t_2,t_3) \in \mathbb{R}^{\{1,2,3\}} : t_1 + t_2 + t_3 \leq 1, t_1 \leq 2, t_2 \leq \frac{5}{4}, t_3 \leq 1\right\}.
\end{aligned}
$$

Thus, player 3 (the banker) is needed as a catalyst. Players 1 and 2 may then agree to share part of their resources (pair of gloves) with player 3 in exchange of his services.

In particular, the Harsanyi value proposes a payoff of $\left(\frac{2}{5}, \frac{2}{5}, \frac{1}{5}\right)$. *For example, players 1 and 2 sell their shoes to player 3 at a exchange rate of 5 pairs for 4 currency units.*

The consistent value, however, proposes a payoff of $\left(\frac{1}{2}, \frac{3}{8}, \frac{1}{8}\right)$, *i.e. since player 2 has the low production ability, he is the one who has to pay player 3.*

Finally, the Shapley NTU value proposes a payoff of $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$. *For example, players 1 and 2 sell their shoes to player 3 at a exchange rate of 1 pair for 1 currency unit. Notice that this payoff is the same players would have agreed upon player 1 should initially have enough money.*

It may be argued that, since player 3 is not a dummy in the game V *(the final payoff of* $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$ *is not attainable without him), he must receive more than 0. However, player 3 does not contribute with any additional production capability. He just provides the other players with money so that trade may freely happen. We may want to incentive the production of goods and not the lending of money. Thus, player 3 should not get profit from the simple fact to have money when others do not have it. In this context, the Shapley NTU value seems a much fairer allocation.*

In our pure exchange economy, two conditions must hold:

1. The farmers have limited liability. The Factories have unlimited liability.
2. Production in Factories is much more efficient than in farms. Thus, it is optimal (in the sense of maximizing aggregate utility) for the Factories to hold *all* the non-monetary commodities.

The first condition implies that the farmers may be in an inferior position with respect to the Factories. If every player had unlimited liability, we would be in a transfer utility (TU) context, and our mechanism would implement the Shapley value. In Example 1, players 1 and 2 play the role of farmers, and player 3 is the Factory.

The second condition implies that the farms produce not for domestic consumption, but for selling to the Factories. Efficiency may be achieved with only money assigned to farmers. In example 1, the consumptions $z_1 = \left(\frac{1}{2}, 0, 0\right), z_2 = \left(\frac{1}{2}, 0, 0\right), z_3 = (0, 1, 1)$, which held the Shapley NTU value, maximize the aggregate utility and give player 3 (the Factory) all the gloves.

In our mechanism, there is a *society* whose members (we call them *active players*) have agreed on the way to share their resources (i.e. they have agreed on a *consumption sharing rule* or simply a *rule*). There is also a set of players (*passive players*) who have had the chance to join the society but they have rejected to do so. The rest of the players are called *candidates*. Candidates sequentially negotiate their admission to the society. The process of negotiation is as follows. The candidate may simply join the society as it is, *i.e.* he agrees on the rule. In this case, the society gets a new member and next candidate is called upon. If the candidate does not agree on the rule, he may propose a new rule and even suggest some of the passive players to join the society with him. If every member of the proposed new society accepts this offer, this new society forms with the suggested rule. Otherwise, the candidate becomes a passive player and next candidate begins negotiations. Once there are no more candidates, the coalition of active players is formed and its members get the payoff given by the agreed single value. The passive players get nothing.

Notice that a passive player is, somehow, out of the game at least some candidate suggests him to join the society.

In section 2, we present the notation and in section 3, we present formally the mechanism and prove that every subgame perfect equilibrium yields the Shapley NTU value.

2 Pure-exchange economies and the Shapley NTU value

We consider *pure-exchange economies* with l *commodities* and $n = n_f + n_F$ *players*, $n_f \geq 0$ of them *farmers* and $n_F \geq 1$ of them *factories*. The set of farmers is denoted by N_f and the set of factories by N_F. We assume without loss of generality $N := N_f \cup N_F = \{1, 2, ..., n\}$.

1. A *consumption* z_i for player $i \in N_f$ (resp. N_F) is a pair (x_i, y_i) such that $x_i \in \mathbb{R}_+$ (resp. $\mathbb{R}$) and $y_i \in \mathbb{R}_+^{l-1}$. Player i is characterized by an *initial endowment* $z_i^0 = \left(x_i^0, y_i^0\right) \in \mathbb{R}_+^l$ and a utility function $u_i : \mathbb{R}_+^l \to \mathbb{R}$ (resp. $u_i : \mathbb{R} \times \mathbb{R}_+^{l-1} \to \mathbb{R}$) such that $u_i(z_i) = x_i + u_i'(y_i)$ for some continuous, nondecreasing function $u_i' : \mathbb{R}_+^{l-1} \to \mathbb{R}$ satisfying $u_i'\left(y_i^0\right) = 0$ (this is a normalization condition without consequence).

Notice that the additivity separability and linearity in x_i of u_i permits utility transfers among players. However, the nonnegativeness of x_i when $i \in N_f$ restricts these transfers when farmers are involved (they have limited liability).

Given a *coalition* $S \subset N$, we denote by Ω^S the set of *feasible consumptions* for players in S, *i.e.*

$$\Omega^S := \left\{ z = (z_i)_{i \in S} : z_i \in \mathbb{R}^l_+ \forall i \in N_f \cap S, z_i \in \mathbb{R} \times \mathbb{R}^{l-1}_+ \forall i \in N_F \cap S, \sum_{i \in S} z_i \leq \sum_{i \in S} z_i^0 \right\}$$

2. There exists a $y^M \in \mathbb{R}^{l-1}_+$ with $y_i^M = 0$ for all $i \in N_f$ such that

$$\sum_{i \in N} u_i' \left(y_i^M \right) = \max \left\{ \sum_{i \in N} u_i' \left(y_i \right) : (x, y) \in \Omega^N \text{ for some } x \in \mathbb{R}^N \right\}.$$

We assume endowments to be known by the planner. Furthermore, the planner is capable of forcing any feasible consumption. However, the planner does not know the particular utility functions.

We denote by $E(N)$ the set of all economies $e = \left(N_f, N_F, z^0, (u_i)_{i \in N} \right)$ satisfying 1 and 2. A *consumption sharing rule* is a function γ which assigns to each $e \in E(N)$ a feasible consumption $\gamma(e) \in \Omega^N$. Let Γ be the set of all consumption sharing rules on $E(N)$ for some N. In particular, we define $\gamma^0 \in \Gamma$ by

$$\gamma^0 (e) = z^0$$

for all $e = \left(N_f, N_F, z^0, (u_i)_{i \in N} \right) \in E(N)$.

Fix $e = \left(N_f, N_F, z^0, (u_i)_{i \in N} \right) \in E(N)$. We can define the associated *non-transferable utility game* (*NTU game*) as a pair (N, V^e) where V^e is a set-valued correspondence (called *characteristic function*) that assigns to every coalition $S \subset N$ a subset $V(S) \subset \mathbb{R}^S$ with represents the utility that players in V can get by themselves by cooperating, *i.e.*

$$V^e(S) := \left\{ (u_i(z_i))_{i \in S} \in \mathbb{R}^S : z \in \Omega^S \right\}.$$

Next properties can be easily deduced for V^e:

(A1) For each $S \subset N$, the set $V^e(S)$ is nonempty, closed, convex and *bounded above* (*i.e.*, for each $t \in \mathbb{R}^S$, the set $\{t' \in V^e(S) : t' \geq t\}$ is compact).

(A2) For each $i \in N$, $\max \{t : t \in V^e(\{i\})\} = x_i^0$.

(A3) *Zero-Monotonicity*: For each $S \subset N$, $t \in V(S)$ and $i \notin S$, we have $\left(t, x_i^0\right) \in V(S \cup \{i\})$. In particular, $\left(x_j^0\right)_{j \in S} \in \mathbb{R}^S$ belongs to $V(S)$.

Remark 2 *Comprehensiveness is not required in this example. In particular, the minimum utility a farmer $\beta \in N_f$ can get is $u_\beta(0)$.*

Let Π the set of all orders of players in N. Given $\pi \in \Pi$ and $i \in N$, we define P_i^π as the set of players who come before i in the order π. Namely:

$$P_i^\pi := \{j \in N : \pi(j) < \pi(i)\}.$$

For notational convenience, we denote $P_{n+1}^\pi := N$.

Let $\lambda = (\lambda_i)_{i \in N} \in \mathbb{R}_{++}^N$ and let $S \subset N$, we define

$$v^{\lambda,e}(S) := \max\left\{\sum_{i \in S} \lambda_i u_i(z_i) : z \in \Omega^S\right\} = \max\left\{\sum_{i \in S} \lambda_i t_i : t \in V^e(S)\right\}$$

for all $S \subset N$ when this maximum exists. Notice that, for $\lambda = 1_N$,

$$v^{1_N,e}(N) = \sum_{i \in N} x_i^0 + \sum_{i \in N} u_i'\left(y_i^M\right).$$

Assume $v^{\lambda,e}(S)$ exists for every $S \subset N$. Let $\pi \in \Pi$. We define $d^\pi(\lambda, e) \in \mathbb{R}^N$ as the vector given by

$$d_i^\pi(\lambda, e) = \frac{1}{\lambda_i}\left[v^{\lambda,e}\left(P_i^\pi \cup \{i\}\right) - v^{\lambda,e}\left(P_i^\pi\right)\right]$$

for all $i \in N$.

Notice that $\sum_{i \in N} \lambda_i d_i^\pi(\lambda, e) = v^{\lambda,e}(N)$.

The *Shapley value* (Shapley, 1953) of $\left(N, v^{\lambda,e}\right)$ is the average of all these $d^\pi(\lambda, e)$'s:

$$Sh(\lambda, e) := \frac{1}{|\Pi|}\sum_{\pi \in \Pi} d^\pi(\lambda, e) \in \mathbb{R}^N.$$

Clearly, $\sum_{i \in N} \lambda_i Sh_i(\lambda, e) = v^{\lambda,e}(N)$.

A point $t \in V^e(N)$ is a *Shapley NTU value* (Shapley, 1969) of (N, V^e) if there exists a vector $\lambda \in \mathbb{R}_{++}^N$ such that $t = Sh(\lambda, e)$.

Notice that the Shapley NTU value is defined on the space of utilities (which is not checkable by the planner), wherever the consumption sharing rules are defined on the space of commodities.

Next results are of interest:

Proposition 3 *Given $\lambda \in \mathbb{R}_{++}^N$ and $\pi \in \Pi$, $d_i^\pi(\lambda, e) \geq x_i^0$ for all $i \in N$.*

Proof. Fix $i \in N$ and let $z^1 \in \Omega^{P_i^\pi}$ such that $v^{\lambda,e}\left(P_i^\pi\right) = \sum_{j \in P_i^\pi} \lambda_j u_j\left(z_j^1\right)$. Then, $z^2 := \left(z^1, z_i^0\right) \in \Omega^{P_i^\pi \cup \{i\}}$ and thus

$$\begin{aligned}
\lambda_i d_i^\pi(\lambda, e) &= v^{\lambda,e}\left(P_i^\pi \cup \{i\}\right) - v^{\lambda,e}\left(P_i^\pi\right) \\
&= v^{\lambda,e}\left(P_i^\pi \cup \{i\}\right) - \sum_{j \in P_i^\pi} \lambda_j u_j\left(z^1\right) \\
&= v^{\lambda,e}\left(P_i^\pi \cup \{i\}\right) - \sum_{j \in P_i^\pi} \lambda_j u_j\left(z^1\right) - \lambda_i u_i\left(z_i^0\right) + \lambda_i x_i^0 \\
&= v^{\lambda,e}\left(P_i^\pi \cup \{i\}\right) - \sum_{j \in P_i^\pi \cup \{i\}} \lambda_j u_j\left(z^2\right) + \lambda_i x_i^0 \\
&\geq \lambda_i x_i^0.
\end{aligned}$$

■

Proposition 4 *There exists a unique Shapley NTU value of* $\left(N, V^{\lambda,e}\right)$, *and it can be obtained with* $\lambda = (1, ..., 1)$.

Proof. Let $1_N := (1, ..., 1) \in \mathbb{R}_{++}^N$. By Proposition 3, we know that $Sh\left(1_N, e\right) \geq x^0$. We prove that $Sh\left(1_N, e\right) \in V(N)$.

Clearly, $v^{1_N,e}(N) = \sum_{i \in N} x_i^0 + \sum_{i \in N} u_i'\left(y_i^M\right)$. We take $\left(x, y^M\right) \in \Omega^N$ with x given by:

$$x_i = Sh_i\left(1_N, e\right) - u_i'\left(y_i^M\right).$$

For $i \in N_f$, $u_i'\left(y_i^M\right) = u_i'(0) \leq 0$ (notice that u' is nondecreasing) and so $x_i \geq x_i^0 \geq 0$. Hence, $\left(x, y^M\right)$ is actually feasible. Moreover, $u_i\left(x_i, y_i^M\right) = Sh_i\left(1_N, e\right)$ and thus $Sh\left(1_N, e\right) \in V(N)$.

Let $\lambda \in \mathbb{R}_{++}^N$ such that $Sh(\lambda, e) \in V^e(N)$.

Let $\alpha \in N_F$. Assume there exists $\beta \in N$ such that $\lambda_\alpha < \lambda_\beta$. Given any $M > 0$ we will find a $z = (x, y) \in \Omega^{N_F}$ such that $\sum_{i \in N_F} \lambda_i u_i(z_i) \geq M$. We can take $x \in \mathbb{R}^N$ such that $x_\alpha = -\frac{M}{\lambda_\beta - \lambda_\alpha}$, $x_j = \frac{M}{\lambda_\beta - \lambda_\alpha}$ and $x_k = 0$ for all $k \in N \backslash \{i, j\}$. Clearly, $\left(x, y^0\right) \in \Omega^N$. Moreover,

$$\sum_{i \in N} \lambda_i u_i\left(x_i, y_i^0\right) = \sum_{i \in N} \lambda_i u_i\left(0, y_i^0\right) + \frac{M}{\lambda_\beta - \lambda_\alpha}\left(\lambda_\beta - \lambda_\alpha\right) = M.$$

Thus, there not exists $v^{\lambda,e}(N)$. This contradiction proves $\lambda_i = \lambda_j$ for all $j \in N_F$ and $\lambda_i \geq \lambda_j$ for all $j \in N_f$.

Assume now there exists $\beta \in N_f$ such that $\lambda_\alpha > \lambda_\beta$.

Let $z^\lambda \in \Omega^N$ such that $Sh_i(\lambda, e) = u_i\left(z_i^\lambda\right)$ for all $i \in N$. Since $\sum_{i \in N} \lambda_i Sh_i(\lambda, e) = v^{\lambda,e}(N)$, we deduce

$$\sum_{i \in N} \lambda_i u_i\left(z_i^\lambda\right) = v^{\lambda,e}(N) = \max\left\{\sum_{i \in N} \lambda_i u_i(z_i) : z \in \Omega^N\right\}.$$

Furthermore, by Proposition 3, $u_i\left(z_i^\lambda\right) \geq x_i^0$ for all $i \in N$. Let $z^* = \left(x^*, y^M\right)$ such that $x_i^* = u_i\left(z_i^\lambda\right) - u_i'\left(y_i^M\right)$ for all $i \in N \backslash \{\alpha, \beta\}$, $x_\alpha^* = u_\alpha\left(z_\alpha^\lambda\right) + u_\beta\left(z_\beta^\lambda\right) - u_\alpha'\left(y_\alpha^M\right)$ and $x_\beta^* = -u_\beta'\left(y_\beta^M\right)$. Since $u_i'\left(y_i^M\right) = u_i'(0) \leq 0$ for all $i \in N_f$ and $u_\alpha\left(z_\alpha^\lambda\right) + u_\beta\left(z_\beta^\lambda\right) \geq x_\alpha^0 + x_\beta^0 \geq 0$, we deduce that $z^* \in \Omega^N$. Furthermore,

$$\begin{aligned}
\sum_{i\in N} \lambda_i u_i\left(z_i^*\right) &= \sum_{i\in N} \lambda_i \left[x_i^* + u_i'\left(y_i^M\right)\right] \\
&= \sum_{i\in N\backslash\{\alpha,\beta\}} \lambda_i u_i\left(z_i^\lambda\right) + \lambda_\alpha\left[u_\alpha\left(z_\alpha^\lambda\right) + u_\beta\left(z_\beta^\lambda\right)\right] \\
&= \sum_{i\in N} \lambda_i u_i\left(z_i^\lambda\right) + \left(\lambda_\alpha - \lambda_\beta\right) u_\beta\left(z_\beta^\lambda\right)
\end{aligned}$$

Since $\sum_{i\in N} \lambda_i u_i\left(z_i^\lambda\right)$ is maximum and $u_\beta\left(z_\beta^\lambda\right) \geq x_\beta^0 \geq 0$, we deduce that either $u_\beta\left(z_\beta^\lambda\right) = 0$ or $\lambda_\alpha \leq \lambda_\beta$. If $u_\beta\left(z_\beta^\lambda\right) = 0$, player β is a null player[3] in both $\left(N, v^{\lambda,e}\right)$ and $\left(N, v^{1_N,e}\right)$. Thus, $Sh_\beta\left(N, v^{\lambda,e}\right) = Sh_\beta\left(N, v^{1_N,e}\right) = 0$ and for every no-null player $i \in N$, $\lambda_i = \lambda_\alpha$. Hence, $Sh\left(N, v^{\lambda,e}\right) = Sh\left(N, v^{1_N,e}\right)$. ■

>From now on, we denote $d^\pi(1_N, e)$, $Sh(1_N, e)$ and $v^{1_N,e}$ as $d^\pi(e)$, $Sh(e)$ and v^e, respectively.

Proposition 5 *For each $S \subset N$ and $i \notin S$*

$$v^e(S \cup \{i\}) \geq v^e(S) + x_i^0.$$

Proof. Let $z^* = (x^*, y^*) \in \Omega^S$ such that $\sum_{j\in S} u_j\left(z_j^*\right) = v^e(S)$. Clearly, $\sum_{j\in S} x_j^* = \sum_{j\in S} x_j^0$. Furthermore, $z^{**} := \left(z^*, z_i^0\right) \in \Omega^{S\cup\{i\}}$. Thus

$$\begin{aligned}
v^e(S) &= \sum_{j\in S} x_j^0 + \sum_{j\in S} u_j'\left(y_j^*\right) \\
&= \sum_{j\in S\cup\{i\}} x_j^0 + \sum_{j\in S} u_j'\left(y_j^*\right) + u_i'\left(y_j^0\right) - x_i^0 \\
&= \sum_{j\in S\cup\{i\}} u_j\left(y_j^{**}\right) - x_i^0 \\
&\leq v^e(S \cup \{i\}) - x_i^0.
\end{aligned}$$

■

Corollary 6 *For any $S, T \subset N$ such that $S \subset T$*

$$v^e(S) \leq v^e(T).$$

[3] A *null player* in (N, v) is a player $i \in N$ such that $v(S \cup \{i\}) = v(S)$ for all $S \subset N \backslash \{i\}$. The Shapley value gives 0 to any null player. Furthermore, the Shapley value for the other players does not change if we add or remove a null player from the game.

3 The non-cooperative mechanism

Players should form a society. First, an order of the players is randomly chosen. Assume the order is $(12...n)$. Player 1 should then present a rule[4] $\gamma \in \Gamma$. No restrictions (apart from feasibility) are imposed on γ. Player 2 may either *agree* on γ and join the society, or *disagree* on γ and propose a new rule $\widetilde{\gamma}$ to player 1. If player 1 accepts (he *votes 'yes'*), the society $\{1,2\}$ forms with the new rule $\widetilde{\gamma}$, and turn passes to player 3. If player 2 rejects (he *votes 'no'*), he remains out of the society and turn passes to player 3.

In general, when turn reaches player i, he faces a society $S \subset P_i^\pi$ with certain rule γ, and a set of players $W = P_i^\pi \backslash S$ who have chosen to stay out of the society. Players in S, W and $N \backslash P_i^\pi$ are called *active players*, *passive players* and *candidates*, respectively. Player i must then either agree to join the society (in that case, player i becomes an active player and turn passes to candidate $i+1$) or disagree and propose both a new rule $\widetilde{\gamma}$ and a new society $\widetilde{S} \subset P_i^\pi \cup \{i\}$ which includes himself and all the members of the old one (i.e. $S \cup \{i\} \subset \widetilde{S}$). The members of $\widetilde{S} \backslash \{i\}$ vote sequentially whether they accept or reject this proposal. If all of them vote 'yes', the new society $\widetilde{S}$ forms with the new value (we say then that the proposal is accepted), and turn passes to candidate $i+1$. If at least one member of $\widetilde{S} \backslash \{i\}$ votes 'no', player i becomes a passive player and turn passes to candidate $i+1$.

Once there are no more candidates, we have a society $S \subset N$ of active players, a rule γ for the society, and a set $W = N \backslash S$ of passive players. Then, every player $i \in S$ receives $\gamma_i(S)$ and every player in W keeps his initial endowment z_i^0. This means that the final payoff for each player $i \in S$ is $u_i \gamma_i(S)$ and[5] the final payoff for each player $i \in N \backslash S$ is $u_i\left(z_i^0\right) = x_i^0$.

We now describe the mechanism $M(e)$ formally. We first describe the games $M(e,\pi,i,W,\gamma)$ and $\widetilde{M}(e,\pi,i,W,\gamma)$. $M(e,\pi,i,W,\gamma)$ is the subgame which begins when, given the order π, turn reaches player i and he faces a society of active players $S = P_i^\pi \backslash W$ with a proposed rule $\gamma \in \Gamma$, and a set of passive players W. $\widetilde{M}(e,\pi,i,W,\gamma)$ is the subgame which arises after player i disagrees in the subgame $M(e,\pi,i,W,\gamma)$.

> Let $\pi \in \Pi$ be an order of the players. We can assume without loss of generality that $\pi = (12...n)$. Given $i \in N \cup \{n+1\}$, $\gamma \in \Gamma$ and $W \subset P_i^\pi$, we inductively define the mechanisms $M(e,\pi,i,W,\gamma)$ and $\widetilde{M}(e,\pi,i,W,\gamma)$ as follows.
>
> In both $M(e,\pi,n+1,W,\gamma)$ and $\widetilde{M}(e,\pi,n+1,W,\gamma)$, every player $i \in N \backslash W$ receives $u_i \gamma_i(N \backslash W)$ and every $i \in W$ receives $u_i\left(z_i^0\right) = x_i^0$.
>
> Assume both $M(e,\pi,j,W',\gamma')$ and $\widetilde{M}(e,\pi,j,W',\gamma')$ are defined for all $j > i$, $\gamma' \in \Gamma$ and $W' \subset P_j^\pi$.

[4] >From now on, we use the term *rule* instead of the more cumbersome *consumption sharing rule*.

[5] We write $u_i \gamma_i(S)$ instead of $u_i(\gamma_i(S))$.

In $\widetilde{M}(e,\pi,i,W,\gamma)$, player i proposes a rule $\widetilde{\gamma} \in \Gamma$ and a set $\widetilde{W} \subset W$. If all the members of $P_i^\pi \backslash \widetilde{W}$ vote 'yes' – they are asked in some prespecified order – then the mechanism $M\left(e,\pi,i+1,\widetilde{W},\widetilde{\gamma}\right)$ is played. If at least one member of $P_i^\pi \backslash \widetilde{W}$ votes 'no', the mechanism $M(e,\pi,i+1,W\cup\{i\},\gamma)$ is played.

In $M(e,\pi,i,W,\gamma)$, player i can either agree or disagree on (W,γ). If he disagrees, $\widetilde{M}(e,\pi,i,W,\gamma)$ is played. If he agrees, $M(e,\pi,i+1,W,\gamma)$ is played.

The mechanism $M(e)$ consists in choosing randomly an order $\pi' \in \Pi$, being each order equally likely to be chosen, and playing the game $\widetilde{M}\left(e,\pi',\pi'^{-1}(1),\emptyset,\gamma^0\right)$.

Clearly, for any set of pure (mixed) strategies, this mechanism terminates in finite time. Thus, the (expected) payoffs at termination are well-defined.

Let $\pi \in \Pi$. From now on, we assume without loss of generality that $\pi = (12...n)$.

Theorem 7 *There exists at least a subgame perfect Nash equilibrium (SPNE) in the negotiation mechanism $M(e)$. Moreover, the only expected final payoff in any SPNE is the Shapley NTU value of the game (N,V^e).*

Proof. First, we prove that there exists a SPNE. Then, we prove that every SPNE yields the Shapley NTU value.

We consider the following set of strategies:

In the subgame $M(e,\pi,n,W,\gamma)$, player n agrees on (W,γ) if and only if

$$u_n\gamma_n(N\backslash W) \geq v^e(N) - \sum_{i\in P_n^\pi\backslash W} u_i\gamma_i\left(P_n^\pi\backslash W\right) - \sum_{i\in W} x_i^0.$$

In the subgame $\widetilde{M}(e,\pi,n,W,\gamma)$, player n proposes $(\emptyset,\widetilde{\gamma})$ such that $\widetilde{\gamma}(N) = \left(x,y^M\right)$ with x given by

$$\begin{array}{ll} x_i = u_i\gamma_i\left(P_n^\pi\backslash W\right) - u_i'\left(y_i^M\right) & \text{for all } i \in P_n^\pi\backslash W \\ x_i = x_i^0 - u_i'\left(y_i^M\right) & \text{for all } i \in W \\ x_n = v^e(N) - \sum\limits_{i\in P_n^\pi\backslash W} u_i\gamma_i\left(P_n^\pi\backslash W\right) - \sum\limits_{i\in W} x_i^0 - u_n'\left(y_n^M\right) & \end{array}$$

We check that $\widetilde{\gamma}(N)$ is a feasible consumption; *i.e.* $\sum\limits_{i\in N} x_i \leq \sum\limits_{i\in N} x_i^0$ and $x_i \geq 0$ for all $i \in N_f$:

$$\sum_{i\in N} x_i = v^e(N) - \sum_{i\in N} u_i'\left(y_i^M\right) = \sum_{i\in N} x_i^0.$$

Let $\beta \in N_f$. We have

- If $\beta \in P_n^\pi \backslash W$, then

$$\begin{aligned} x_\beta &= u_\beta \gamma_\beta \left(P_n^\pi \backslash W\right) - u'_\beta (0) \\ &= \gamma_{\beta x} \left(P_n^\pi \backslash W\right) + u'_\beta \gamma_{\beta y} \left(P_n^\pi \backslash W\right) - u'_\beta (0) \\ &\geq \gamma_{\beta x} \left(P_n^\pi \backslash W\right) \\ &\geq 0. \end{aligned}$$

- If $\beta \in W$, then $x_\beta = x_\beta^0 - u'_\beta (0) \geq x_\beta^0 - u'_\beta \left(y_\beta^0\right) = x_\beta^0 \geq 0$.

- If $\beta = n$, then

$$\begin{aligned} x_n &= v^e (N) - \sum_{i \in P_n^\pi \backslash W} u_i \gamma_i \left(P_n^\pi \backslash W\right) - \sum_{i \in W} x_i^0 - u'_n \left(y_n^M\right) \\ &= \sum_{i \in N \backslash W} x_i^0 + \sum_{i \in P_n^\pi} u'_i \left(y_i^M\right) - \sum_{i \in P_n^\pi \backslash W} u_i \gamma_i \left(P_n^\pi \backslash W\right) \\ &\geq \sum_{i \in N \backslash W} x_i^0 + \sum_{i \in P_n^\pi} u'_i \left(y_i^M\right) - \sum_{i \in P_n^\pi \backslash W} x_i^0 - \sum_{i \in P_n^\pi \backslash W} u'_i \gamma_{iy} \left(P_n^\pi \backslash W\right) \\ &\geq x_n^0 \geq 0. \end{aligned}$$

Thus, $\widetilde{\gamma} (N)$ is a feasible consumption.

In the subgame $\widetilde{M} (e, \pi, n, W, \gamma)$, assume player n proposes $\left(\widetilde{W}, \widetilde{\gamma}\right)$ and $i \in P_\alpha^\pi \backslash \widetilde{W}$. Then, player i votes 'yes' if and only if

$$u_i \widetilde{\gamma}_i \left(N \backslash \widetilde{W}\right) \geq u_i \gamma_i \left(P_n^\pi \backslash W\right).$$

Fix $i \in N$. Assume we have defined the strategies of the players in $M (e, \pi, j, W, \gamma)$ and $\widetilde{M} (e, \pi, j, W, \gamma)$ for any $j > i$ and any (W, γ). We denote by $a (j, W, \gamma)$ the final payoff in the subgame $M (e, \pi, j, W, \gamma)$ when players follow these strategies. This value is well-defined for any $j > i$ and any (W, γ).

We now describe the strategies in $M (e, \pi, i, W, \gamma)$ and $\widetilde{M} (e, \pi, i, W, \gamma)$.

In $M (e, \pi, i, W, \gamma)$, player i agrees if and only if

$$a_i (i+1, W, \gamma) \geq v^e \left(P_{i+1}^\pi\right) - \sum_{j \in P_i^\pi} a_j (i+1, W \cup \{i\}, \gamma). \tag{1}$$

In the subgame $\widetilde{M} (e, \pi, i, W, \gamma)$, player i proposes $(\emptyset, \widetilde{\gamma})$ such that $\widetilde{\gamma} (N) = \left(x, y^M\right)$ with x given by

$$\left.\begin{array}{ll} x_j = a_j (i+1, W \cup \{i\}, \gamma) - u'_j \left(y_j^M\right) & \text{for all } j \in P_i^\pi \\ x_i = v^e \left(P_{i+1}^\pi\right) - \sum\limits_{j \in P_i^\pi} a_j (i+1, W \cup \{i\}, \gamma) - u'_i \left(y_i^M\right) & \\ x_j = d_j^\pi (v^e) - u'_j \left(y_j^M\right) & \text{for all } j > i. \end{array}\right\} \tag{2}$$

We check later (Claim (II) below) that this consumption $\widetilde{\gamma}(N)$ is feasible.

In the subgame $\widetilde{M}(e,\pi,i,W,\gamma)$, assume player i proposes $\left(\widetilde{W},\widetilde{\gamma}\right)$ and $j \in P_i^\pi \backslash \widetilde{W}$. Then, player j votes 'yes' if and only if

$$a_j\left(i+1,\widetilde{W},\widetilde{\gamma}\right) \geq a_j\left(i+1, W \cup \{i\}, \gamma\right). \tag{3}$$

It is straightforward to check that, under these strategies, player 1 proposes $(\emptyset,\gamma^\pi)$ with $u_i\gamma^\pi(N) = d_i^\pi(v)$ for all $i \in N$ and the rest of players agree on it. Society is then formed with all the players and the final outcome is $d^\pi(v)$. Hence the final expected final outcome is the Shapley NTU value.

In order to check that these strategies form a SPNE, we prove three claims:

Claim (I): Given $i \in N$, $W \subset P_i^\pi$ and $\gamma \in \Gamma$

$$a_j(i,W,\gamma) = x_j^0$$

for all $j \in W$.

We proceed by induction on i. For $i = n$, the result is straightforward. Assume the result holds for $i+1$ and any $W \subset P_i^\pi$, $\gamma \in \Gamma$. In the subgame $M(e,\pi,i,W,\gamma)$, given $j \in W$, three cases may occur:

1. Player i agrees. Then, $a_j(i,W,\gamma) = a_j(i+1,W,\gamma)$, which is x_j^0 by induction hypothesis.
2. Player i disagrees, proposes $(\emptyset,\widetilde{\gamma})$ given as in (2), and this proposal is accepted. Then, the rest of the players agree too and $a_j(i,W,\gamma) = u_j\widetilde{\gamma}_j(N) = a_j(i+1,W\cup\{i\},\gamma)$, which is x_j^0 by induction hypothesis.
3. Player i disagrees, proposes $(\emptyset,\widetilde{\gamma})$ given as in (2), and this proposal is rejected. Then, $a_j(i,W,\gamma) = a_j(i+1,W\cup\{i\},\gamma)$, which is x_j^0 by induction hypothesis.

Claim (II): Given $i \in N$, $W \subset P_i^\pi$ and $\gamma \in \Gamma$

$$\sum_{j \in P_i^\pi} a_j(i,W,\gamma) \leq v^e\left(P_i^\pi \backslash W\right).$$

We proceed by induction on i. For $i = n$, the result is straightforward. Assume the result holds for $i+1$ and any $W \subset P_i^\pi$, $\gamma \in \Gamma$. In the subgame $M(e,\pi,i,W,\gamma)$, two cases may occur:

1. Player i agrees. Then

$$a(i,W,\gamma) = a(i+1,W,\gamma)$$

and

$$a_i\left(i+1,W,\gamma\right)\geq v^e\left(P_{i+1}^{\pi}\right)-\sum_{j\in P_i^{\pi}}a_j\left(i+1,W\cup\{i\},\gamma\right).$$

Thus, using induction hypothesis, Claim (I) and monotonicity of (N,v):

$$\begin{aligned}\sum_{j\in P_i^{\pi}}a_j\left(i,W,\gamma\right) &= \sum_{j\in P_{i+1}^{\pi}}a_j\left(i+1,W,\gamma\right)-a_i\left(i+1,W,\gamma\right)\\ &\leq v^e\left(P_{i+1}^{\pi}\backslash W\right)-v^e\left(P_{i+1}^{\pi}\right)+\sum_{j\in P_i^{\pi}}a_j\left(i+1,W\cup\{i\},\gamma\right)\\ &\leq v^e\left(P_{i+1}^{\pi}\backslash W\right)-v^e\left(P_{i+1}^{\pi}\right)+v^e\left(P_{i+1}^{\pi}\backslash\left(W\cup\{i\}\right)\right)-a_i\left(i+1,W\cup\{i\},\gamma\right)\\ &\leq v^e\left(P_{i+1}^{\pi}\backslash\left(W\cup\{i\}\right)\right)-a_i\left(i+1,W\cup\{i\},\gamma\right)-\sum_{j\in W}x_j^0\\ &\leq v^e\left(P_i^{\pi}\backslash W\right)-\sum_{j\in W}x_j^0\\ &\leq v^e\left(P_i^{\pi}\backslash W\right).\end{aligned}$$

2. Player i disagrees. Then

$$a_j\left(i,W,\gamma\right)=a_j\left(i+1,W\cup\{i\},\gamma\right)$$

for all $j\in P_i^{\pi}$.
Thus, using induction hypothesis and Claim (I):

$$\begin{aligned}\sum_{j\in P_i^{\pi}}a_j\left(i,W,\gamma\right) &= \sum_{j\in P_i^{\pi}}a_j\left(i+1,W\cup\{i\},\gamma\right)\\ &\leq v\left(P_{i+1}^{\pi}\backslash\left(W\cup\{i\}\right)\right)-a_i\left(i+1,W\cup\{i\},\gamma\right)\\ &\leq v\left(P_i^{\pi}\backslash W\right).\end{aligned}$$

Now, we check that $\widetilde{\gamma}\left(N\right)=\left(x,y^M\right)$ given by (2) is feasible:

$$\sum_{j\in N}x_j=v^e\left(P_{i+1}^{\pi}\right)+\sum_{j>i}d_j^{\pi}\left(v^e\right)-\sum_{j\in N}u_j'\left(y_j^M\right)=v^e\left(N\right)-\sum_{j\in N}u_j'\left(y_j^M\right)=\sum_{j\in N}x_j^0.$$

Given $\beta\in N_f$:

1. • If $\beta\in P_i^{\pi}$,

$$x_\beta=a_\beta\left(i+1,W\cup\{i\},\gamma\right)-u_\beta'\left(y_\beta^M\right)\geq u_\beta\left(0\right)-u_\beta'\left(0\right)=0.$$

- If $\beta = i$,

$$\begin{aligned} x_i &= v^e\left(P_{i+1}^{\pi}\right) - \sum_{j \in P_i^{\pi}} a_j\left(i+1, W \cup \{i\}, \gamma\right) - u_i'\left(y_i^M\right) \\ &\geq a_i\left(i+1, W \cup \{i\}, \gamma\right) - u_i(0) \geq 0. \end{aligned}$$

- If $\beta > i$,

$$x_\beta \geq d_\beta^{\pi}(e) - u_\beta'(0) \geq x_\beta^0 \geq 0.$$

Claim (III): $a_i(i, W, \gamma) \geq x_i^0$ for all $i \in N$, $W \subset P_i^{\pi}$ and $\gamma \in \Gamma$.

In the subgame $M(e, \pi, i, W, \gamma)$, two things may happen:

1. Player i agrees. Then:

$$\begin{aligned} a_i(i, W, \gamma) &= a_i(i+1, W, \gamma) \\ &\geq v^e\left(P_{i+1}^{\pi}\right) - \sum_{j \in P_i^{\pi}} a_j\left(i+1, W \cup \{i\}, \gamma\right) \\ &= v^e\left(P_{i+1}^{\pi}\right) - \sum_{j \in P_{i+1}^{\pi}} a_j\left(i+1, W \cup \{i\}, \gamma\right) \\ &\geq x_i^0 + v^e\left(P_{i+1}^{\pi} \backslash (W \cup \{i\})\right) - \sum_{j \in P_{i+1}^{\pi}} a_j\left(i+1, W \cup \{i\}, \gamma\right) \\ &\geq x_i^0. \end{aligned}$$

2. Player i disagrees. Then:

$$a_i(i, W, \gamma) = v^e\left(P_{i+1}^{\pi}\right) - \sum_{j \in P_i^{\pi}} a_j\left(i+1, W \cup \{i\}, \gamma\right) \geq x_i^0.$$

We now prove that these strategies form a SPNE.

Assume we are in the subgame $\widetilde{M}(e, \pi, i, W, \gamma)$ and player i proposes $\left(\widetilde{W}, \widetilde{\gamma}\right)$ with $j \in P_i^{\pi} \backslash \widetilde{W}$.

If some player after player j is bound to vote 'no' should turn reach him, player j is indifferent between voting 'yes' or 'no'. Assume then the offer is bound to be accepted should player j vote 'yes'. By doing so, and given the strategies of the rest of the players, player j gets $a_j\left(i+1, \widetilde{W}, \widetilde{\gamma}\right)$. By rejecting, however, player i gets $a_j(i+1, W \cup \{i\}, \gamma)$. Thus, it is optimal for player i to vote 'yes' if and only if (3) holds.

Assume now we are in the subgame $\widetilde{M}(e, \pi, i, W, \gamma)$ and assume player i changes his strategy and proposes a different $\left(\widetilde{W}, \widetilde{\gamma}\right)$. If (3) does not hold for some $j \in P_i^\pi \backslash \widetilde{W}$, this player will vote 'no' and, by Claim (I), the final payoff for player i is x_i^0, *i.e.* not more than with the original strategy.

Assume then (3) holds for all $j \in P_\alpha^\pi \backslash \widetilde{W}$. The proposal is then accepted and the final payoff for player i is at most $a_i \left(i+1, \widetilde{W}, \widetilde{\gamma}\right)$. However, by using (3), Claim (I), Claim (II) and Claim (III):

$$\begin{aligned} a_i \left(i+1, \widetilde{W}, \widetilde{\gamma}\right) &= \sum_{j \in P_{i+1}^\pi} a_j \left(i+1, \widetilde{W}, \widetilde{\gamma}\right) - \sum_{j \in P_i^\pi} a_j \left(i+1, \widetilde{W}, \widetilde{\gamma}\right) \\ &\leq v^e \left(P_{i+1}^\pi \backslash \widetilde{W}\right) - \sum_{j \in P_i^\pi \backslash W} a_j \left(i+1, W \cup \{i\}, \gamma\right) - \sum_{j \in W} x_j^0 \\ &= v^e \left(P_{i+1}^\pi \backslash \widetilde{W}\right) - \sum_{j \in P_i^\pi} a_j \left(i+1, W \cup \{i\}, \gamma\right) \\ &\leq v^e \left(P_{i+1}^\pi\right) - \sum_{j \in P_i^\pi} a_j \left(i+1, W \cup \{i\}, \gamma\right) \end{aligned}$$

and thus player i does not improve his final payoff.

Finally, assume we are in the subgame $M(e, \pi, i, W, \gamma)$.

If (1) holds and player i disagrees on (W, γ), he will get at most

$$v^e \left(P_{i+1}^\pi\right) - \sum_{j \in P_i^\pi} a_j \left(i+1, W \cup \{i\}, \gamma\right) \tag{4}$$

which is not less than what he would get by agreeing. Thus, he will not improve his final payoff by deviating.

If (1) does not hold and player i agrees on (W, γ), he will get less than (4), which is the payoff he obtains by not deviating. Thus, it is optimal for him to disagree.

We now prove that every SPNE has the Shapley NTU value as expected final outcome. Assume we are in an SPNE. Let $b \in V^e(N)$ be the expected final payoff. Let $b(\pi') \in V^e(N)$ be the expected final payoff conditioned to $\pi' \in \Pi$ be the chosen order. Thus,

$$b = \frac{1}{|\Pi|} \sum_{\pi' \in \Pi} b(\pi').$$

We prove that $b_i(\pi) = d_i^\pi(e)$ for all $i \in N$.

Given $i \in N$, $W \subset P_i^\pi$ and $\gamma \in \Gamma$, let $b(i, W, \gamma) \in V^e(N)$ be the expected final payoff in the subgame $M(e, \pi, i, W, \gamma)$, and let $\widetilde{b}(i, W, \gamma) \in V^e(N)$ be the expected final payoff in the subgame $\widetilde{M}(e, \pi, i, W, \gamma)$. Notice that, by the mechanism definition, $b_i(i, W, \gamma) \geq \widetilde{b}_i(i, W, \gamma)$.

Given $S \subset N$, we define

$$S^\pi := \{i \in S : P_i^\pi \subset S\}.$$

Thus, players in S^π are the first players out of S who come together in the order π. We also define

$$\Gamma^\pi := \left\{\gamma \in \Gamma : \gamma(S) = \left(\gamma_j(S^\pi)\right)_{j\in S^\pi} \times \left(z_j^0\right)_{j\in S\setminus S^\pi} \text{ for all } S \subset N\right\}.$$

Thus, Γ^π is the set of rules which do not share the resources of the players after the first 'gap' in the coalition (with respect to π).

We proceed by a series of claims.

Claim (A): Assume we are in the subgame $\widetilde{M}(e,\pi,i,W,\gamma)$ for some (W,γ) such that $\gamma \in \Gamma^\pi$. If the proposal of player i is rejected, the final payoff of any player $j \in P_{i+1}^\pi$ is $u_j\gamma_j(P_i^\pi\setminus W)$ (if $j \in P_i^\pi\setminus W$) or x_j^0 (if $j \in W\cup\{i\}$). Namely:

$$\gamma \in \Gamma^\pi \Longrightarrow \begin{cases} b_j(i+1, W\cup\{i\},\gamma) = u_j\gamma_j(P_i^\pi\setminus W) & \text{for all } j \in P_i^\pi\setminus W \\ b_j(i+1, W\cup\{i\},\gamma) = x_j^0 & \text{for all } j \in W\cup\{i\}. \end{cases}$$

Claim (B): Assume player i proposes $\left(\widetilde{W},\widetilde{\gamma}\right)$ in the subgame $\widetilde{M}(e,\pi,i,W,\gamma)$ for some (W,γ). If $b_j\left(i+1,\widetilde{W},\widetilde{\gamma}\right) > b_j(i+1, W\cup\{i\},\gamma)$ for all $j \in P_i^\pi$, then the proposal is accepted.

Claim (C): Assume we are in the subgame $M(e,\pi,i,W,\gamma)$ for some (W,γ). Then, the final payoff for player i is not less than $v^e\left(P_{i+1}^\pi\right) - \sum_{j\in P_i^\pi\setminus W} u_j\gamma_j(P_i^\pi\setminus W) - \sum_{j\in W} x_j^0$. Namely

$$b_i(i,W,\gamma) \geq \widetilde{b}_i(i,W,\gamma) \geq v^e\left(P_{i+1}^\pi\right) - \sum_{j\in P_i^\pi\setminus W} u_j\gamma_j(P_i^\pi\setminus W) - \sum_{j\in W} x_j^0$$

for all (W,γ).

Claim (D): Player i can assure himself a final expected payoff of at least $d_i^\pi(e)$. Namely,

$$\begin{aligned} b_i(j,W,\gamma) &\geq d_i^\pi(e) \\ \widetilde{b}_i(j,W,\gamma) &\geq d_i^\pi(e) \end{aligned}$$

for all $j \leq i$ and all (W,γ).

Claim (E): Assume we are in the subgame $\widetilde{M}(e,\pi,i,W,\gamma)$ for some (W,γ) such that $\gamma \in \Gamma^\pi$. Then, any player $j \in P_i^\pi$ gets a final payoff of $u_j\gamma_j(P_i^\pi\setminus W)$ (if $j \in P_i^\pi\setminus W$) or x_j^0 (if $j \in W$). Namely,

$$\gamma \in \Gamma^\pi \Longrightarrow \begin{cases} \widetilde{b}_j(i,W,\gamma) = u_j\gamma_j(P_i^\pi\setminus W) & \text{for all } j \in P_i^\pi\setminus W \\ \widetilde{b}_j(i,W,\gamma) = x_j^0 & \text{for all } j \in W. \end{cases}$$

Claim (F): Assume we are in the subgame $M(e,\pi,i,W,\gamma)$ for some (W,γ) such that $\gamma\in\Gamma^\pi$. If player i disagrees on γ, his final payoff is $v^e(P_i^\pi\cup\{i\}) - \sum_{j\in P_i^\pi\backslash W} u_j\gamma_j(P_i^\pi\backslash W) - \sum_{j\in W} x_j^0$. Namely,

$$\gamma\in\Gamma^\pi \Longrightarrow \widetilde{b}_i(i,W,\gamma) = v^e\left(P_{i+1}^\pi\right) - \sum_{j\in P_i^\pi\backslash W} u_j\gamma_j(P_i^\pi\backslash W) - \sum_{j\in W} x_j^0.$$

Claim (G): Assume we are in the subgame $M(e,\pi,i,W,\gamma)$ for some (W,γ) such that $\gamma\in\Gamma^\pi$ and

$$u_k\gamma_k(N) > v^e\left(P_{k+1}^\pi\right) - \sum_{j\in P_k^\pi\backslash W} u_j\gamma_j(P_k^\pi\backslash W) - \sum_{j\in W} x_j^0 \qquad (5)$$

for all $k\geq i$. Then player i agrees on γ.

We prove these claims by induction on i.

Proof of Claim (A) for $i=n$: Trivial, since the final payoff in case of rejection is

$$b(n+1,W\cup\{n\},\gamma) = \left(u_j\gamma_j(P_n^\pi\backslash W)\right)_{j\in P_n^\pi\backslash W}\times\left(x_j^0\right)_{j\in W\cup\{n\}}.$$

Proof of Claim (B) for $i=n$: Assume players in $P_n^\pi\backslash\widetilde{W}$ vote in the order $j_1,j_2,...,j_l$. If turn reach player j_l, *i.e.* there has been no previous rejection, it is optimal for him to vote 'yes'. If turn reach player j_{l-1}, he anticipates player j_l's reaction. Thus, it is also optimal for him to vote 'yes'. By going backwards, we prove the result.

Proof of Claim (C) for $i=n$: Assume Claim (C) does not hold for $i=n$. Namely, there exists $r>0$ such that

$$\widetilde{b}_n(n,W,\gamma) = v^e(N) - \sum_{j\in P_n^\pi\backslash W} u_j\gamma_j(P_n^\pi\backslash W) - \sum_{j\in W} x_j^0 - r.$$

Since player n can easily assure himself a final payoff of x_n^0, we deduce that

$$v^e(N) - \sum_{j\in P_n^\pi\backslash W} u_j\gamma_j(P_n^\pi\backslash W) - \sum_{j\in W} x_j^0 - r \geq x_n^0. \qquad (6)$$

Let $\varepsilon\in(0,r)$ and assume player n changes his strategy so that he disagrees on (W,γ) and proposes $(\emptyset,\widetilde{\gamma})$ such that $\widetilde{\gamma}(N)=\left(x,y^M\right)$ with x given by

$$\begin{aligned}
x_j &= u_j\gamma_j(P_n^\pi\backslash W) + \frac{\varepsilon}{(n-1)} - u_j'\left(y_j^M\right) && \text{for all } j\in P_n^\pi\backslash W\\
x_j &= x_j^0 + \frac{\varepsilon}{(n-1)} - u_j'\left(y_j^M\right) && \text{for all } j\in W\\
x_n &= v^e(N) - \sum_{j\in P_n^\pi\backslash W} u_j\gamma_j(P_n^\pi\backslash W) - \sum_{j\in W} x_j^0 - \varepsilon - u_n'\left(y_n^M\right).
\end{aligned}$$

This rule is feasible:

$$\sum_{j\in N} x_j = v^e(N) - \sum_{j\in N} u_j'\left(y_j^M\right) = \sum_{j\in N} x_j^0.$$

Let $j \in N_f$. If $j \neq n$, by arguments similar to those used in the proof of (2), we can prove that $x_j \geq 0$. Assume $n \in N_f$. By (6)

$$\begin{aligned} x_n &= v^e(N) - \sum_{j\in P_n^\pi \backslash W} u_j\gamma_j\left(P_n^\pi \backslash W\right) - \sum_{j\in W} x_j^0 - \varepsilon - u_n'(0) \\ &\geq x_n^0 + r - \varepsilon - u_n'(0) > 0 \end{aligned}$$

Hence, the rule is feasible.

Furthermore, it is straightforward to check that conditions of Claim (B) are satisfied and thus $(\emptyset, \widetilde{\gamma})$ is accepted. Hence, the final payoff of player n is bigger than $\widetilde{b}_n(n, W, \gamma)$. This contradiction proves Claim (C) for $i = n$.

Proof of Claim (D) for $i = n$: Since turn eventually reaches player n and $b_n(n, W, \gamma) \geq \widetilde{b}_n(n, W, \gamma)$, it is enough to prove that $\widetilde{b}_n(n, W, \gamma) \geq d_n^\pi(e)$ for any (W, γ). By Claim (C),

$$\widetilde{b}_n(n, W, \gamma) \geq v^e(N) - \sum_{j\in P_n^\pi \backslash W} u_j\gamma_j\left(P_n^\pi \backslash W\right) - \sum_{j\in W} x_j^0$$

let $z^* \in \Omega^{P_n^\pi}$ be given by $z_j^* = \gamma_j\left(P_n^\pi \backslash W\right)$ for all $j \in P_n^\pi \backslash W$ and $z_j^* = z_j^0$ for all $j \in W$:

$$= v^e(N) - \sum_{j\in P_n^\pi} u_j(z^*) \geq v^e(N) - v^e\left(P_n^\pi\right) = d_n^\pi(e).$$

Proof of Claim (E) for $i = n$: Assume Claim (E) does not hold for $i = n$. Assume first there exists a player $j \in P_n^\pi \backslash W$ such that $\widetilde{b}_j(n, W, \gamma) < u_j\gamma_j\left(P_n^\pi \backslash W\right)$ or a player $j \in W$ such that $\widetilde{b}_j(n, W, \gamma) < x_j^0$. This means that the offer of n is accepted, but this is not possible because any player j in $P_n^\pi \backslash W$ or W, by rejecting, assures himself a payoff of $u_j\gamma_j\left(P_n^\pi \backslash W\right)$ or x_j^0, respectively. This contradiction proves that $\widetilde{b}_j(n, W, \gamma) \geq u_j\gamma_j\left(P_n^\pi \backslash W\right)$ for all $j \in P_n^\pi \backslash W$ and $\widetilde{b}_j(n, W, \gamma) \geq x_j^0$ for all $j \in W$.

Assume now there exists either a player $j_0 \in P_n^\pi \backslash W$ such that $\widetilde{b}_{j_0}(n, W, \gamma) = u_{j_0}\gamma_{j_0}\left(P_n^\pi \backslash W\right) + r$ or a player $j_0 \in W$ such that $\widetilde{b}_{j_0}(n, W, \gamma) = x_{j_0}^0 + r$ with $r > 0$. This means that the proposal of n (say, $\left(\widetilde{W}, \widetilde{\gamma}\right)$) is accepted and thus his final payoff is $\widetilde{b}_n(n, W, \gamma) = u_n\widetilde{\gamma}_n\left(N \backslash \widetilde{W}\right)$. Furthermore

$$\sum_{j\in P_n^\pi} \widetilde{b}_j(n, W, \gamma) \geq \sum_{j\in P_n^\pi \backslash W} u_j\gamma_j\left(P_n^\pi \backslash W\right) + \sum_{j\in W} x_j^0 + r$$

and so

$$u_n\widetilde{\gamma}_n\left(N\backslash\widetilde{W}\right)\leq v^e\left(N\right)-\sum_{j\in P_n^\pi\backslash W}u_j\gamma_j\left(P_n^\pi\backslash W\right)-\sum_{j\in W}x_j^0-r.$$

Let $\varepsilon\in(0,r)$. Assume player n changes his strategy and proposes $\left(\emptyset,\widetilde{\widetilde{\gamma}}\right)$ with $\widetilde{\widetilde{\gamma}}(S)=0$ for all $S\subsetneq N$ and $\widetilde{\widetilde{\gamma}}(N)=\left(x,y^M\right)$ such that:

$$\begin{aligned}
x_j &= u_j\gamma_j\left(P_n^\pi\backslash W\right)+\frac{\varepsilon}{n-1}-u_j'\left(y_j^M\right) && \text{for all } j\in P_n^\pi\backslash W\\
x_j &= x_j^0+\frac{\varepsilon}{n-1}-u_j'\left(y_j^M\right) && \text{for all } j\in W\\
x_n &= v^e\left(N\right)-\sum_{j\in P_n^\pi\backslash W}u_j\gamma_j\left(P_n^\pi\backslash W\right)-\sum_{j\in W}x_j^0-\varepsilon-u_n'\left(y_n^M\right)
\end{aligned}$$

By the same arguments used in the proof of Claim (C) for $i=n$, it is not difficult to check that $\widetilde{\widetilde{\gamma}}(N)$ is feasible. Furthermore, $\widetilde{\widetilde{\gamma}}\in\Gamma^\pi$. Moreover, by Claim (B) this proposal is accepted by players in P_n^π and thus the final payoff for player n is $u_n\widetilde{\widetilde{\gamma}}_n(N)>u_n\widetilde{\gamma}_n\left(N\backslash\widetilde{W}\right)$. This contradiction proves Claim (E) for $i=n$

Proof of Claim (F) for $i=n$: Assume player n rejects (W,γ):

$$\widetilde{b}_n\left(n,W,\gamma\right)=\sum_{j\in N}\widetilde{b}_j\left(n,W,\gamma\right)-\sum_{j\in P_n^\pi}\widetilde{b}_j\left(n,W,\gamma\right)$$

by Claim (E) applied to $i=n$:

$$\begin{aligned}
&= \sum_{j\in N}\widetilde{b}_j\left(n,W,\gamma\right)-\sum_{j\in P_n^\pi\backslash W}u_j\gamma_j\left(P_n^\pi\backslash W\right)-\sum_{j\in W}x_j^0\\
&\leq v\left(N\right)-\sum_{j\in P_n^\pi\backslash W}u_j\gamma_j\left(P_n^\pi\backslash W\right)-\sum_{j\in W}x_j^0.
\end{aligned}$$

By Claim (C), equality holds and thus Claim (F) holds for $i=n$.

Proof of Claim (G) for $i=n$: Assume

$$u_n\gamma_n\left(N\right)>v^e\left(N\right)-\sum_{j\in P_n^\pi\backslash W}u_j\gamma_j\left(P_n^\pi\backslash W\right)-\sum_{j\in W}x_j^0.$$

Then, it is optimal for n to accept γ and obtain $\gamma_n(N)$, since by Claim (F) his maximal payoff after rejection is not more that this. Thus, Claim (G) holds for $i=n$.

Assume now Claims (A), (B), (C), (D), (E), (F) and (G) hold for $j>i$

Proof of Claim (A): Clearly, any player $j \in P_i^\pi \backslash W$ can assure himself a payoff of $u_j \gamma_j (P_i^\pi \backslash W)$ by rejecting any new proposal. Notice that, since $\gamma \in \Gamma^\pi$, the final payoff for player j is not affected by new players joining the society by agreeing on (W, γ). Similarly, any player $j \in W \cup \{i\}$ can assure himself a payoff of x_j^0. Hence

$$\left.\begin{array}{ll} b_j (i+1, W \cup \{i\}, \gamma) \geq u_j \gamma_j (P_i^\pi \backslash W) & \text{for all } j \in P_i^\pi \backslash W \\ b_j (i+1, W \cup \{i\}, \gamma) \geq x_j^0 & \text{for all } j \in W \cup \{i\}. \end{array}\right\} \quad (7)$$

Furthermore, by induction hypothesis applied to Claim (D), we know that

$$b_j (i+1, W \cup \{i\}, \gamma) \geq d_j^\pi (e)$$

for all $j > i+1$; and, by induction hypothesis applied to Claim (C)

$$b_{i+1} (i+1, W \cup \{i\}, \gamma) \geq v^e \left(P_{i+2}^\pi\right) - \sum_{j \in P_i^\pi \backslash W} u_j \gamma_j (P_i^\pi \backslash W) - \sum_{j \in W \cup \{i\}} x_j^0.$$

Thus

$$\begin{aligned} & \sum_{j \in P_{i+1}^\pi} b_j (i+1, W \cup \{i\}, \gamma) \\ \leq \quad & v^e (N) - b_{i+1} (i+1, W \cup \{i\}, \gamma) - \sum_{j > i+1} b_j (i+1, W \cup \{i\}, \gamma) \\ \leq \quad & v^e (N) - v^e \left(P_{i+2}^\pi\right) + \sum_{j \in P_i^\pi \backslash W} u_j \gamma_j (P_i^\pi \backslash W) + \sum_{j \in W \cup \{i\}} x_j^0 - \sum_{j > i+1} d_j^\pi (e) \\ = \quad & \sum_{j \in P_i^\pi \backslash W} u_j \gamma_j (P_i^\pi \backslash W) + \sum_{j \in W \cup \{i\}} x_j^0. \end{aligned}$$

Thus, equalities hold in (7).

Proof of Claim (B): If the proposal is rejected, any player $j \in P_i^\pi \backslash \widetilde{W}$ receives $b_j (i+1, W \cup \{i\}, \gamma)$. If the proposal is accepted, any player $j \in P_i^\pi \backslash \widetilde{W}$ receives $b_j \left(i+1, \widetilde{W}, \widetilde{\gamma}\right)$. Thus, the result is straightforward.

Proof of Claim (C): Assume Claim (C) does not hold. Namely, there exists a $r > 0$ such that

$$\widetilde{b}_i (i, W, \gamma) = v^e \left(P_{i+1}^\pi\right) - \sum_{j \in P_i^\pi \backslash W} u_j \gamma_j (P_i^\pi \backslash W) - \sum_{j \in W} x_j^0 - r.$$

Since player i can easily assure himself a payoff of x_i^0, we deduce that

$$v^e \left(P_{i+1}^\pi\right) - \sum_{j \in P_i^\pi \backslash W} u_j \gamma_j (P_i^\pi \backslash W) - \sum_{j \in W} x_j^0 - r \geq x_i^0. \quad (8)$$

Let $\varepsilon \in (0, r)$ and assume player i changes his strategy so that he disagrees on (W, γ) and proposes $(\emptyset, \widetilde{\gamma})$ such that

$$\begin{aligned} &\widetilde{\gamma}(P_k^\pi) = \arg\max \left\{ \sum_{j \in P_k^\pi} u_j(z) : z \in \Omega^{P_k^\pi} \right\} && \text{for all } k > i \\ &\widetilde{\gamma}(S) = \widetilde{\gamma}(S^\pi) && \text{for all } S \subsetneq N. \end{aligned}$$

Notice that $\sum_{j \in P_k^\pi} u_j \gamma_j(P_k^\pi) = v^e(P_k^\pi)$ for all $k > i$. Finally, $\widetilde{\gamma}(N) = \left(x, y^M\right)$ with x given by

$$\begin{aligned} x_j &= b_j(i+1, W \cup \{i\}, \gamma) + \frac{\varepsilon}{(n-1)} - u_j'\left(y_j^M\right) && \text{for all } j \in P_i^\pi \\ x_i &= v^e\left(P_{i+1}^\pi\right) - \sum_{j \in P_i^\pi} b_j(i+1, W \cup \{i\}, \gamma) - \varepsilon - u_i'\left(y_i^M\right) \\ x_j &= d_j^\pi(e) + \frac{\varepsilon}{(n-1)} - u_j'\left(y_j^M\right) && \text{for all } j > i \end{aligned}$$

This $\widetilde{\gamma}(N)$ is feasible:

$$\sum_{j \in N} x_j = v^e\left(P_{i+1}^\pi\right) - \sum_{j \in N} u_j'\left(y_j^M\right) + \sum_{j > i} d_j^\pi(e) = v^e(N) - \sum_{j \in N} u_j'\left(y_j^M\right) = \sum_{j \in N} x_j^0.$$

Let $j \in N_f$. If $j \neq i$, by arguments similar to those used in the proof of (2), we can prove that $x_j \geq 0$ for all $j \in N_f$. Assume $i \in N_f$

$$\begin{aligned} x_i &= v^e\left(P_{i+1}^\pi\right) - \sum_{j \in P_i^\pi} b_j(i+1, W \cup \{i\}, \gamma) - \varepsilon - u_i'\left(y_i^M\right) \\ &\geq v^e\left(P_{i+1}^\pi\right) - v^e(N) + \sum_{j \geq i} b_j(i+1, W \cup \{i\}, \gamma) - \varepsilon - u_i'\left(y_i^M\right) \\ &\geq v^e\left(P_{i+1}^\pi\right) - v^e(N) + x_i^0 + \sum_{j > i} b_j(i+1, W \cup \{i\}, \gamma) - \varepsilon - u_i'\left(y_i^M\right). \end{aligned}$$

By induction hypothesis applied to Claim (C) and Claim (D)

$$x_i \geq v^e\left(P_{i+1}^\pi\right) - \sum_{j \in P_i^\pi \setminus W} u_j \gamma_j(P_i^\pi \setminus W) - \sum_{j \in W} x_j^0 - \varepsilon - u_n'\left(y_i^M\right). \tag{9}$$

Since $y_i^M = 0$ and by (8)

$$x_i \geq x_i^0 + r - \varepsilon - u_n'(0) > 0.$$

Hence, $\widetilde{\gamma}(N)$ is feasible. Furthermore, $\widetilde{\gamma} \in \Gamma^\pi$.

If players in P_i^π accept $(\emptyset, \widetilde{\gamma})$, we check that condition (5) of Claim (G) with $\gamma = \widetilde{\gamma}$ is satisfied for $i+1, ..., n$. Given $(\emptyset, \widetilde{\gamma})$ and $k > i$

$$u_k \widetilde{\gamma}_k(N) > d_k^\pi(e) = v^e\left(P_{k+1}^\pi\right) - v^e(P_k^\pi) = v^e\left(P_{k+1}^\pi\right) - \sum_{j \in P_k^\pi} u_j \widetilde{\gamma}_j(P_k)$$

Hence, any player in $\{i+1,...,n\}$ is bound to agree on $(\emptyset,\widetilde{\gamma})$ once the turn reaches him. We conclude then that $b_j(i+1,\emptyset,\widetilde{\gamma}) = u_j\widetilde{\gamma}_j(N)$ for all $j \in N$.

We check that the condition of Claim (B) hold. Given $j \in P_i^\pi$,

$$b_j(i+1,\emptyset,\widetilde{\gamma}) = u_j\widetilde{\gamma}_j(N) > b_j(i+1, W \cup \{i\}, \gamma).$$

Thus, the proposal is accepted and the final payoff of player i is

$$b_i(i+1,\emptyset,\widetilde{\gamma}) = u_i\widetilde{\gamma}_i(N) = x_i + u_i'\left(y_i^M\right)$$

by (9)

$$\geq v^e\left(P_{i+1}^\pi\right) - \sum_{j\in P_i^\pi\backslash W} u_j\gamma_j\left(P_i^\pi\backslash W\right) - \sum_{j\in W} x_j^0 - \varepsilon$$

which is bigger than $\widetilde{b}_i(i,W,\gamma)$. This contradiction proves Claim (C) for $i=n$.

Proof of Claim (D): Given (W,γ), it is enough to prove that $\widetilde{b}_i(i,W,\gamma) \geq d_i^\pi(e)$. By Claim (C),

$$b_i(i,W,\gamma) \geq v^e\left(P_{i+1}^\pi\right) - \sum_{j\in P_i^\pi\backslash W} u_j\gamma_j\left(P_i^\pi\backslash W\right) - \sum_{j\in W} x_j^0$$

let $z^* \in \Omega^{P_i^\pi}$ be given by $z_j^* = \gamma_j\left(P_i^\pi\backslash W\right)$ for all $j \in P_i^\pi\backslash W$ and $z_j^* = z_j^0$ for all $j \in W$:

$$= v^e\left(P_{i+1}^\pi\right) - \sum_{j\in P_i^\pi} u_j(z^*) \geq v^e\left(P_{i+1}^\pi\right) - v^e\left(P_i^\pi\right) = d_i^\pi(e).$$

Proof of Claim (E): Assume we are in $\widetilde{M}(e,\pi,i,W,\gamma)$ with $\gamma \in \Gamma^\pi$. By Claim (A), any player $j \in P_i^\pi\backslash W$, by voting 'no', can assure himself a payoff of $u_j\gamma_j\left(P_i^\pi\backslash W\right)$. Similarly, any player $j \in W$ can assure himself a payoff of x_j^0 by voting 'no'. Hence

$$\left.\begin{array}{ll}\widetilde{b}_j(i,W,\gamma) \geq u_j\gamma_j\left(P_i^\pi\backslash W\right) & \text{for all } j \in P_i^\pi\backslash W \\ \widetilde{b}_j(i,W,\gamma) \geq x_j^0 & \text{for all } j \in W.\end{array}\right\}$$

Furthermore, by Claim (C)

$$\widetilde{b}_i(i,W,\gamma) \geq v^e\left(P_{i+1}^\pi\right) - \sum_{j\in P_i^\pi\backslash W} u_j\gamma_j\left(P_i^\pi\backslash W\right) - \sum_{j\in W} x_j^0$$

and, by Claim (D)

$$\widetilde{b}_j(i,W,\gamma) \geq d_j^\pi(e)$$

for all $j > i$. Hence

$$\begin{aligned}
\sum_{j\in P_i^\pi}\widetilde{b}_i(i,W,\gamma) &= \sum_{j\in N}\widetilde{b}_i(i,W,\gamma)-\widetilde{b}_i(i,W,\gamma)-\sum_{j>i}\widetilde{b}_j(i,W,\gamma)\\
&\leq v^e(N)-v^e\left(P_{i+1}^\pi\right)+\sum_{j\in P_i^\pi\setminus W}u_j\gamma_j\left(P_i^\pi\setminus W\right)+\sum_{j\in W}x_j^0-\sum_{j>i}d_j^\pi(e)\\
&= \sum_{j\in P_i^\pi\setminus W}u_j\gamma_j\left(P_i^\pi\setminus W\right)+\sum_{j\in W}x_j^0.
\end{aligned}$$

Thus, Claim (E) holds.

Proof of Claim (F): Assume we are in $\widetilde{M}(e,\pi,i,W,\gamma)$ with $\gamma\in\Gamma^\pi$. By Claim (E)

$$\left.\begin{array}{ll}
\widetilde{b}_j(i,W,\gamma)=u_j\gamma_j\left(P_i^\pi\setminus W\right) & \text{for all } j\in P_i^\pi\setminus W\\
\widetilde{b}_j(i,W,\gamma)=x_j^0 & \text{for all } j\in W.
\end{array}\right\}$$

Furthermore, by Claim (C)

$$\widetilde{b}_i(i,W,\gamma)\geq v^e\left(P_{i+1}^\pi\right)-\sum_{j\in P_i^\pi\setminus W}u_j\gamma_j\left(P_i^\pi\setminus W\right)-\sum_{j\in W}x_j^0$$

and, by Claim (D)

$$\widetilde{b}_j(i,W,\gamma)\geq d_j^\pi(e)$$

for all $j > i$. Hence

$$\begin{aligned}
\widetilde{b}_i(i,W,\gamma) &\leq v^e(N)-\sum_{j\in N\setminus\{i\}}\widetilde{b}_j(i,W,\gamma)\\
&\leq v^e(N)-\sum_{j\in P_i^\pi\setminus W}u_j\gamma_j\left(P_i^\pi\setminus W\right)-\sum_{j\in W}x_j^0-\sum_{j>i}d_j^\pi(e)\\
&= v^e\left(P_{i+1}^\pi\right)-\sum_{j\in P_i^\pi\setminus W}u_j\gamma_j\left(P_i^\pi\setminus W\right)-\sum_{j\in W}x_j^0.
\end{aligned}$$

By Claim (C), equality holds.

Proof of Claim (G): Assume (5) holds for all $k\geq i$. If player i agrees on (W,γ), by induction hypothesis applied to Claim (G), players $i+1,...,n$ all agree on (W,γ) and the final payoff is $u_j\gamma_j(N)$ for all $j\in N$. Then, it is optimal for player i to accept (W,γ) and obtain $u_i\gamma_i(N)$, since by Claim (F) his maximal payoff after rejection is not more that this. Thus, Claim (G) holds.

We now prove that the final expected payoff for player $i\in N$ is $d_i^\pi(e)$.

By Claim (D), we know that $b_i(\pi) \geq d_i^{\pi}(e)$ for all $i \in N$. This means that $\sum\limits_{i \in N} b_i(\pi) \geq v^e(N)$. Since $b \in V(N)$, we have $\sum\limits_{i \in N} b_i = v^e(N)$ and thus $b_i(\pi) = d_i^{\pi}(e)$ for all $i \in N$.

We have proved that $b_i(\pi) = d_i^{\pi}(e)$ for all $i \in N$ and all $\pi \in \Pi$. Thus

$$z_i = \frac{1}{|\Pi|} \sum_{\pi \in \Pi} d_i^{\pi}(e) = Sh_i(e).$$

■

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