A Topological Approach to Quitting Games

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Abstract

This paper presents a question of topological dynamics and demonstrates that its affirmation would establish the existence of approximate equilibria in all quitting games. A quitting game is an un-discounted stochastic game with finitely many players where every player has only two moves, to end the game with certainty or to allow the game to continue. If nobody ever acts to end the game, all players receive payoffs of 0.

Key words: Stochastic Games, Topological Dynamics, Viability Theory

1 Introduction and Background

A stochastic game is played in stages. At every stage the game is in some state of the world, known by all players. The action combination that was chosen by all the players, together with the current state, determine the stage payoff that each player receives and the probability distribution according to which the new state of the game is chosen. The past history of moves and states is known by all players.

For any $\epsilon \geq 0$, an *ϵ-equilibrium* in a game is a set of strategies, one for each player, such that no player can gain in payoff by more than ϵ by choosing a different strategy, given that all the other players do not change their strategies. An *equilibrium* is an ϵ-equilibrium for $\epsilon = 0$. We say that approximate equilibria exist if for every ϵ there exists an ϵ-equilibrium.

An outstanding open question of game theory is whether all stochastic games with finitely many players, states and moves have approximate equilibria. The interest in this question has been made acute by the proof by N. Vieille (2000 a,b,c) of the existence of approximate equilibria for all two-person stochastic games with finitely many states and moves.

In this paper, we consider a special class of stochastic games called *quitting games*, and introduced in full generality by Solan and Vieille (2001). In a quitting game each player has only two moves, c for continue and q for quit. As soon as one or more of the players at any stage chooses q, the game stops and the players receive their payoffs, which are determined by the subset of players that choose simultaneously the move q. As long as no player has stopped the game, all players receive a payoff of zero.

Examples of quitting games were studied first by Flesch, Thuijsman, and Vrieze (1997). Interest in quitting games is due largely to their discovery of a game with cyclic symmetry with respect to the players such that for all sufficiently small $\epsilon > 0$ and all ϵ equilibria the future expected payoffs conditioned by the event that nobody has ended the game yet change dramatically with the progression of stages, even when restricting to those stages which are reached with a probability of at least one-half.

It is not clear why quitting games should have approximate equilibria, and the existing results concerning this question are limited.

E. Solan (1999) proved that all three player quitting games have approximate equilibria (and this proof concerns a broader class of stochastic games). There is a proof of approximate equilibria by Solan and Vieille (2001) for a subclass of quitting games involving very restricted conditions on the payoffs.

There are proofs by Solan and Vieille (2002) and by Solan and Vohra (2001, 2002) of the existence of approximate correlated equilibria for quitting games. The Solan and Vieille proof is for all stochastic games with finitely many states and moves, and the Solan and Vohra proof is for an intermediate class of games, however showing a special type of approximate correlated equilibrium. Correlated equilibria are equilibria with respect to an extended game (including a correlation device) and therefore these results do not pertain directly to the existence of equilibria of the original quitting game.

There is a four player example by Solan and Vieille (2001) with approximate equilibria and with the same property mentioned above pertaining to the Flesch, Thuijsman, and Vrieze example (1997) but with the additional property that there is some triple $\epsilon > 0$, $\delta > 0$ and $\gamma > 0$ such that every ϵ-equilibrium has a stage reached with a probability of at least γ where some player quits the game with a probability of at least ρ.

The complexity of quitting games lies in the potentially large number of players. Even with four players, it is not clear why all quitting games should have approximate equilibria. We describe a simple four player quitting game. The players are represented modulo 4. They are paired in two teams, Player 1 with Player 3, Player 2 with Player 4. If Player i quits alone he gives himself a payoff of 1, gives his partner Player $i+2$ a payoff of 100, and gives the other two players payoffs of 0. If Player $i+2$ quits and his partner Player i doesn't quit then Player i receives 100, regardless of who else might quit. Whenever Players i and $i+2$ do not quit but one or both of the other two players quits then Players i and $i+2$ receive 0. If both Players i and $i+1$ quit, then Player i receives a payoff of -1 no matter who else might quit. If Player i quits, Player $i+1$ does not quit and either Player $i-1$ or Player $i+2$ quits, then Player i receives 99 (also if both $i-1$ and $i+2$ quit). This exhausts all cases. For any proposed ϵ-equilibrium one must ask why the player partnered with the one who stops with the highest probability (with respect to the start of the game) would ever wish to stop the game. If the answer is indeed that he should never choose the move q, then his partner who stops with the highest probability should either stop the game immediately or he should be the only player who stops the game. Neither behavior would describe an ϵ equilibrium for any positive ϵ between 0 and 1/2. On the other hand we think this game belongs to a class of quitting games whose existence of approximate equilibria can be proven similarly to Theorem 3 below.

Our study of quitting games brought us to what could be considered a relatively new topic of dynamic systems. For want of a better term, we call it "Discrete-Time Viability Theory". Conventional Viability Theory concerns continuous-time dynamic processes with some control mechanism and the ability of these processes to stay within given sets; see Aubin (1991). First we give a general description of Discrete-Time Viability Theory, and then present the question that relates this topic to quitting games.

Assume that E is an Euclidean space and F is a correspondence from E to E, meaning that it can be represented as a subset of $E \times E$ (where the projection of F to the first copy of E can be perceived to be the domain and the projection of F to the second copy can be perceived to be the range). For every $x \in E$ define $\overline{F}(x) := \{y \mid (x, y) \in F\}$. A *forward orbit* of the correspondence F is a sequence $(x_0, x_1, \ldots)$ of points of E such that for every non-negative integer $n \geq 0$ we have $(x_n, x_{n+1}) \in F$. An *extended forward orbit* is a sequence $((x_{j,0}, x_{j,1}, \ldots) \mid 0 \leq j < Q)$ of forward orbits, possibly with $Q = \infty$, such that for every j with $j + 1 < Q$ we have $\lim_{k\to\infty} x_{j,k} = x_{j+1,0}$. The extended forward orbit has bounded variation if $\sum_{j<Q} \sum_{i=0}^{\infty} ||x_{j,i} - x_{j,i-1}|| < \infty$, and otherwise it has unbounded variation. By the cluster points of an extended orbit we mean the cluster points of the last orbit (if it exists) or the limit of some subsequence of points where the first index goes to infinity. An extended forward orbit converges to a point if this point is the only cluster point of the orbit. Bounded variation implies convergence, but the converse doesn't hold.

If X is defined as a subset of a Euclidean space E, ∂X will stand for the boundary of X relative to E. The distance in Euclidean space will be the Euclidean distance.

We assume two types of conditions on a correspondence F.

1) There is a subset $C \subseteq E$ with a property that from every point in ∂C the correspondence F contains a motion back into the set C (meaning that for every $x \in \partial C$ there is a $y \in C$ with $(x, y) \in F$),

2) There is some kind of continuity in the definition of F (giving F closure and relating its definition on the interior of C to its definition on the boundary of C).

The topic concerns under what explicit conditions does there exist a forward orbit or an extended forward orbit (possibly with additional properties) for F. With most existing literature on discrete-time dynamic systems the iterations are well defined for trivial reasons and the main questions concern their properties or behavior. Our concerns may have a closer relation to algebraic topology and fixed point theory, though explicitly they are problems of dynamics. A good reference on iterating correspondences is McGehee (1992).

Two examples are given in the last section that demonstrate how there can be no orbits in some contexts if one does not require that some of the motions from the boundary of the set C are small.

A homeomorphism between two topological spaces X and Y is a bijective map that demonstrates that X and Y are topologically equivalent. This means additionally that the collection of open sets of X are mapped bijectively to the collection of open sets of Y.

A *homotopy* is a continuous map $h : X \times [0, 1] \to Y$, where X and Y are topological spaces. Two functions $f : X \to Y$ and $g : X \to Y$ are *homotopic* if f and g both can be represented as $h_t := h(\cdot, t)$ for some $t \in [0, 1]$ (and some homotopy h). If Y can be embedded in a convex space then the homotopy $h : X \times [0, 1] \to Y$ is a *straight line* homotopy if for every $x \in X$ and $t \in [0, 1]$ $h(x, t) = th(x, 1) + (1 - t)h(x, 0)$.

A set X is *contractible* if there is a homotopy $h : X \times [0, 1] \to X$ and a point $y \in X$ such that for all $x \in X$ $h(x, 1) = y$ and $h(x, 0) = x$.

For any finite set A by $\mathbf{R}^A$ we mean the real vector space of dimension $|A|$ whose coordinates are in the set A. For any $r \in \mathbf{R}^A$ and $a \in A$ by r^a we mean the a coordinate of r. Let $\overline{0}$ stand for the origin of any Euclidean space. For every non-negative integer n define $S^n := \{r \in \mathbf{R}^{n+1} \mid ||r|| = 1\}$. Given any subset B of S^n we define the *convex cone generated by* B to be the set $\{r \in R^{n+1} \mid r = \sum_{i=1}^k \lambda_i b_i$, for every $1 \leq i \leq k \;\; b_i \in B$ and $\lambda_i \geq 0\}$.

In this paper we show (Theorem 2) that all quitting games have approximate equilibria if the following question can be affirmed:

Question 1: Let $C \subseteq R^n$ be a union of finitely many compact and convex polytopes (intersections of finitely many half-spaces) $C_1, C_2, \ldots C_k$, each of dimension n. Let $J : C \times [0, 1] \to \mathbf{R}^n \times \mathbf{R}^n$ be a homotopy such that
1) for every $x \in \partial C$ and all $t \in [0, 1]$ $J(x, t) = (x, x)$ and for all $x \in C$ $J(x, 0) = (x, x)$, and
2) conversely $(x, x) \in J(C, 1)$ implies that $x \in \partial C$.

Let V be a compact neighborhood of ∂C, and let G be a compact subset of $V \times \mathbf{R}^n$. Let F be a compact subset of $\mathbf{R}^n \times \mathbf{R}^n$ and define $F_\delta \subseteq F$ by $(x,y) \in F_\delta$ if and only if $(x,y) \in F$ and $||y-x|| \leq \delta$. Assume that
3) F contains both $J(C,1)$ and G, and
that there is a $\gamma > 0$ such that
4) $F_\gamma \subseteq G$.
Furthermore assume that
5) for every $x \in V$ the set $\overline{G}(x)$ is contractible and contains x,
6) if the distance between $x \in V$ and $\partial C \cap C_i$ is no more than γ then there is a $y \in \overline{G}(x)$ with dist $(y, C_i) \leq$dist (x, C_i) such that $||y-x|| \geq \gamma$ and the closed line segment from x to y is in $\overline{G}(x)$,
7) if $\lim_{i \to \infty} \frac{y_i - x_i}{||y_i - x_i||} = s \in S^{n-1}$ for some pair of sequences x_i and y_i both converging to a point $x \in \partial C$ with $(x_i, y_i) \in J(C,1)$ and $y_i \neq x_i$ for every i then the direction s is in the convex cone generated by directions $\frac{y-x}{||y-x||}$ satisfying $(x,y) \in G$, $||y-x|| \geq \gamma$ and that the entire closed line segment from x to y is in C, and
8) C is contractible.
Does there exist an extended forward orbit of F with unbounded variation?

What is the connection between quitting games and the topological question?

There is a strong connection between quitting games and another area of game theory usually not associated with stochastic games – structure theorems used to establish stability properties of one-shot games. We remind the readers of the main theorem of Kohlberg and Mertens, (1986). Let N be a finite player set, $(A^j \mid j \in N)$ the finite sets of actions for the players, X the space of all $|A^1| \times \ldots \times |A^{|N|}|$ matrices with vector payoff entries from $\mathbf{R}^N$. For any $x \in X$ let Γ_x be the one stage game defined by the matrices determined by x. Let $\tilde{A}$ be $\prod_{j \in N} \Delta(A^j)$, the strategy space, (where $\Delta(A^j)$ is the simplex of probability distributions on A^j). Let $E \subseteq X \times \tilde{A}$ be the correspondence defined by $\overline{E}(x) := \{y \in \tilde{A} \mid y$ is an equilibrium of the game $\Gamma_x\}$. Let $\pi : X \times \tilde{A} \to X$ be the canonical projection. The structure theorem of Kohlberg and Mertens states that there is a straight line homotopy $H(\cdot,\cdot)$ from $X \times [0,1]$ to $X \times \tilde{A}$ such that $\pi \circ H(x,0) = x$ for all $x \in X$, the image of $H(\cdot,1)$ is exactly the graph of the correspondence E, and the function H can be extended continuously to the one-point compactification of X (meaning that for every compact set $C \subseteq X$ there is an $R > 0$ large enough that if the norm $||x||$ exceed R then for all $t \in [0, 1]$ the point $H(x, t)$ does not lie over C). Here we have slightly modified the statement of the structure theorem, using the fact that $\tilde{A}$ is convex.

For a quitting game, we can consider the following matrix: in all positions where at least someone has chosen q the corresponding payoff vector is placed. Where all players choose the move c we place a variable vector payoff $x \in \mathbf{R}^N$ that represents the future expected payoff on the next stage given that nobody chose to quit. We could consider what the Kohlberg-Mertens structure theorem could say about the equilibrium correspondence that lies over this subspace isomorphic to $\mathbf{R}^N$, in particular what happens when this correspondence is iterated indefinitely.

There are two problems with applying the Kohlberg-Mertens structure theorem as suggested above. First, we must understand how the equilibrium correspondence (described by the structure theorem) behaves on such subspaces of X. More critical is how the equilibrium correspondence behaves on subsets of vectors that are realized through long term play. Second, if some player can receive more than a payoff of zero by stopping the game alone then the part of the equilibrium correspondence where every player chooses c with certainty is useless to the construction of an approximate equilibrium. Removing these parts of the equilibrium correspondence may destroy important topological properties.

To overcome the two above mentioned problems of applying the Kohlberg-Mertens structure theorem, we prove a new version of the structure theorem that is especially suited to quitting games. In particular, we marginalize those points of the equilibrium correspondence that involve zero probability for the move q. This marginalization is the key step in proving that an affirmation of Question 1 implies the existence of approximate equilibria.

We are agnostic concerning both Question 1 and the existence of approximate equilibria in quitting games. At the present, we tend to think that there are counter-examples to both, however that a game theoretical counter-example is very difficult to find due precisely to its connection to this question and its topological contents. We present the topological question with the hope that it will be the key to understanding approximate equilibria in quitting games – that the existence of approximate equilibria would be proven best by an affirmation of the topological question (and furthermore that a generalization of the topological question could be central to proving approximate equilibrium existence for all multi-player stochastic games) or that the topological question would be refuted best by a quitting game counter-example.

The rest of this paper is organized as follows. The next section presents the formal model of quitting games and defines more precisely the challenge of proving the existence of approximate equilibria. The third section proves our version of the structure theorem. The fourth section establishes the connection between Question 1 and the existence of approximate equilibria. The fifth section considers questions related to Question 1.

2 The Model and the Challenge

In this section we formulate the problem of equilibrium existence as a problem of dynamic systems. Primarily we discuss the work of Solan and Vieille (2001), re-formulating their work in the context of what will follow.

If ϕ is a function taking values in $\mathbf{R}^A$, by ϕ^a we mean the function $\pi_a \circ \phi$, where π_a stands for the projection to the a-coordinate.

2.1 Players, strategies and payoffs

Let N be the set of players. Due to the above mentioned result of E. Solan, we could assume that $|N| \geq 4$; however, for the sake of completeness we will assume only that $|N| \geq 2$. Each player has exactly two moves, q and c, q for "quit" and c for "continue".

For every player let $[0,1]$ stand for her strategy space in a one stage game, with the quantity $p \in [0,1]$ representing the probability that she chooses to end the game (with the move q). N is the set of players, and $[0,1]^N$ stands for the product of the strategy spaces of all the players in a one stage game. Since $\overline{0} \in \mathbf{R}^N$ stand for the origin, $\overline{0} \in [0,1]^N$ means that all players choose the move c with certainty.

A strategy profile for the players is a sequence of probability vectors $(p_i \mid i = 0, 1, 2, \ldots)$ such that for every stage i $p_i \in [0,1]^N$. p_i^j stands for the probability that Player j will stop the game (with the move q) at stage i.

The payoffs are defined as follows. For every non-empty subset $A \subseteq N$ of players there is a payoff vector $v(A) \in \mathbf{R}^N$. At the first stage that any player chooses the move q and A is the non-empty subset of players that choose q, the players receive the payoff $v(A)$. This means that Player i receives $v(A)^i \in \mathbf{R}$. If nobody plays the move q throughout all stages of play, then all players receive 0. The quantity M is defined to be $40 + 3 \max_{i\in N\ ,\ \emptyset\neq A\subseteq N} \ |v(A)^i|$.

Define $\chi \in \mathbf{R}^N$ to be that vector such that for every $j \in N$ χ^j is the min-max value for Player j (the upper bound for what Player j can obtain in response to all strategy choices of the other players). The importance of the min-max value χ^j is that it represents the ability of the players to punish Player j with pre-determined strategies (for example as part of an approximate equilibrium). Because the other players are limited in their ability to coordinate their moves, this min-max value could be strictly greater than the max-min value when there are at least two other players.

2.2 Equilibrium correspondences

For every $r \in \mathbf{R}^N$ let Γ_r be the one stage game where Player $j \in N$ receives the payoff r^j if all players choose the move c.

For every $r \in \mathbf{R}^N$ and $p \in [0,1]^N$, let $a^j(p)$ be the expected payoff for Player j if he chooses q against the strategies $(p^k\ |k \neq j)$ and let $b^j(p,r)$ be the expected payoff for Player j from the move c, given that the other players choose the strategies $(p^k\ |k \neq j)$ and he will receive the payoff r^j if everyone chooses the move c (meaning that the game Γ_r is played. One can calculate $a^j(p)$ and $b^j(p,r)$ easily. We have

$$a^j(p) = \sum_{A\subseteq N\backslash\{j\}} v(A \cup \{j\})^j \prod_{k\in A} p^k \prod_{k\neq j,\ k\notin A} (1-p^k)$$

and

$$b^j(p,r) = r^j \prod_{k\neq j} (1-p^k) + \sum_{\emptyset\neq A\subseteq N\backslash\{j\}} v(A)^j \prod_{k\in A} p^k \prod_{k\neq j,\ k\notin A} (1-p^k).$$

Every strategy profile $p = (p_i \mid i = 0,1,2,\ldots)$ defines payoff vectors $(r_i \in \mathbf{R}^N \mid i = 0,1,2,\ldots)$ for the players. r_i^j is the future expected payoff for player j before the moves are made at the stage i, conditioned on the fact that all players chose c at all stages before i. This means that r_i is the expected payoff vector for the quitting game that begins at stage i.

Define a function $q : [0,1]^N \to [0,1]$ by $q(p) := 1 - \prod_{j\in N}(1-p^j)$. The function q is the total probability that at least one player chooses the move q.

A strategy profile $p = (p_i \mid i = 0, 1, 2, \ldots)$ is a sequence of *perfect one-shot ϵ equilibria* if for every stage i and every player j the following holds:
i) if $p_i^j > 0$, then $a^j(p_i) \geq b^j(p_i, r_{i+1}(p)) - \epsilon$,
ii) if $p_i^j < 1$ then $b^j(p_i, r_{i+1}(p)) \geq a^j(p_i) - \epsilon$,
iii) for every $i \geq 0$ $r_i^j(p) \geq \chi^j - \epsilon$.
The strategy profile is absorbing if
iv) $\sum_{i=0}^{\infty} q(p_i) = \infty$.

We want to consider correspondences generated by moving backward from stage $i+1$ to stage i through an approximate equilibrium of the one-shot game. For any $\epsilon, \rho \geq 0$ we construct correspondences $E_{\epsilon,\rho} \subseteq \mathbf{R}^N \times [0,1]^N$ and $F_{\epsilon,\rho} \subseteq \mathbf{R}^N \times \mathbf{R}^N$ in the following way. We set

$$\overline{E}_{\epsilon,\rho}(r) := \{p \in [0,1]^N \mid p^j > 0 \Rightarrow a^j(p) \geq b^j(p,r) - \epsilon,$$

$$p^j < 1 \Rightarrow b^j(p,r) \geq a^j(p) - \epsilon, \quad q(p) \geq \rho\}.$$

For every $r \in \mathbf{R}^N$ and $p \in [0,1]^N$ define a new member of $\mathbf{R}^N$, namely

$$f(r,p) := r \prod_{j \in N} (1 - p^j) + \sum_{\emptyset \neq A \subset N} v(A) \prod_{j \in A} p^j \prod_{j \notin A} (1 - p^j).$$

We define $\overline{F}_{\epsilon,\rho}(r) := \{f(r,p) \mid p \in E_{\epsilon,\rho}(r)\}$. For every $r \in \mathbf{R}^N$ $\overline{E}_{\epsilon,\rho}(r)$ is a subset of the ϵ equilibria of the game Γ_r with at least a ρ probability that somebody chooses to quit; $\overline{F}_{\epsilon,\rho}(r)$ are their corresponding payoffs.

2.3 Basic results

Lemma A: An absorbing sequence of perfect one-shot ϵ equilibria generates a γ equilibrium for the relation $\epsilon = c\gamma^6$ for some $c > 0$ that is determined by the number of players and a bound on the absolute value of all payoffs.

Proof: Solan and Vieille (2001) proved this result without Condition iii) but with the condition that every player quitting alone receives a positive payoff, (meaning $v^j > 0$ for all $j \in N$). However their additional condition is used in their proof only to demonstrate that $r_i^j > \chi^j - \gamma$ for all $i \geq 0$ and $j \in N$. $\square$

The underlying justification for Lemma A (explicit in Solan and Vieille 2001) is the following: either over some long period of near certain quitting this is due almost exclusively to the actions of a single player, or over all long periods of near certain quitting this is due to the actions of at least two players. If the former is true, then there will an approximate equilibrium resulting from the quitting behavior of this one actor, and enforced by punishment in the event that this player refuses to end the game. If the latter is true, then the passivity of any player cannot prevent an end to the game and the stage for stage equilibrium property will imply a sufficient cumulative equilibrium property.

Remark 1: For the sake of proving that there is a sequence of perfect one-shot 2ϵ equilibria we can assume that $||x - y|| > 0$, $(x, y) \in \overline{F}_{\epsilon,0}$ and x in $\{z \mid \forall j\ z_i^j \leq \chi^j - \epsilon\}$ imply that $\epsilon\ q(p) \leq ||x - y||$ for the corresponding strategies $p \in [0, 1]^N$. If this were not true then $q(p) \geq \frac{||x-y||}{\epsilon}$ would result in expected payoffs conditioned on quitting within a Euclidean distance of ϵ from x. Since x is in $\{z \mid \forall j\ z_i^j \geq \chi^j - \epsilon\}$, the stationary strategy profile generated by p would be a sequence of perfect one-shot 2ϵ equilibria.

The next three propositions are useful for proving that an affirmation of Question 1 implies the existence of approximate equilibria. The argument of the following proposition is similar to one in Solan and Vieille (2001) concerning cyclic orbits.

Proposition A If for all positive $\epsilon > 0$ the correspondence $F_{\epsilon,0}$ has an extended forward orbit in $\{x \mid \forall j\ x_i^j \geq \chi^j - \epsilon\}$ with unbounded variation then the quitting game has approximate equilibria.

Proof: We assume the existence of an extended forward orbit $((x_{l,0}, x_{l,1}, \ldots) \mid 0 \leq l < Q)$ of $F_{\epsilon/3,0}$ with unbounded variation in $\{x \mid \forall j\ x_i^j \geq \chi^j - \epsilon/3\}$. Let $(p_{l,i}| \mid i = 1, 2, \ldots)$ be the corresponding strategies in $[0, 1]^N)$ (such that $x_{l,i+1} = f(x_{l,i}, p_{l,i})$).

Case 1; There is a sequence $(x_{l,0}, x_{l,1}, \ldots)$ such that $\sum_{i=0}^{\infty} ||x_{l,i} - x_{l,i+1}|| = \infty$:

Let x be any cluster point of this sequence. $\sum_{i=0}^{\infty} q(p_{l,i}) < \infty$ would imply $\sum_{i=0}^{\infty} ||x_{l,i} - x_{l,i+1}|| < \infty$, so assume $\sum_{i=0}^{\infty} q(p_{l,i}) = \infty$, which implies that $\prod_{i=0}^{\infty}(1 - q(p_{l,i})) = 0$. Let $x_{l,m}$ and $x_{l,n}$ be any two points in this sequence with $m < n$ such that both vectors are within $\epsilon/8$ of x and $\prod_{m \leq i < n}(1 - q(p_{l,i})) < \frac{\epsilon}{8M}$. Observe what happens by reversing the order of the strategies between $x_{l,m}$ and $x_{l,n}$ and repeating them cyclically and infinitely, meaning $(\tilde{p}_0 = p_{l,n-1},\ \tilde{p}_1 = p_{l,n-2}, \ldots, \tilde{p}_{n-m-1} = p_{l,m},\ \tilde{p}_{n-m} = p_{l,n-1}, \ldots)$, and define

$(\tilde{r}_i \mid i = 0, 1, \ldots)$ to be the vectors representing the expected payoffs as generated above. For every $k = 0, 1, \ldots$ $\tilde{r}_{k(n-m)}$ will be within $\epsilon/8$ of $x_{l,n}$, since the probability of quitting from the deleted part of the sequences does not exceed $\frac{\epsilon}{8M}$. But then for ever $k = 0, 1, \ldots$ and $0 \leq i < n-m$ $||\tilde{r}_{k(n-m)+i} - x_{l,n-i}|| \leq \epsilon/2$, since $||x_{l,m} - x_{l,n}|| \leq \epsilon/4$. After discarding the stages where $\overline{0}$ was the corresponding strategy we have a cyclic orbit of $F_{\epsilon,\delta}$ for some positive δ. The rest follows by Lemma A.

Case 2; $\sum_{i=0}^{\infty} ||x_{l,i} - x_{l,i+1}|| < \infty$ for every $l < \infty$:
Let x be any cluster point of the sequence $(x_{0,0}, x_{0,1}, \ldots)$. As with Case 1 let $x_{m,0}$ and $x_{n,0}$ be any two points in this sequence such that both are within $\epsilon/5$ of x and $\prod_{m<l<n} \prod_{i=0}^{\infty} (1 - q(p_{l,i})) < \frac{\epsilon}{30M}$. By Remark 1 we can assume without loss of generality that $\sum_{i=0}^{\infty} q(p_{l,i}) < \infty$ for all $l < \infty$. For every $m \leq l < n$ with $\sum_{i=0}^{\infty} ||x_{l,i} - x_{l,i+1}|| > 0$ let n_l be a positive integer large enough so that $\prod_{i=0}^{n_l} (1 - q(p_{l,i})) < (1 + \frac{\epsilon}{30M}) \prod_{i=0}^{\infty} (1 - q(p_{l,i}))$. Exactly as with Case 1, reversing the order of the corresponding strategies $(p_{l,i} \mid m \leq l < n, 0 \leq i \leq n_l)$, and dropping the ones with zero probability of quitting, will generate a cyclic orbit of $F_{\epsilon,\delta}$ for some positive δ. $\square$.

Proposition B: Assume that each payoff in $(v(A) \mid \emptyset \neq A \subseteq N)$ has been changed by no more than $\delta > 0$. For every $\epsilon \geq 0$ an absorbing sequence of perfect one-shot ϵ equilibria for the resulting game is also an absorbing sequence of perfect one-shot $\epsilon + \delta$ equilibria for the original game.

Proof: Because each r_i^j determined by the strategy profile cannot change by more than δ, it follows directly from the conditions defining the perfect one-shot $\epsilon + \delta$ equilibria. $\square$.

Proposition C: If for any $x \in \mathbf{R}^N$ with $x^j \geq \chi^j - \epsilon$ for all $j \in N$ there is a member p of $E_{\epsilon,1}(x)$ (with $q(p) = 1$), then there is an ϵ'-equilibrium for any $\epsilon' > 2\epsilon$.

Proof: The players are requested to play p, following by the punishment of any player j with $p^j = 1$ who did not choose q. $\square$

If there is a one stage equilibrium as described in Proposition C we say that it is an *instant* ϵ^+-equilibrium.

3 The Structure Theorem for Quitting Games

3.1 The Theorem

Define $\tilde{E}_{0,0}$ to be the subset of $E_{0,0}$ such that the $p \in [0,1]^N$ probability vector obeys $q(p) < 1$.

The vector $v \in \mathbf{R}^N$ is defined by $v^i := v(\{i\})^i$ for every $i \in N$. For every Player $j \in N$ define the set $W_j := \{r \mid r^j \leq v^j\}$. Define $W := \cup_{j \in N} W_j = \{r \mid r^j \leq v^j \text{ for some } j \in N\} = \mathbf{R}^N \backslash \{r \mid r^j > v^j \text{ for all } j \in N\}$. ∂W will be $W \cap \{x \mid \forall j \in N \ x^j \geq v^j\}$, also an unbounded set. ∂W is the set where for at least one $j \in N$ $x^j = v^j$ and otherwise $x^k \geq v^k$ for all $k \neq j$.

The main goal of this section is to prove the following structure theorem for quitting games.

Theorem 1: There is a straight line homotopy $H(\cdot,\cdot)$ from $\mathbf{R}^N \times [0,1]$ to $\mathbf{R}^N \times [0,1]^N$ such that for all $x \in X$
1)$H(x,0) = (x,\overline{0})$,
2) the image of $H(\cdot,1)$ is the set $\tilde{E}_{0,0}$,
3) for all x in the closure of the complement of W and for all $t \in [0,1]$ we have $H(x,t) = (x,\overline{0})$.
If there is no instant δ^+ equilibrium for some $\delta > 0$ then
4) there exists an $R > 0$ such that for any $x \notin [-R,R]^N$ and $t \in [0,1]$ we have that $H(x,t) \notin \{x \in \mathbf{R}^N \mid \chi^j - \frac{99\delta}{100} \leq x^j \leq |N|M + 1\} \times [0,1]^N$.

The part of the equilibrium correspondence that is useless for understanding the approximate equilibria of quitting games is that coming from the complement of W. Its topological marginalization is the third property. The fourth property is analogous to the property in Kohlberg and Mertens (1986) that the homotopy can be extended continuously to the one-point compactification, and it plays a critical role in this paper.

3.2 The map ϕ

We fix any ϵ with $0 < \epsilon \leq 1$. (Although ϵ plays no significant role in this section, its use as a variable will be important later.) We define a map ϕ from $\tilde{E}_{0,0}$ to $\mathbf{R}^N$ in the following way. Given any $(x,p) \in \tilde{E}_{0,0} \subseteq \mathbf{R}^N \times [0,1]^N$, we define for every $j \in N$

$$\phi^j(x,p) := f^j(x,p) - \frac{5|N|^2 M^3}{\epsilon^2} \frac{p^j}{(1-p^j)^{|N|}} + M \sum_{k \neq j} p^k.$$

Because we consider only those equilibria with $q(p) < 1$, the map ϕ is well defined and continuous. The most dramatic aspect of the definition of ϕ is the reduction in the value of ϕ^j for a player j who chooses q with significant probability. On the other hand, the dominance of M over all the payoffs from the game implies that if a player j chooses c with certainty then increasing the probabilities that the other players choose q will increase the value of ϕ^j.

3.3 The bijectivity of ϕ

Lemma 1: ϕ is injective. Furthermore $(x, \overline{0}) \in \tilde{E}_{0,0}$ if and only if $x \in \{y \in \mathbf{R}^N \mid y^j \geq v^j \ \forall j \in N\}$, (the closure of the complement of W), and if so then $\phi(x, \overline{0}) = x$.

Proof: Let (x,p) and $(\hat{x}, \hat{p})$ be two distinct equilibria in $\tilde{E}_{0,0}$. Clearly if $p = \hat{p}$ then $q(p) \neq 1$ implies that $\phi(x,p) = \phi(\hat{x}, \hat{p})$ if and only if $x = \hat{x}$. Therefore we assume that $p \neq \hat{p}$, and we assume that $j \in N$ is a player such that $|p^j - \hat{p}^j| = \max_{k \in N} |p^k - \hat{p}^k|$. Without loss of generality we assume that $\hat{p}^j > p^j$, and let $t := \hat{p}^j - p^j$. Supposing that $\phi(x,p) = \phi(\hat{x}, \hat{p})$, we will show that Player j with (x,p) has a clear preference for choosing q, a contradiction.

First, we compare what happens when Player j in both situations chooses the move q. We get $f^j(x,p) \geq a^j(p) > a^j(\hat{p}) - (1-(1-t)^{|N|-1})M = f^j(\hat{x}, \hat{p}) - (1-(1-t)^{|N|-1})M$. The first inequality follows because with (x,p) Player j does not choose q with certainty; the second inequality follows because t is the largest difference in probability used by any player and all differences in payoffs are less than $2M/3$; the equality at the end follows because with $(\hat{x}, \hat{p})$ Player j chooses q with some positive probability less than one.

Look at the consequences of $\phi^j(x,p) = \phi^j(\hat{x}, \hat{p})$ from the definition of map ϕ and the quantity $t = \hat{p}^j - p^j > 0$. Looking at the $\frac{5|N|^2 M^3}{\epsilon^2} \frac{p^j}{(1-p^j)^{|N|}}$ term defining $\phi^j(x,p)$ we have

$$\frac{5|N|^2 M^3}{\epsilon^2} \frac{\hat{p}^j}{(1-\hat{p}^j)^{|N|}} - \frac{5|N|^2 M^3}{\epsilon^2} \frac{p^j}{(1-p^j)^{|N|}} \geq \frac{5|N|^2 M^3}{\epsilon^2} \frac{t}{(1-t)^{|N|}}.$$

After ignoring the $\epsilon \leq 1$ we have from the definition of ϕ^j and the equality $\phi^j(x,p) = \phi^j(\hat{x},\hat{p})$ that $f^j(\hat{x},\hat{p}) - f^j(x,p) \geq 5|N|^2M^3\frac{t}{(1-t)^{|N|}} - M\sum_{k\neq j}|\hat{p}^k - p^k| \geq 5|N|^2M^3\frac{t}{(1-t)^{|N|}} - M(|N|-1)t$. Together with the last paragraph we have $M(|N|-1)t + (1-(1-t)^{|N|-1})M > 5|N|^2M^3\frac{t}{(1-t)^{|N|}}$. From $\frac{|N|}{(1-t)^{|N|}} > |N|-1$ and dropping a power of M and $|N|$ we conclude that $(1-(1-t)^{|N|-1}) > 4|N|M\frac{t}{(1-t)^{|N|}}$.

For a contradiction we need only show that $4|N|t + (1-t)^{2|N|-1} - (1-t)^{|N|} > 0$ for all $0 < t \leq 1$. If $t = 0$ there is an equality. We get our claim by taking the derivative in t of this function and showing that this derivative must be positive for all the values $0 \leq t \leq 1$. The derivative is $4|N| - 2|N|(1-t)^{2|N|-2} + (1-t)^{2|N|-2} + |N|(1-t)^{|N|-1}$, which is at least $2|N|$ for all $0 \leq t \leq 1$. Injectivity is proven.

If $x^j \geq v^j$ for all $j \in N$, then there exists at least one equilibrium in $\tilde{E}_{0,0}(x)$, namely the strategy $\overline{0}$; by the definition of ϕ we have that $\phi(x,\overline{0}) = x$. If $x^j < v^j$ for some $j \in N$, then $(x,\overline{0})$ cannot be in $\tilde{E}_{0,0}$, since Player j would prefer choosing q over the move c. □

Now the term $\sum_{k\neq j} p^k M$ in the definition of ϕ will play a critical role in proving that ϕ is surjective.

Lemma 2: ϕ is surjective, meaning that it is onto $\mathbf{R}^N$. Furthermore, ϕ^{-1} is continuous, (meaning with the injectivity and continuity of ϕ that it is a homeomorphism).

Proof: Let $x \in \mathbf{R}^N$ be arbitrary, and let $\zeta := 1 + \max\{0, -v^j, -x^j \mid j \in N\}$. Take any $0 < t < 1$ such that $\frac{t\epsilon^2}{5M^3|N|^2(1-t)^{|N|}} > 2\zeta + 2|N|M$.

Next consider the set $Y_t := \{(x,p) \mid p \in [0,t]^N, \forall j \in N\ a^j(p) = b^j(p,x)\}$. Because $t < 1$, given any $p \in [0,t]^N$ we have a well defined and unique x with $(x,p) \in Y_t$. (For a fixed p and any player j one knows from $t < 1$ that there is a value $\hat{x}^j \in \mathbf{R}$ high enough and a value $\tilde{x}^j \in \mathbf{R}$ low enough so that if the other players acted according to p in the game $\Gamma_{\hat{x}}$ then Player j would prefer c but Player j would prefer to choose q in the game $\Gamma_{\tilde{x}^j}$. Between $\tilde{x}^j$ and $\hat{x}^j$ we find our unique x^j.) Define for all $p \in [0,t]^N$ $x(p)$ to be that x such that $(x(p),p) \in Y_t$. We see also from $t < 1$ that $x(p)$ is continuous in p. Consider what happens when ϕ is applied to the set Y_t. We define $\tilde{\phi} : [0,t]^N \rightarrow \mathbf{R}^N$ by $\tilde{\phi}(p) := \phi(x(p),p)$.

The definition of t assures with any $p \in [0,t]^N$ that $p^j = t$ implies $\tilde{\phi}(p)^j <$

$-\zeta - 1$. This follows from the definition of ϕ^j and the fact that $a^j(p) = b^j(p, x(p)) = f^j(x(p), p)$ must be within $-M/3$ and $M/3$.

Assuming that x is not already in $\tilde{\phi}([0,t]^N)$ to show that x is in the image of ϕ it suffices to find a $y \in \tilde{\phi}([0,t]^N)$ such that $x^j \geq y^j$ for all $j \in N$ and $x^j > y^j$ implies that $(\tilde{\phi}^{-1}(y))^j = 0$. Setting $\hat{p} = \tilde{\phi}^{-1}(y)$ we would have $(x, \hat{p})$ in $\tilde{E}_{0,0}$ since any player j satisfying $x^j > y^j$ would choose the move c with certainty.

For every $p \in [0,t]^N$ define support $(p) := \{j \in N \mid p^j > 0\}$.

Claim A: For every $p \in [0,t]^N$ and every vector $s \in \mathbf{R}^{\text{ support}(p)}$ with $s^j \geq 0$ for all $j \in \text{support}(p)$ and $s^j \neq \overline{0}$ there is a point $\hat{p} \in [0,t]^N$ with $\text{support}(\hat{p}) \subseteq \text{support}(p)$ such that the vector $\tilde{\phi}(\hat{p}) - \tilde{\phi}(p)$ when restricted to the support(p) coordinates is a multiple of s by some scaler λ with $0 < \lambda \leq 1$, and furthermore if $j \notin \text{support}(p)$ then the jth coordinate of $\tilde{\phi}(\hat{p}) - \tilde{\phi}(p)$ is negative.

Claim B: For every $y \in \tilde{\phi}([0,t]^N)$ and $z \in [-\zeta, \infty)^N$ such that $z^j \leq y^j$ for all $j \in N$ it follows that z is also in $\tilde{\phi}([0,t]^N)$.

Claim C: $\tilde{\phi}$ is a homeomorphism from $[0,t]^N$ to its image.

Proof of the Claims: Notice from the polynomial functions $a^j : [0,1]^N \to \mathbf{R}$ that the function $\tilde{\phi}$ is rational, hence smooth (continuous partial derivatives of all degrees). By considering maximal changes in the payoffs relative to changes in probability vectors we have $M/3 < \frac{\partial^j \tilde{\phi}}{\partial p^i} < 5M/3$ for all $i \neq j$ and $\frac{\partial^j \tilde{\phi}}{\partial p^j} < -4|N|^2 M^3$ for all $j \in N$. These bounds on the partial derivatives confirm all three claims. Claims A follows from arbitrary small decreases in the values of the coordinates in the support of p and the fact that the desired direction of the vector s lies in the interior of the convex cone generated by the directions defined by the partial derivatives. Claim B results from the same argument and small increases in all coordinates of p. Claim C follows from Lemma 1 and the fact, easy to confirm, that the Jacobian determinant is bounded uniformly far away from zero (on the negative side if $|N|$ is odd).

We assume without loss of generality that $x \notin \tilde{\phi}([0,t]^N)$. Define a function w from $\tilde{\phi}([0,t]^N) \cap \prod_{j \in N} [-\zeta, x^j]$ to $\mathbf{R}$ by $w(z) := \max_{\, j \in N \ , \ z^j < x^j} (\tilde{\phi}^{-1}(z))^j$. By Claim B, we know that $(-\zeta, -\zeta, \ldots, -\zeta)$ is in $\tilde{\phi}([0,t]^N)$, and therefore the domain of w is not empty. By Claim C $\tilde{\phi}^{-1}$ is a continuous function, therefore w is a lower-semi-continuous function and a minimum value $\tilde{w} \geq 0$

is obtained. If $\tilde{w} = 0$, then we are done. For the sake of contradiction, we suppose that $\tilde{w}$ is positive. From Claim A we can find another $\hat{z}$ in the domain with an even smaller value for w, a contradiction. $\square$

3.4 The Homotopy

Lemma 3: Let ϕ be defined by some $0 < \epsilon \leq 1$. For any $0 < \delta \leq 1$ if there is no instant δ^+-equilibrium then there is an $R > \frac{200M^4|N|^3}{\epsilon^2}$ such that if $|x^j| \geq R$ for some player j, $x = \phi(y,p)$ and $z = ty + (1-t)x$ for some $0 \leq t \leq 1$ then $z \notin \{\xi \in \mathbf{R}^N \mid \forall j \in N \ \chi^j - \frac{99\epsilon}{100} \leq \xi^j \leq M|N| + 1\}$. Furthermore if $x^j \geq R$ for some player j then $y^j > R - M|N| > M|N| + 1$ and $p^j = 0$.

Proof: By the compactness of $E_{0,0} \cap \{y \in \mathbf{R}^N \mid \forall j \in N \ \chi^j - \epsilon \leq y^j \leq 2M|N|\} \times [0,1]^N$ and the continuity of ϕ^{-1} the $[0,1]$ coordinates for all players must obtain a maximum value. Since there is no instant δ^+ equilibrium this value must be strictly less than one. If the difference between this value and one is less than $\frac{1}{20}(\frac{\delta}{2M|N|})^{|N|}$, then let w equal this difference. If not, then let w equal $\frac{1}{20}(\frac{\delta}{2M|N|})^{|N|}$. Let R equal $\frac{10M^4|N|^3}{\epsilon^2 w^{|N|}}$.

Case 1; $x^j \geq R$ for some player j: By the definition of ϕ $f^j(y,p) \geq R - (|N|-1)M > |N|M + 1$. Since all payoffs of the game are below $M/3$ it follows that $p^j = 0$ and $y^j > |N|M + 1$.

Case 2; $x^j \leq -R$ and $y^k \leq \chi^k - \delta$ for a pair of players j and k (with $j = k$ ok): The definition of ϕ implies that $x^k < (|N|-1)M + y^k < |N|M$. By the very large size of R we have either $z^k < \chi^k - \frac{99\delta}{100}$ or $z^j < \chi^j - \frac{99\delta}{100}$.

Case 3; $x^j \leq -R$ for some player j, $y^k > \chi^k - \delta$ for all players k and $y^l \leq 2|N|M$ for all players l satisfying $p^l \leq 1 - \frac{\delta}{2|N|M}$:

By the definition of R we know that $p^j > 1 - w$. Let A be the set of players who quit with a probability of at least $1 - \frac{\delta}{2|N|M}$, (with j belonging to A). Consider a new probability tuple $\hat{p} \in [0,1]^N$ defined by $\hat{p}^m = p^m$ if $m \notin A$ and $\hat{p}^m = 1$ if $m \in A$. With $\hat{\chi} \in \mathbf{R}^N$ defined by $\hat{\chi}^m := \chi^m - \delta$ we claim that $(\hat{\chi}, \hat{p})$ is an instant δ^+ equilibrium (and therefore Case 3 is not possible). For any player m in A the payoff from quitting does not change by more than $\frac{(|N|-1)\delta}{3|N|}$. Since Player m should quit with certainty with $\hat{p}$ the assumption $y^m > \hat{\chi}^m$ is sufficient to assure that there is no advantage from not quitting of more than $\frac{(|N|-1)\delta}{3|N|} < \delta/3$ over quitting. On the other hand if m is not in

A then we have assumed that $y^m \leq 2|N|M$. By the same argument as above $|a^m(\hat{p}) - a^m(p)| \leq \frac{(|N|-1)\delta}{3|N|}$. On the other hand $\chi^m - \delta < y^m \leq 2|N|M$ and $p^j > 1 - w$ imply that $|b^m(\hat{\chi}, \hat{p}) - b^m(y, p)| \leq \frac{(|N|-1)\delta}{3|N|} + 2|N|Mw < \delta/2$.

Case 4; $x^j \leq -R$ for some player j, $y^k > \chi^k - \delta$ for all players k and $y^l > 2|N|M$ for some player l satisfying $p^l \leq 1 - \frac{\delta}{2|N|M}$:

Let $1 - \xi$ be the largest probability with which any player chooses the move q, and let m be a player who chooses q with probability $1 - \xi$. By the definition of ϕ we have $x^m \leq |N|M - \frac{5M^3|N|^2}{\epsilon^2 \xi^{|N|}}$. On the other hand $y^m < M + \frac{M}{\xi^{|N|-1}}$, since otherwise Player m would prefer not to choose q. Due to the definition of ϕ we have $x^m < -R + M|N|$ and $\xi \leq w$. Due to the above comparison between x^m and y^m to avoid $z^m < \chi^m - \frac{99\epsilon}{100}$ it is necessary for t to be at least $1 - \frac{\epsilon^2 \xi}{3M^2|N|^2}$, meaning that t must be at least $1 - \frac{\epsilon^2 w}{3M^2|N|^2}$. By looking at the l coordinate we have $x^l \geq 2|N|M - \frac{5M^3|N|^2 p^l}{\epsilon^2(1-p^l)^{|N|}}$ and therefore with our assumption on t being at least $1 - \frac{\epsilon^2 \xi}{3M^2|N|^2}$ and $p^l \leq 1 - \frac{\delta}{2|N|M}$ we have $z^l \geq 2|N|M - \frac{5M^3|N|^2}{\epsilon^2(\frac{\delta}{2|N|M})^{|N|}} \frac{\epsilon^2 w}{3M^2|N|^2}$. But with w less than or equal to $\frac{1}{20}(\frac{\delta}{2M|N|})^{|N|}$ we get $z^l \geq 2|N|M - M/4$.

The last part of the lemma was proven in Case 1. □

Proof of Theorem 1: We define the map $H : \mathbf{R}^N \times [0,1] \to R^N \times [0,1]^N$ by $H(x,t) := (1-t)(x, \overline{0}) + t(\phi^{-1}(x))$. The continuity of H follows from the continuity of ϕ^{-1}, proven in Lemma 2. The first two properties follow from the definition of H. The third property follows from Lemma 1. The fourth property follows from Lemma 3. q.e.d.

3.5 More properties of ϕ

The following proposition is helpful to understanding the correspondence ϕ. From the continuity of ϕ and ϕ^{-1} and from Theorem 1 if $\alpha \in W$ maps by ϕ^{-1} to $(\beta, p) \in \tilde{E}_{0,0}$ and p is very close to but not equal to $\overline{0}$ then we know that α is close to but not in ∂W, and vice versa. Now we quantify these relations, (which we will need later).

For any $p \in [0,1]^N$ and $j \in N$ define $p_{-j} \in [0,1]^N$ by $p^j_{-j} = 0$ and $p^k_{-j} = p^k$ for $k \neq j$.

Proposition 1: Let ϕ be defined by $0 < \epsilon \leq 1$. Assume that there is no

instant δ^+ equilibrium and let R be large enough to satisfy Theorem 1. Let $\overline{w} > 0$ be small enough so that $\phi^{-1}([-R-1, R+1]^N) \subseteq \mathbf{R}^N \times [0, 1-\overline{w}]^N$. Assume that α is in $W \cap [-R-1, R+1]^N \backslash \partial W$ with $\alpha = \phi(\beta, p)$ and $c = f(\beta, p)$. It follows that
1) $\alpha^j \leq v^j$ implies that Player j chooses q with positive probability and $p^j \geq \frac{\epsilon^2 \overline{w}^{|N|-1}}{5|N|^2 M^3}(\frac{M}{3}\sum_{k\neq j} p^k + v^j - \alpha^j) \geq \frac{\epsilon^2 \overline{w}^{|N|-1}}{5|N|^2 M^3}(q(p_{-j})\frac{M}{3} + v^j - \alpha^j)$,
2) $\alpha^j \leq \beta^j$ implies that Player j chooses q with positive probability and $p^j \geq \frac{\epsilon^2 \overline{w}^{|N|-1}}{5|N|^2 M^3}(\frac{M}{3}\sum_{k\neq j} p^k + \beta^j - \alpha^j) \geq \frac{\epsilon^2 \overline{w}^{|N|-1}}{5|N|^2 M^3}(q(p_{-j})\frac{M}{3} + \beta^j - \alpha^j)$,
3) $p^j > 0$ and $\alpha^j \geq v^j$ implies that $q(p_{-j}) > 0$ with $\sum_{k\neq j} p^k \geq (\alpha^j - v^j)/(2M)$,
4) $\alpha^j \geq \beta^j$ implies that $q(p_{-j}) > 0$ with $\sum_{k\neq j} p^k \geq (\alpha^j - \beta^j)/(2M)$,
5) if α is within $\gamma > 0$ of ∂W then $q(p) < \frac{\gamma \epsilon^2}{4|N|M^3}$.

Proof: 1) Because the difference between any two payoffs is less than $2M/3$, the payoff for Player j if she chooses q is at least $v^j - \frac{2M}{3}\sum_{k\neq j} p^k$. Since c^j cannot be less than this quantity we have $c^j + M\sum_{k\neq j} p^k \geq v^j + \frac{M}{3}\sum_{k\neq j} p^k$. Since $1-\overline{w}$ is an upper bound for the probability that any player quits, the definition of $\phi(\beta, p) = \alpha$ implies that $\alpha^j \geq v^j + \frac{M}{3}\sum_{k\neq j} p^k - p^j \frac{5|N|^2 M^3}{\epsilon^2 \overline{w}^{|N|-1}}$. Furthermore, $p^j = 0$ would imply with Theorem 1 that $\sum_{k\neq j} p^k > 0$, and therefore also $\alpha^j > v^j$.

2) Because the difference between any two payoffs is less than $2M/3$, the payoff for Player j if she chooses c is at least $\beta^j - \frac{2M}{3}\sum_{k\neq j} p^k$, and so $c^j + M\sum_{k\neq j} p^k \geq \beta^j + \frac{M}{3}\sum_{k\neq j} p^k$. The rest of the argument follows identically to that of 1), but with β replacing v.

3) Because the difference between any two payoffs is less than $2M/3$, the payoff for Player j if she chooses q is no more than $v^j + \frac{2M}{3}\sum_{k\neq j} p^k$, which implies that $c^j + M\sum_{k\neq j} p^k \leq v^j + \frac{5M}{3}\sum_{k\neq j} p^k$. This implies that $\alpha^j \leq v^j + \frac{5M}{3}\sum_{k\neq j} p^k$. Furthermore $q(p_{-j}) = 0$ would imply with Theorem 1 that $p^j > 0$ and $\alpha^j < c^j = v^j$.

4) The payoff for Player j if she chooses c is no more than $\beta^j + \frac{2M}{3}\sum_{k\neq j} p^k$. The rest of the argument follows identically to that of 3), but with β replacing v.

5) Let t be $\max_{j\in N} p^j$, and let k be a player such that $p^k = t$. By the initial assumption we have $\alpha^k \geq v^k - \gamma$. $\alpha^k \leq c^k - \frac{5|N|^2 M^3 t}{\epsilon^2 (1-t)^{|N|}} + (|N|-1)Mt$ follows from the definition of t and ϕ. We have $a^k(p) = c^k < v^k + \frac{2}{3}(|N|-1)Mt$ from considering what happens when Player k chooses q. But all three inequalities

together imply that $\frac{5|N|^2M^3t}{\epsilon^2(1-t)^{|N|}} < \gamma + \frac{5}{3}(|N|-1)Mt$ and $\frac{4|N|^2M^3t}{\epsilon^2} < \frac{4|N|^2M^3t}{\epsilon^2(1-t)^{|N|}} < \gamma$. The rest follows by the definition of t. □

4 From Topological Question to Approximate Equilibrium Existence

4.1 The set-up

There are two main problems with the correspondence $F_{0,0}$.

First, in most interesting cases of quitting games (where there are no 0 equilibria) all extended forward orbits of $F_{0,0}$ will have bounded variation and converge to points in ∂W. To get around this problem we glue to $F_{0,0}$ a correspondence contained within $F_{\delta,0}$ for some small $\delta > 0$ and defined on points near to ∂W. This new correspondence will involve only very small motions but enough to allow, either sometimes or always, for the existence of unbounded variation extended forward orbits. The same idea was contained in Solan and Vieille (2001), but without the application of a structure theorem.

Second, we need a theoretical context, that of correspondences defined on compact sets, and the relevant set W is unbounded. We intersect W with a set whose boundary is far enough away from the origin that the vectors of this boundary cannot have anything to do with payoffs associated with the quitting game, (here the quantity R from Lemma 3 is used). We extend our sub-correspondence of $F_{\delta,0}$ to these distant points in a way so that all extended forward orbits of unbounded variation must move toward and stay with vectors that are relevant to the quitting game. Also we want all relevant properties of the correspondence found on ∂W to hold on the new boundary that includes points far away from ∂W.

The only serious problem with this approach concerns a player who would never choose to quit alone in the quitting game. Recall the definition of the vectors $v, \chi \in \mathbf{R}^N$. Assume that $v^j < \chi^j$ for some player j and assume that the vector x satisfies $x^j = v^j$ and $x^k > v^k$ for all players $k \neq j$. There will be a small positive $\delta > 0$ such that Player j quitting alone with probability δ would represent an equilibrium in $\overline{E}_{0,\delta}$. However, in a quitting game Player j would never be motivated to act in this way, since this player would be opting for a payoff below what she could guarantee herself! Therefore we must avoid applying $F_{0,\delta}$ to such vectors x, and this is our main complication. Even worse, let $v(\{j\})^k \geq v^k$ for all $k \neq j$. Then Player j quitting alone on each stage with some small $\delta > 0$ would satisfy all Properties but iii of a sequence of perfect one-shot δM equilibria.

For any positive $\delta > 0$ define a player $j \in N$ to be a δ-*normal* player when $\chi^j < v^j + \delta$. A δ-normal player j can be punished effectively (relative to the quantity δ) for not ending the game alone. A *normal* player is a δ-normal player for all positive $\delta > 0$, and define $\hat{N}$ to be the set of normal players. A player is normal if and only if by quitting alone he receives at least his min-max payoff. A player that is not normal will be called an *abnormal* player. Keep in mind that almost everything that follows would be significantly simpler if there were only normal players.

Remark 2: For any non δ-normal player j we know that $v^j \leq -\delta$, since otherwise the other players could try to punish j by never ending the game, and we would have $\chi^j \leq \max\{0, v^j\}$, a contradiction. As a consequence, if no player was normal then all players choosing c at all stages would be an equilibrium.

For every subset Q of the player set N with $|Q| \geq 2$ consider the $|Q| \times |Q|$ matrix A_Q defined by the entries $A_Q(i,j) := v(\{j\})^i - v^i$ (where the diagonal entries are zero because $v^j = v(\{j\})^j$).

The Assumptions:

From now on we assume that $\epsilon > 0$ is fixed. Until we reach the conclusion of this section, we proceed with the following Assumptions 1 through 5. These assumptions will be justified by the conclusion where we will prove (Theorem 2) that an affirmation of Question 1 implies that all quitting games have approximate equilibria.

Assumption 1: there is at least one normal player.
Assumption 2: the determinants of all the matrices A_Q (with $|Q| \geq 2$) are not zero.
Assumption 3: all ϵ-normal players are normal players.
Assumption 4: $||x - y|| > 0$, $x \in \{z \mid \forall j\ z^j \geq \chi^j - \epsilon\}$ and $(x, y) \in F_{\epsilon,0}$ imply that $\epsilon\ q(p) \leq ||x - y||$ for the corresponding $p \in [0,1]^N$ with $y = f(x, p)$.
Assumption 5: there is no ϵ^+ instant equilibrium.

We need a game in generic form (Assumption 2) to satisfy Property 5 of

Question 1. We fix a positive quantity ρ that is less than $\frac{\epsilon}{20|N|M}$ and small enough so that for every choice of Q (with $|Q| \geq 2$) and any choice of real numbers $d_{i,j}$ with $|d_{i,j}| \leq 2\rho M|N|$ (and with $i = j$ allowed) the determinants of the matrices B defined by $B(i,j) := v(\{j\})^i - v^i + d_{i,j}$ are not zero and for every $r \in S^{Q-1} := \{v \in \mathbf{R}^Q \mid |v| = 1\}$ and for all such matrices B the Euclidean norm of the vector Br is at least ρ. (This is possible from the continuity of the determinant function, the compactness of S^{Q-1}, and that $Br = 0$ for any r in the sphere S^{Q-1} implies that the determinant of B is zero.)

Define the map ϕ according to the choice of ϵ. Define the positive quantity R according to Lemma 3 (justified by Assumption 5), define $\overline{w} > 0$ according to Proposition 1, and define $\hat{w}$ to be $\frac{\epsilon^2 \overline{w}^{|N|}}{5M^3|N|^2}$.

4.2 Two lemmatta concerning χ

Lemma 4:

1) For any abnormal player j $v(\{k\})^j \geq \chi^j$ for every $k \neq j$.

2) If p is in $[0,\rho]^N$ then the probability that at least two players choose q simultaneously is no more than $\rho(|N|-1)$ times the probability that at least some player chooses q.

3) For any abnormal player j the payoff to Player j conditioned on quitting from any $p \in [0,\rho]^N$ with $p^j = 0$ is at least $\chi^j - M(|N|-1)\rho \geq \chi^j - \epsilon/20$.

Proof:

1) Consider the stationary strategy where Player k chooses q with probability ρ and all other players (other than k) choose c with certainty. By choosing q Player j would receive no more than $v^j + \rho M < \chi^j$ (by the abnormality of j). By the definition of χ it must follow that $v(\{k\})^j \geq \chi^j$.

2) By the definition of $\rho \leq \frac{1}{2|N|}$ we have $(1-\rho)^{|N|} \geq 1/2$. The probability that some player chooses q alone is $\sum_j p^j \prod_{k\neq j}(1-p^k)$. The probability that two players choose q together is no more than $\sum_{j\neq k} p^j p^k = \frac{1}{2}\sum_j p^j \sum_{k\neq j} p^k$ (because in the second summation each distinct pair appears twice). Comparing $\sum_{k\neq j} p^k$ with $\prod_{k\neq j}(1-p^k)$ for each fixed j we have $\sum_{k\neq j} p^k \leq (|N|-1)\rho$ and $\prod_{k\neq j}(1-p^k) \geq 1/2$.

3) It follows directly from Parts 1) and 2). $\square$.

Recall the definition of p_{-j} from the third section.

Lemma 5: Assume that $(x, y) \in F_{0,0}$. $x^j \geq \chi^j$ implies that $y^j \geq \chi^j$. Given that $y = f(x, p)$ with $(x, p) \in E_{0,0}$ then $x^j < \chi^j$ implies that $y^j - x^j \geq q(p_{-j})(\chi^j - x^j)$.

Proof: The first part follows directly from the definition of χ. If the second part were not true, then the payoff to Player j conditioned by the other players quitting (according to p_{-j}) would be strictly less than χ^j. $y^j < \chi^j$ and $p \in E_{0,0}$ would imply that Player j gets strictly less than χ^j by quitting against p_{-j}. p_{-j} would be a way to hold Player j down to a payoff below χ^j, a contradiction. $\square$

4.3 The set C

For any player j define C_j to be the set $C_j := W_j \cap [-R-1, R+1]^N$. For any pair j, k of abnormal players define $C_{j,k} := C_j \cap C_k$. Define the set C to be

$$C := \bigcup_{j \in \hat{N}} C_j \quad \bigcup_{j \neq k,\ j,k \in N \setminus \hat{N}} C_{j,k}.$$

We cannot exclude quitting behavior involving only abnormal players, since all equilibrium behavior could involve some stages where two or more abnormal players choose q with positive probability and all normal players choose c with certainty. On the other hand, we must exclude quitting behavior involving only one abnormal player. This is behind the special construction $C_{j,k}$ for pairs of abnormal players.

For every abnormal player j define the open set $D_j := \{x \in \mathbf{R} \mid x^k > v^k$ for all $k \neq j\}$. Define the subset $\hat{W} \subseteq W$ to be $\{x \mid x^j = v^j$ for some normal player $j,\ \forall k \neq j\ x^k \geq v^k\}$, (and equal to W if all players are normal). Define the set D to be the closure of $\partial C \backslash (\partial W \cap \hat{W})$ and define U to be $\partial C \cap \partial W \cap \hat{W}$, so that $\partial C = U \cup D$. Define $\hat{C}$ to be $\cup_{j \in N} C_j$ and define $\hat{D}$ to be $\cup_{k \in N \setminus \hat{N}} D_j$.

The following three lemmatta are useful for understanding the geometry of the set C.

Lemma 6: The set C is equal to $\hat{C} \backslash \hat{D}$. The set $\partial \hat{D}$ is the union of the two sets

$$\partial \hat{D}^1 := \cup_{j \in N \setminus \hat{N} \text{ abnormal}} \{x \mid x^k = v^k \text{ for some normal } k, \forall l \notin \{j,k\}\ x^l \geq v^l\}$$

and $\partial \hat{D}^2 := \{x \mid x^j \leq v^j,\ x^k = v^k$ for some distinct abnormal pair

$$j, k \in N \backslash \hat{N} \text{ and } \forall l \notin \{j, k\}\ x^l \geq v^l\}.$$

Proof: 1) Let x be in C_j for a normal j. $x^j \leq v^j$ implies that $x \notin \hat{D}$.

Let x be in $C_{j,k}$ for a pair j, k of abnormal players. $x^j \leq v^j$ and $x^k \leq v^k$ imply that x cannot be in D_l for any l.

Let x be in $\hat{C}$. If $x^l \leq v^l$ for some normal l then x is in $C_l \backslash \hat{D}$. If $x^l > v^l$ for all normal l then the only way to avoid $\hat{D}$ if for $x^j \leq v^j$ and $x^k \leq v^k$ for some abnormal pair j, k (and then x is in $C_{j,k}$).

2) $\partial D_j = \{x \mid x^k = v^k$ for some $k \neq j$, $x^l \geq v^l$ for all $l \notin \{j, k\}\}$, and any point in $\partial \hat{D}$ must be in ∂D_j for some abnormal j. By increasing all coordinates we enter the interior of D_j. The only question is whether there is a sufficiently small δ so that by changing all coordinates by no more than δ we must stay inside of $\hat{D}$. Consider five cases for a point in D_j:

Case 1; k is normal: By decreasing the coordinate x^k we leave the set $\hat{D}$.

Case 2; k is abnormal and $x^j \leq v^j$: By decreasing both coordinates x^j and x^k we leave the set $\hat{D}$.

Case 3; k is abnormal, $x^j > v^j$, and $x^l = v^l$ for some normal l: This is equivalent to Case 1, but with l replacing k.

Case 4; k is abnormal, $x^j > v^j$, $x^l > v^l$ for all normal l, and $x^m = v^m$ for at least two distinct abnormal $m = m_1, m_2$: By decreasing the coordinates of m_1 and m_2 we leave the set $\hat{D}$.

Case 5; k is abnormal, $x^j > v^j$, $x^l > v^l$ for all normal l, and $x^m = v^m$ for at most one abnormal m: If such an m exists then by choosing $\delta := \frac{1}{2} \min_{n \neq m} |x^n - v^n|$ and by changing every coordinate by no more than δ one remains in the set D_m. If such an m does not exist the same holds true for D_m for all abnormal m.

Notice that Cases 1 through 4 define the set $\partial \hat{D}^1 \cup \partial \hat{D}^2$. A point satisfying Case 5 is in the interior of $\hat{D}$. □

Lemma 7: ∂C is the union of the following sets:
$B_j^1 := \{x \mid x^j = v^j, \forall k \neq j\ v^k \leq x^k \leq R + 1\}$, j normal,
$B_j^2 := \{x \mid x^j = -R - 1, \forall k \neq j\ -R - 1 \leq x^k \leq R + 1\}$, j normal,
$B_{j,k}^3 := \{x \mid x^j = -R - 1,\ -R - 1 \leq x^k \leq v^k,\ \forall l \notin \{j, k\}\ -R - 1 \leq x^l \leq R + 1\}$, j abnormal, $k \in N$ either normal or abnormal, $k \neq j$,

$B^4_{j,k} := \{x \mid -R-1 \le x^j \le v^j,\ x^k = v^k,\ \forall l \notin \{j,k\}\ v^l \le x^l \le R+1\}$, j abnormal, $k \in N$ either normal or abnormal, $k \neq j$,
$B^5_{j,k} := \{x \mid x^j = R{+}1,\ -R{-}1 \le x^k \le v^k,\ \forall l \notin \{j,k\}\ -R{-}1 \le x^l \le R{+}1\}$, j either normal or abnormal, k normal, $k \neq j$,
$B^6_{j,k,l} := \{x \mid x^j = R{+}1,\ -R{-}1 \le x^k \le v^k,\ x^l \le v^l,\ \forall m \notin \{j,k,l\}\ -R{-}1 \le x^m \le R+1\}$, j either normal or abnormal, k,l abnormal, j,k,l distinct.
Furthermore every point of ∂C is in the closure of the interior of C.

Proof: First we show that all the listed sets are in ∂C and that any point of any of these sets is arbitrarily close to a point in the interior of C.

Let x be any point in B^1_j for a normal j. By increasing all coordinates of x one leaves C and by decreasing slightly all the coordinates one enters the interior of C_j.

Let x be any point in B^2_j for a normal j. By decreasing the x^j coordinate one leaves C and by increasing the x^j coordinate slightly and (if necessary) moving the other coordinates one enters the interior of C_j.

Let x be any point in $B^3_{j,k}$ for a distinct pair j,k with j abnormal. By decreasing the x^j coordinate one leaves C and by increasing the x^j coordinate slightly, and, if necessary, moving the x^k coordinate away from either $-R-1$ or v^k and moving the other coordinates, if necessary, one enters the interior of $C_{j,k}$ if k is abnormal or the interior of C_k if k is normal.

Let x be any point in $B^4_{j,k}$ for a distinct pair $j,k \in N$ with j abnormal. By increasing the coordinates of all players other than j one leaves the set C. By decreasing the x^k coordinate slightly, and moving, if necessary, x^j away from $-R-1$ or v^j (in the direction of the average) and moving, if necessary, the other coordinates one enters the interior of $C_{j,k}$ if k is abnormal or the interior of C_k if k is normal.

Let x be any point in $B^5_{j,k}$ for a distinct pair $j,k \in N$ with k normal. By increasing the x^j coordinate one leaves the set C. By decreasing the x^j coordinate slightly, moving, if necessary, the x^k coordinate away from $-R$ or v^k, and moving the other coordinates, if necessary, one enters the interior of C_k.

Let x be any point in $B^6_{j,k,l}$ for a mutually distinct triple $j,k,l \in N$ with k and l abnormal. By increasing the x^j coordinate one leaves the set C. By increasing the x^j coordinate slightly, moving, if necessary, the x^k and x^l coordinates away from $-R$ or either v^k or v^l, and moving, if necessary, the other coordinates, one enters the interior of $C_{k,l}$.

Next we show that the boundary of C is contained in the union of these sets.

Recognize $\partial\hat{C}$ as the union of

$$\partial\hat{C}^1 := \{x \mid x^j = v^j \text{ for some } j \in N,\ \forall l \neq j\ v^l \leq x^l \leq R+1\},$$

$$\partial\hat{C}^2 := \{x \mid x^j = -R-1 \text{ for some } j \in N,\ \forall l \neq j\ -R-1 \leq x^l \leq R+1\}$$

$$\text{and } \partial\hat{C}^3 := \{x \mid \text{ for some } j \in N\ x^j = R+1,$$

$$\text{for some } k \neq j\ -R-1 \leq x^k \leq v^k, \forall l \notin \{j,k\}\ -R-1 \leq x^l \leq R+1\}.$$

With $C = \hat{C}\backslash\hat{D}$ from Lemma 6 and $\hat{D}$ an open set we have $\partial C = (\partial\hat{C}\backslash\hat{D}) \cup (\partial\hat{D} \cap \hat{C})$. First we look at $\partial\hat{C}\backslash\hat{D}$.

If j is normal then $\{x \mid x^j = v^j\ \forall l \neq j\ v^l \leq x^l \leq R+1\}$, the appropriate part of $\partial\tilde{C}^1$, contains nothing of $\hat{D}$, and it is equal to B^1_j.

If j is abnormal then $x \in \{x \mid x^j = v^j\ \forall l \neq j\ v^l \leq x^l \leq R+1\}$ avoids $\hat{D}$ if and only if there is a second player k with $x^k = v^k$. This places x in $B^4_{j,k}$ if k is abnormal and in B^1_k if k is normal.

If j is normal then $\{x \mid x^j = -R-1,\ \forall l \neq j\ -R-1 \leq x^l \leq R+1\}$, the appropriate part of $\partial\hat{C}^2$, has no intersection with $\hat{D}$, and is equal to the set B^2_j.

If j is abnormal then $\{x \mid x^j = -R-1,\ \forall l \neq j\ -R-1 \leq x^l \leq R+1\}$ has an intersection only with D_j. To remove D_j we need at least one k, normal or abnormal, such that $x^k \leq v^k$. This yields the set $B^3_{j,k}$.

Let j,k be any pair of distinct players with k abnormal. The set $\{x \mid x^j = R+1,\ -R-1 \leq x^k \leq v^k,\ \forall l \notin \{j,k\}\ -R-1 \leq x^l \leq R+1\}$ has no intersection with $\cup_{l\neq k} D_l$. If x is a member of this set then $x \notin D_k$ if and only if $x^l \leq v^l$ for some $l \notin \{j,k\}$. If l is normal then x is in $B^5_{j,l}$. If l is abnormal then x is in $B^6_{j,k,l}$.

Let j,k be any pair of distinct players with k normal. The set $\{x \mid x^j = R+1,\ -R-1 \leq x^k \leq v^k,\ \forall l \notin \{j,k\}\ -R-1 \leq x^l \leq R+1\}$ has no intersection with $\hat{D}$ and it defines the set $B^5_{j,k}$.

Next we look at $\partial\hat{D} \cap \hat{C}$ and use Lemma 6.

For every abnormal j and normal k look at $\{x \mid x^k = v^k\ \forall l \notin \{j,k\}\ x^l \geq v^l\} \cap \hat{C}$, the appropriate part of $\partial\hat{D}^1$. Its intersection with $\hat{C}$ is $\{x \mid -R-1 \leq x^j \leq R+1,\ x^k = v^k\ \forall l \neq j\ v^l \leq x^l \leq R+1\}$. A point x in this set is in $B^4_{j,k}$ if $x^j \leq v^j$ and is in B^1_k if $x^j \geq v^j$.

For any pair j,k of abnormal players look at $\{x \mid x^j \leq v^j,\ x^k = v^k\ \forall l \notin \{j,k\}\ x^l \geq v^l\}$, the appropriate part of $\partial \hat{D}^2$. Its intersection with $\hat{C}$ is $B^4_{j,k}$. $\Box$

Lemma 8: For every pair j,k of abnormal players $\partial C_{j,k} \cap \partial C \subseteq D$ and $U = \cup_{j \text{ normal}} B^1_j$.

Proof: $U = \cup_{j \text{ normal}} B^1_j$ follows directly from the definition of U. Next we confirm that all the other sets of Lemma 7 are already in D.

For trivial reasons all the sets B^2_l and B^3_l are contained already in the set D. Likewise for the other sets (other than the B^1_j) there is some player n such that $x^n < v^n$ already or $x^n = v^n$ and x^n can be decreased slightly while staying in the set.

For any pair j,k of abnormal players we have $\partial C_{j,k}$ equal to the union of the three sets $\partial C^1_{j,k} := \{x \in [-R-1, R+1]^N \mid x^j \leq v^j,\ x^k = v^k\}$, $\partial C^2_{j,k} := \{x \in [-R-1, R+1]^N \mid x^k \leq v^k,\ x^j = v^j\}$, and $\partial C^3_{j,k} := \partial C_{j,k} \cap \{x \in [-R-1, R+1]^N \mid |x^l| = R+1 \text{ for some } l \in N\}$. We confirm that the intersection of these sets with U is always in the set D. By the above, we need only check the intersection with the sets B^1_j.

Let x be a point in $\partial C_{j,k}$ that is also in B^1_l for some normal $l \in \hat{N}$. We must have $x^j = v^j$ and $x^k = v^k$. By adding any $\delta > 0$ to the coordinates for all players other than j or k we get a point in ∂C that is not in $\hat{W} \cap \partial W$. $\Box$

4.4 The centering function, the homotopy, γ, and F

Define a special function $z : \mathbf{R}^N \to \mathbf{R}^N$ by
$z^j(x) = v^j - \epsilon/2$ if j is normal,
$z^j(x) = v^j$ if j is abnormal and $x^j \leq v^j$,
$z^j(x) = x^j$ if j is abnormal and $v^j \leq x^j \leq |N|M$,
$z^j(x) = |N|M$ if j is abnormal and $v^j \geq |N|M$.
We call z the *centering function.*

Define a correspondence $Z \subseteq \mathbf{R}^N \times \mathbf{R}^N$ by $\overline{Z}(x) := \{y \mid z^j(x) \leq y^j \leq M|N| \text{ for all abnormal } j \text{ and } |y^k - z^k(x)| \leq |N|M\rho \text{ for all normal } k\}$. We call Z the *centering correspondence.*

Define the functions $b_D : C \to [0,1]$ and $b_U : C \to [0,1]$ by $b_D(x) = \max\{0, 1 - \frac{1}{2\rho} \text{ dist } (x, D)\}$ and $b_U(x) = \max\{0, 1 - \frac{10|N|}{\rho} \text{ dist } (x, U)\}$. Define the functions $\lambda : C \to [0,1]$, $b_1 : C \to [0,1]$ and $b_2 : C \to [0,1]$ by $\lambda(x) :=$

$\max\{0, 1-2b_D(x)\}$, $b_1(x) := 2(1-b_U(x))\min\{b_D(x), 1-b_D(x)\}$ and $b_2(x) := 1 - b_1(x) - \lambda(x)$. Notice that $\max\{0, 1-2b_D(x)\} + 2\min\{b_D(x), 1-b_D(x)\}$ is equal to 1 if $b_D(x) \le 1/2$ and equal to $2 - 2b_D(x) \le 1$ if $b_D(x) \ge 1/2$, so that b_2 is well defined.

Define $\pi_1 : \mathbf{R}^N \times [0,1]^N \to \mathbf{R}^N$ and $\pi_2 : \mathbf{R}^N \times [0,1]^N \to [0,1]^N$ to be the canonical projections. Define the functions $\psi_1 : C \to \mathbf{R}^N$ and $\psi_2 : C \to \mathbf{R}^N$ by $\psi_1(x) := \pi_1 \circ \phi^{-1}(x)$ and $\psi_2(x) := f(\pi_1 \circ \phi^{-1}(x), \pi_2 \circ \phi^{-1}(x))$.

Define $J : C \times [0,1] \to E \times E$ by

$$J(x,t) := (1-t)(x,x) + t\lambda(x)(\psi_1(x), \psi_2(x)) + tb_1(x)(x, z(x)) + tb_2(x)(x,x).$$

Next we must define the correspondence G. For every normal $j \in \hat{N}$ define V_j to be the set $\{x \in [-M, R+1+\rho]^N \mid$ for every $k \in N$ $x^k \ge v^k - \rho$ and $x^j \le v^j + \rho\}$. Similarly define the sets $V_O = \cup_{j \in \hat{N}} V_j$, $V_D := \{x \mid$ distance $(x, D) \le 3\rho\}$ and $V := V_D \cup V_O$. Define a correspondence $G_O \subseteq V_O \times \mathbf{R}^N$ by $\overline{G}_O(x) = \{f(x,p) \mid p^j = 0$ if $x \notin V_j$ and otherwise $0 \le p^j \le \rho$ if $x \in V_j\}$. If $x \in V_O \backslash V_D$ then $G(x) := G_O(x)$.

More complicated is to define G on V_D. For every normal player j define the vector $u_j \in \mathbf{R}^N$ by $u_j^k := v(\{j\})^k$. For every subset A of normal players define the set $G_A := \{x \mid$ dist $(x,$ convex hull $(\{u_j^k \mid j \in A\})) \le M|N|\rho\}$, a convex set. For every $x \in V_0$ define $A(x) = \{j \mid x \in V_j\}$. Define a correspondence $G^* \subseteq V_O \times \mathbf{R}^N$ by $\overline{G^*}(x) = \{(1-\xi)x + \xi y \mid y \in G_{A(x)},\ 0 \le \xi \le \rho|A(x)|\}$. If $x \in V_D \backslash V_O$ then $\overline{G}(x)$ is convex hull $(Z(x) \cup \{x\})$. If $x \in V_D \cap V_O$ then define $\overline{G}(x)$ to be convex hull $(Z(x) \cup G^*(x))$.

We need to show that G is compact. It suffices to show that if $x \in V_0 \cap V_D$ then $\overline{G}_O(x) \subseteq \overline{G^*}(x)$. This follows directly from Lemma 4.

With Assumption 5 and the small positive quantity $\overline{w}$ defined from Proposition 1 (with $\delta = \epsilon$), we define $\hat{w}$ to be $\frac{\epsilon^2 \overline{w}^{|N|}}{5|N|^2 M^3}$ and define γ to be $\frac{\epsilon\rho}{10|N|}\hat{w}^2$.

Define the correspondence F to be $J(C,1) \cup G$.

That points near the boundary of $[-R-1, R+1]^N$ will not be relevant to defining the correspondence F near to cluster points of any unbounded variation extended forward orbit is easy to believe. Harder to believe is that the same will be true for points near to the set D (either from $J(C,1)$ or from G). Indeed that is what we will show. By truncating the orbit so that it starts at a point near to a cluster point we will get an extended forward orbit of unbounded variation defined entirely by $F_{0,0}$ and G_O. The connection to Proposition A will be established by the following lemma.

Lemma 9: If $(x, y) \in G$ and the distance from x to D is more than 3δ or $(x, y) = J(\alpha, 1)$ and the distance from α to D is at least 2ρ then $(x, y) \in F_{\epsilon/3,0}$.

Proof: If $(x, y) = J(\alpha, 1)$ then by the definition of J and ϕ we have $(x, y) \in E_{0,0}$. Otherwise assume that $(x, y) \in G_O$. Since $x^j \geq v^j - \rho$ for all players j, by not quitting player j receives at least $v^j - \rho - M|N|\rho$. On the other hand, any player j who quits will receive a value within $\rho + M|N|\rho$ of v^j. The rest follows by $\rho + M|N|\rho < \epsilon/6$ and the definition of V_j. $\square$

4.5 Satisfaction

We must show that F as defined above satisfies all the conditions of Question 1. First we prove three simple lemmatta.

Until the end of this paper, if $J(\alpha, 1) = (x, y)$ and there is no ambiguity then we will use the notations $\lambda := \lambda(\alpha)$, $b_1 := b_1(\alpha)$, $b_2 := b_2(\alpha)$, $x = (1 - \lambda)\alpha + \lambda\beta$ with $\phi(\beta, p) = \alpha$, $c = f(\beta, p)$, and $y = b_1 z(x) + b_2 x + \lambda c$. If $q(p) > 0$ then define $d := d(\alpha) \in \mathbf{R}^N$ by $c = (1 - q(p))\beta + q(p)d$.

Lemma 10: Assume that $(x, y) = J(\alpha, 1)$ with $\rho < \text{dist }(\alpha, D) < 2\rho$. If $\beta^j \geq \chi^j - \epsilon/2$ for all abnormal players then $|\alpha^k| > R$ for some player k.

Proof:

For the sake of contradiction, assume that $|\alpha^m| \leq R$ for all players m, which means by the definition of D and Lemma 8 that $\alpha^j \leq v^j + 2\rho$ for some abnormal player j. Due to $\beta^j \geq \chi^j - \epsilon/2 \geq v^j + \epsilon/2$ and Proposition 1 Player j chose q with positive probability and there must be some other player k such that $p^k \geq \frac{\epsilon}{2M|N|}$ and therefore $\alpha^k < v^k - 2\rho$. This is possible only if k is also abnormal, (otherwise x is too deep in the interior of C). But then by switching roles, we have $\alpha^l < v^l - 2\rho$ for some player $l \neq k$, a contradiction to dist $(\alpha, \partial C) < 2\rho$. $\square$

Lemma 11: Assume that $(x, p) \in E_{0,0}$ and $\phi(x, p) = \alpha \in C$. If $p^j > 0$ for an abnormal j then there is a player k such that $p^k \geq \hat{w}Mp^j/3$.

Proof: By the definition of C $\alpha^k \leq v^k$ for at least one $k \neq j$. $p^k > 0$ follows by Part 1 of Proposition 1, and also $p^k \geq \hat{w}\frac{M}{3}\sum_{l \neq k} p^l \geq \hat{w}\frac{M}{3}p^j$. $\square$

Lemma 12: If $(x, y) = J(\alpha, 1)$, $-M/3 \leq x^l \leq M|N| + 1$ for all players l, then $|\alpha^k| > R$ for some player k is possible only if $\alpha^k < -R$ and then

$\lambda \geq 1 - \frac{\epsilon^2}{4|N|M^2}$.

Proof: If $\alpha^k > R$ for some k then from Lemma 3 $p^k = 0$ and $x^k \geq R - |N|M$, a contradiction.

Next consider the possibility that $\alpha^k < -R$ for some player k. Let t equal $\max_{j \in N} p^j$, and let l be such a player obtaining this maximum. We defined R so large that t must be at least 9/10. We have that $\alpha^l \leq \frac{-5tM^3|N|^2}{(1-t)^N \epsilon^2} + (|N| - 1)Mt + M/3 \leq \frac{-4M^3|N|^2 t}{(1-t)^N \epsilon^2}$, but also $\alpha^l \leq -R + |N|M$, since otherwise it would be impossible for $\alpha^k \leq -R$ for some other player k. On the other hand we have $\beta^l \leq v^l + \frac{2(|N|-1)Mt}{3(1-t)^{N-1}} + M/3$, since otherwise Player l would prefer to choose the move c. With $x^l \geq -M/3$ and $\lambda\beta^l + (1-\lambda)\alpha^l = x^l$ we are done. $\square$

Proposition 2: The correspondence F satisfies all the conditions of Question 1.

Proof: J is continuous because all functions defining it are continuous.

Property 1: $J(x, 0) = (x, x)$ for all $x \in C$ follows by definition. $x \in \partial C$ implies that $x \in D$ or $x \in U$. If $x \in D$ $J(x, t) = (x, x)$ follows because $b_2 = 1$. If $x \in U$ then $b_1 = 0$, and then regardless of the values of b_2 and λ $J(x, t) = (x, x)$ follows by Lemma 1.

Property 3: It follows by definition.

Property 4 $G \subseteq F$ is by definition, so we need only to prove that if $(x, y) = J(\alpha, 1)$ and $||x - y|| \leq \gamma$ then (x, y) is also in G.

Case 1; dist $(\alpha, D) \geq 2\rho$ **and** $\beta^j > \chi^j - \epsilon/2$ **for all players** j**:** We have $x = \beta$ and $(x, y) \in F_{0,0}$. By Assumption 4 the p with $(x, p) \in E_{0,0}$ and $y = f(x, p)$ satisfies $q(p) \leq \frac{||y-x||}{\epsilon} \leq \frac{\gamma}{\epsilon} < \rho/(|N|M)$. It is not possible for $x^j \leq v^j - \rho$ for any player, since otherwise from $q(p) \leq \frac{\gamma}{\epsilon}$ Player j would prefer to choose q. Likewise from the size of $q(p)$ if j is any player who chooses q with positive probability then $x^j < v^j + \rho$. We have $x^j - v^j \geq \epsilon/2$ for all abnormal players j, and therefore every abnormal player chooses c with certainty.

Case 2; dist $(\alpha, D) \geq 2\rho$**,** $\beta^j \geq \chi^j - \epsilon/2$ **for all abnormal** j**, and** $\beta^k < \chi^k - \epsilon/2$ **for some normal** k**:** Since $\chi^k \leq v^k$ we have $\beta^k < v^k - \epsilon/2$. $q(p_{-k}) > \frac{\epsilon}{2M}$ is necessary to prevent Player k from preferring the move q.

But then by Lemma 5 $y^k \geq x^k + \frac{\epsilon^2}{4M}$, a contradiction.

Case 3; dist $(\alpha, D) \leq \rho$**:** By definition $x = \alpha$ and $y = b_1 z(x) + b_2 x$ is in G.

Case 4; $\rho <$ **dist** $(\alpha, D) < 2\rho$ **and** $b_1 \geq 1/10$**:** If $|\alpha^k| > R$ for some player k then with $b_1 \geq 1/10$ it would follow that $|y^k - x^k| > R/30$. So we continue with the assumption that $|\alpha^k| \leq R$ for all players k. This implies that $\alpha^k \geq v^k - 2\rho$ for all normal players k. Since $z^k(\alpha) = v^k - \epsilon/2$ for every normal player k, $b_1 \geq 1/10$ and $||x - y|| \leq \gamma$ would imply that $c^k \geq \beta^k + \epsilon/20 - 20\rho$ for all normal players k. This is possible only if $q(p) \geq \frac{\epsilon}{20M}$. From our assumption that $|\alpha^k| \leq R$ for all players k, at most only one player, an abnormal player j, can satisfy $\alpha^j < v^j - 2\rho$. This implies that $\alpha^j \leq v^j - \frac{M^2|N|}{6\epsilon} < v^j - \frac{10M}{\epsilon}$. But to prevent $y^j - x^j > \gamma$ we must have $c^j < \beta^j - M/\epsilon$. If $\beta^j \leq \chi^j$ this would mean that by quitting Player j receives a payoff far less than any payoff defining the game, a contradiction. On the other hand, if $\beta^j \geq \chi^j$ then $q(p_{-j}) \geq \epsilon/M$ and there must be a second player l with $\alpha^l < v_l - 2\rho$, also a contradiction.

Case 5; dist $(\alpha, D) > \rho$**,** $b_1 \leq 1/10$**, and dist** $(\alpha, U) \leq \frac{\rho}{10|N|}$**.**

dist $(\alpha, U) \leq \frac{\rho}{10|N|}$ implies that there is a normal player k such that $|\alpha^k - v^k| \leq \frac{\rho}{10|N|}$ and that $\alpha^j \geq v^j - \frac{\rho}{10|N|}$ for all $j \in N$. But then by Lemma 7 dist $(\alpha, D) \geq \rho$ implies that $\alpha^j \geq v^j + 9\rho/10$ for all abnormal j. By Part 5 of Proposition 1 and the containment of U in ∂W we have $q(p) \leq \frac{\epsilon^2\rho}{40M^3|N|^2}$. By Part 3 or Part 4 of Proposition 1 $p^j = 0$ follows if α^j or β^j is at least $v^j + \frac{\epsilon^2\rho}{10|N|^2M^2}$ (including the case of j abnormal). Therefore from $q(p) \leq \frac{\epsilon^2\rho}{40M^3|N|^2}$ and $\alpha^j \geq v^j - \frac{\rho}{10|N|}$ for all $j \in N$ we have $|\beta^j - \alpha^j| \leq \rho/(5|N|)$ for all j (if $p^j = 0$ then it follows directly from the definition of ϕ and if $p^j > 0$ then it follows indirectly, from $\beta^j \leq v^j + \frac{\epsilon^2\rho}{10|N|^2M^2}$). Furthermore $p^j = 0$ for any abnormal player j implies that $\beta^j \leq \alpha^j$, meaning also that $x^j \leq \alpha^j$.

If $\lambda = 1$ then $y = f(x, p)$ and we are done. Otherwise with the formula $y = \lambda q(p) d + b_1 z(\alpha) + b_2 \alpha + \lambda(1 - q(p))\beta$ we need to represent y as $\lambda q(p) d' + b_1 z' + (b_2 + \lambda(1 - q(p))x$ for some $d' \in G_{A(x)}$ and $z' \in \overline{Z}(x)$. With $x = \lambda\beta + (1 - \lambda)\alpha$ and $\lambda + b_1 + b_2 = 1$ re-write y as $y = \lambda q(p)(d + x - \beta) + b_1(z(\alpha) + x - \alpha) + (b_2 + \lambda(1 - q(p)))x$. $||\beta - \alpha||$ no more than $\rho/5$ implies the same for $||\beta - x||$ and $||\alpha - x||$. By Lemma 4 we have $d + x - \beta$ in $G_{A(x)}$. Since $x^j \leq \alpha^j$ for every abnormal j we have $z(\alpha) + x - \alpha$ in $\overline{Z}(x)$.

Case 6; $\rho <$ dist (α, D), $\beta^j \leq \chi^j - \epsilon/2$ for some abnormal j, $b_1 \leq 1/10$, and dist $(\alpha, U) \geq \frac{\rho}{10|N|}$: It follows that $b_2 = 0$ and $\lambda \geq 9/10$. With $U \subseteq \partial W$ by Proposition 1 we have $q(p) \geq \frac{\rho \hat{w}}{10|N|}$. But then by Lemma 11 we have also $q(p_{-j}) \geq \frac{\rho \hat{w}^2 M}{30|N|}$. With $\beta^j \leq \chi^j - \epsilon/2$ Lemma 5 implies that $y^j > x^j + \gamma$, a contradiction.

Case 7; $\rho <$ dist $(\alpha, D) < 2\rho$, $\beta^j > \chi^j - \epsilon/2$ for all abnormal j, and dist $(\alpha, U) \geq \frac{\rho}{10|N|}$: Part 1 of Proposition 1 implies that $q(p) \geq \hat{w}\rho/(10|N|)$. By Lemma 10 we must assume that $|\alpha^k| > R$ for some player k.

First assume that x^l is not between $-M/3$ and $|N|M + 1$ for some player l. The centering function and all payoffs for Player l in the game give values between $1 - M/3$ and $|N|M$. dist $(\alpha, U) \geq \frac{\rho}{10|N|}$ and $\rho <$ dist (α, D) would imply that $b_2 = 0$ and therefore $|y^l - x^l| \geq \hat{w}\rho/(10|N|)$, a contradiction.

Next assume that $x \in [-M/3, |N|M + 1]^N$. By Lemma 12 we know that $\alpha^k < -R$, $q(p) \geq 9/10$ and $\lambda \geq 1 - \frac{\epsilon^2}{4|N|M^2}$. If there is some player l with $\beta^l \geq 2|N|M$ then clearly $y^l - x^l < M/2$. Otherwise by Assumption 5 and the definition of R $\beta^l < \chi^l - \epsilon$ follows for some player l, who by our assumption must be a normal player. But then by Lemma 5 $c^l > \beta^l + \epsilon/2$ and $z^l(\alpha) > \alpha^l + 1$ with $b_2 = 0$ implies that $y^l > x^l + \epsilon/2$, a contradiction.

Property 2 With Property 4 proven, $(x, x) \in J(C, 1)$ is possible only if $(x, x) \in G$, as $y = x$ is included in the assumption $||y - x|| < \gamma$. Cases 2,4, 6 and 7 of the proof of Property 4 led to contradictions. Therefore we need only consider Cases 1,3, and 5 of the above proof of Property 4.

Case 1: By Assumption 4 the corresponding $p \in [0, 1]^N$ must be $p = \overline{0}$. Theorem 1 implies $x \in \partial C$.

Case 3: If suffices to show that $x \neq z(x)$ for all such x. $z^k(x) = v^k - \epsilon/2$ for any normal player k and therefore the only possibility for $z(x) = x$ is if $x^j \leq -R$ for some player j. But then x^j is not equal to $z^j(x)$.

Case 5: Assuming $x \notin \partial C$ and dist $(\alpha, D) > \rho$ we have $\lambda q(p) > 0$. If there is an abnormal player j with $\beta^j < \chi^j - \epsilon/20$ then by Lemma 4 and $||\alpha - \beta|| < \rho$ $y^j > x^j$ cannot be avoided. On the other hand if $\beta^j \geq \chi^j - \epsilon/20$ for all abnormal j then by Lemma 10 $|\alpha^k| \geq R$ for some player k. By the assumption that α is close to U we have $\alpha^k \geq R$ and then $y^k < x^k$ holds.

Property 5 $(x, x) \in G$ for all $x \in V$ follows by the definition of G, so we need only check that $\overline{G}(x)$ is always contractible.

If $x \in V_D$ then it follows because all convex sets are contractible.

If $x \in V_O \backslash V_D$ then demonstrating Property 5 is more difficult. Let A be the set of normal players j satisfying $|x^j - v^j| \leq \rho$ and assume that $|A| \geq 2$. Let P^A_ρ be the subset of $[0, \rho]^N$ defined by $P^A_\rho := \{p \mid p^j = 0$ if $j \notin A$ and $0 \leq p^j \leq \rho$ if $j \in A\}$. Define the map $\phi_x : P^A_\rho \to \mathbf{R}^N$ by $\phi_x(p) := f(x, p)$, and notice that $\overline{G}(x)$ is equal to the image of ϕ_x. If there is an positive quantity $\xi > 0$ such that for all pairs $p_1, p_2 \in P^A_\rho$ $|\phi_x(p_1) - \phi_x(p_2)| \geq \xi |p_1 - p_2|$ then one shows that $\overline{G}(x)$, the image of ϕ_x, is contractible to x with the homotopy $h(y, t) := \phi_x((1-t)\phi^{-1}_x(y))$ (as then this function is well defined and continuous). The above inequality works with $\xi = \rho$, and follows by elementary calculus, as we demonstrate now. Assume that $r \in S^{Q-1}$ is defined by $r := \frac{\phi_x(p_1) - \phi_x(p_2)}{|\phi_x(p_1) - \phi_x(p_2)|}$. For every $i \in A$ the difference $\phi^i_x(p_1) - \phi^i_x(p_2)$ must equal $w_i \cdot r$ for some vector $w_i \in \mathbf{R}^A$ satisfying $w^j_i = \frac{\partial^i \phi_x}{\partial j}(p)$ for some p on the straight path between p_1 and p_2. Because the probability of any player other than $i \in A$ quitting is never more than $(|N| - 1)\rho$, the partial derivative $\frac{\partial^i \phi_x}{\partial j}$ throughout the domain P^A_ρ is never further than $|N| M \rho$ from $v(\{j\})^i - v^i$ (an additional quantity ρ comes from the fact that $|x^i - v^i| \leq \rho$). The w^j_i form a matrix $W(i, j)$ with $\phi_x(p_1) - \phi_x(p_2) = |p_1 - p_2| W r$, and by Assumption 2 (and the choice of ρ) we have $||\phi_x(p_1) - \phi_x(p_2)|| \geq \rho ||p_1 - p_2||$ as desired. Finally if $|A| = 1$ then $\overline{G}(x)$ is a line segment.

Property 6: We will define an appropriate pair $(x, y) \in G$.

Case 1; x is within γ of $U \cap C_j$ for some normal j and $x^l \geq \chi^l - \epsilon/2$ for all abnormal l: x is in V_j and we let Player j quit alone with probability ρ. Since $x^j \geq v^j - \gamma$ for all players j, by Assumption 4 $||y - x||$ is at least $\rho\epsilon$. If $x^j \leq v^j$ then $y^j \leq v^j$. If $x^j > v^j$ then $y^j < x^j$. Furthermore if $|x^k| > R$ for any player k then $|y^k| < |x^k|$.

Case 2; x is within γ of $U \cap C_j$ for some normal j, and $x^l < \chi^l - \epsilon/2$ for some abnormal l: x is in V_j and we let Player j quit alone with probability ρ. By Lemma 5 $y^l - x^l \geq \rho\epsilon/2$. The rest of the argument is identical to Case 1.

Case 3; x is within γ of $D \cap C_j$ for some normal j, and $x^l < \chi^l - \epsilon/2$ for some l: If $x \in V_j$ then one could proceed as in Case 2, letting Player j quit alone with probability ρ. Otherwise if $x \notin V_j$ then we define $y := z(x)$.

Since $z^j(x) = v^j - \epsilon/2$ (and the other coordinates for z are within $-M$ and $M|N|$) $y = z(x)$ is a point of the interior of C_j at least a distance of $\epsilon/2$ from ∂C_j.

Case 4; x is within γ of $\partial C \cap \partial C_{k,l}$ for some pair k, l of abnormal players: By Lemma 8 we can replace $\partial C \cap \partial C_{k,l}$ by $D \cap \partial C_{k,l}$ and therefore $x \in V_D$. We define $y := z(x)$. If $x^k \leq v^k$ and $x^l \leq v^l$ then $y = z(x)$ is also in $C_{k,l}$. If $x^k > v^k$ and $x^l \leq v^l$ then $\gamma \geq \text{dist } (x, C_{k,l}) \geq |x^k - v^k| = \text{dist } (y, C_{k,l})$. If $x^k > v^k$ and $x^l > v^l$ then $\gamma \geq \text{dist } (x, C_{k,l}) \geq \sqrt{(x^k - v^k)^2 + (x^l - v^l)^2} = \text{dist } (y, C_{k,l})$. $|x^j| \geq R$ for any j implies $|y^j - x^j| > M$. Otherwise we have $|y^j - x^j| \geq \epsilon/3$ for any normal player j.

Property 7: Let $x_i = (1 - \lambda_i)\alpha_i + \lambda_i \beta_i$, with $\alpha_i = \phi(\beta_i, p_i)$.

Claim D: The α_i converge to x.

Proof of Claim D: Due to the compactness of C, by taking a convergent subsequence we can assume that the sequence α_i converges to an α not equal to x and that the λ_i converge to a $\lambda > 0$. With $\alpha = \phi(\beta, p)$ by the continuity of ϕ^{-1} we have that the β_i converge to β and that the p_i converge to p. Given the proof of Property 4, we can assume further (by taking a subsequence) that either Cases 1, 3, and 5 of the proof of Property 4 applies to the entire sequence. We can ignore Case 3, since for this case $\alpha_i = x_i$ by definition.

If Case 1 is valid for the sequence then Assumption 4 and Theorem 1 imply that the α_i converge to x.

By taking a further subsequence we can assume that Case 5 but not Case 1 applies for the entire sequence, meaning that for all i $\rho < \text{dist } (\alpha_i, D) < 2\rho$. Supposing that $\alpha \neq \beta$, by Theorem 1 we can assume that p is not equal to $\overline{0}$. With $\lambda q(p) > 0$, Lemma 5 and Lemma 11 we know that $\beta^j \geq \chi^j - \epsilon/2$ for all abnormal j, since otherwise y_i would be at least $x_i + \lambda \epsilon \hat{w} q(p)/4$ for every large enough i. But then by Lemma 10 and the distance to U we have $\alpha^k > R$ for some k. This would imply that the y_i^k would be less than $x_i^k - \lambda q(p) M$ for all sufficiently large i, a contradiction. Claim D is proven.

Notice that if $x \in \partial C$ then $||z(x) - x|| \geq \epsilon/3$ and the closed line segment between x and $z(x)$ is inside of C_j for a normal j if $x \in C_j$ and is inside of $C_{j,k}$ for a distinct pair of abnormal j, k if $x \in C_{j,k}$.

With Claim D proven, consider the two cases of 1) dist $(x, D) < \rho$ or 2) $x \in U$ and dist $(x, D) \geq \rho$. In the former case for large enough i the $\frac{y_i - x_i}{||y_i - x_i||}$

must be defined by the line segment between x and $z(x)$. For the latter case if all the λ_i are zero for sufficiently large i then the same argument applies. Otherwise by Claim D, $U \subseteq \partial W$ and Theorem 1 the p_i converge to $\overline{0}$. Define $B(x) := \{j \mid x^j = v^j\}$; by Claim D and Proposition 1 for large enough i $p_i^j = 0$ for all $j \notin B(x)$. dist $(x, D) \geq \rho$ implies that $x^j \geq v^j + \rho$ for all abnormal players j, so $B(x)$ contains only normal players. According to the proof of Case 1 or Case 2 of Property 6 to define a $y \in \overline{G}(x)$ every member of $B(x)$ can choose to quit alone with probability ρ and the resulting vector will be in C with a distance of at least γ from x. Whether or not x is in V_D convergence to a member of the convex cone follows from the p_i converging to $\overline{0}$ and the second part of Lemma 4.

Property 8: It follows because the set C is contractible to the point $(-R, -R, \ldots, -R) \in \mathbf{R}^N$. □

4.6 Application

If all players were normal then one could skip Steps i through v of the following proposition. First we prove two more lemmatta.

Lemma 13; If $(x, y) = J(\alpha, 1)$**, for at least one abnormal player** j $\beta^j \geq \chi^j - \epsilon/2$ **and** $\alpha^j < \beta^j$**,** $|\alpha^k| \leq R$ **for all players** k**, and** $x^l \geq v^l - \gamma$ **for all abnormal players** l**, then** $\lambda \geq 1 - \frac{\epsilon^2}{6M^2|N|}$**:**

Proof: Since $\alpha^j < \beta^j$, by Proposition 1 Player j must have chosen q with positive probability. But with $\beta^j \geq \chi^j - \epsilon/2 \geq v^j + \epsilon/2$, this is possible only if $q(p_{-j}) \geq \epsilon/(2M)$. dist $(\alpha, D) \geq 2\rho$ would imply $\lambda = 1$. To avoid dist $(\alpha, D) \geq 2\rho$ we need an abnormal player k such that $p^k \geq q(p) - \frac{\epsilon}{4M}$ and $\beta^k \leq v^k + \epsilon/7$ (since otherwise there would be another player n with $\alpha^n < v^n - 2\rho$). This implies that $\alpha^k < v^k - \frac{M^2|N|}{\epsilon}$. With $x^k \geq v^k - \gamma$ and $x^k = \lambda\beta^k + (1-\lambda)\alpha^k$ we are done. □

Lemma 14: If $(x, y) = J(\alpha, 1)$, $x^l \geq v^l - \gamma$ for all abnormal players l and $x^j < \chi^j - \epsilon/5$ for an abnormal j then $y^j - x^j \geq \lambda\,(\chi^j - x^j - \epsilon/10)q(p_{-j})$.

Proof:

Case 1; $\beta^j < \chi^j - \epsilon/10$**:** It follows by Lemma 5 and $\alpha^k \leq |N|M$.

Case 2; $\beta^j \geq \chi^j - \epsilon/10$ **and** $\alpha^j < \beta^j$**:** By Lemma 5 with either Lemma 12 or Lemma 13 it follows that $x^j \geq \chi^j - \epsilon/6$. □.

Proposition 3: Any extended forward orbit of unbounded variation of the correspondence F reaches and stays with points $(x,y) \in F$ satisfying $x^j \geq \chi^j - \epsilon$ for all players $j \in N$ and if $J(\alpha,1) = (x,y)$ then dist $(\alpha, D) \geq 2\rho$ and if $(x,y) \in G$ then $x \in V_O \backslash V_D$.

Proof: By observing the second (image) part of F we know that for every abnormal player the coordinates of the cluster points of any orbit of unbounded variation are between $1 - M/3$ and $M|N|$, while for normal players they are between $1 - M/3$ and $-1 + M/3$. Let $(x_{l,i} \mid l < Q,\ i = 0, 1, \ldots)$ be the extended forward orbit. Define the set T to be the cluster points of the orbit, and define T' to be the subset of points in the orbit that are within a distance of γ from T.

Step i; Show that if j is an abnormal player and $x \in T$ then $x^j \geq v^j$:

Assume that $(x,y) \in F$. It suffices to show that if $x^j \geq v^j$ then $y^j \geq v^j$ and if $x^j < v^j$ then $y^j - x^j \geq \gamma(v^j - x^j)||y - x||$. From Lemma 4 this is true for the case of $(x,y) \in G$. Next we consider the case that $(x,y) = J(\alpha, 1)$. If dist $(\alpha, D) \leq \rho$ then it follows by the definition of z. So for what follows assume that dist $(\alpha, D) > \rho$.

Case 1; $\alpha^j \geq v^j$ and $x^j \geq v^j$: If $\beta^j \geq \chi^j$ then $c^j \geq \chi^j > v^j$ and $y^j \geq v^j$. If $\beta^j < \chi^j$ then $c^j < M/3$ and $\alpha^j < c^j + (|N| - 1)M < |N|M$ and $c^j \geq \beta^j$ would imply that $y^j \geq x^j$.

Case 2; $\alpha^j \geq v^j$ and $x^j < v^j$: It follows that $\beta^j \leq x^j$ and from $c^j \leq \beta^j + 2M/3$ we have that $\alpha^j \leq |N|M$. Furthermore $\lambda(v^j - \beta^j) \geq v^j - x^j$ holds from the equality $x^j = (1-\lambda)\alpha^j + \lambda\beta^j$. With $\beta^j < v^j$ something must prevent Player j from preferring to choose the move q, and therefore $q(p_{-j}) \geq \frac{v^j - \beta^j}{M}$. From Lemma 5 we have $c^j - \beta^j \geq q(p_{-j})(\chi^j - \beta^j) \geq q(p_{-j})\epsilon$. Putting everything together we get $y^j - x^j \geq \lambda(c^j - \beta^j) \geq \lambda q(p_{-j})\epsilon \geq \epsilon\lambda\frac{v^j - \beta^j}{M} \geq \frac{\epsilon}{M}(v^j - x^j)$.

Case 3; $\alpha^j < v^j$ and $\beta^j < \chi^j$: It follows by Lemma 5, $z^j(\alpha) = v^j$ and Lemma 11.

Case 4; $\alpha^j < v^j$ and $\beta^j \geq \chi^j$: If dist $(\alpha, D) \geq 2\rho$ then $\beta = x$ and $y = f(x,p)$, and the claim follows directly from Lemma 5. So now we assume that dist $(\alpha, D) < 2\rho$. By Part 2 of Proposition 1 $p^j > 0$ and by $\beta^j \geq v^j + \epsilon$ we have $q(p_{-j}) \geq \epsilon/M$. This implies that $\alpha^k < v^k - \frac{3M^2}{\epsilon}$ for some player

k, of course with $p^k > 0$. If $\beta^k \leq M$ then $\lambda \geq 1 - \frac{\epsilon}{3M}$ follows and Lemma 5 implies that $y^j \geq \chi^j - \epsilon/3 > v^j$. If $\beta^k \geq M$ then $q(p_{-k}) \geq 1/2$ and $\alpha^l < v^l - M$ for another $l \neq k$; with dist $(\alpha, D) < 2\rho$ we have $|\alpha^m| > R$ for some $m \in N$. Lemma 12 applies with $\alpha^m < -R$, $b_2 = 0$, and $\lambda \geq 1 - \frac{\epsilon^2}{4|N|M^2}$. $y^j \geq \chi^j - \epsilon/3 > v^j$ follows also by Lemma 5. Step i is proven.

Step ii; Show that if j is an abnormal player and $x^j \geq \chi^j - \epsilon/5$ for some $x \in T'$ then all points following x in the orbit also satisfy the same condition:

Case 1; $(x, y) \in G$: It follows by Lemma 4 and the definition of z.

Case 2; $(x, y) = J(\alpha, 1)$, $\rho <$ dist (α, D) and $\beta^j \leq \chi^j$: First notice that $\beta^j \leq \chi^j$ implies that $c^j \leq M/3$ and $\alpha^j \leq N|M|$. $c^j \geq \beta^j$ follows by Lemma 5 and then also $y^j \geq x^j$ from $z^j(\alpha) = \alpha^j$.

Case 3; $(x, y) = J(\alpha, 1)$, $\rho <$ dist (α, D), $\beta^j \geq \chi^j$ and $\alpha^j \geq \beta^j$: $c^j \geq \chi^j$ follows by Lemma 5 and then also $y^j \geq \chi^j$ from $z^j(\alpha) \geq \alpha^j \geq \chi^j$.

Case 4; $(x, y) = J(\alpha, 1)$, $\rho <$ dist (α, D), $\alpha^j < \beta^j$, $\beta^j \geq \chi^j$ and $|\alpha^k| > R$ for some player k: By Lemma 5 $c^j \geq \beta^j$ and then $y^j \geq \chi^j - \epsilon/5$ follows by Lemma 12 and Step i.

Case 5; $(x, y) = J(\alpha, 1)$, $\rho <$ dist (α, D), $\alpha^j < \beta^j$, $\beta^j \geq \chi^j$ and $|\alpha^k| \leq R$ for all players k: It follows by Lemma 13, Lemma 5 and Step i.

Case 6; $(x, y) = J(\alpha, 1)$ and dist $(\alpha, D) \leq \rho$: It follows by the definition of the centering function, and this completes the proof of Step ii.

Step iii; Show for all $x \in T'$ that $x^j \geq \chi^j - \epsilon/5$ for all abnormal j

If $(x_{l,i}, x_{l,i+1}) \in F$ is defined by $J(\alpha_{l,i}, 1)$ then let $\lambda_{l,i} := \lambda(\alpha_{l,i})$ be the weight given to the strategies $p_{l,i}$, should they exist. If $(x_{l,i}, x_{l,i}) \in F$ is defined by G then let $1 - \lambda_{l,i}$ be the weight given to the centering correspondence, with $q_{l,i} \in [0, 1]$ the quantity with $0 \leq q_{l,i} \leq \rho|A(x_{l,i})|$ such that $q_{l,i}\lambda_{l,i}$ is the weight given to the $G_{A(x_{l,i})}$ correspondence. If $(x_{l,i}, x_{l,i+1}) \in J(C, 1)$ then define $(q_{l,i})_{-j}$ to be $q((p_{l,i})_{-j})$. Otherwise if $(x_{l,i}, x_{l,i+1}) \in G$ define $(q_{l,i})_{-j}$ to be $q_{l,i}$ (as the correspondence G uses the quitting behavior of only normal players).

With Lemma 4, Lemma 14 and Step ii it suffices to show for every abnormal j that $\sum_{0 \leq l < N} \sum_{i=1}^{\infty} \lambda_{l,i}(q_{l,i})_{-j} = \infty$. But by Lemma 11 it suffices to show that $\sum_{0 \leq l < N} \sum_{i=1}^{\infty} \lambda_{l,i} q_{l,i} = \infty$.

If $\sum_{0 \leq l < N} \sum_{i=1}^{\infty} \lambda_{l,i} q_{l,i} < \infty$, then by unbounded variation there would be a stage l, i after which the orbit would be defined almost entirely by the centering function z or the centering correspondence Z. But $z^k(x)$ is $v^k - \epsilon/2$ for all normal players k and all vectors x – therefore unbounded variation would imply that after some stage l_0, i_0 all x in the orbit would satisfy $x^k \leq v^k - \frac{\epsilon}{3}$ for all normal k with $-M \leq x^j \leq |N|M + \rho$ for all players j. The distance from D implies that from this stage onward $J(C, 1)$ defines the orbit. For the stages after l_0, i_0 there are three cases:
Case 1) $\alpha^j < -R$ for some player j,
Case 2) $\alpha^j \geq -R$ for all players j and $\lambda < 1$, or
Case 3) $\alpha^j \geq -R$ for all players j and $\lambda = 1$.
For Case 1) we have $q(p) \geq 9/10$. For Case 2) $\alpha^k \geq v^k - 2\rho$ and $\beta^k \leq v^k - \frac{\epsilon}{3}$ (from $x^k \leq v^k - \epsilon/3$) for all normal players k implies (by Proposition 1) that $q(p) \geq \epsilon/3M$ (to prevent any normal player from preferring the move q). For Case 3) we have also $q(p) \geq \epsilon/(3M)$ for the same reason (since $\beta = x$). In all three cases our assumption of a finite sum implies that the $\lambda_{l,i}$ must converge to zero as l goes to infinity, so that after some stage Case 3 is not possible. It is not possible that there is an infinite subsequence satisfying Case 1), since $-M \leq x^l \leq M|N| + \rho$ for all players l and $x \in T'$ implies that $\lambda > 9/10$. But also an infinite subsequence satisfying Case 2) is not possible, since the λ_i converging to zero would imply that there is some stage l_1, i_1 after l_0, i_0 such that the $x^k_{l,i}$ stay above $v^k - 3\rho$ for all normal players k. Step iii is proven.

Since $D \cap \{x \mid x^j \geq \chi^j - \epsilon/2 \geq v^j + \epsilon/2$ for all abnormal $j\} \cap [-R, R]^N$ is empty, we conclude that $x \in T'$ and $(x, y) \in G$ imply that dist $(x, D) > 5\rho$ and $(x, y) \in G_O$.

Step iv; Show that if $(x, y) = J(\alpha, 1)$ **with** $x \in T'$ **and** $\rho <$ **dist** $(\alpha, D) < 2\rho$ **then that there is some** $j \in N$ **with** $\alpha^j < -R$**:**

By Lemma 3 $\alpha^k \geq R$ for any player k is not possible. If $\beta^j \geq \chi^j - \epsilon/2$ for all abnormal j then $\alpha^j < -R$ for some player j follows by Lemma 10. Otherwise by Step iii we must assume that $\alpha^j > \chi^j - \epsilon/5$ and $\beta^j < \chi^j - \epsilon/2$ for some abnormal j. By Proposition 1 we have $q(p_{-j}) > \frac{\epsilon}{10M}$. To avoid $\alpha^k < v^k - 2\rho$ for some normal player k or the same for at least two abnormal players we must assume that some abnormal player k other than j chooses q with probability at least $\frac{\epsilon}{20M}$. But then we can switch roles: whether $\beta^k \geq \chi^k - \epsilon/2$ or $\alpha^k > \chi^k - \epsilon/5$ there is a player other than k choosing q

with probability at least $\frac{\epsilon}{20M}$, meaning that indeed $\alpha^l < v^l - 2\rho$ for at least two distinct players l.

Step v; For every normal player j and every $x \in T'$ show that $x^j > \chi^j - 4\epsilon/5$.

As with Step i, given a normal player j, an $x \in T'$ and y following x in this orbit, we will show that either $x^j \geq \chi^j - 4\epsilon/5$ implies that $y^j \geq \chi^j - 4\epsilon/5$ or that $y^j - x^j \geq \gamma||y-x||$.

As before, this follows easily if $(x,y) \in G$ and $x \in V_O$, since $v^j \geq \chi^j$, the probability of someone else quitting does not exceed $(|N|-1)\rho$, $z^j(x) = v^j - \epsilon/2$, and we must assume that $x^j \geq v^j - \rho$. Likewise if $(x,y) \in G$ and $x \in V_D \backslash V_O$ then it follows by $z^j(x) = v^j - \epsilon/2 \geq \chi^j - \epsilon/2$. Likewise it follows if $(x,y) = J(\alpha,1)$ and dist $(\alpha, D) \leq \rho$. Also if $(x,y) = J(\alpha,1)$ and dist $(\alpha, D) \geq 2\rho$ then $\beta = x$ and it follows by Lemma 5 and $v^j \geq \chi^j$.

Finally consider $(x,y) = J(\alpha,1)$ and $\rho <$dist $(\alpha, D) < 2\rho$. By Part v $\alpha^k < -R$ for some k, meaning that $b_2 = 0$. $\lambda \geq 1 - \frac{\epsilon^2}{4|N|M^2}$ follows by Lemma 12.

Case 1; $\beta^j \leq \chi^j - 3\epsilon/5$ **and** $\alpha^j \leq \chi^j - 3\epsilon/5$**:** With $\chi^j \leq v^j$ to prevent Player j from preferring to quit $q(p_{-j})$ must be at least $\frac{9\epsilon}{10M}$. $y^j - x^j \geq \epsilon^2/(2M)$ follows by $b_2 = 0$, $\lambda \geq 1 - \frac{\epsilon^2}{4|N|M^2}$, Lemma 5 and the definition of $z^j(x)$.

Case 2; $\beta^j \leq \chi^j - 3\epsilon/5$ **and** $\alpha^j > \chi^j - 3\epsilon/5$**:** By the definition of ϕ $p^j < \frac{\epsilon^2}{5M^2|N|}$. But by $x^k < -R$ for some player k we have $q(p_{-j}) \geq 9/10$. This implies that $c^j \geq \beta^j + \epsilon/2$. $\beta^j \leq \chi^j - 3\epsilon/5$ implies that $\alpha^j - z^j(\alpha) < |N|M$ and then $y^j > x^j + \epsilon/4$ follows from $\lambda \geq 1 - \frac{\epsilon^2}{4|N|M^2}$.

Case 3; $\beta^j \geq \chi^j - 3\epsilon/5$**:** Lemma 5 implies that $c^j \geq \chi^j - 3\epsilon/5$. $z^j(\alpha) = v^j - \epsilon/2$ and $b_2 = 0$ suffice for $y^j \geq \chi^j - 3\epsilon/5$.

Step vi; Show that dist $(\alpha, D) \geq 2\rho$ for all $\alpha \in C$ with $(x,y) = J(\alpha, 1)$ and $x \in T'$.

For the sake of contradiction, suppose that it is not true for some $\alpha \in C$. By Step iv there is some player k with $\alpha^k < -R$. With $x = (1-\lambda)\alpha + \lambda\beta$ and $x \in [-M, M|N|+1]^N$ Lemma 3 implies that $x^j \leq \chi^j - 99\epsilon/100$ for some player j. This would be a contradiction to either Step iii or Step v. $\square$

4.7 Conclusion

Theorem 2: An affirmation of Question 1 also affirms the existence of approximate equilibria in quitting games.

Proof: We start with any quitting game with at least two players.

First, is there at least one normal player? If there are no normal players, by Remark 2 there is an ϵ equilibrium of the game for every positive ϵ. Otherwise, we can make Assumption 1, that there is at least one normal player.

Second, we choose any $\delta > 0$ and change no payoff by more than δ to satisfy Assumption 2.

Third, we choose an $\epsilon > 0$ smaller than δ and also small enough so that all ϵ-normal players are normal players, namely Assumption 3. By Proposition B and either Proposition A and Lemma A it suffices to show with the altered game either that there exists an extended forward orbit of $F_{\epsilon,0}$ with unbounded variation in $\{x \mid x^j \geq \chi^j - \epsilon \text{ for all players } j\}$ or that there is an absorbing sequence of perfect one-shot 2ϵ equilibria.

Fourth, is there an absorbing sequence of perfect one-shot 2ϵ equilibria generated by a stationary strategy profile? If so then we are done. Otherwise by Remark 1 we can proceed with Assumption 4.

Fourth, is there an instant ϵ^+ equilibrium? If so, by Proposition C the game has a 3ϵ equilibrium (and also an absorbing sequence of 2ϵ perfect equilibria generated by stationary strategies). Otherwise, we proceed with Assumption 5.

With all five assumptions, the map ϕ, the set C (with its composing sets C_j and $C_{j,k}$), the homotopy J, the closed neighborhood V, and the quantity γ are defined as above. By Proposition 2 (and the affirmation of Question 1) there is an extended forward orbit of F with unbounded variation.

We can restrict ourselves to that part of the orbit that starts within a distance of γ from a cluster point of the orbit, and the remaining part will still be of unbounded variation. By Proposition 3 and Lemma 9 this orbit is also an orbit of $F_{\epsilon,0}(x)$ with the desired property with regard to the values χ^j for all players $j \in N$. q.e.d.

5 Related Questions

The following theorem demonstrates the intuitive basis for believing in an affirmation of Question 1.

Theorem 3: Assume Properties 1, 2,3, and 6 of Question 1 (not necessarily assuming Properties 4,5,7 and 8) and assume additionally that $J(C, 1)$ is the graph of a continuous function from C to $\mathbf{R}^n$. The conclusion of Question 1 is affirmed.

Proof: By Property 1 we can extend our continuous function whose graph is $J(C, 1)$ to a continuous function $g : \mathbf{R}^n \to \mathbf{R}^n$ with $g(x) = x$ for all $x \notin C$. Let Q be the compact set defined by the union of ∂C with the points outside of C that are within a distance of γ from C. Let x be any member of Q. If $y \in \overline{F}(x) \cap Q$ satisfying Property 6 is also in Q, then we could start again at the point y. Otherwise, let the y in $\overline{G}(x)$ satisfying Property 6 be in the interior of C. We look at the continuum from x to y in $\overline{G}(x)$.

Case 1; $g^k(y)$ is in C for all $k \geq 0$ and $g^k(y)$ does not converge to any point: This case is obvious.

Case 2; $g^k(y)$ is in C for all $k \geq 0$ and $g^k(y)$ converges: By Property 2 it must converge to a point in ∂C, and then we can continue from this point. A variation of γ is obtained in the motion from x to y.

Case 3; $g^k(y)$ is not in C for some $k > 0$:
If $g^k(y)$ is in Q, then we can continue with y (with a variation of γ from x to y). Otherwise, assuming that $g^k(y)$ is not in $Q \cup C$, by the continuity of g^k there is some z in the line segment from x to y such that $g^k(z) \in \partial Q \backslash C$. We can continue with $g^k(z)$. By Property 6 either an unbounded variation is obtained from motions to other points in Q or a variation of γ is obtained from a motion back to the set C. □

We believe that this proof can be generalized quite far, at least to the case of $J(C, 1)$ lying entirely in $C \times \mathbf{R}^n$. It is the multi-dimensional complexity of motions taking place outside of $C \times \mathbf{R}$ which causes us to doubt that Question 1 can be affirmed. Properties 4,5, and 7 could be very important to affirming Question 1, as they connect G to $J(C, 1)$. The potential importance of Property 8 is suggested by Question 2.

With respect to Question 1, motion from points outside of C back into C

are problematic. It suggests that a set larger than the original contractible C should be used to formulate an appropriate question of Discrete-Time Viability Theory.

Question 2: Let C be any connected and compact subset of an Euclidean space E. Let $f : C \to E$ be a continuous function such that f and the identity map on C are homotopic and such that all the functions f_t in the homotopy (with f_0 the identity, $f_1 = f$) have the property that $f_t(\partial C) \subseteq C$.
Does there exist a forward orbit for the function f?

First consider the following example, which is **not** a counter-example to Question 2. This example was shown to me by Tamas Wiandt.

Example 1: Let E be the circle and let C be the compact set of angles between 0 and $\frac{6\pi}{5}$. Let $f : C \to C$ be the rotation by the angle $\frac{4\pi}{5}$, (namely 2/5 of the way around the circle). The function f takes both end points of C back into C, and yet from any start in C four or less iterations of the function f result in entering the compliment of C. Of course E is not a Euclidean space, and the problem with translating this example to one dimensional space is that a "point at infinity" is not allowed in an Euclidean space.

The following counter-example to Question 1 was shown to me by Massimo Gobbino.

Example 2: Let (r, θ) be polar coordinates in $\mathbf{R}^2$, the Euclidean plane. Let the sets A_1, A_2 be defined by
$A_1 := \{(r, \theta) \mid 1 \leq r \leq 2\}$,
$A_2 := \{(r, \theta) \mid 2 \leq r \leq 3, 0 \leq \theta \leq \frac{6\pi}{5}\}$.
Let $C = A_1 \cup A_2$ and define $g(r) := \frac{r+3}{2}$ if r is between 1 and 2 and $g(r) := \frac{7-r}{2}$ if r is between 2 and 3. Define $f(r, \theta) := (g(r), \theta + \frac{4\pi}{5})$.

After a few iterations all points of an orbit of f will have a radius strictly between 2 and 3. With regard to the angle we have the identical situation of Example 1, implying that there can be no forward orbit. Lastly it is easy to confirm the desired homotopy property through the level of radius 2.

Example 2 contrast strongly with the well know context of fixed point theory. If $C = D^n := \{x \in \mathbf{R}^n \mid ||x|| \leq 1\}$ and $f : C \to \mathbf{R}^n$ is a continuous function such that $f(\partial C) \subseteq C$ then there is a simple proof for the existence of a fixed point of f. Let $r : \mathbf{R}^n \to D^n$ be the nearest point retraction defined by $r(x) = x$ if $x \in D^n$ and otherwise $r(x) = \frac{x}{||x||}$. Let $Q > 1$ be so large that the image of f lies entirely in $D_Q := \{x \mid ||x|| \leq Q\}$. Consider the function $f \circ r : D_Q \to D_Q$. By Brouwer's Fixed Point Theorem $f \circ r$ has a fixed point x. $x \notin D^n$ is a contradiction, since $r(x) \in \partial D^n = S^{n-1}$ and $f(S^{n-1}) \subseteq D^n$. And with $x \in D^n$ we have $x = f \circ r(x) = f(x)$.

It would be tempting to define a correspondence F based on a quitting game and a choice of $\epsilon > 0$ so that all forward orbits must have unbounded variation and the existence of a forward orbit implies that there is a sequence of perfect one-shot ϵ equilibrium. Perhaps there is a way to do this, but the following question and counter-example generate some doubt concerning this approach.

Question 3: Let E be a Euclidean space and C a contractible subset of E of the same dimension. Let $G \subseteq C \times E$ be a compact set such that
1) for every $c \in C$ $\{y \in E \mid (c, y) \in G\}$ is a non-empty convex set, and
2) for every $c \in C$ all of the extremal points of $G(c) := \{y \mid (c, y) \in G\}$ are in C.
Let $J : G \times [0, 1] \to E \times E$ be a homotopy such that
3) for every $c \in \partial C$ and $(c, y) \in G$ $J((c, y), t) = (c, y)$ for all $t \in [0, 1]$ and
4) $J((c, y), 0) = (c, y)$ for all $(c, y) \in G$.
Define $F := J(G, 1) \subseteq E \times E$.
Does there exist a forward orbit of F?

The following example was constructed by Massimo Gobbino and myself.

Example 3: Let E be R^2. Let W_1 be $[-10, 0] \times [-10, 10]$ and let W_2 be $[-10, 10] \times [-10, 0]$. Let C be $W_1 \cup W_2$. Let $d : C \to [0, 1]$ be the function defined by $d(x, y) = \min\{1, \text{Euclidean distance}((x, y), \partial C)\}$. For the set W_1 define the function $f_1(x, y) = d(x, y)(8, 8) + (1 - d(x, y))(8, -2)$. Define the correspondence F such that $\overline{F}(x, y) = \{f_1(x, y)\}$ if $(x, y) \in W_1 \backslash W_2$, $\overline{F}(x, y) = \{(-2, 8)\}$ if $(x, y) \in W_2 \backslash W_1$, and otherwise $\overline{F}(x, y) =$ convex hull $(\{f_1(x, y), (-2, 8)\})$ if $(x, y) \in W_1 \cap W_2$.

Starting at $(-2, 8)$, we move outside of C to $(8, 8)$ in one step. Starting in $W_2 \backslash W_1$, we move to $(-2, 8)$. Starting at $W_1 \backslash W_2$, we move to W_2 or outside of C in one step. Finally, from any point of C in one step we move to either $W_1 \backslash W_2$, $W_2 \backslash W_1$, or to the compliment of C. The homotopy property can be satisfied by a translation of the point $(0, 0)$ to the point $(8, 8)$.

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