On the Existence of Pure Strategy Monotone Equilibria in Asymmetric First-Price Auctions*

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October 2002

Abstract

We demonstrate the existence of pure strategy equilibria in monotone bidding functions in first-price auctions with asymmetric bidders, interdependent values and affiliated one-dimensional signals. Our proof sidesteps the precisely two ways that single-crossing can fail, which we identify here. We also provide a private value example suggesting that the assumption of one-dimensional signals is essential.

*Reny gratefully acknowledges financial support from the National Science Foundation (SES-9905599).

1. Introduction

There is by now a large literature on first-price auctions. While initial efforts centered around the symmetric bidder case (e.g., Milgrom and Weber (1982)), attention has begun to shift toward the even more challenging—and in practice often very relevant—case of asymmetric bidders. A key difference between the two cases is that only the symmetric bidder setting admits closed-form expressions for equilibrium bid functions. Because of this, analysis of equilibrium bidding behavior in asymmetric first-price auctions requires an implicit characterization of equilibrium through first-order necessary conditions for optimal bidding.[1] But if an equilibrium fails to exist, such an analysis is vacuous.

Our objective here is to provide conditions ensuring the existence of a pure strategy equilibrium in nondecreasing bid functions for asymmetric first-price auctions with affiliated private information and interdependent values. As a by-product, we therefore provide a foundation for the first-order approach to analyzing equilibrium bidding behavior in such auctions.

Recent work on the question of equilibrium existence in first-price auctions can be found in Athey (2001), Bresky (1999), Lebrun (1996, 1999), Lizzeri and Persico (2000), Jackson and Swinkels (2001), Maskin and Riley (2000), and Reny (1999).[2] But there appears to be a common difficulty. The above papers restrict attention either to two bidders, symmetric bidders, independent signals, private values, or common values. That is, the most general case involving three or more asymmetric bidders with affiliated signals and interdependent values is not covered. The reason for this is that standard proof techniques rely on the following *single-crossing condition* (SCC) exploited with great ingenuity in Athey (2001):

If the others employ nondecreasing bid functions and one's payoff from a high bid is no smaller than that from a lower bid, then the high bid remains as good as the lower one when one's signal rises.[3]

However, even when bidders' signals are affiliated, SCC can fail (see Section 3) unless there is but a single other bidder (as in the two-bidder case), or all signals

[1] See, for example, Bajari (1997).

[2] For conditions ensuring *uniqueness* in two-bidder settings, see Lizzeri and Persico (2000), Maskin and Riley (1996) and Rodriguez (2000). Under more restrictive conditions, Maskin and Riley (1996) obtain some uniqueness results for more than two bidders. See also Bajari (1997) and Lebrun (1999).

[3] This is only "half" of the condition. The other half is obtained by reversing the roles of "high(er)" and "low(er)," and replacing "rises" with "falls." See Section 2 for a formal definition.

are symmetric and bidders employ the same bidding function (as in the symmetric case), or signals are independent, or values are either purely private or purely common. It is for this reason that a general result is not yet at hand.

Our main insight is that there are only two ways that SCC can fail for a bidder. The first is when both of the bids the bidder employs are individually irrational, yielding a negative expected payoff. The second is when one of the bidder's bids ties one of the opponents' bids with positive probability. That is, we show that subject to no ties and individual rationality, SCC holds. More precisely, we establish the following *individually rational tieless single-crossing condition* (IRT-SCC) for first-price auctions:

If the others employ nondecreasing bid functions and one's payoff from a high bid is non-negative and no smaller than that from a lower bid, then the high bid remains as good as the lower one when one's signal rises so long as neither bid ties a positive bid of any opponent with positive probability.[4]

Now, standard proofs that monotone pure equilibria exist in first-price auctions begin by restricting bidders to finite grids of bids. This renders their otherwise discontinuous payoffs, continuous, and so their best reply correspondences are rendered nonempty-valued and upper hemicontinuous. These proofs then establish single-crossing, which, as Athey (2001) demonstrates, suffices for the existence of a pure monotone equilibrium in the finite bid setting. One then takes the limit of such equilibria as the finite grid of bids becomes dense in $\mathbb{R}_+$ to obtain a pure monotone equilibrium of the auction with unrestricted bids.

We too shall follow this standard line of proof, but we must tread somewhat more carefully to avoid the two failures of SCC. We begin by restricting bidders to finite grids of bids with the property that the zero bid is the only bid common to distinct bidders. Consequently, in our restricted auction game, by construction, no ties can occur at positive bids. This avoids one of the ways that SCC can fail.

The remaining failure of SCC, occurring when bidders employ individually irrational bids, does not in fact pose any difficulty. This is because standard proof techniques employ SCC only to show that, when the others use nondecreasing bidding functions, a bidder's *best reply* correspondence, as a function of his signal, is (in an appropriate sense) nondecreasing. But because best replies are a fortiori individually rational, SCC need then only hold when a bidder employs individually

[4]Once again, this only half of the condition and the other half is obtained as before. See Section 2 for a formal definition.

rational bids. Hence, the more permissive condition IRT-SCC suffices to establish the required monotonicity of best reply correspondences, given our choice of the finite bid sets. Thus, the novel part of our proof centers around the demonstration that IRT-SCC holds in quite general first-price auction environments. With this result in hand, the existence of a pure monotone equilibrium in our auction with carefully chosen finite bid sets follows from arguments due to Athey (2001).

The final step in our proof is again standard. We consider a sequence of monotone pure equilibria of the restricted auction games as the finite grids of bids become dense in $\mathbb{R}_+$ and show that any limit point of this sequence is a pure monotone equilibrium of the first-price auction with unrestricted bids.

The remainder of the paper is organized as follows. Section 2 describes the class of first-price auctions covered here, provides the assumptions we maintain throughout, and contains our main result. This section also provides a discussion of Athey's (2001) single crossing condition (SCC) and introduces our individually rational tieless single crossing condition (IRT-SCC). Section 3 provides examples of the two ways Athey's (2001) single crossing condition can fail. Section 4 provides a sketch of the proof of IRT-SCC. Section 5 provides a private value example suggesting that one-dimensionality of the bidders' signals is essential for the existence of monotone pure strategy equilibria in the class of first-price auctions studied here. All proofs are contained in the appendix.

2. The Model and Main Result

Consider the following first-price auction game. There is a single object for sale and $N \geq 2$ bidders. Each bidder i receives a private signal $s_i \in [0,1]$. The joint density of the bidders' signals is $f : [0,1]^N \to \mathbb{R}_+$. After receiving their signals, each bidder i submits a nonnegative sealed bid. The highest bid greater or equal to the public reserve price $r \geq 0$ wins the object, with ties broken randomly and uniformly.

If the vector of signals is $s = (s_1, ..., s_N)$ and bidder i wins the object with a bid of b_i, then bidder i's payoff is given by $u_i(b_i, s)$. All other bidders receive a payoff of zero. This specification allows for a variety of attitudes toward risk, as well as a variety of payment rules.

We shall maintain the following assumptions. For all bidders $i = 1, ..., N$:

A.1 (i) $u_i : \mathbb{R}_+ \times [0,1]^N \to \mathbb{R}$ is measurable, $u_i(b_i, s)$ is bounded in $s \in [0,1]^N$ for each $b_i \in \mathbb{R}_+$ and continuous in b_i for each s.

(ii) There exists $\tilde{b} \geq 0$ such that $u_i(b_i, s) < 0$ for all $b_i > \tilde{b}$ and all $s \in [0,1]^N$.

(iii) $u_i(0, s) \geq 0$ for all $s \in [0,1]^N$

(iv) For every $b_i \geq 0$, $u_i(b_i, s)$ is nondecreasing in s_{-i} and strictly increasing in s_i.

(v) $u_i(\bar{b}_i, s) - u_i(\underline{b}_i, s)$ is nondecreasing in s whenever $\bar{b}_i > \underline{b}_i$.

A.2 (i) $f(s)$ is measurable and strictly positive on $[0,1]^N$.

(ii) $f(s \vee s')f(s \wedge s') \geq f(s)f(s')$ for all $s, s' \in [0,1]^N$, where $\vee$ and $\wedge$ denote componentwise maximum and minimum, respectively.

Remark 1. *It is not necessary that $u_i(b_i, s)$ decrease in b_i, only that it is eventually negative for large enough b_i. Thus, while we require the winner to be the highest bidder, we do not require the winner to pay his bid, nor even an amount that is an increasing function of his bid.*[5] *It is important, however, that the winner's payment depend only upon his own bid.*

Remark 2. *Note that A.1(v) is satisfied automatically when bidder i is risk neutral and the winner must pay his bid because in this case $u_i(b_i, s) = w_i(s) - b_i$ and so the difference expressed in A.1(v) is constant in s. More generally, if $u_i(b_i, s) = U_i(w_i(s) - b_i)$, then A.1(v) holds when $w_i(s)$ is nondecreasing in s and $U_i'' \leq 0$ (i.e. bidder i is risk averse).*

Remark 3. *Assumption A.2 (i) rules out moving supports, and A.2(ii) requires the bidders' signals to be affiliated (see MW).*

Given a vector of bids $b = (b_1, ..., b_N)$, let $v_i(b, s)$ denote bidders i's expected payoff when the vector of signals is s. That is

$$v_i(b, s) = \begin{cases} \frac{1}{m} u_i(b_i, s), & \text{if } m = \#\{j : b_j = b_i = \max_k b_k \geq r\} \geq 1 \\ 0, & \text{otherwise.} \end{cases}$$

[5] So, for example, we are able to cover settings in which the auction rules favor some bidders by allowing them to pay only a fraction of their bid should they win.

Note that this specification implies that a lone bid equal to the reserve price is a winning bid.

Throughout, upper case letters will denote random variables and lower case letters will denote their realizations. A pure strategy for bidder i is a measurable (bid) function $\mathbf{b}_i : [0,1] \to \mathbb{R}_+$. Given a vector of pure strategies $\mathbf{b} = (\mathbf{b}_1, ..., \mathbf{b}_N)$, let $V_i(\mathbf{b})$ denote bidder i's (ex-ante) expected payoff in the auction. That is,

$$V_i(\mathbf{b}) = E[v_i(\mathbf{b}(S), S)],$$

where $\mathbf{b}(S)$ denotes the random vector $(\mathbf{b}_1(S_1), ..., \mathbf{b}_N(S_N))$ and the expectation is taken with respect to f. It will also be convenient to define bidder i's interim payoff. Accordingly, let $V_i(b_i, \mathbf{b}_{-i} \mid s_i)$ denote bidder i's expected payoff conditional on his signal s_i and given that he bids b_i and the others employ the strategies $\mathbf{b}_{-i}$. That is,[6]

$$V_i(b_i, \mathbf{b}_{-i} \mid s_i) = E[v_i(b_i, \mathbf{b}_{-i}(S_{-i}), s_i, S_{-i}) \mid S_i = s_i].$$

A pure strategy equilibrium is an N-tuple of pure strategies $\mathbf{b}^* = (\mathbf{b}_1^*, ..., \mathbf{b}_N^*)$ such that for all bidders i, $V_i(\mathbf{b}^*) \geq V_i(\mathbf{b}'_i, \mathbf{b}^*_{-i})$ for all pure strategies $\mathbf{b}'_i$.

Our interest lies in establishing, for any first-price auction game, the existence of a pure strategy equilibrium in which each bidder's bid function is nondecreasing in his signal. We shall refer to this as a *monotone pure strategy equilibrium.* This brings us to our main result.

Theorem 2.1. *All first-price auction games satisfying assumptions A.1 and A.2 possess a monotone pure strategy equilibrium.*

The proof of Theorem 2.1 is in the appendix and consists of two main steps. The first step establishes that a monotone equilibrium exists when bidders are restricted to finite sets of bids with no positive bids in common, while the second step shows that the limit of such equilibria, as the sets of permissible bids become dense in $\mathbb{R}_+$, is an equilibrium when, for all bidders, any nonnegative bid is feasible.

The novelty of our approach lies in the first step, where standard techniques have up to now failed. For example, it would be straightforward to establish the existence of a monotone equilibrium with finite bid sets if one could establish the single-crossing condition employed in Athey (2001). One could then simply appeal directly to Athey's Theorem 1.

[6] All statements involving conditional probabilities are made with respect to the following version of the conditional density: $f(s_{-i}|s_i) = f(s)/\int_{[0,1]^{N-1}} f(s_i, s_{-i}) ds_{-i}$, which, by A.2(i), is well defined.

In the context of our first-price auction, Athey's (2001) single-crossing condition is as follows. For any bidder i, any bids b_i and b'_i, and any nondecreasing bid functions $\mathbf{b}_j$ for all bidders $j \neq i$, the following must hold:

SCC. If $V_i(b'_i, \mathbf{b}_{-i} \,|\, s_i) \geq V_i(b_i, \mathbf{b}_{-i} \,|\, s_i)$ then this inequality is maintained when s_i rises if $b'_i > b_i$, while it is maintained when s_i falls if $b'_i < b_i$.

Unfortunately, Athey's (2001) result cannot be applied because, for arbitrary finite or infinite bid sets, SCC can fail in two ways (see Section 3). First, SCC can fail when there are ties at positive bids and this is why we must approximate the bidders' common continuum bid set $\mathbb{R}_+$ with finite bid sets whose only common bid is zero.

Second, SCC can fail if a bidder employs an individually irrational bid. But this failure of SCC does not pose a problem because Athey's (2001) techniques nonetheless apply. To see this, recall that Athey employs SCC only to establish that when the others use monotone strategies, a bidder's best reply correspondence, as a function of his signal, is increasing in the strong set order.[7,8] However, being a property of *best replies*, the required monotonicity can in fact be established precisely as in Athey (2001) *so long as SCC holds for bids that are best replies*. That is, the failure of SCC for individually irrational bids is immaterial for establishing strong set order monotonicity of the best reply correspondence.

Consequently, the existence of a monotone pure strategy equilibrium can be established in a first-price auction with our particular finite bid set approximation if SCC can be established whenever ties at positive bids are absent and bids are best replies.

We in fact establish a stronger form of single-crossing, which, a fortiori, suffices for our purposes. The following *individually rational tieless single-crossing condition* (IRT-SCC) requires that, in addition to the absence of ties, one of the two relevant bids be individually rational (neither bid is required to be a best reply).

[7] That is, if a high bid is best at a low signal and a low bid is best at a high signal, then both bids are best at both signals. (Milgrom and Shannon (1994) introduced the strong set order into the economics literature and, using it, established a number of important comparative statics results.)

[8] Athey's (2001) convexity results then apply and existence follows, as Athey shows, from Kakutani's theorem.

Definition 2.2. *A first-price auction satisfies IRT-SCC if for each bidder i, and all pairs of bids $b_i, b_i' \in \mathbb{R}_+$, the following condition is satisfied for all nondecreasing bid functions $\mathbf{b}_j : [0,1] \to \mathbb{R}_+$ of the other bidders such that $\Pr(\mathbf{b}_j > 0$, and $\mathbf{b}_j = b_i$ or $b_i') = 0$ for all $j \neq i$.*

IRT-SCC. *Suppose $V_i(b_i', \mathbf{b}_{-i} \mid s_i) \geq 0$. If $V_i(b_i', \mathbf{b}_{-i} \mid s_i) \geq V_i(b_i, \mathbf{b}_{-i} \mid s_i)$, then this inequality is maintained when s_i rises if $b_i' > b_i$, while it is maintained when s_i falls if $b_i' < b_i$.*

The main contribution leading to the proof of Theorem 2.1 is the following proposition. Its proof can be found in the appendix.

Proposition 2.3. *Under assumptions A.1 and A.2, IRT-SCC holds.*

Remark 4. It can in fact be shown that, given individual rationality, the single-crossing inequality, IRT-SCC, holds even if ties occur at the *higher* of the two bids b_i' and b_i. It is only ties at the *lower* of the two bids that cause single-crossing to fail. But we shall not pursue this further here.

We next illustrate the two ways that SCC can fail.

3. The Two Failures of Single-Crossing

In each of the two examples below, there are three bidders and the joint distribution of their signals is as follows.

Bidders $i = 1, 2$ have signals, s_i, that are i.i.d. uniform on $[0,1]$. These are drawn first. Bidder 3's signal, s_3, is drawn from $[0,1]$ conditional on 1's signal according to the density

$$g(s_3|s_1) = \begin{cases} 1, & \text{if } s_1 \leq 1/2 \\ 2/3, & \text{if } s_1 > 1/2 \text{ and } s_3 \leq 1/2 \\ 4/3 & \text{if } s_1 > 1/2 \text{ and } s_3 > 1/2 \end{cases}$$

Thus, 3's signal is uniform on $[0,1]$ if $s_1 \leq 1/2$. If $s_1 > 1/2$, then 3's signal is twice as likely to be above $1/2$ as below $1/2$, but is otherwise uniformly distributed on each of the two halves of the interval $[0,1]$. So defined, the bidders' signals are affiliated.

The examples will be constructed so that SCC fails for bidder 1. Consequently, bidders 2 and 3 can, for example, be given private values. In each example, Bidder 1's utility will take the quasilinear form $u_1(b,s) = w_1(s) - b$, where, for $v_0 \leq v_1 \leq v_2 \leq v_3$,

$$w_1(s_1,s_2,s_3) = \begin{cases} v_3, & \text{if } (s_2,s_3) \in [\frac{1}{2},1] \times [\frac{1}{2},1] \\ v_2, & \text{if } (s_2,s_3) \in [\frac{1}{2},1] \times [0,\frac{1}{2}) \\ v_1, & \text{if } (s_2,s_3) \in [0,\frac{1}{2}) \times [\frac{1}{2},1] \\ v_0, & \text{if } (s_2,s_3) \in [0,\frac{1}{2}) \times [0,\frac{1}{2}) \end{cases}$$

Figure 3.1 illustrates both the distribution of the others' signals conditional on 1's signal, and bidder 1's value for the good, $w_1(s_1,s_2,s_3)$, as a function the other bidders' signals.[9] In each panel, the numbers in square brackets are the probabilities of each of the four regions conditional on 1's signal. In panel (a) 1's signal is low (i.e., 1/2 or less), while in panel (b) it is high (i.e., above 1/2). In both panels, the joint density of s_2 and s_3 is uniform within each region.

The two failures of SCC result from two distinct specifications of the values v_0, v_1, v_2, and v_3.

3.1. The First Failure: Individually Irrational Bids

Consider the following values.

$$v_0 = 1, \quad v_1 = 2, \quad v_2 = 8, \quad v_3 = 9$$

Suppose also that bidders 2 and 3 each employ a strictly increasing bidding function that specifies a bid of 5 at the signal 1/2 and a bid of 6 at the signal 1. Now consider two signals, $\underline{s}_1$ and $\bar{s}_1$, for bidder 1 such that $\underline{s}_1 < 1/2 < \bar{s}_1$. Given the bid functions of bidders 2 and 3, bidder 1 is indifferent between bidding 5 and 6 when his signal is $\underline{s}_1$, but that bidding 5 is strictly better than bidding 6 when 1's signal increases to $\bar{s}_1$ (see below). This of course violates SCC.

[9] Note that $w_1(s)$ is nondecreasing in s, but, contrary to A.1, it is independent of s_1. This is for simplicity only. Adding εs_1 to $w_1(s)$, for $\varepsilon > 0$ small enough, renders 1's utility strictly increasing in his signal and maintains the failure of SCC in both examples.

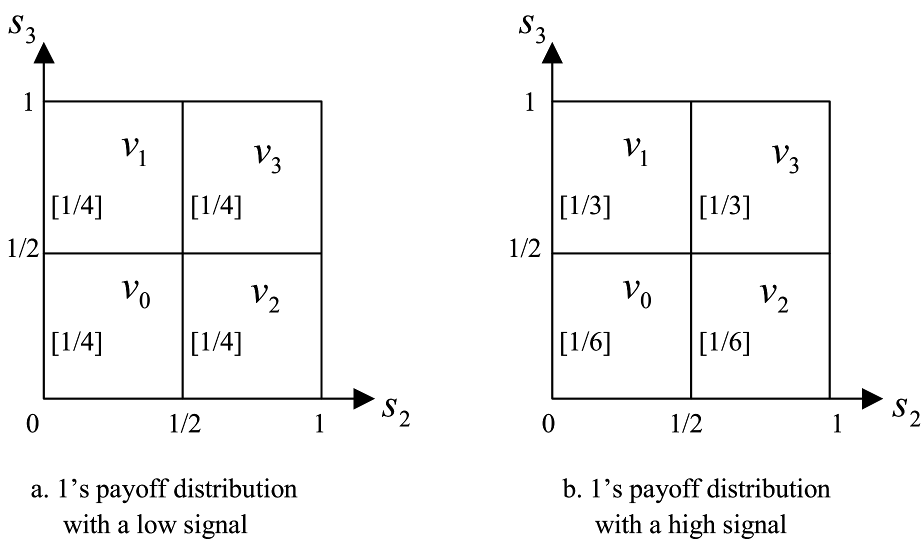

Figure 3.1: 1's Payoff Distribution

Furthermore, both bids, 5 and 6, are individually irrational for bidder 1 whether his signal is high or low. As we have already indicated, and as our proof establishes, without ties in bids this is the only way that single-crossing can fail. The requisite calculations follow.

$$
\begin{aligned}
V_1(b_1 = 6, \mathbf{b}_2, \mathbf{b}_3 | \underline{s}_1) &= \tfrac{1}{4}(9 + 8 + 2 + 1) - 6 &&= -1 \\
V_1(b_1 = 5, \mathbf{b}_2, \mathbf{b}_3 | \underline{s}_1) &= \tfrac{1}{4}(1 - 5) &&= -1 \\
V_1(b_1 = 6, \mathbf{b}_2, \mathbf{b}_3 | \bar{s}_1) &= (\tfrac{1}{3}9 + \tfrac{1}{6}8 + \tfrac{1}{3}2 + \tfrac{1}{6}1) - 6 &&= -\tfrac{5}{6} \\
V_1(b_1 = 5, \mathbf{b}_2, \mathbf{b}_3 | \bar{s}_1) &= \tfrac{1}{6}(1 - 5) &&= -\tfrac{2}{3}
\end{aligned}
$$

3.2. The Second Failure: Ties at Positive Bids

Consider now the following values.

$$v_0 = 0, \quad v_1 = 0, \quad v_2 = 336, \quad v_3 = 336$$

Suppose this time that bidders 2 and 3 bid zero when their signal is 1/2 or lower and bid 120 when their signal is above 1/2. Consequently, bidders 2 and 3 bid zero and 120 with positive probability each.

Consider again two signals, $\underline{s}_1$ and $\bar{s}_1$, for bidder 1 such that $\underline{s}_1 < 1/2 < \bar{s}_1$. Direct calculations now establish that, given the bidding functions of bidders 2 and 3, bidder 1's unique best reply among the bids $\{0, 120, 167\}$ is 167 when his signal is $\underline{s}_1$, but his unique best reply when his signal increases to $\bar{s}_1$ is 120. Thus SCC is again violated. However, this time the chosen bids, 120 and 167, are individually rational. The relevant calculations are as follows.

$$
\begin{aligned}
V_1(b_1 = 167, \mathbf{b}_2, \mathbf{b}_3 | \underline{s}_1) &= \tfrac{1}{4}(0 + 0 + 336 + 336) - 167 &&= 1 \\
V_1(b_1 = 120, \mathbf{b}_2, \mathbf{b}_3 | \underline{s}_1) &= \tfrac{1}{4}(0 - 120) + \tfrac{1}{4}(\tfrac{1}{2})(336 - 120) && \\
&\quad + \tfrac{1}{4}(\tfrac{1}{2})(0 - 120) + \tfrac{1}{4}(\tfrac{1}{3})(336 - 120) &&= 0 \\
V_1(b_1 = 0, \mathbf{b}_2, \mathbf{b}_3 | \underline{s}_1) &= \tfrac{1}{4}(\tfrac{1}{3})(0 - 0) &&= 0 \\
V_1(b_1 = 167, \mathbf{b}_2, \mathbf{b}_3 | \bar{s}_1) &= \tfrac{1}{6}336 + \tfrac{1}{3}336 - 167 &&= 1 \\
V_1(b_1 = 120, \mathbf{b}_2, \mathbf{b}_3 | \bar{s}_1) &= \tfrac{1}{6}(0 - 120) + \tfrac{1}{6}(\tfrac{1}{2})(336 - 120) && \\
&\quad + \tfrac{1}{3}(\tfrac{1}{2})(0 - 120) + \tfrac{1}{3}(\tfrac{1}{3})(336 - 120) &&= 2 \\
V_1(b_1 = 0, \mathbf{b}_2, \mathbf{b}_3 | \bar{s}_1) &= \tfrac{1}{6}(\tfrac{1}{3})(0 - 0) &&= 0
\end{aligned}
$$

4. IRT-SCC: A Sketch of the Proof

We now provide a sketch of the proof of IRT-SCC. To keep things simple, we shall consider the case of three bidders, 1, 2, and 3, and establish only part of IRT-SCC for bidder 1. Assumptions A.1 and A.2 are, of course, in force.

Consider two bids, $\bar{b}_1 > \underline{b}_1 > 0$ for bidder 1, and suppose that bidders $j = 2, 3$ each employ a strictly increasing bidding function, $\mathbf{b}_j$, such that $\mathbf{b}_j(0) = 0$, $\mathbf{b}_j(\bar{s}_j) = \bar{b}_1$ and $\mathbf{b}_j(\underline{s}_j) = \underline{b}_1$ for some signals $\bar{s}_j > \underline{s}_j$.[10] Because the joint

[10] Because we allow asymmetric bidders, the bidding functions employed by bidders 2 and 3 needn't be identical. Consequently, we assume neither that $\underline{s}_2 = \underline{s}_3$ nor that $\bar{s}_2 = \bar{s}_3$. The assumption that these bidding functions are strictly increasing, rather than merely nondecreasing, is made only to simplify this proof sketch. The proof given in the appendix requires only that these functions do not induce ties at either $\bar{b}_1$ or $\underline{b}_1$.

density over signals is everywhere strictly positive, both bids of bidder 1 win with strictly positive probability regardless of his signal. Note also that ties occur with probability zero. Throughout this section, the strategies of bidders 2 and 3 will remain fixed, and all statements about 1's payoff are against these fixed strategies.

Consider two signals for bidder 1, one high, $\bar{s}_1$, and one low, $\underline{s}_1$. We shall content ourselves with showing that if $\bar{b}_1$ is individually rational and at least as good as $\underline{b}_1$ for bidder 1 when his signal is low, then $\bar{b}_1$ remains at least as good as $\underline{b}_1$ for bidder 1 when his signal is high.

Now, $\underline{b}_1$ is either individually rational or individually irrational for bidder 1 at $\bar{s}_1$. Consider first the case in which it is individually irrational. Hence, by definition, 1's payoff from bidding $\underline{b}_1$ at $\bar{s}_1$ is negative. Now, because we are assuming that $\bar{b}_1$ is individually rational at $\underline{s}_1$, 1's payoff from bidding $\bar{b}_1$ is nonnegative at $\underline{s}_1$. Therefore, because $u_1(\bar{b}_1, s)$ is nondecreasing in s, and because the signals are affiliated, $\bar{b}_1$ is also individually rational at $\bar{s}_1$.[11] Consequently, bidding $\bar{b}_1$ is at least as good as bidding $\underline{b}_1$ when 1's signal is high, which is our desired conclusion. So, in the remainder of this proof-sketch we may assume that we are in the other case, namely, that in which $\underline{b}_1$ is individually rational for bidder 1 at $\bar{s}_1$.

Consult Figure 4.1. The figure identifies four regions of the joint signal space of bidders 2 and 3. In region A_0, both bids, $\bar{b}_1$ and $\underline{b}_1$, are winning for bidder 1, while in regions A_1, A_2, and A_3, only the higher bid, $\bar{b}_1$, is winning for bidder 1. Consequently, as shown in the figure, the ex-post difference in 1's payoff from $\bar{b}_1$ versus $\underline{b}_1$, which we shall denote by $\Delta(s)$, is $u_1(\bar{b}_1, s) - u_1(\underline{b}_1, s)$ in region A_0, while it is simply $u_1(\bar{b}_1, s)$ in the other three regions.

Let $A = [0, \bar{s}_2] \times [0, \bar{s}_3]$ denote the union of the four regions, i.e., the event that $\bar{b}_1$ is a winning bid. When (s_2, s_3) is outside A, both bids lose and so $\Delta(s)$ is zero. Consequently, given his signal s_1, the difference in 1's payoff from bidding $\bar{b}_1$ versus $\underline{b}_1$ is

$$\Pr(A|s_1)E(\Delta(S)|A, s_1).$$

Now, because $\bar{b}_1$ wins with positive probability regardless of 1's signal, $\Pr(A|s_1) > 0$. Hence, if we could show that $E(\Delta(S)|A, s_1)$ were nondecreasing in s_1 we would be done. The following question thus arises. When are conditional expectations of functions of affiliated random variables monotone in their conditioning variables?

[11] This follows from the monotonicity result due to Milgrom and Weber (1982) described below.

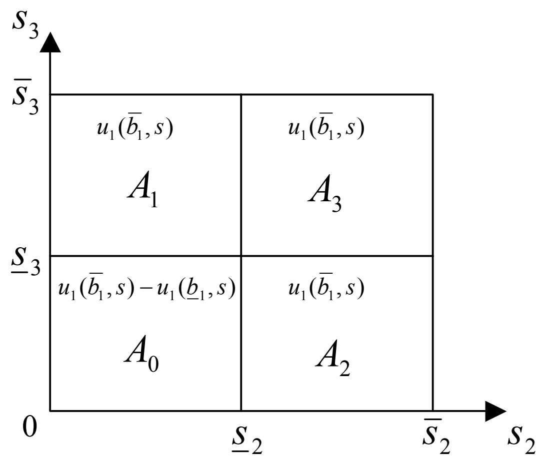

Figure 4.1: Payoff difference from $b_1 = \bar{b}_1$ vs. $b_1 = \underline{b}_1$

An important and well-known theorem due to Milgrom and Weber (1982) states that if $X_1, ..., X_n$ are affiliated and $\phi(x_1, ..., x_n)$ is a nondecreasing real-valued function, then the expectation of ϕ, conditional on any number of events of the form $X_k \in [a_k, b_k]$, is nondecreasing in all the a_k and b_k, and where $a_k = b_k$ is permitted.

So, for example, if the ex-post difference, $\Delta(s)$, in 1's payoff from bidding $\bar{b}_1$ versus $\underline{b}_1$ were nondecreasing in $s = (s_1, s_2, s_3)$ across the four regions of the figure we'd be done.

Now, by assumption, *within* each of the four regions, $\Delta(s)$ is, for fixed s_1, nondecreasing in s_2 and s_3, and for any fixed s_2 and s_3 it is nondecreasing in s_1. Unfortunately, $\Delta(s)$ need not be nondecreasing across all four regions because it can quite easily happen that a negative value of $u_1(\underline{b}_1, s)$ near the upper border of region A_0 renders $u_1(\bar{b}_1, s) - u_1(\underline{b}_1, s)$ strictly greater than $u_1(\bar{b}_1, s)$ just across that border into region A_1, say, so that $\Delta(s)$ falls as (s_2, s_3) increases from region A_0 into region A_1. This is the essential difficulty that must be overcome.

The idea is to instead consider the *average* values of $\Delta(s)$ in each of the four regions, rather than $\Delta(s)$ itself. We will see that these average values are nondecreasing and this will allow us to apply Milgrom and Weber's monotonicity result to the desired effect.

First, let us write $E(\Delta(S)|A, s_1)$ as the sum of the conditional expectations over the four regions, i.e.,

$$
\begin{aligned}
E(\Delta(S)|A,,s_1) &= \Pr(A_0|A,s_1)E(\Delta(S)|A_0,s_1)\\
&\quad + \Pr(A_1|A,s_1)E(\Delta(S)|A_1,s_1)\\
&\quad + \Pr(A_2|A,s_1)E(\Delta(S)|A_2,s_1)\\
&\quad + \Pr(A_3|A,s_1)E(\Delta(S)|A_3,s_1).
\end{aligned}
$$

As already remarked, $\Delta(s)$ is nondecreasing in $s \in [0,1] \times A_k$ for each $k = 0,1,2,3$. Consequently, Milgrom and Weber's monotonicity result implies $E(\Delta(S)|A_k,\underline{s}_1) \leq E(\Delta(S)|A_k,\bar{s}_1)$ for each k. Hence, we may write

$$
\begin{aligned}
E(\Delta(S)|A,\underline{s}_1) &\leq \Pr(A_0|A,\underline{s}_1)E(\Delta(S)|A_0,\bar{s}_1)\\
&\quad + \Pr(A_1|A,\underline{s}_1)E(\Delta(S)|A_1,\bar{s}_1)\\
&\quad + \Pr(A_2|A,\underline{s}_1)E(\Delta(S)|A_2,\bar{s}_1)\\
&\quad + \Pr(A_3|A,\underline{s}_1)E(\Delta(S)|A_3,\bar{s}_1).
\end{aligned}
$$

Letting $\alpha_k = E(\Delta(S)|A_k,\bar{s}_1)$ denote the average value of $\Delta(s)$ within region A_k given $\bar{s}_1$, we may write the above expression more succinctly as

$$E(\Delta(S)|A,\underline{s}_1) \leq \Pr(A_0|A,\underline{s}_1)\alpha_0+\Pr(A_1|A,\underline{s}_1)\alpha_1+\Pr(A_2|A,\underline{s}_1)\alpha_2+\Pr(A_3|A,\underline{s}_1)\alpha_3.$$

Consider now the following step function over the four regions of the figure:

$$h(s_2,s_3) = \alpha_k, \text{ if } (s_2,s_3) \in A_k.$$

Then we may rewrite once more the above inequality as

$$E(\Delta(S)|A,\underline{s}_1) \leq E(h(S_2,S_3)|A,\underline{s}_1).$$

Suppose, for the moment, that the step-function h is nondecreasing over all four regions. Then, by Milgrom and Weber's monotonicity result, we'd have

$$
\begin{aligned}
E(\Delta(S)|A,\underline{s}_1) &\leq E(h(S_2,S_3)|A,\underline{s}_1)\\
&\leq E(h(S_2,S_3)|A,\bar{s}_1)\\
&= E(\Delta(S)|A,\bar{s}_1),
\end{aligned}
$$

where the last equality follows from the definitions of h and the α_k, and we'd have proven the desired monotonicity of $E(\Delta(S)|A,s_1)$ in s_1. Hence, it suffices to show that h is nondecreasing, or, equivalently, that $\alpha_0 \leq \alpha_k \leq \alpha_3$ for $k = 1,2$.

Now, for $k = 1, 2$,

$$\begin{aligned}\alpha_k &= E(u_1(\bar{b}_1, S)|A_k, \bar{s}_1) \\ &\leq E(u_1(\bar{b}_1, S)|A_3, \bar{s}_1) \\ &= \alpha_3,\end{aligned}$$

where the inequality follows from Milgrom and Weber's monotonicity result. It therefore remains only to show that $\alpha_0 \leq \alpha_k$ for $k = 1, 2$.

It is here where the individual rationality of the bid $\underline{b}_1$ at $\bar{s}_1$ is needed. Note that a bid of $\underline{b}_1$ wins precisely when the others' signals are in region A_0, and this occurs with positive probability. Hence, $\underline{b}_1$ is individually rational at $\bar{s}_1$ if and only if

$$E(u_1(\underline{b}_1, S)|A_0, \bar{s}_1) \geq 0.$$

We then have that for $k = 1, 2$.

$$\begin{aligned}\alpha_k &= E(u_1(\bar{b}_1, S)|A_k, \bar{s}_1) \\ &\geq E(u_1(\bar{b}_1, S)|A_0, \bar{s}_1) \\ &= E(u_1(\bar{b}_1, S) - u_1(\underline{b}_1, S)|A_0, \bar{s}_1) + E(u_1(\underline{b}_1, S)|A_0, \bar{s}_1) \\ &\geq E(u_1(\bar{b}_1, S) - u_1(\underline{b}_1, S)|A_0, \bar{s}_1) \\ &= E(\Delta(S)|A_0, \bar{s}_1) \\ &= \alpha_0,\end{aligned}$$

where the first inequality follows from Milgrom and Weber's monotonicity result and the second follows from the individual rationality of the bid $\underline{b}_1$ at $\bar{s}_1$.

5. Multi-Dimensional Signals: A Private Value Counterexample

We now provide a private value example suggesting that Theorem 2.1 fails if the bidders' signals are not one-dimensional. The example possesses a unique equilibrium, which is pure and non-monotone.

The example is only suggestive because, while it satisfies the multi-dimensional signal analogue of assumption A.1, it involves several extreme distributional specifications. For example, some signals are discrete random variables rather than continuous ones, and some signals are perfectly correlated with others. Consequently, the joint signal distribution has no density function and so, formally, A.2

fails. However, the signals in our example are affiliated, and we conjecture that no smoothed nearby example satisfying A.1 and A.2 will possess a monotone pure strategy equilibrium either.[12] But this remains an open question.

There are three bidders, 1,2,3. Bidder 1 receives the two-dimensional signal $S_1 = (X, Y)$, while bidders 2 and 3 each receive the *same* two-dimensional signal $S_2 = S_3 = (Y, Z)$, where X and Y are i.i.d. random variables taking on the values 0 and 1 with probability 1/2 each, and Z is independently and uniformly distributed on $[0, 1] \cup [2, 3]$. Consequently, the six real random variables $(S_1, S_2, S_3) = (X, Y, Y, Z, Y, Z)$ are affiliated.

The bidders have private values and quasilinear utilities. Bidder 1's value is

$$v_1(x, y) = 6x,$$

while bidders 2 and 3 have identical values $v_2(y, z) = v_3(y, z) = v(y, z)$, where

$$v(y,z) = \begin{cases} 7 & \text{if } y = 1,\ z \in [2,3] \\ z & \text{otherwise.} \end{cases}$$

Proposition 5.1. *The above first-price auction example possesses a unique equilibrium (up to ex ante probability zero events) which is pure and non-monotone. Indeed, the equilibrium is: For $y = 0, 1$ and a.e. z, $b_1(0, y) = 0$, $b_1(1, 0) = 3$, $b_1(1, 1) = 1$; $b_j(y, z) = v(y, z)$ $j = 2, 3$.*

The proof of the proposition is in the appendix, but the argument is straightforward. Because bidders 2 and 3 have identical values, and because their signals are also identical, their identical values are common knowledge between them. Consequently, a standard Bertrand competition argument establishes that bidders 2 and 3 must each bid their value in equilibrium. Hence, bidders 2 and 3 employ monotone pure strategies. It remains only to find bidder 1's best reply.

When 1's signal $S_1 = (x, y) = (0, y)$, bidder 1's unique best reply is to bid zero because his value is $v_1 = 6x = 0$. The interesting case is when $x = 1$.

When $S_1 = (x, y) = (1, 0)$, bidder 1's value is $v_1 = 6$ and he knows that $Y = 0$. Consequently, he knows that the common bid of bidders 2 and 3 is $v(0, Z) = Z$, which is uniformly distributed on $[0, 1] \cup [2, 3]$, and a straightforward calculation establishes that 1's unique best reply is to bid 3.

[12] Arbitrarily nearby examples satisfying A.1 and A.2 exist.

However, when $S_1 = (x, y) = (1, 1)$, bidder 1's value is again $v_1 = 6$, but he now knows that $Y = 1$. Consequently, he knows that bidders 2 and 3 each bid $v(1, Z) = 7$ if $Z \in [2, 3]$ while they each bid $v(1, Z) = Z$ if $Z \in [0, 1]$. Clearly, it would be suboptimal for bidder 1 to bid 7 or more since his value is only 6. Consequently, bidder 1 will bid less than 7 and so can condition on the event that 2 and 3 bid less than 7 as well. But, conditional on bidding less than 7, bidders 2 and 3 submit a common bid that is uniformly distributed on $[0, 1]$. Another straightforward calculation establishes that bidder 1's unique best reply now is to bid 1.

Hence, bidder 1's unique equilibrium bidding function is non-monotone, falling from a bid of 3 to a bid of 1 when his signal increases from $(1, 0)$ to $(1, 1)$.

One reason for the failure of monotonicity here is the failure of affiliation to be inherited by monotone functions of multi-dimensional affiliated random variables. Specifically, even though the random variables Y and Z are independent and hence affiliated, the random variables Y and $v(Y, Z)$ are *not* affiliated, despite the fact that $v(Y, Z)$ is nondecreasing. To see this simply observe that

$$\frac{\Pr(v(Y, Z) \in [2, 3]|Y = 0)}{\Pr(v(Y, Z) \in [0, 1]|Y = 0)} = 1 > 0 = \frac{\Pr(v(Y, Z) \in [2, 3]|Y = 1)}{\Pr(v(Y, Z) \in [0, 1]|Y = 1)}.$$

Consequently, in the example, one of the dimensions of bidder 1's signal, Y, is not affiliated with the equilibrium bids of the other bidders.

A. APPENDIX

Proof of Proposition 2.3. To establish IRT-SCC, fix $\bar{b}_i > \underline{b}_i$ and, for all bidders but i, fix nondecreasing bid functions, $\mathbf{b}_{-i}$, satisfying $\Pr(\mathbf{b}_j > 0$, and $\mathbf{b}_j = \bar{b}_i$ or $\underline{b}_i) = 0$ for all $j \neq i$. Because the IRT-SCC inequality holds trivially when the reserve price, r, strictly exceeds $\bar{b}_i$, assume without loss that $\bar{b}_i \geq r$.

Let $V_i(b_i, \mathbf{b}_{-i} \,|\, D, s_i)$ denote i's expected payoff, conditional on an event D (a Borel subset of $[0,1]^N$) and the signal s_i, from bidding b_i when the others employ the given strategies $\mathbf{b}_{-i}$.

Let A denote the event that bidder i's bid of $\bar{b}_i$ is among the highest, i.e.,

$$A = \bigcap_{j \neq i} \{s \in [0,1]^N : \mathbf{b}_j(s_j) \leq \bar{b}_i\}.$$

If $\Pr(A|s_i) = 0$ for some $s_i \in [0,1]$, then (because $f > 0$ on $[0,1]^N$) $\Pr(A|s_i) = 0$ for all $s_i \in [0,1]$ and the IRT-SCC inequality holds trivially. Hence, we may assume that $\Pr(A \,|\, s_i) > 0$ for all $s_i \in [0,1]$. So, because for $b_i \leq \bar{b}_i$, $V_i(b_i, \mathbf{b}_{-i} \,|s_i) = \Pr(A \,|\, s_i) V_i(b_i, \mathbf{b}_{-i} \,|\, A, s_i)$, the IRT-SCC inequality will hold if for $\bar{s}_i > \underline{s}_i$

$$V_i(\bar{b}_i, \mathbf{b}_{-i} \,|\, A, \underline{s}_i) - V_i(\underline{b}_i, \mathbf{b}_{-i} \,|\, A, \underline{s}_i) \geq 0 \text{ implies } V_i(\bar{b}_i, \mathbf{b}_{-i} \,|\, A, \bar{s}_i) - V_i(\underline{b}_i, \mathbf{b}_{-i} \,|\, A, \bar{s}_i) \geq 0, \tag{A.1}$$

when $V_i(\bar{b}_i, \mathbf{b}_{-i} \,|\, A, \underline{s}_i) \geq 0$, and

$$V_i(\bar{b}_i, \mathbf{b}_{-i} \,|\, A, \bar{s}_i) - V_i(\underline{b}_i, \mathbf{b}_{-i} \,|\, A, \bar{s}_i) \leq 0 \text{ implies } V_i(\bar{b}_i, \mathbf{b}_{-i} \,|\, A, \underline{s}_i) - V_i(\underline{b}_i, \mathbf{b}_{-i} \,|\, A, \underline{s}_i) \leq 0, \tag{A.2}$$

when $V_i(\underline{b}_i, \mathbf{b}_{-i} \,|\, A, \bar{s}_i) \geq 0$.

Now, $\bar{b}_i > \underline{b}_i \geq 0$ implies $\Pr(\mathbf{b}_j = \bar{b}_i) = \Pr(\mathbf{b}_j > 0$, and $\mathbf{b}_j = \bar{b}_i) \leq \Pr(\mathbf{b}_j > 0$, and $\mathbf{b}_j = \bar{b}_i$ or $\underline{b}_i) = 0$ for all $j \neq i$. But $f > 0$ on $[0,1]^N$ then implies $\Pr(\mathbf{b}_j = \bar{b}_i|s_i) = 0$ for all $s_i \in [0,1]$. Hence, $V_i(\bar{b}_i, \mathbf{b}_{-i} \,|\, A, s_i) = E(u_i(\bar{b}_i, S)|A, s_i)$. Moreover, by A.1 (iv) and Theorem 5 in Milgrom and Weber (1982) (henceforth MW Thm. 5), $E(u_i(\bar{b}_i, S)|A, s_i)$ is nondecreasing in s_i, and so

$$V_i(\bar{b}_i, \mathbf{b}_{-i} \,|\, A, \underline{s}_i) \geq 0 \text{ implies } V_i(\bar{b}_i, \mathbf{b}_{-i} \,|\, A, \bar{s}_i) \geq 0.$$

Consequently, if $V_i(\bar{b}_i, \mathbf{b}_{-i} \,|\, A, \underline{s}_i) \geq 0$ and $V_i(\underline{b}_i, \mathbf{b}_{-i} \,|\, A, \bar{s}_i) < 0$, then (A.1) holds simply because the second difference is positive, being the difference between a nonnegative and a negative number. Hence, it suffices to establish (A.1) and (A.2) when $V_i(\underline{b}_i, \mathbf{b}_{-i} \,|\, \bar{s}_i) \geq 0$.

We shall in fact show more than this, namely, that if

$$V_i(\underline{b}_i, \mathbf{b}_{-i} \,|\, \bar{s}_i) \geq 0, \tag{A.3}$$

then

$$V_i(\bar{b}_i, \mathbf{b}_{-i} \,|\, A, \underline{s}_i) - V_i(\underline{b}_i, \mathbf{b}_{-i} \,|\, A, \underline{s}_i) \leq V_i(\bar{b}_i, \mathbf{b}_{-i} \,|\, A, \bar{s}_i) - V_i(\underline{b}_i, \mathbf{b}_{-i} \,|\, A, \bar{s}_i) \tag{A.4}$$

To see this, define for $j \neq i$, $\chi_j(s_j) = 0$ if $b_j(s_j) \in [0, \underline{b}_i]$ and $\chi_j(s_j) = 1$ if $b_j(s_j) \in (\underline{b}_i, \bar{b}_i]$. Partition A into subevents as follows. For each $x \in \{0,1\}^{N-1}$ let

$$A(x) = \{s : \chi_j(s_j) = x_j, \forall j \neq i\}$$

be the event that bidders $j \neq i$ such that $x_j = 0$ submit bids in $[0, \underline{b}_i]$, while bidders $j \neq i$ such that $x_j = 1$ submit bids in $(\underline{b}_i, \bar{b}_i]$.[13] Consequently, $\{A(x)\}_{x \in \{0,1\}^{N-1}}$ is a partition of A.[14]

Note that for any $x \in \{0,1\}^{N-1}$, our assumption that $f > 0$ implies that if $\Pr(A(x)|A, s_i)$ is zero for some s_i, then it is zero for all s_i.

The event $A(\mathbf{0})$ is that in which all bidders $j \neq i$ bid weakly below $\underline{b}_i$. Now, if $\Pr(A(\mathbf{0})|A, s_i)$ is zero for some s_i, it is zero for all s_i, so that bidder i's bid of $\underline{b}_i$ wins with probability zero regardless of his signal. In this case, (A.4) reduces to $E(u_i(\bar{b}_i, S)|A, \underline{s}_i) \leq E(u_i(\bar{b}_i, S)|A, \bar{s}_i)$, which follows from MW Thm. 5. Hence, we may assume that $\Pr(A(\mathbf{0})|A, s_i) > 0$ for every $s_i \in [0,1]$.

Now, if bidder i bids $\underline{b}_i$ and $\underline{b}_i > 0$ then $\Pr(\mathbf{b}_j = \underline{b}_i) = \Pr(\mathbf{b}_j > 0$, and $\mathbf{b}_j = \underline{b}_i) \leq \Pr(\mathbf{b}_j > 0$, and $\mathbf{b}_j = \bar{b}_i$ or $\underline{b}_i) = 0$ for all $j \neq i$. Hence, because $f > 0$ on $[0,1]^N$, $\Pr(\mathbf{b}_j = \underline{b}_i|A(\mathbf{0}), s_i) = 0$ for all $s_i \in [0,1]$. That is, conditional on the others bidding weakly below $\underline{b}_i$, and conditional on any s_i bidder i wins with probability one with a bid of $\underline{b}_i$. Hence, i's expected payoff would be $E(u_i(\underline{b}_i, S)|\, A(\mathbf{0}), s_i)$.

On the other hand, if bidder i bids $\underline{b}_i$ and $\underline{b}_i = r = 0$, then $\Pr(\mathbf{b}_j = \underline{b}_i|A(\mathbf{0}), s_i) = 1$. That is, conditional on the others bidding $\underline{b}_i = 0$ and conditional on s_i, bidder i ties with every other bidder with probability one with a bid of zero. In this case, i's expected payoff would be $\frac{1}{N}E(u_i(\underline{b}_i, S)|\, A(\mathbf{0}), s_i)$.

[13] The components of the $N-1$ dimensional vector x are numbered in ascending order excluding i. For example, if $i = 2$ and $N = 4$, then $x = (x_1, x_3, x_4)$. This is equivalent to writing x_{-i} when $x = (x_1, x_2, x_3, x_4)$, but is notationally less burdensome in what follows.

[14] The reader might wonder why we do not employ a more succinct notation for the subevents. For example, for $J \subseteq N\backslash\{i\}$, letting $A(J) = \{s : b_j(s_j) \in [0, \underline{b}_i]$ if $j \notin J \cup \{i\}$, and $b_j(s_j) \in (\underline{b}_i, \bar{b}_i]$ if $j \in J\}$ produces the same partition of events as J varies over all subsets of $N\backslash\{i\}$. But the reader will see shortly that we will make use of the ordering of the vectors x in $\{0,1\}^{N-1}$ and the notation $A(x)$ will then prove especially useful.

Finally, as we have already shown, because $\bar{b}_i > 0$, $\Pr(\mathbf{b}_j = \bar{b}_i) = 0$ for all $j \neq i$. So, because $f > 0$, $\Pr(\mathbf{b}_j = \bar{b}_i | A, s_i) = 0$ for all $s_i \in [0,1]$. That is, a bid of $\bar{b}_i$ wins with probability one conditional on any s_i and conditional on A, the event that all other bidders bid weakly below $\bar{b}_i$. This gives i an expected payoff of $E(u_i(\bar{b}_i, S)|\, A, s_i)$

Hence, we may write the left-hand side of (A.4) as

$$V_i(\bar{b}_i, \mathbf{b}_{-i} \,|\, A, \underline{s}_i) - V_i(\underline{b}_i, \mathbf{b}_{-i} \,|\, A, \underline{s}_i) =$$

$$\Pr(A(\mathbf{0})|A, \underline{s}_i) E[u_i(\bar{b}_i, S) - \lambda u_i(\underline{b}_i, S)|\, A(\mathbf{0}), \underline{s}_i]$$

$$+ \sum_{x \in \{0,1\}^{N-1} \setminus \{\mathbf{0}\}} \Pr(A(x)|A, \underline{s}_i) E[u_i(\bar{b}_i, S)|\, A(x), \underline{s}_i],$$

where $\lambda = 1$ if $\underline{b}_i = \max(\underline{b}_i, r) > 0$, $\lambda = 1/N$ if $\underline{b}_i = r = 0$ and $\lambda = 0$ if $\underline{b}_i < r$. If some probability in the sum above is zero, define the associated conditional expectation to be any finite number.

Because we have assumed that both $u_i(\bar{b}_i, s) - u_i(\underline{b}_i, s)$ and $u_i(\bar{b}_i, s)$ are nondecreasing in s, their convex combination, $u_i(\bar{b}_i, s) - \lambda u_i(\underline{b}_i, s)$, is also nondecreasing in s. Consequently, because the S_k are affiliated, we have (by MW Thm.5)

$$\begin{aligned}
V_i(\bar{b}_i, \mathbf{b}_{-i} \,|\, A, \underline{s}_i) - V_i(\underline{b}_i, \mathbf{b}_{-i} \,|\, A, \underline{s}_i) = \\
& \Pr(A(\mathbf{0})|A, \underline{s}_i) E[u_i(\bar{b}_i, S) - \lambda u_i(\underline{b}_i, S)|\, A(\mathbf{0}), \underline{s}_i] \\
& + \sum_{x \in \{0,1\}^{N-1} \setminus \{\mathbf{0}\}} \Pr(A(x)|A, \underline{s}_i) E[u_i(\bar{b}_i, S)|\, A(x), \underline{s}_i] \\
\leq\ & \Pr(A(\mathbf{0})|A, \underline{s}_i) E[u_i(\bar{b}_i, S) - \lambda u_i(\underline{b}_i, S)|\, A(\mathbf{0}), \bar{s}_i] \\
& + \sum_{x \in \{0,1\}^{N-1} \setminus \{\mathbf{0}\}} \Pr(A(x)|A, \underline{s}_i) E[u_i(\bar{b}_i, S)|\, A(x), \bar{s}_i].
\end{aligned} \tag{A.5}$$

Define $h : \{0,1\}^{N-1} \to \mathbb{R}_+$ as follows: For $x = \mathbf{0}$ define

$$h(\mathbf{0}) = E[u_i(\bar{b}_i, \bar{s}_i, S_{-i}) - \lambda u_i(\underline{b}_i, \bar{s}_i, S_{-i})|\, A(\mathbf{0}), \bar{s}_i].$$

Next, for all nonzero x define

$$h(x) = \max_{\mathbf{0} \leq y \leq x, \Pr(A(y)|\underline{s}_i)>0} E[u_i(\bar{b}_i, S)|\, A(y), \bar{s}_i], \tag{A.6}$$

where the maximum is always well defined because $\Pr(A(\mathbf{0})|\underline{s}_i) > 0$. Hence, by MW Thm.5, $\Pr(A(x)|A, \underline{s}_i) > 0$ implies

$$h(x) = E[u_i(\bar{b}_i, S)|\, A(x), \bar{s}_i]. \tag{A.7}$$

So, we may rewrite the inequality expressed in (A.5) as

$$V_i(\bar{b}_i, \mathbf{b}_{-i} \,|\, A, \underline{s}_i) - V_i(\underline{b}_i, \mathbf{b}_{-i} \,|\, A, \underline{s}_i) \leq \sum_{x \in \{0,1\}^{N-1}} \Pr(A(x)|A, \underline{s}_i) h(x). \tag{A.8}$$

We shall argue that the sum on the right-hand side of (A.8) is nondecreasing in i's signal. To see this, note first that given (A.6), $h(x)$ is nondecreasing on $\{0,1\}^{N-1} \backslash \{\mathbf{0}\}$. So, h will be nondecreasing on all of $\{0,1\}^{N-1}$ if $h(x) \geq h(\mathbf{0})$ for all nonzero x. But this follows from the fact that for any $\mathbf{0} \leq y \leq x$ such that $\Pr(A(y)|A, \underline{s}_i) > 0$ (and hence $\Pr(A(y)|A, \bar{s}_i) > 0$),

$$\begin{aligned}
E[u_i(\bar{b}_i, S)|\, A(y), \bar{s}_i] &\geq E[u_i(\bar{b}_i, S)|\, A(\mathbf{0}), \bar{s}_i] \\
&= E[u_i(\bar{b}_i, S) - \lambda u_i(\underline{b}_i, S)|\, A(\mathbf{0}), \bar{s}_i] \\
&\quad + E[\lambda u_i(\underline{b}_i, S)|\, A(\mathbf{0}), \bar{s}_i] \\
&= h(\mathbf{0}) + E[\lambda u_i(\underline{b}_i, S)|\, A(\mathbf{0}), \bar{s}_i] \\
&= h(\mathbf{0}) + V_i(\underline{b}_i, \mathbf{b}_{-i} \,|\, \bar{s}_i) / \Pr(A(\mathbf{0}) \,|\, A, \bar{s}_i) \\
&\geq h(\mathbf{0}),
\end{aligned}$$

where the first inequality follows from MW Thm.5, and the last inequality follows from (A.3). Thus, we have established that h is nondecreasing.

For each vector of signals s, define

$$\phi(s) = \begin{cases} h(x), & \text{if } s \in A(x) \text{ for } x \in \{0,1\}^{N-1} \\ 0, & \text{otherwise} \end{cases}$$

So defined, $\phi(s)$ is nondecreasing in s on A. Consequently, by MW Thm. 5, $E(\phi(S)|A, S_i = s_i)$ is nondecreasing in s_i. Observing that

$$E(\phi(S)|A, S_i = s_i) = \sum_{x \in \{0,1\}^{N-1}} \Pr(A(x)|A, s_i) h(x)$$

establishes that the sum on the right-hand side of (A.8) is nondecreasing in i's signal. Consequently, from (A.8) we have

$$\begin{aligned} V_i(\bar{b}_i, \mathbf{b}_{-i} \,|\, A, \underline{s}_i) - V_i(\underline{b}_i, \mathbf{b}_{-i} \,|\, A, \underline{s}_i) &\leq \sum_{x \in \{0,1\}^{N-1}} \Pr(A(x)|A, \underline{s}_i) h(x) \\ &\leq \sum_{x \in \{0,1\}^{N-1}} \Pr(A(x)|A, \bar{s}_i) h(x) \\ &= V_i(\bar{b}_i, \mathbf{b}_{-i} \,|\, A, \bar{s}_i) - V_i(\underline{b}_i, \mathbf{b}_{-i} \,|\, A, \bar{s}_i), \end{aligned}$$

where the final equality follows from (A.7) and because $\Pr(A(x)|A, \bar{s}_i) = 0$ whenever $\Pr(A(x)|A, \underline{s}_i) = 0$. Thus, we have established that (A.3) implies (A.4). ■

Proof of Theorem 2.1. PART 1. In this first part of the proof we shall focus attention on a slightly modified first-price auction game. There are two modifications. First, we restrict the bidders to finite sets of nonnegative bids. Each finite set contains the zero bid, but no two sets have any positive bids in common. This means that ties can only occur at bids of zero. Second, we restrict the bidders' strategies so that each bidder must bid zero when his signal is in $[0, \varepsilon)$, where $\varepsilon \in (0, 1)$ is fixed. Because the joint density of signals is strictly positive on $[0, 1]^N$, every bid $b \geq r$ wins with strictly positive probability regardless of one's signal. Consequently, such bids must earn non-negative expected utility in equilibrium. We wish to show the following.

Under the two modifications above, a monotone pure strategy equilibrium exists.

To establish this, it would suffice to verify the single-crossing condition (SCC) employed in Athey (2001). We could then appeal to Athey's Theorem 1. However, as we have seen, SCC does not hold in our setting. Fortunately, Athey's existence proof goes through without any changes if the following, more permissive, *best-reply* single-crossing condition holds in our modified auction.

BR-SCC. If b_i' is a best reply for s_i against $\mathbf{b}_{-i}$, and b_i is any other feasible bid, then the inequality $V_i(b_i', \mathbf{b}_{-i} \,|\, s_i') \geq V_i(b_i, \mathbf{b}_{-i} \,|\, s_i')$ holds for all $s_i' > s_i$ if $b_i' > b_i$, while it holds for all $s_i' < s_i$ if $b_i' < b_i$.

Hence, if we can show that BR-SCC holds in our modified auction, then it possesses a monotone pure strategy equilibrium.[15] We now show that this is indeed the case.

So, suppose that in our modified auction b_i' is a best reply for s_i against the others' monotone strategy $\mathbf{b}_{-i}$, and b_i is any other feasible bid for i. If $s_i \in [0, \varepsilon)$, then $b_i' = b_i = 0$ and we are done. Hence, we may suppose that $s_i \geq \varepsilon$. Now, because the zero bid is available for i and b_i' is a best reply for s_i, we must have $V_i(b_i', \mathbf{b}_{-i} \mid s_i) \geq \max(V_i(b_i, \mathbf{b}_{-i} \mid s_i), V_i(0, \mathbf{b}_{-i} \mid s_i)) \geq 0$. Hence, in addition to being as good as b_i, b_i' is individually rational for i at s_i.

Also, because the only common bid available to distinct bidders is zero, $\Pr(\mathbf{b}_j > 0$, and $\mathbf{b}_j = b_i'$ or $b_i) = 0$ for all $j \neq i$. Hence, by Proposition 2.3 we may conclude that $V_i(b_i', \mathbf{b}_{-i} \mid s_i') \geq V_i(b_i, \mathbf{b}_{-i} \mid s_i')$ holds for all $s_i' > s_i$ if $b_i' > b_i$, while it holds for all $s_i' < s_i$ if $b_i' < b_i$.

This establishes BR-SCC for our modified auction and so we conclude that it possesses a monotone pure strategy equilibrium.[16]

PART 2. In this second part of the proof, we consider a sequence of monotone equilibria of the modified auctions from Part 1. For $n = 1, 2, ...$, let G^n denote the modified auction in which $\varepsilon = 1/n$ and bidder i's finite set of bids is denoted by B_i^n. Further, suppose that $B_i^n \supseteq B_i^{n-1}$ and that $B_i^\infty = \cup_n B_i^n$ is dense in $\mathbb{R}_+$. Let $\hat{\mathbf{b}}^n$ denote a monotone pure strategy equilibrium of G^n. Without loss we may suppose that no bid $\hat{\mathbf{b}}_i^n(s_i)$ is strictly between zero and r because equilibrium will be preserved if all such bids are redefined to be zero. Also, by A.1 (ii), there exists $\tilde{b} \geq 0$ such that $u_i(b, s) < 0$ for all $b > \tilde{b}$ and all $s \in [0,1]^N$. Consequently, because all bids equal to or above r win with positive probability, because the zero bid is available in G^n and because, by A.1(iii), $u_i(0, s) \geq 0$ for all $s \in [0,1]^N$, each $\hat{\mathbf{b}}_i^n \geq 0$ is bounded above by $\tilde{b}$. By Helley's Theorem, we may assume without loss that $\hat{\mathbf{b}}^n(s) \to \hat{\mathbf{b}}(s)$ for every $s \in [0,1]^N$, where each $\hat{\mathbf{b}}_i$ is nondecreasing on $[0,1]$.[17] We shall argue that $\hat{\mathbf{b}}$ is a monotone equilibrium of the first-price auction game.

[15] While our first restriction, that bids are taken from finite sets, matches Athey's finite action set environment, our second restriction, that bidders must bid zero when their signals are less than ε, is not present in Athey's (2001) treatment. Nonetheless, Athey's results easily go through with this restriction in place because it has no effect on her convexity arguments.

[16] Note that our example of the second failure of SCC in Section 5 demonstrates that BR-SCC fails if one allows the bidders' finite bid sets to have positive bids in common.

[17] Most versions of Helley's theorem state that convergence can be guaranteed at all continuity points of each $\hat{\mathbf{b}}_i$. However, because signals are one-dimensional, and each $\hat{\mathbf{b}}_i$ is nondecreasing, there are at most countably many discontinuity points and so convergence can in fact be guaranteed everywhere by suitably modifying $\hat{\mathbf{b}}_i$ at these countably many points and considering a subsequence if necessary.

To do so, we first establish that, given $\hat{\mathbf{b}}$, the probability that any two distinct bidders each submit the highest bid weakly above r is zero.

Suppose, by way of contradiction, that the probability that the bid $\bar{b} \geq r$ is the highest bid and is submitted simultaneously by two distinct bidders is positive.

For every bidder i, define $\underline{s}_i = \inf\{s_i | \hat{\mathbf{b}}_i(s_i) \geq \bar{b}\}$ and $\bar{s}_i = \sup\{s_i | \hat{\mathbf{b}}_i(s_i) \leq \bar{b}\}$.[18] Note then that $\hat{\mathbf{b}}_i(s_i) = \bar{b}$ for all $s_i \in (\underline{s}_i, \bar{s}_i)$, and there is a subset, I, containing at least two bidders such that

$$\Pr(s_i \in (\underline{s}_i, \bar{s}_i), \forall i \in I \text{ and } s_i < \bar{s}_i, \forall i) > 0. \tag{A.9}$$

Hence, $\bar{s}_i > 0$ for all bidders $i = 1, 2, ..., N$, and $\underline{s}_i < \bar{s}_i$ for all bidders $i \in I$.

For distinct bidders i and j, and $s_i \in [0, 1]$, let $s_j^n(s_i)$ denote the supremum of those $s_j \in [0, 1]$ such that $\hat{\mathbf{b}}_j^n(s_j) \leq \hat{\mathbf{b}}_i^n(s_i)$.[19] Because $s_j^n(s_i)$ is nondecreasing in s_i, we may assume without loss (by Helley's Theorem) that $s_j^n(s_i) \to \hat{s}_j(s_i)$ for every $s_i \in [0, 1]$.

Note that $\hat{s}_j(s_i) \leq \bar{s}_j$ for every $s_i \in (\underline{s}_i, \bar{s}_i)$. To see this, fix $s_i \in (\underline{s}_i, \bar{s}_i)$ and $\tilde{s}_j > \bar{s}_j$. By the definition of $\bar{s}_j$, we have $\hat{\mathbf{b}}_j(\tilde{s}_j) > \bar{b} = \hat{\mathbf{b}}_i(s_i)$. Consequently, for n large enough, $\hat{\mathbf{b}}_j^n(\tilde{s}_j) > \hat{\mathbf{b}}_i^n(s_i)$, so that $s_j^n(s_i) \leq \tilde{s}_j$. Taking the limit yields $\hat{s}_j(s_i) \leq \tilde{s}_j$, and because $\tilde{s}_j > \bar{s}_j$ is arbitrary, we conclude that $\hat{s}_j(s_i) \leq \bar{s}_j$.

Consider $s_i \in (\underline{s}_i, \bar{s}_i)$. Because for no n is $\hat{\mathbf{b}}_i^n(s_i)$ strictly between zero and r, $\hat{\mathbf{b}}_i^n(s_i) \to \hat{\mathbf{b}}_i(s_i) = \bar{b} \geq r$ implies $\hat{\mathbf{b}}_i^n(s_i) \geq r$ for all n large enough.

Because in G^n ties in bids can occur with positive probability only at the bid zero, i's payoff at $\hat{\mathbf{b}}^n$ when his signal is $s_i \in (\underline{s}_i, \bar{s}_i)$ and his bid (greater or equal to r for n large enough) is among the highest, is equal to $E(u_i(\hat{\mathbf{b}}_i^n(s_i), S) | S_i = s_i, \hat{\mathbf{b}}_j^n(S_j) \leq \hat{\mathbf{b}}_i^n(s_i), \forall j \neq i)$, unless $\hat{\mathbf{b}}_i^n(s_i) = 0$ in which case his payoff is $1/N$th as large. In either case, for every $s_i \in (\underline{s}_i, \bar{s}_i)$, and for n large enough, we must have,

$$\begin{aligned} 0 &\leq E(u_i(\hat{\mathbf{b}}_i^n(s_i), S) | S_i = s_i, \hat{\mathbf{b}}_j^n(S_j) \leq \hat{\mathbf{b}}_i^n(s_i), \forall j \neq i) \\ &= E(u_i(\hat{\mathbf{b}}_i^n(s_i), S) | S_i = s_i, S_j \leq s_j^n(s_i), \forall j \neq i) \\ &\to E(u_i(\bar{b}, S) | S_i = s_i, S_j \leq \hat{s}_j(s_i), \forall j \neq i), \end{aligned}$$

where the inequality follows because, in G^n, all bids greater or equal to r win with positive probability and the zero bid is available and yields at least zero utility;

[18] Define $\underline{s}_i = 1$ if the set in its definition is empty, and define $\bar{s}_i = 0$ if the set in its definition is empty.

[19] Such s_j's always exist because our restriction requires $\hat{\mathbf{b}}_j^n(s_j) = 0$ for all $s_j \in [0, 1/n]$.

and where the limit follows if $\hat{s}_j(s_i) > 0$, $\forall j \neq i$. So, for every $s_i \in (\underline{s}_i, \bar{s}_i)$ such that $\hat{s}_j(s_i) > 0 \ \forall j \neq i$,

$$0 < E(u_i(\bar{b}, S)|S_i = s_i, S_j \leq \hat{s}_j(s_i), \forall j \neq i) \leq E(u_i(\bar{b}, S)|S_i = s_i, S_j \leq \bar{s}_j, \forall j \neq i), \tag{A.10}$$

where the strict inequality follows because u_i is strictly increasing in s_i, and the weak inequality follows from MW Thm. 5 because $\hat{s}_j(s_i) \leq \bar{s}_j$.

Next, we wish to argue that $\hat{s}_j(s_i) = \bar{s}_j$ for every distinct i, j and every $s_i \in (\underline{s}_i, \bar{s}_i)$. So, assume by way of contradiction that $\hat{s}_j(s_i) < \bar{s}_j$ for some i, j and some $s_i \in (\underline{s}_i, \bar{s}_i)$. Then because f is strictly positive on $[0,1]^N$, and $\bar{s}_k > 0$ for all k,

$$\Pr(S_k \leq \bar{s}_k, \forall k \neq i|s_i) > \Pr(S_k \leq \hat{s}_k(s_i), \forall k \neq i|s_i). \tag{A.11}$$

Also, if $\hat{s}_j(s_i) < \bar{s}_j$, then for all $s_i' \in (\underline{s}_i, s_i)$ we have $\hat{s}_j(s_i') < \bar{s}_j$. Consequently, by (A.10) and (A.11),

$$\begin{aligned}&\Pr(S_k \leq \bar{s}_k, \forall k \neq i|s_i')E(u_i(\bar{b}, S)|S_i = s_i', S_k \leq \bar{s}_k, \forall k \neq i) > \\ &\Pr(S_k \leq \hat{s}_k(s_i'), \forall k \neq i|s_i)\lambda E(u_i(\bar{b}, S)|S_i = s_i', S_k \leq \hat{s}_k(s_i'), \forall k \neq i),\end{aligned} \tag{A.12}$$

for any $\lambda \in [0,1]$ if $\hat{s}_k(s_i') > 0$, $\forall k \neq i$. By defining the right-hand side to be zero when $\hat{s}_k(s_i') = 0$ for some k, this strict inequality holds whether or not $\hat{s}_k(s_i') > 0$ $\forall k \neq i$, because the left-hand side is strictly positive.[20]

Observe that, because $u_i(\cdot, s)$ is continuous by A.1(i), the left-hand side of (A.12) is the limit as $n \to \infty$ and then as $\delta \downarrow 0$, of s_i''s payoff when he bids $\bar{b} + \delta$ in G^n. Similarly, having defined the right-hand side to be zero when $\hat{s}_j(s_i') = 0$ ensures that the right-hand side is the limit of s_i''s payoff when he bids $\hat{\mathbf{b}}_i^n(s_i')$ in G^n, where λ is understood to be $1/N$ in case $\hat{\mathbf{b}}_i^n(s_i') = 0$ for n large enough, while $\lambda = 1$ otherwise.

So, if for every m, $b_m \in B_i^m$ is a feasible bid for i in G^m and $b_m \downarrow \bar{b}$ (such a sequence exists because B_i^m becomes dense in $\mathbb{R}_+$), then for all $s_i' \in (\underline{s}_i, s_i)$

[20] Strict positivity follows from (A.11) and the relations, taken from above, that for all $s_i \in (\underline{s}_i, \bar{s}_i)$, $0 \leq E(u_i(\hat{\mathbf{b}}_i^n(s_i), S)|S_i = s_i, \hat{\mathbf{b}}_j^n(S_j) \leq \hat{\mathbf{b}}_i^n(s_i), \forall j \neq i) = E(u_i(\hat{\mathbf{b}}_i^n(s_i), S)|S_i = s_i, S_j \leq s_j^n(s_i), \forall j \neq i) \leq E(u_i(\hat{\mathbf{b}}_i^n(s_i), S)|S_i = s_i, S_j \leq \bar{s}_j, \forall j \neq i) \to E(u_i(\bar{b}, S)|S_i = s_i, S_j \leq \bar{s}_j, \forall j \neq i)$, because $\bar{s}_j > 0$ for all j. Hence, $E(u_i(\bar{b}, S)|S_i = s_i, S_j \leq \bar{s}_j, \forall j \neq i) > 0$ for all $s_i \in (\underline{s}_i, \bar{s}_i)$ because u_i is strictly increasing in s_i.

$$
\begin{aligned}
&\lim_m \lim_n V_i(b_m, \hat{\mathbf{b}}^n_{-i}|s'_i) \\
= \quad & \Pr(S_j \le \bar{s}_j, \forall j \ne i|s'_i)E(u_i(\bar{b}, S)|S_i = s'_i, S_j \le \bar{s}_j, \forall j \ne i) \\
> \quad & \Pr(S_j \le \hat{s}_j(s'_i), \forall j \ne i|s_i)\lambda E(u_i(\bar{b}, S)|S_i = s'_i, S_j \le \hat{s}_j(s'_i), \forall j \ne i) \\
= \quad & \lim_n V_i(\hat{\mathbf{b}}^n_i(s'_i), \hat{\mathbf{b}}^n_{-i}|s'_i).
\end{aligned}
$$

Consequently, we may choose a subsequence n_m of n such that $n_m \ge m$ for all m, and such that for all $s'_i \in (\underline{s}_i, s_i)$

$$\lim_m \left[V_i(b_m, \hat{\mathbf{b}}^{n_m}_{-i}|s'_i) - V_i(\hat{\mathbf{b}}^{n_m}_i(s'_i), \hat{\mathbf{b}}^{n_m}_{-i}|s'_i)\right] = \phi(s'_i) > 0,$$

where $\phi(s'_i) = \Pr(S_j \le \bar{s}_j, \forall j \ne i|s'_i)E(u_i(\bar{b}, S)|S_i = s'_i, S_j \le \bar{s}_j, \forall j \ne i) - \Pr(S_j \le \hat{s}_j(s'_i), \forall j \ne i|s_i)\lambda E(u_i(\bar{b}, S)|S_i = s'_i, S_j \le \hat{s}_j(s'_i), \forall j \ne i)$ is the difference between the left-hand side and right-hand side in (A.12).

Hence, letting f_i denote the marginal of f on bidder i's signal, we have, by Lebesgue's dominated convergence theorem,

$$\lim_m \int_{\underline{s}_i}^{s_i} \left[V_i(b_m, \hat{\mathbf{b}}^{n_m}_{-i}|s'_i) - V_i(\hat{\mathbf{b}}^{n_m}_i(s'_i), \hat{\mathbf{b}}^{n_m}_{-i}|s'_i)\right] f_i(s'_i)ds'_i = \int_{\underline{s}_i}^{s_i} \phi(s'_i)f_i(s'_i)ds'_i > 0,$$

so that for m large enough,

$$V_i(b_m, \hat{\mathbf{b}}^{n_m}_{-i}|s'_i) > V_i(\hat{\mathbf{b}}^{n_m}_i(s'_i), \hat{\mathbf{b}}^{n_m}_{-i}|s'_i)$$

for a positive f_i measure of signals $s'_i \in (\underline{s}_i, s_i)$. But because $n_m \ge m$ implies that B^{n_m} contains B^m and so also that $b_m \in B^{n_m}$, this contradicts $\hat{\mathbf{b}}^{n_m}$ being an equilibrium in G^{n_m}.

Therefore, we must have $\hat{s}_j(s_i) = \bar{s}_j$ for every $s_i \in (\underline{s}_i, \bar{s}_i)$ and every distinct i, j. But this implies that for $s_i \in (\underline{s}_i, \bar{s}_i)$ and $s_j \in (\underline{s}_j, \bar{s}_j)$, $\hat{\mathbf{b}}^n_j(s_j) \le \hat{\mathbf{b}}^n_i(s_i)$ and (reversing the roles of i and j) $\hat{\mathbf{b}}^n_j(s_j) \ge \hat{\mathbf{b}}^n_i(s_i)$, for all n large enough. Recalling that, in G^n, ties in bids can occur only at a bid of zero, we may conclude that for all distinct i, j, every $s_i \in (\underline{s}_i, \bar{s}_i)$ and every $s_j \in (\underline{s}_j, \bar{s}_j)$, $\hat{\mathbf{b}}^n_j(s_j) = \hat{\mathbf{b}}^n_i(s_i) = 0$ for all n large enough.

Choose $i, j \in I$. Then, for every $s_i \in (\underline{s}_i, \bar{s}_i)$,

$$\begin{aligned}
&\Pr(S_k \leq \bar{s}_k, \forall k \neq i | s_i) E(u_i(0, S) | S_i = s_i, S_k \leq \bar{s}_k, \forall k \neq i) = \\
&\Pr(S_k \leq \hat{s}_k(s_i), \forall k \neq i | s_i) E(u_i(0, S) | S_i = s_i, S_k \leq \hat{s}_k(s_i), \forall k \neq i) > \quad \text{(A.13)} \\
&\Pr(S_k \leq \hat{s}_k(s_i), \forall k \neq i | s_i) \frac{1}{N} E(u_i(0, S) | S_i = s_i, S_k \leq \hat{s}_k(s_i), \forall k \neq i) > 0,
\end{aligned}$$

the third line being strictly positive because $u_i(0, s) \geq 0$ is strictly increasing in s_i, and because $\Pr(S_k \leq \hat{s}_k(s_i), \forall k \neq i | s_i) = \Pr(S_k \leq \bar{s}_k, \forall k \neq i | s_i) > 0$. The strict inequality in the second line, being N times the third line, then follows.

Observe now that the first line of (A.13) is the limit as $n \to \infty$ and then as $\delta \downarrow 0$,of s_i's payoff when he bids δ in G^n, while the third line is the limit of his payoff when he bids $\hat{\mathbf{b}}_i^n(s_i) = 0$ in G^n. Hence, as before, for n large enough, bidder i has a profitable deviation in G^n against $\hat{\mathbf{b}}_{-i}^n$, a contradiction. We conclude that, given $\hat{\mathbf{b}}$, the probability that distinct bidders submit the highest bid weakly above r is zero.

We now complete the proof by showing that $\hat{\mathbf{b}}$ is an equilibrium. Note first that because $\hat{\mathbf{b}}$ involves no ties at bids that win with positive probability, each $V_i(\mathbf{b})$ is continuous (in the topology of pointwise convergence) at $\hat{\mathbf{b}}$.

Assume by way of contradiction that $\hat{\mathbf{b}}$ is not an equilibrium. Then some bidder i has a profitable deviation, $\tilde{\mathbf{b}}_i$ say, which, without loss, is bounded (by A.1(ii)). Now, $\tilde{\mathbf{b}}_i$, being measurable, is the pointwise limit of a sequence, $\tilde{\mathbf{b}}_i^n$, of simple functions.[21] Further, we may assume without loss that, for every n, the joint strategy $(\tilde{\mathbf{b}}_i^n, \hat{\mathbf{b}}_{-i})$ involves no ties at bids that win with positive probability and consequently that V_i is continuous at $(\tilde{\mathbf{b}}_i^n, \hat{\mathbf{b}}_{-i})$.[22]

By Lebesgue's dominated convergence theorem, $\tilde{\mathbf{b}}_i^n$ is a profitable deviation for i for some n. But this implies that at least one of the bids, $\bar{b}$, in the range of $\tilde{\mathbf{b}}_i^n$ is a profitable deviation for every member of a positive f_i-measure set, S_i, of i's signals. Hence, the strategy

$$\bar{\mathbf{b}}_i(s_i) = \begin{cases} \hat{\mathbf{b}}_i(s_i), & \text{if } s_i \in [0,1] \backslash S_i \\ \bar{b}, & \text{if } s_i \in S_i \end{cases} \quad \text{(A.14)}$$

[21] A simple function is one that takes on finitely many values.

[22] This is because each $\hat{\mathbf{b}}_j$ induces a distribution of bids with at most countably many mass points. One can therefore choose the finitely many values taken on by each $\tilde{\mathbf{b}}_i^n$ to be distinct from all such mass points.

is a profitable deviation for bidder i, and V_i is continuous at $(\bar{\mathbf{b}}_i, \hat{\mathbf{b}}_{-i})$

So, altogether we have that

$$V_i(\bar{\mathbf{b}}_i, \hat{\mathbf{b}}_{-i}) > V_i(\hat{\mathbf{b}})$$

and V_i is continuous at both $(\bar{\mathbf{b}}_i, \hat{\mathbf{b}}_{-i})$ and $\hat{\mathbf{b}}$, where $\bar{\mathbf{b}}_i$ is defined by (A.14).

Choose a sequence $b_n \to \bar{b}$ so that $b_n \in B^n$ for every n. Then the sequence of bidding functions

$$\bar{\mathbf{b}}_i^n(s_i) = \begin{cases} \hat{\mathbf{b}}_i^n(s_i), & \text{if } s_i \in [0,1] \backslash S_i \\ b_n, & \text{if } s_i \in S_i \end{cases}$$

converges pointwise to $\bar{\mathbf{b}}_i$. Furthermore, for every n, $\bar{\mathbf{b}}_i^n$ is a feasible strategy for i in G^n, and

$$\lim_n V_i(\bar{\mathbf{b}}_i^n, \hat{\mathbf{b}}_{-i}^n) = V_i(\bar{\mathbf{b}}_i, \hat{\mathbf{b}}_{-i}) > V_i(\hat{\mathbf{b}}) = \lim V_i(\hat{\mathbf{b}}^n).$$

Consequently, for n large enough,

$$V_i(\bar{\mathbf{b}}_i^n, \hat{\mathbf{b}}_{-i}^n) > V_i(\hat{\mathbf{b}}^n),$$

contradicting $\hat{\mathbf{b}}^n$ being an equilibrium of G^n. We conclude that $\hat{\mathbf{b}}$ is an equilibrium. ■

Proof of Theorem 5.1. Consider any equilibrium b_1, b_2, b_3 (not necessarily pure). We first establish a number of claims.

1. $b_1(0, y) = 0$ for $y = 0, 1$.

   Proof. Assume by way of contradiction that there exist y and δ, $0 < \delta < 1$ s.t. $\Pr(b_1(0, y) \geq \delta) > 0$. In this event, bidder 1 must lose with probability 1, since otherwise his payoff in equilibrium would be negative, which is impossible in equilibrium. Hence,

   $$\Pr\left(b_2(y, z) \vee b_3(y, z) > \delta\right) = 1, \text{ a.e. } z$$

   $$\text{and in particular, a.e. } z \in [0, \delta).$$

   Consequently, for a positive measure of $z \in [0, \delta)$ one of bidder 2 or 3 wins the auction and pays more than his value for the object, earning a negative payoff, which is a contradiction.

2. For $j = 2, 3$, let $e_j(y, z)$ be the equilibrium payoff of bidder j with signal (y, z). We claim that $e_j(y, z) = 0 \quad y = 0, 1$ and a.e. z.

   Proof. Suppose by way of contradiction that, for example, $e_2(y, z) > 0$ for $y = 0$ or 1 and a positive measure of z. Then $\exists \delta > 0$ and a set, A, of z's of positive measure s.t. for all $z \in A$, $\Pr(b_2(y, z) \leq v(y, z) - \delta) > 0$. Against this, and given (1), bidder 3 with signal (y, z), where $z \in A$, can guarantee a positive payoff by bidding $v(y, z) - \delta/2$. Hence, $e_3(y, z) > 0$, a.e. $z \in A$. Consequently, for $j = 2, 3$, and a.e. $z \in A$, the least point of the support of $b_j(y, z)$, denoted $\underline{b}_j(y, z)$, must satisfy $\underline{b}_2(y, z) = \underline{b}_3(y, z) = \underline{b}(y, z) < v(y, z)$. (We must have $\underline{b}_2(y, z) = \underline{b}_3(y, z)$ because $\underline{b}_2(y, z) < \underline{b}_3(y, z)$ would imply $e_2(y, z) = 0$ and $\underline{b}_3(y, z) < \underline{b}_2(y, z)$ would imply $e_3(y, z) = 0$.)

   Hence, for a.e. $z \in A$, there can be no atom of $b_j(y, z)$ at $\underline{b}(y, z)$ since otherwise, $\underline{b}(y, z) + \epsilon$ would be a profitable deviation from $\underline{b}(y, z)$ for sufficiently small $\epsilon > 0$. But the absence of atoms implies that the bid $\underline{b}(y, z)$ loses with probability 1, implying $e_2(y, z) = 0$ for a.e. $z \in A$, a contradiction.

   The claim now follows because equilibrium requires $e_j(y, z) \geq 0$ a.e. z.

3. For $j = 2, 3$, $\underline{b}_j(y, z) \geq v(y, z) \quad y = 0, 1$ and a.e. z.

   Proof. For example, $\underline{b}_2(y, z) < v(y, z)$ for a positive measure of z's implies $e_3(y, z) > 0$ for a positive measure of z's, contradicting (2).

4. For $j = 2, 3$, $\Pr(b_j(y, z) = v(y, z)) = 1 \qquad y = 0, 1$ and a.e. z.

   Proof. Let $\overline{b}(y, z) = \overline{b}_2(y, z) \vee \overline{b}_3(y, z)$. If, for $\delta > 0$, some y, and a positive measure of z, $\overline{b}(y, z) = \overline{b}_2(y, z) = v(y, z) + \delta \geq \overline{b}_3(y, z)$, then with probability at least $\frac{1}{2}$ (for $x = 0$) $\times \frac{1}{2}$ (for the tie) bidder 2 wins when his signal is (y, z), and so obtains a negative payoff for a positive measure of his signals in equilibrium which is impossible. A similar contradiction obtains by reversing the roles of bidders 2 and 3. The claim now follows from (3).

We may conclude from (4) that if there is an equilibrium, then bidders 2 and 3 must each bid their values with probability one, as stated in the proposition. Note that in the presence of bidder 3's strategy, bidder 2's strategy is a best reply regardless of the behavior of bidder 1 (and similarly for bidder 2's strategy in the presence of 3's). Consequently, it remains only to verify that the proposition specifies a best reply for bidder 1 against the strategies of bidders 2 and 3 and that this best reply is unique.

Clearly, it is uniquely optimal for bidder 1 to bid zero when $x = 1$. Hence, it remains only to show that deviations for bidder 1 when $x = 1$ are strictly worse than following the given strategy. Now, if $y = 0$, he is supposed to bid 3, winning for sure and obtaining a payoff of 3. Clearly any deviation to $3+\epsilon$ is inferior while bidding $3 - \epsilon$ yields:

$$\left.\begin{array}{rll} (3-\epsilon)(1-\frac{\epsilon}{2}) = 3 - \frac{\epsilon}{2} - \frac{\epsilon^2}{2} & < 3, & \text{if } 0 < \epsilon \leq 1 \\ \frac{1}{2}(3+\epsilon) = \frac{3}{2} + \frac{\epsilon}{2} & \leq \frac{5}{2}, & \text{if } 1 \leq \epsilon \leq 2 \\ \frac{1}{2}(3+\epsilon)(3-\epsilon) & \leq \frac{5}{2}, & \text{if } 2 \leq \epsilon \leq 3 \end{array}\right\} \begin{array}{l} \text{strictly suboptimal for} \\ \text{every value of } \varepsilon > 0 \end{array}$$

For $x = 1$ and $y = 1$, bidder 1 is supposed to bid 1 yielding an expected payoff of $\frac{1}{2} \times 4 = 2$. Clearly, for $\varepsilon > 0$, any deviation to $1 + \epsilon$ is inferior, while bidding $1 - \epsilon$ yields: $\frac{1}{2}(4+\epsilon)(1-\epsilon) < 2$. ■

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