The consistent coalitional value*

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Abstract

We introduce a value for NTU games with coalition structure. This value coincides with the consistent value for trivial coalition structures, and with the Owen value for TU games with coalition structure. Furthermore, we present two characterizations: the first one using a consistency property and the second one using balanced contributions' properties.

1. Introduction

Some of the most important issues of cooperative game theory are to define "good" values, to study which interesting properties are satisfied by these values, and to obtain axiomatic characterizations using some of these properties.

In cooperative games with transferable utility (TU games), Shapley (1953) introduces the Shapley value. He characterizes it as the only value satisfying efficiency, null player, symmetry, and additivity. Later, several authors obtain new characterizations of the Shapley value using other properties. For instance, Myerson (1980) characterizes the Shapley value using balanced contributions and Hart and Mas-Colell (1989) characterize it by consistency.

There are several extensions of TU games. The most natural one is the extension to games without transferable utility (NTU games). Another extension

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applies to TU games with coalition structure, which study situations where players are partitioned into several groups. This model was considered by Aumann and Dréze (1974) and Owen (1977). Of course, a third extension is to NTU games with coalition structure. Since in TU games the Shapley value has a lot of interesting properties, several authors propose, in these extended models, values which are generalizations of the Shapley value.

In NTU games the Harsanyi value (Harsanyi (1963)), and the Shapley NTU value (Shapley (1969)), are generalizations of the Shapley value. Maschler and Owen (1989, 1992) define the consistent value for hyperplane games and NTU games. The main idea behind this generalization is to maintain (as far as possible) the consistency property of the Shapley value. Later, Hart and Mas-Colell (1996) develop a bargaining mechanism which implements the consistent value. They also characterize it by means of balanced contributions.

Owen (1977) introduces a generalization of the Shapley value, called the Owen value, for TU games with coalition structure. He characterizes his value using axioms similar to those used by Shapley (1953). Later, Winter (1992) characterizes the Owen value using the consistency property and Calvo, Lasaga, and Winter (1996) do so using properties of balanced contributions.

In the volumes 2 and 3 of the handbook of game theory with economic applications, chapters 37 ("Coalition structures" by *J.* Greenberg), 53 ("The Shapley value" by *E.* Winter), 54 ("Variations of the Shapley value" by *D.* Monderer and *D.* Samet), and 55 ("Values of non-transferable utility games" by *R. P.* McLean), it is possible to find surveys of this literature.

It is of our interest to know whether the consistent value and the Owen value can be generalized the same way. Then, we introduce a new value, called the consistent coalitional value. It can be characterized in two ways: by the consistency property and by the balanced contributions properties. We must note that our characterizations generalize the results about consistency obtained by Maschler and Owen (1989) for the consistent value and Winter (1992) for the Owen value. They also generalize the results about balanced contributions obtained by Hart and Mas-Colell (1996) for the consistent value and Calvo *et al.* (1996) for the Owen value. We believe these characterizations make the consistent coalitional value a proper generalization of the consistent and the Owen value for NTU games with coalition structure.

Furthermore, Vidal-Puga (2002) proposes a non-cooperative game for which the consistent coalitional value arises as equilibrium payoff. His results are similar to those presented by Hart and Mas-Colell (1996) for the consistent value.

The paper is organized as follows. In Section 2 we introduce the notation and some previous results. In Section 3 we define the consistent coalitional value. In Section 4 we study which properties are satisfied by this value. In Section 5 we present two axiomatic characterizations. Section 6 is devoted to some concluding remarks. Finally, in the Appendix, we present the proofs of the results obtained in the paper.

2. Definitions and Previous Results

Given a set A, $|A|$ denotes the cardinal of A. Given $x, y \in \mathbb{R}^N$, we say $y \leq x$ when $y_i \leq x_i$ for each $i \in N$ and $x * y$ is the scalar product $\sum\limits_{i \in N} x_i y_i$. We denote $\mathbb{R}_+^N = \left\{x \in \mathbb{R}^N : x_i \geq 0, \forall i\right\}$ and $\mathbb{R}_{++}^N = \left\{x \in \mathbb{R}^N : x_i > 0, \forall i\right\}$. We say that $x \in \mathbb{R}^N$ is *normalized* if $\sum\limits_{i \in N} \max\left\{x_i, -x_i\right\} = 1$. Given $\lambda \in \mathbb{R}^N$ a vector orthogonal to some surface on $\mathbb{R}^N$, we say that λ is *orthonormal* if it is normalized.

A *game without transferable utility*, or simply an *NTU game*, is a pair (N, V) where $N = \{1, 2, ..., n\}$ is the set of players and V is a correspondence (*characteristic function*) which assigns to each coalition $S \subset N$ a subset $V(S) \subset \mathbb{R}^S$. This subset represents all the possible payoffs that members of S can obtain for themselves when play cooperatively. For $S \subset N$, if there is no ambiguity, we maintain the notation V when refering to the application V restricted to S as player set. We also denote $\overline{S} = N \backslash S$.

Following Maschler and Owen (1992) we impose the next conditions on the function V:

$(A1)$ For each $S \subset N$, the set $V(S)$ is comprehensive (*i.e.*, if $x \in V(S)$ and $y \in \mathbb{R}^S$ with $y \leq x$, then $y \in V(S)$) and bounded above (*i.e.*, for each $x \in \mathbb{R}^S$, the set $\{y \in V(S) : y \geq x\}$ is compact).

$(A2)$ For each $S \subset N$, the boundary of $V(S)$, which we denote by $\partial V(S)$, is smooth (on each point of the boundary there exists an unique outward orthonormal vector) and nonlevel (the outward vector on each point of $\partial V(S)$ has its coordinates positive). We denote these orthonormal vectors as $\lambda^S = \left(\lambda_i^S\right)_{i \in S}$.

$(A3)$ These λ_i^S are continuous functions on $\partial V(S)$.

$(A4)$ There exists a positive number δ, such that for each $S \subset N$ and $i \in S$, $\lambda_i^S > \delta$.

$(A5)$ For each $S \subset N$, the origin $0_S = (0, ..., 0) \in \mathbb{R}^S$ belongs to $V(S)$.

Property $(A5)$ is a normalization and does not affect our results.

We denote by $NTU(N)$ the set of NTU games over N and by NTU the set of all NTU games.

We now introduce two particular subclasses of NTU games studied in this paper.

We say that (N, V) is a *game with transferable utility* (or *TU game*) if there exists a function $v : 2^N \to \mathbb{R}$, called the characteristic function, satisfying $v(\emptyset) = 0$ and for each $S \subset N$, $V(S) = \left\{ x \in \mathbb{R}^S : \sum_{i \in S} x_i \leq v(S) \right\}$. Usually we represent a TU game as the pair (N, v). We denote by $TU(N)$ the set of TU games over N and by TU the set of all TU games.

We say that (N, V) is a *hyperplane game* if for all $S \subset N$ there exists $\lambda^S \in \mathbb{R}^S_{++}$ satisfying

$$V(S) = \left\{ x \in \mathbb{R}^S : \lambda^S * x \leq v(S) \right\} \tag{2.1}$$

for some $v : 2^N \to \mathbb{R}$.

Notice that each TU game is a hyperplane game (just take $\lambda_i^S = 1$ for each $S \subset N$ and $i \in S$).

A *coalition structure* C over N is a partition of the player set, *i.e.*, $C = \{C_1, C_2, ..., C_m\} \subset 2^N$ where $\bigcup_{C_q \in C} C_q = N$ and $C_q \cap C_r = \emptyset$ when $q \neq r$. We denote by (N, V, C) an NTU game (N, V) with coalition structure C over N. We denote $CNTU(N)$ as the set of NTU games with coalition structure over N ($CTU(N)$ for TU games) and by $CNTU$ the set of all NTU games with coalition structure (CTU for TU games).

Given $S \subset N$ we denote by C_S the structure C restricted to the players in S, *i.e.*, $C_S = \{C_q \cap S\}_{C_q \in C}$. Notice that this implies that C_S may have less or the same number of coalitions as C. For simplicity we use C_{-i} instead of $C_{N \setminus \{i\}}$.

A *payoff configuration* for (N, V) is a set of payoffs $x = \left(x^S\right)_{S \subset N}$ with $x^S \in V(S)$ for all $S \subset N$.

Given G a subset of $CNTU$ (or NTU), a *value* Γ on G is a correspondence which assigns to each $(N, V, C) \in G$ a subset $\Gamma(N, V, C) \subset V(N)$. We say that $\left(\Gamma^S\right)_{S \subset N}$ is a payoff configuration associated to Γ if $\Gamma^S \in \Gamma(S, V, C_S)$ for all $S \subset N$. When several NTU games or coalition structures are involved we write $\Gamma^S(V)$, $\Gamma^S(C)$, or $\Gamma^S(V, C)$ instead of Γ^S.

If $\Gamma(N, V, C)$ is a single point of $V(N)$ for all $(N, V, C) \in G$ we say that Γ is a *single value*. Of course each single value has a unique payoff configuration associated. Usually we write Γ^N instead of $\Gamma(N, V, C)$.

We denote by ϕ^N (or $\phi^N(v)$) the *Shapley value* (Shapley (1953)) of the TU game (N, v).

For TU games with coalition structure ϕ^N (or $\phi^N(v, C)$) denotes the *Owen value* (Owen (1977)), which is a generalization of the Shapley value (when $C = \{N\}$ or $C = \left\{\{i\}_{i \in N}\right\}$, the Owen value coincides with the Shapley value).

We now present the consistent value for NTU games following Maschler and Owen (1989, 1992).

Let Π be the set of all permutations over N. Given $\pi \in \Pi$, we define the set of *predecessors* of i under π as

$$P(\pi, i) = \{j \in N : \pi(j) < \pi(i)\}.$$

We call the *marginal contribution* of player $i \in N$ to the game V in the permutation π to

$$d_i(\pi) = \max\left\{x_i : \left((d_j(\pi))_{j \in P(\pi,i)}, x_i\right) \in V\left(P(\pi, i) \cup \{i\}\right)\right\}.$$

So, $d_i(\pi)$ is the maximum that player i can obtain in $V(S)$ after his predecessors obtain their respective $d_j(\pi)$'s. We denote $d(\pi) = (d_i(\pi))_{i \in N}$.

Given a hyperplane game (N, V), the *consistent value* Ψ^N (or $\Psi^N(V)$) (Maschler and Owen (1989)) is the vector of expected marginal contributions, where each $\pi \in \Pi$ is equally likely, *i.e.*

$$\Psi^N = \frac{1}{n!} \sum_{\pi \in \Pi} d(\pi).$$

Notice that each $d(\pi)$ is an efficient vector (it belongs to the boundary of $V(N)$). Since we are dealing with hyperplane games, this boundary is flat and the consistent value is also an efficient value.

Maschler and Owen (1989) prove that, given $i \in N$,

$$\Psi_i^N = \frac{1}{|N|\lambda_i^N}\left(\sum_{j \in N \setminus \{i\}} \lambda_i^N \Psi_i^{N \setminus \{j\}} + v(N) - \sum_{j \in N \setminus \{i\}} \lambda_j^N \Psi_j^{N \setminus \{i\}}\right). \tag{2.2}$$

One way to extend a hyperplane solution to the general class of NTU games with convex $V(S)$'s is to pass arbitrary hyperplanes to the various sets $V(S)$. These hyperplanes determine a hyperplane game to which we know the solution. If this solution belongs to $V(N)$ we say that this is a solution to the NTU game

(N, V). This is the way adopted by Maschler and Owen (1992) for extending the consistent value to the class of NTU games.

Formally, given an NTU game (N, V) we say that (N, V') is a supporting hyperplane game for (N, V) if for each $S \subset N$,

$$V'(S) = \left\{ x \in \mathbb{R}^S : \lambda^S * x \leq v(S) \right\}$$

where λ^S is orthonormal to the boundary of $V(S)$ and $v(S) = \max \left\{ \lambda^S * x : x \in V(S) \right\}$. Notice that $V(S) \subset V'(S)$.

Given an NTU game (N, V) a payoff configuration x is a *consistent value* for (N, V) if there exists a supporting hyperplane game for (N, V) such that $x^S = \Psi^S(V')$ for all $S \subset N$. It is known that the consistent value is not a single value.

3. The Consistent Coalitional Value

In this section we define the consistent coalitional value for NTU games. We first define it in hyperplane games by generalizing the expression (2.2) of Ψ to situations with coalition structure.

Given a hyperplane game (N, V, C), the *consistent coalitional value* Υ^N (or $\Upsilon^N(V, C)$) is the only vector satisfying the following two conditions:

For all $C_q \in C$,

$$\sum_{j \in C_q} \lambda_j^N \Upsilon_j^N =$$

$$\frac{1}{|C|} \left[\sum_{C_r \in C \backslash C_q} \left(\sum_{j \in C_q} \lambda_j^N \Upsilon_j^{N \backslash C_r} \right) + v(N) - \sum_{C_r \in C \backslash C_q} \left(\sum_{j \in C_r} \lambda_j^N \Upsilon_j^{N \backslash C_q} \right) \right]. \tag{3.1}$$

For all $i \in C_q \in C$,

$$\Upsilon_i^N =$$

$$\frac{1}{|C_q| \lambda_i^N} \left(\sum_{j \in C_q \backslash \{i\}} \lambda_i^N \Upsilon_i^{N \backslash \{j\}} + \sum_{j \in C_q} \lambda_j^N \Upsilon_j^N - \sum_{j \in C_q \backslash \{i\}} \lambda_j^N \Upsilon_j^{N \backslash \{i\}} \right). \tag{3.2}$$

Remark 1. It is straightforward to prove that Υ is well-defined and $\sum_{j \in N} \lambda_j^N \Upsilon_j^N = v(N)$.

Since Υ is a single value, there is only one consistent coalitional payoff configuration $\Upsilon = \left(\Upsilon^S\right)_{S\subset N}$, which satisfies that $\Upsilon^S = \Upsilon^S\left(V, C_S\right) \in \partial V\left(S\right)$ for all $S \subset N$.

The generalization of Υ to NTU games is done analogously to the consistent value. For an NTU game with coalition structure (N, V, C), we take for each coalition $S \subset N$ a orthonormal vector λ^S to the boundary of $V(S)$. Let (N, V', C) be the resulting hyperplane game and let $\Upsilon = \left(\Upsilon^S\right)_{S\subset N}$ be the consistent coalitional payoff configuration associated with (N, V', C). If $\Upsilon^S \in V\left(S\right)$ for all $S \subset N$, we say that Υ is a *consistent coalitional payoff configuration.*

In the next theorem we prove the existence of consistent coalitional payoff configurations.

Theorem 1: Every NTU game has a consistent coalitional payoff configuration.

Proof. See the Appendix.

If $C = \{N\}$ or $C = \left\{\{i\}_{i\in N}\right\}$ the consistent coalitional value Υ coincides with the consistent value Ψ. When $C = \{N\}$, (3.2) coincides with (2.2) and when $C = \left\{\{i\}_{i\in N}\right\}$, (3.1) coincides with (2.2) . Since Υ is the only value in hyperplane games satisfying (2.2) we conclude that $\Upsilon = \Psi$. Now it is straightforward to conclude that $\Upsilon = \Psi$ in NTU games. Moreover, for TU games with coalition structure the consistent coalitional value coincides with the Owen value (we will see it later in Corollary 1). Thus, the consistent coalitional value is a generalization of the consistent value and the Owen value for NTU games with coalition structure.

We know that the Shapley value, the consistent value, and the Owen value are obtained as an average of marginal contributions depending on equal-likely permutations. Thus, it seems reasonable to generalize these values in the same way.

We say that a permutation $\pi \in \Pi$ is *admissible* with respect to C if given $i, j \in C_q \in C$ and $k \in N$ such that $\pi(i) < \pi(k) < \pi(j)$ then $k \in C_q$. We denote by Π^C the set of all permutations over N admissible with respect to C.

Given a hyperplane game (N, V, C), the *random order coalitional value* $\digamma^N$ (or $\digamma^N(V, C)$) is defined as the vector of expected marginal contributions when all the admissible permutations with respect to C are equally likely, *i.e.*

$$\digamma^N = \frac{1}{|\Pi^C|}\sum_{\pi\in\Pi^C} d(\pi).$$

We can extend, as in the case of the consistent coalitional value, the random order coalitional $\digamma$ to NTU games. Using arguments similar to those used with Υ we can prove that $\digamma$ is a single value in hyperplane games but not in general. Moreover, $\digamma$ also generalizes the consistent value and the Owen value.

In TU games McLean (1991) defines the random order coalitional structure values. $\digamma$ is the natural generalization to NTU games of McLean's values when all the admissible permutations are equal-likely and the rest of permutations have probability 0. It is remarkable that Maschler and Owen (1992) even suggest the name random order value instead of consistent value.

The definition of Υ is not so intuitive as the definition of $\digamma$. Nevertheless, we believe that Υ is a more suitable value for hyperplane games (and NTU games) than $\digamma$. We will prove later that Υ satisfies more interesting properties. Moreover, Υ can be characterized generalizing axiomatic characterizations of the Owen value and the consistent value.

We now compute Υ and $\digamma$ in the following example.

Example 1. (Owen (1972)). Let (N, V, C) be such that $N = \{1, 2, 3\}$ and

$$\begin{aligned}
V(\{i\}) &= \{x_i \in \mathbb{R}^{\{i\}} : x_i \leq 0\},\ \forall i \in N, \\
V(\{1,2\}) &= \{(x_1, x_2) \in \mathbb{R}^{\{1,2\}} : x_1 + 4x_2 \leq 1, x_1 \leq 1, x_2 \leq \frac{1}{4}\}, \\
V(\{1,3\}) &= \{(x_1, x_3) \in \mathbb{R}^{\{1,3\}} : x_1 \leq 0, x_3 \leq 0\}, \\
V(\{2,3\}) &= \{(x_2, x_3) \in \mathbb{R}^{\{2,3\}} : x_2 \leq 0, x_3 \leq 0\},
\end{aligned}$$

and

$$V(N) = \{x \in \mathbb{R}^N : \sum_{i \in N} x_i \leq 1; x_i \leq 1\ \forall i \in N; x_i + x_j \leq 1\ \forall i, j \in N\}.$$

If $C = \{\{1, 2\}, \{3\}\}$, we obtain that

$$\Upsilon^N = \left(\tfrac{13}{32}, \tfrac{13}{32}, \tfrac{6}{32}\right) \text{ and } \digamma^N = \left(\tfrac{8}{16}, \tfrac{5}{16}, \tfrac{3}{16}\right).$$

However, for $C = \{\{1\}, \{2, 3\}\}$ both values coincide because

$$\Upsilon^N = \digamma^N = \left(\frac{8}{16}, \frac{5}{16}, \frac{3}{16}\right).$$

This example shows that Υ and $\digamma$ are different.

4. Properties

In this section we present several desirable properties and study which of them are satisfied by the consistent coalitional value.

Some of these properties are well-known in the literature of NTU games. Others are introduced in this paper generalizing properties of TU games. We present the definitions for single values. The definition for payoff configurations associated with general values is straightforward.

We say that a value Γ satisfies *efficiency (EF)* if for each $(N, V, C) \in CNTU$, $\Gamma^N \in \partial V(N)$.

Remark 2. Since V satisfies $(A2)$ we conclude that if Γ satisfies efficiency then for each $(N, V, C) \in CNTU$ and $S \subset N$, there exists $\lambda^S \in \mathbb{R}^S_{++}$ satisfying $\lambda^S * \Gamma^S = v(S)$ where $v(S) = \max\left\{\lambda^S * x : x \in V(S)\right\}$. Of course, the reciprocal statement is also true.

Given $(N, V, C) \in CNTU$ we say that two players $i, j \in N$ are *symmetric* if two properties hold:

For each $S \subset N \backslash \{i, j\}$, if $x \in V(S \cup \{i\})$, $y_j = x_i$, and $y_k = x_k$ for each $k \in S$ then, $y \in V(S \cup \{j\})$.

For each $S \supset \{i, j\}$, if $x \in V(S)$, $y_i = x_j$, $y_j = x_i$, and $x_k = y_k$ for each $k \in S\backslash \{i, j\}$ then, $y \in V(S)$.

We say that a value Γ satisfies *individual symmetry (IS)* if for each pair of symmetric players $i, j \in C_q \in C$,

$$\Gamma^N_i = \Gamma^N_j.$$

We now present the property of covariance in hyperplane games following Maschler and Owen (1989). Let (N, V, C) and $(N, \widetilde{V}, C)$ be two hyperplane games such that for each $S \subset N$,

$$V(S) = \left\{x \in \mathbb{R}^S : \lambda^S * x \leq v(S)\right\} \text{ and } \widetilde{V}(S) = \left\{x \in \mathbb{R}^S : \widetilde{\lambda}^S * x \leq \widetilde{v}(S)\right\}.$$

We say that (N, V, C) and $(N, \widetilde{V}, C)$ are *equivalent under a linear transformation of player i's utility* if there exist two constants $a \in \mathbb{R}_{++}$ and $b \in \mathbb{R}$ such that for all $S \subset N$: $\widetilde{\lambda}^S_i = \dfrac{\lambda^S_i}{a}$, $\widetilde{\lambda}^S_j = \lambda^S_j$ if $j \neq i$, $\widetilde{v}(S) = v(S) + \dfrac{b\lambda^S_i}{a}$ if $i \in S$, and

$\widetilde{v}(S) = v(S)$ if $i \notin S$. Notice that if (N, V, C) and $\left(N, \widetilde{V}, C\right)$ are equivalent under a linear transformation of player i's utility, then $\widetilde{x} \in V(S)$ if and only if there exists $x \in V(S)$ satisfying: $\widetilde{x}_i = ax_i + b$ and $\widetilde{x}_j = x_j$ if $j \in S \setminus \{i\}$.

We say that a value Γ satisfies *covariance (COV)* if, given two hyperplane games (N, V, C) and $(N, \widetilde{V}, C)$, equivalent under a linear transformation of some player i's utility,

$$\begin{aligned} \Gamma_i^N\left(\widetilde{V}, C\right) &= a\Gamma_i^N(V, C) + b \text{ and} \\ \Gamma_j^N\left(\widetilde{V}, C\right) &= \Gamma_j^N(V, C) \text{ if } j \in N \setminus \{i\}. \end{aligned}$$

Thus, covariance just states that, if we linearly change player i's utility function, his final payoff changes the same way, while other players' payoffs remain constant.

Hart and Mas-Colell (1989) characterize the Shapley value as the only value on TU games satisfying consistency and other properties. They say that a value Γ satisfies consistency ($CONS$) if and only if for each $(N, v) \in TU, S \subset N$, and $i \in S$,

$$\Gamma_i^N(v) = \Gamma_i^S(v_S)$$

where $v_S(T) = v\left(T \cup \overline{S}\right) - \sum_{j \in \overline{S}} \Gamma_j^N(v)$ for all $T \subset S$.

Winter (1992) extends the definition of consistency to TU games with coalition structure. He say that a value Γ satisfies *consistency* ($CONS$) if and only if for each $(N, v, C) \in CTU$, $C_q \in C$, $S \subset C_q$, and $i \in S$,

$$\Gamma_i^N(v, C) = \Gamma_i^S(v_S, \{S\})$$

where $v_S(T) = v\left(T \cup \overline{S}\right) - \sum_{j \in \overline{S}} \Gamma_j^N(v)$ for all $T \subset S$. Notice that Winter's consistency is a generalization of Hart and Mas-Colell's consistency to TU games with coalition structure.

Maschler and Owen (1989) show that if we define the property of consistency of Hart and Mas-Colell (1989) in NTU games as in the TU case, there is no value satisfying consistency and other "basic" properties (for instance, efficiency). Thus they provide a weaker definition of consistency for hyperplane games.

They say that a value Γ satisfies l-consistency if for each hyperplane game (N, V, C) with $l \leq |N|$ and $i \in N$,

$$\sum_{S \subset N, i \in S, |S|=l} \Gamma_i^S (V_S) = \binom{|N|-1}{l-1} \Gamma_i^N (V).$$

where $V_S(T) = \left\{ x \in \mathbb{R}^T : \left(x, \left(\Gamma_i^{T \cup \overline{S}} \right)_{i \in \overline{S}} \right) \in V \left(T \cup \overline{S} \right) \right\}$ for all $T \subset S$.

We now present a generalization to hyperplane games with coalition structure.

Given a value Γ, a hyperplane game (N, V, C), and $S \subset C_q \in C$, the *reduced game* $(S, V_S, \{S\})$ is defined for each $T \subset S$ as follows:

$$V_S(T) = \left\{ x \in \mathbb{R}^T : \left(x, \left(\Gamma_i^{T \cup \overline{S}} \right)_{i \in \overline{S}} \right) \in V \left(T \cup \overline{S} \right) \right\}.$$

It is straightforward to prove that V_S is the hyperplane game given, for each $T \subset S$, by

$$V_S(T) = \left\{ x \in \mathbb{R}^T : \sum_{i \in T} \lambda_i^{T \cup \overline{S}} x_i \leq v \left(T \cup \overline{S} \right) - \sum_{i \in \overline{S}} \lambda_i^{T \cup \overline{S}} \Gamma_i^{T \cup \overline{S}} \right\}.$$

We say that a value Γ satisfies *l-consistency* if for each hyperplane game (N, V, C), $C_q \in C$ with $l \leq |C_q|$, and $i \in C_q$,

$$\sum_{S \subset C_q, i \in S, |S|=l} \Gamma_i^S (V_S) = \binom{|C_q|-1}{l-1} \Gamma_i^N (V).$$

For simplicity we denote $\Gamma_i^S (V_S) = \Gamma_i^S (V_S, \{S\})$ and $\Gamma_i^N (V) = \Gamma_i^N (V, C)$.

We call 2-consistency *bilateral consistency (BCONS).*

Notice that our bilateral consistency generalizes, in the natural way, the consistency of Hart and Mas-Colell (1989), the consistency of Winter (1992), and the bilateral consistency of Maschler and Owen (1989).

Myerson (1980) characterizes the Shapley value using efficiency and balanced contributions. He say that a value Γ satisfies balanced contributions (BC) if for each $(N, v) \in TU$ and $i, j \in N$,

$$\Gamma_i^N - \Gamma_i^{N \setminus \{j\}} = \Gamma_j^N - \Gamma_j^{N \setminus \{i\}}.$$

Calvo *et al.* (1996) generalize the property of balanced contributions for TU games with a coalition structure obtaining two properties: $BCAC$ and $BCAP$[1].

They say that a value Γ satisfies *balanced contributions among coalitions (BCAC)* if for each $(N, v, C) \in CTU$ and $C_q, C_r \in C$ with $q \neq r$,

$$\sum_{j \in C_q} \Gamma_j^N - \sum_{j \in C_q} \Gamma_j^{N \backslash C_r} = \sum_{j \in C_r} \Gamma_j^N - \sum_{j \in C_r} \Gamma_j^{N \backslash C_q}.$$

They say that a value Γ satisfies *balanced contributions among players in the same coalition (BCAP)* if for each $i, j \in C_q \in C$ with $i \neq j$,

$$\Gamma_i^N - \Gamma_i^{N \backslash \{j\}} = \Gamma_j^N - \Gamma_j^{N \backslash \{i\}}.$$

Hart and Mas-Colell (1996) introduce the following generalization of balanced contributions for NTU games.

They say that a value Γ satisfies *average balanced contributions (ABC)* if for each $(N, V) \in NTU$, $S \subset N$, and $i \in S$, there exists $\lambda^S \in \mathbb{R}_{++}^S$ such that

$$\sum_{j \in S \backslash \{i\}} \lambda_i^S \left(\Gamma_i^S - \Gamma_i^{S \backslash \{j\}} \right) = \sum_{j \in S \backslash \{i\}} \lambda_j^S \left(\Gamma_j^S - \Gamma_j^{S \backslash \{i\}} \right).$$

We now introduce the properties of average balanced contributions among coalitions and average balanced contributions among players in the same coalition for NTU games with coalition structure.

We say that a value Γ satisfies *average balanced contributions among coalitions (ABCAC)* if for each $(N, V, C) \in CNTU$, $S \subset N$, and $C'_q = C_q \cap S \in C_S$, there exists $\lambda^S \in \mathbb{R}_{++}^S$ such that

$$\sum_{C'_r \in C_S \backslash C'_q} \left[\sum_{j \in C'_q} \lambda_j^S \left(\Gamma_j^S - \Gamma_j^{S \backslash C'_r} \right) \right] = \sum_{C'_r \in C_S \backslash C'_q} \left[\sum_{j \in C'_r} \lambda_j^S \left(\Gamma_j^S - \Gamma_j^{S \backslash C'_q} \right) \right].$$

We say that a value Γ satisfies *average balanced contributions among players in the same coalition (ABCAP)* if for each $(N, V, C) \in CNTU$, $S \subset N$, $C'_q = C_q \cap S \in C_S$, and $i \in C'_q$, there exists $\lambda^S \in \mathbb{R}_{++}^S$ such that

$$\sum_{j \in C'_q \backslash \{i\}} \lambda_i^S \left(\Gamma_i^S - \Gamma_i^{S \backslash \{j\}} \right) = \sum_{j \in C'_q \backslash \{i\}} \lambda_j^S \left(\Gamma_j^S - \Gamma_j^{S \backslash \{i\}} \right).$$

[1] Even though Calvo et al. (1996) present these two balanced properties as only one, we think that for our paper it is more natural the formulation as two properties.

Notice that our average balanced contributions properties generalize, in the natural way, the balanced properties of Myerson (1980), Calvo *et al.* (1996), and Hart and Mas-Colell (1996).

Before studying the properties satisfied by the consistent coalitional value we need a previous result.

Lemma 1. Given a hyperplane game (N, V, C) and $i \in S \subset C_q \in C$,

$$(S \backslash \{i\}, V_S, \{S \backslash \{i\}\}) = \left(S \backslash \{i\}, V_{S \backslash \{i\}}, \{S \backslash \{i\}\}\right).$$

Proof. This result is due to Maschler and Owen (1989). ■

Maschler and Owen (1989) prove that Ψ satisfies, in hyperplane games, l-consistency for all $l = 1, ..., n$. In the next proposition we obtain a similar result for Υ.

Proposition 1. The consistent coalitional value satisfies, in hyperplane games, l-consistency for each l with $1 \leq l \leq n$.

Proof. See the Appendix.

In the next theorem we study which of these properties are satisfied by the consistent coalitional value.

Theorem 2. The consistent coalitional value satisfies EF, IS, $ABCAC$, and $ABCAP$. Moreover, in hyperplane games it also satisfies COV and $BCONS$.

Proof. See the Appendix.

By Theorem 2 we know that Υ satisfies, in NTU games or hyperplane games, all the interesting properties that the Owen value satisfies in TU games and the consistent coalitional value in NTU games or hyperplane games.

5. Axiomatic characterizations

In this section we present two axiomatic characterizations of the consistent coalitional value. The first one applies to the set of hyperplane games; and we present it by using consistency. The second one applies to the set of NTU games; and we present it by using balanced contributions.

Hart and Mas-Colell (1989) characterize the Shapley value on the class of TU games as the only single value satisfying EF, SYM (symmetry), COV, and $CONS$. Later, Winter (1992) and Maschler and Owen (1989) extend this result in two different ways.

Winter (1992) extends it to the class of TU games with coalition structure. He proves that the Owen value is the only single value satisfying EF, IS, COV, $CONS$, and Game Between Coalitions Property ($GBCP$).

They say that a single value Γ satisfies $GBCP$ if for each TU game (N, v, C) and $C_q \in C$,

$$\sum_{i \in C_q} \Gamma_i(N, v, C) = \Gamma_{C_q}\left(M, v^{[C]}, \{M\}\right),$$

where $M = \{C_1, ..., C_m\}$ and $v^{[C]}(S) = v\left(\bigcup_{C_r \in S} C_r\right)$ for each $S \subset M$. This property says that the amount received by a coalition in the game played by the coalitions (every coalition acts as a single player) coincides with the sum of the amounts received by the members of this coalition in the original game.

This property cannot be exported to hyperplane games.

It is not difficult to check that the proof of Winter's result about the characterization of the Owen value is also valid if we replace $GBCP$ by $BCAC$. Then, the Owen value is the only single value satisfying EF, IS, COV, $CONS$, and $BCAC$.

Maschler and Owen (1989) extend this result to the class of hyperplane games. They prove that the consistent value is the only single value satisfying EF, SYM, COV, and $BCONS$.

In Theorem 3 below we generalize the results of Hart and Mas-Colell (1989), Winter (1992), and Maschler and Owen (1989) to hyperplane games with coalition structure.

Theorem 3: The consistent coalitional value is the only single value on the class of hyperplane games satisfying EF, IS, $COV, BCONS$, and $ABCAC$.

Proof. See the Appendix.

Remark 3. The properties used in this theorem are independent (see the Appendix).

Myerson (1980) characterizes the Shapley value on the class of TU games as the only single value satisfying EF and BC. Later, Calvo *et al.* (1996) and Hart and Mas-Colell (1996) extend this result in two different ways.

Calvo *et al.* (1996) extend it to the class of TU games with coalition structure. They prove that the Owen value is the only single value satisfying EF, $BCAP$, and $BCAC$.

Hart and Mas-Colell (1996) extend Myerson's result to the class of NTU games proving that the consistent value is the only value satisfying EF and ABC.

In Theorem 4 below we generalize the results of Myerson (1980), Calvo *et al.* (1996), and Hart and Mas-Colell (1996) to NTU games with coalition structure.

Theorem 4. The consistent coalitional value is the only value on the class of NTU games with coalition structure satisfying EF, $ABCAC$, and $ABCAP$.

Proof. See the Appendix.

Remark 4. The properties used in this theorem are independent (see the Appendix).

We now prove that the consistent coalitional value generalizes the Owen value.

Corollary 1: For each TU game (N, v, C) the Owen value is the only consistent coalitional value.

Proof. See the Appendix.

The results obtained in this section about the consistent coalitional value and the relation with other values can be summarized in the following table.

Table 1

<table>
<tr><td colspan="4">About consistency</td></tr>
<tr><td colspan="2">Without coalition structure</td><td colspan="2">With coalition structure</td></tr>
<tr><td>TU</td><td>Hyperplane</td><td>TU</td><td>Hyperplane</td></tr>
<tr><td>Shapley</td><td>Consistent</td><td>Owen</td><td>Consistent Coalitional</td></tr>
<tr><td>EF
SYM
COV
$CONS$</td><td>EF
SYM
COV
$BCONS$</td><td>EF
IS
COV
$CONS$
$BCAC$</td><td>EF
IS
COV
$BCONS$
$ABCAC$</td></tr>
<tr><td colspan="4">About balanced contributions</td></tr>
<tr><td colspan="2">Without coalition structure</td><td colspan="2">With coalition structure</td></tr>
<tr><td>TU</td><td>NTU</td><td>TU</td><td>NTU</td></tr>
<tr><td>Shapley</td><td>Consistent</td><td>Owen</td><td>Consistent Coalitional</td></tr>
<tr><td>EF
BC</td><td>EF
ABC</td><td>EF
$BCAC$
$BCAP$</td><td>EF
$ABCAC$
$ABCAP$</td></tr>
</table>

Then, the consistent coalitional value is the right generalization of the Owen value and the consistent value to NTU games with coalition structure if we focus on the properties of consistency and balanced contributions of both values.

6. Concluding remarks

In this paper we present two generalizations of the Owen value and the consistent value for NTU games with coalition structure: the consistent coalitional value and the random order coalitional value.

We now study which of the properties introduced before are satisfied by the random order coalitional value. In the following lemma we prove that the random order coalitional value also satisfies (3.1).

Lemma 2. F satisfies (3.1) in the class of hyperplane games.

Proof. See the Appendix.

Since $\digamma$ and Υ are different (Example 1) we conclude that $\digamma$ does not satisfy (3.2).

Proposition 2. a) The random order coalitional value satisfies EF, IS, COV (in hyperplane games), and $ABCAC$.

b) The random order coalitional value satisfy neither $BCONS$ nor $ABCAP$.

Proof. a) It is trivial to see that $\digamma$ satisfies EF and IS.

Maschler and Owen (1989) show that, for any $\pi \in \Pi$, the vector $d(\pi)$ satisfies COV. Since $\digamma$ is the mean of some of these $d(\pi)$'s, we conclude that $\digamma$ also satisfies COV.

By Lemma 2, $\digamma$ satisfies (3.1). Now, using arguments similar to those used in the proof of Theorem 2 for Υ, we can conclude that $\digamma$ also satisfies $ABCAC$.

b) It is as a consequence of theorems 3 and 4. ■

In TU games, NTU games, and TU games with coalition structure the Shapley value, the consistent value, and the Owen value have two important aspects. Firstly, they have an intuitive definition because they can be computed through the vector of marginal contributions. Secondly, these values can be characterized with nice properties (namely, consistency and balanced contributions).

NTU games with coalition structure generalize the three class of games mentioned before. Nevertheless, in this general framework we can not find a value with an intuitive definition and nice characterizations. The random order coalitional value has an intuitive definition through the vector of marginal contributions but fails in the nice characterizations. The consistent coalitional value has nice characterizations but fails in the intuitive definition.

This fact is not surprising. There are results from TU games that can not be generalized in the same way to NTU games. For instance, the consistent value and the Shapley NTU value are generalizations of the Shapley value. But, whereas the consistent value generalizes the characerizations of the Shapley value based in the properties of consistency and balanced contributions, Aumann (1985) proves that the Shapley NTU value generalizes the classical axiomatization of the Shapley value.

NTU games with coalition structure are studied also by Winter (1991), where he characterizes the Game Coalition Structure Value. This value is a generalization of both, the Harsanyi value for NTU games and the Owen value for TU games with coalition structure. Winter characterizes his value with six axioms:

EF, COV, conditional additivity, independence of irrelevant alternatives, inessential games, and unanimity games. We believe that the properties used in the characterizations of Υ are more natural than the properties used by Winter. For instance, unanimity games says that the value must coincide with the Owen value in unanimity games.

7. Appendix

Proof of Theorem 1. The structure of the proof is analogous to the proof of Theorem 3.3 in Maschler and Owen (1992), where they prove the existence of the consistent value for general NTU games.

We make use of induction to prove the following claim:

> *Given $\left(x^T\right)_{T\subsetneq N}$ with $x^T \in \mathbb{R}^T$ such that, for any $S \subsetneq N$, the collection $\left(x^T\right)_{T\subset S}$ is a consistent coalitional payoff configuration of the game (S, V, C_S), there exists $x^N \in \partial V(N)$ such that $\left(x^T\right)_{T\subset N}$ is a consistent coalitional payoff configuration of (N, V, C).*

For $n = 1$ the claim is trivially true, the collection being the empty set.

Assume now the claim holds for less than n players. Thus, there exists a collection $\left(x^T\right)_{T\subsetneq N}$ such that, for any $S \subsetneq N$, $\left(x^T\right)_{T\subset S}$ is a consistent coalitional payoff configuration of the game (S, V, C_S).

Assume that $z \in \partial V(N)$. For each $T \subsetneq N$, let $\lambda^T = \left(\lambda_i^T\right)_{i\in T}$ be the orthonormal vector outwards x^T. Moreover, $\left(\lambda_i^N\right)_{i\in N}$ is the orthonormal vector outwards z.

Consider the hyperplane game (N, V^z, C) such that, for each $S \subset N$,

$$V^z(S) = \left\{y \in \mathbb{R}^S : \lambda^S * y \leq v\left(S\right)\right\}$$

where $v\left(S\right) = \lambda^S * x^S$ when $S \neq N$ and $v\left(N\right) = \lambda^N * z$.

Let $\left(\Upsilon^S(z)\right)_{S\subset N}$ be the (unique) consistent coalitional payoff configuration for the hyperplane game (N, V^z, C). By definition of V^z, $\Upsilon^S\left(z\right) = x^S$ for all $S \subsetneq N$, independently of the chosen z.

We want to show that there exists a point $x^N \in \partial V(N)$ such that the collection $\left(x^T\right)_{T\subset N}$ is a consistent coalitional payoff configuration for (N, V, C). Notice that it is enough to prove that $\Upsilon^N\left(x^N\right) = x^N$. We make use of a fixed point theorem.

Since Υ satisfies (3.1) and (3.2) and the λ_i^S's are strictly positive and continuous functions, $\Upsilon^N(z)$ is also a continuous function of z.

We define $M = \max\left\{\frac{|x_i^T|}{\delta} : i \in T \subsetneq N\right\}$, where δ is given by (A4).

Given $C_q \in C$, by (3.1),

$$|C| \sum_{j\in C_q} \lambda_j^N \Upsilon_j^N(z) = \sum_{C_r\in C\setminus C_q} \left(\sum_{j\in C_q} \lambda_j^N x_j^{N\setminus C_r}\right) + v(N) - \sum_{C_r\in C\setminus C_q} \left(\sum_{j\in C_r} \lambda_j^N x_j^{N\setminus C_q}\right).$$

By $(A5)$, $v(N) \geq 0$, and since the λ_j^N's are positive,

$$\begin{aligned}
&\geq \sum_{C_r\in C\setminus C_q} \left(\sum_{j\in C_q} \lambda_j^N (-M\delta)\right) - \sum_{C_r\in C\setminus C_q} \left(\sum_{j\in C_r} \lambda_j^N (M\delta)\right) \\
&= -(|C|-1) M\delta \sum_{j\in C_q} \lambda_j^N - M\delta \sum_{j\in N\setminus C_r} \lambda_j^N \\
&\geq -(|C|-1) M\delta - M\delta = -|C| M\delta,
\end{aligned}$$

where the last inequality comes because λ^N is normalized.

So, $\sum_{j\in C_q} \lambda_j^N \Upsilon_j^N(z) \geq -M\delta$ for each $C_q \in C$.

Given $i \in C_q \in C$, by (3.2),

$$\begin{aligned}
|C_q| \Upsilon_i^N(z) &= \sum_{j\in C_q\setminus\{i\}} x_i^{N\setminus\{j\}} + \frac{\sum_{j\in C_q} \lambda_j^N \Upsilon_j^N(z) - \sum_{j\in C_q\setminus\{i\}} \lambda_j^N x_j^{N\setminus\{i\}}}{\lambda_i^N} \\
&\geq \sum_{j\in C_q\setminus\{i\}} (-M\delta) + \frac{-M\delta - \sum_{j\in C_q\setminus\{i\}} \lambda_j^N M\delta}{\lambda_i^N} \\
&= -(|C_q|-1) M\delta - \frac{M\delta}{\lambda_i^N} - \frac{\sum_{j\in C_q\setminus\{i\}} \lambda_j^N M\delta}{\lambda_i^N}
\end{aligned}$$

since $\lambda_i^N > \delta$, λ^N is normalized, and $\delta < 1$,

$$> -(|C_q|-1) M\delta - M - M \sum_{j\in C_q\setminus\{i\}} \lambda_j^N$$

$$
\begin{aligned}
&> -\left(\left|C_q\right|-1\right)M\delta - M - M \\
&> -\left(\left|C_q\right|-1\right)M - 2M \\
&\geq -2\left|C_q\right|M.
\end{aligned}
$$

So, $\Upsilon_i^N(z) > -2M$.

The rest of the proof is analogous to Maschler and Owen's (1992) and we just give a geometric description for the case $n = 2$.

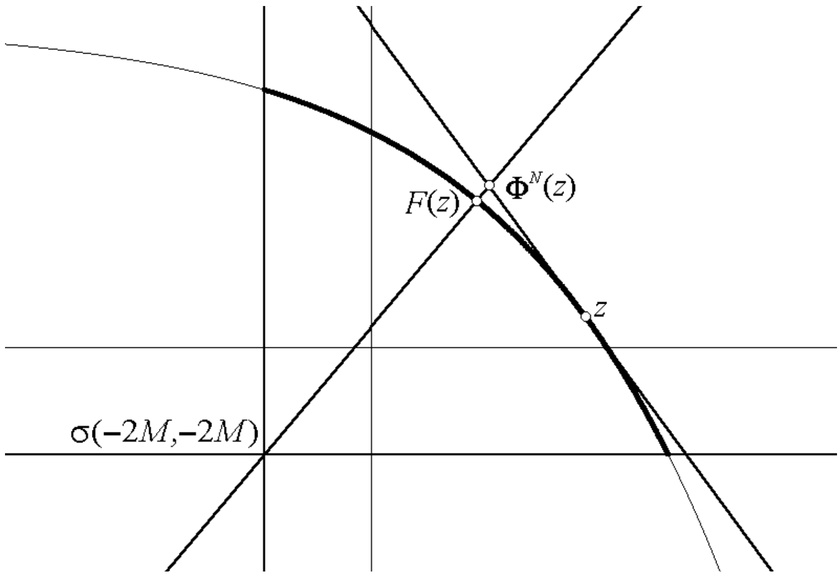

Figure 1

We define $D = \left\{x \in \mathbb{R}^N : x_i \geq -2M \text{ for all } i \in N\right\}$. Given a vector z on $\partial V(N) \cap D$ (which is the thick line in figure 1), we have proved that $\Upsilon^N(z) \in D$; and so the point $F(z)$ obtained by applying a projection centered at $\sigma = (-2M, ..., -2M) \in \mathbb{R}^N$, also belongs to $\partial V(N) \cap D$ (see figure 1). By applying a standard fixed point theorem over the (continuous) function F, we find the desired x^N. ■

Proof of Proposition 1. We proceed by induction on l. The theorem is trivially true for $l = 1$. Assume it is true for at most $l - 1$.

If we apply the induction hypothesis to the game $(N\setminus\{j\}, V, C_{-j})$ with $j \in C_q\setminus\{i\}$ (if $C_q = \{i\}$, the result is trivially true for C_q) then,

$$\sum_{T\subset C_q\setminus\{j\}:i\in T,|T|=l-1}\Upsilon_i^T\left(V_T\right)=\tbinom{|C_q|-2}{l-2}\Upsilon_i^{N\setminus\{j\}}\left(V\right). \tag{7.1}$$

We wish to prove that for each $C_q\in C$ and $i\in C_q$,

$$l\lambda_i^N\sum_{S\subset C_q:i\in S,|S|=l}\Gamma_i^S\left(V_S\right)=l\lambda_i^N\tbinom{|C_q|-1}{l-1}\Gamma_i^N\left(V\right). \tag{7.2}$$

To do so, we analyze the left side of this expression. Assume that $i\in S\subset C_q$ and $|S|=l$. Applying (3.2) to $(S,V_S,\{S\})$, which is also a hyperplane game, we obtain:

$$l\lambda_i^N\Upsilon_i^S\left(V_S\right)=\sum_{j\in S\setminus\{i\}}\lambda_i^N\Upsilon_i^{S\setminus\{j\}}\left(V_S\right)+\sum_{j\in S}\lambda_j^N\Upsilon_j^S\left(V_S\right)-\sum_{j\in S\setminus\{i\}}\lambda_j^N\Upsilon_j^{S\setminus\{i\}}\left(V_S\right).$$

If we compute Υ in the game V_S we obtain that

$$\sum_{j\in S}\lambda_j^N\Upsilon_j^S\left(V_S\right)=v\left(N\right)-\sum_{j\in\overline{S}}\lambda_j^N\Upsilon_j^N\left(V\right).$$

Hence,

$$l\lambda_i^N\Upsilon_i^S\left(V_S\right)=\sum_{j\in S\setminus\{i\}}\lambda_i^N\Upsilon_i^{S\setminus\{j\}}\left(V_S\right)+v\left(N\right)-\sum_{j\in\overline{S}}\lambda_j^N\Upsilon_j^N\left(V\right)-\sum_{j\in S\setminus\{i\}}\lambda_j^N\Upsilon_j^{S\setminus\{i\}}\left(V_S\right).$$

Since there are $\tbinom{|C_q|-1}{l-1}$ possible sets $S\subset C_q$ with $i\in S$ and $|S|=l$,

$$l\lambda_i^N\sum_{S\subset C_q:i\in S,|S|=l}\Upsilon_i^S\left(V_S\right)=$$

$$\sum_{S\subset C_q:i\in S,|S|=l}\left(\sum_{j\in S\setminus\{i\}}\lambda_i^N\Upsilon_i^{S\setminus\{j\}}\left(V_S\right)\right)+\binom{|C_q|-1}{l-1}v\left(N\right)$$
$$-\sum_{S\subset C_q:i\in S,|S|=l}\left(\sum_{j\in\overline{S}}\lambda_j^N\Upsilon_j^N\left(V\right)\right)-\sum_{S\subset C_q:i\in S,|S|=l}\left(\sum_{j\in S\setminus\{i\}}\lambda_j^N\Upsilon_j^{S\setminus\{i\}}\left(V_S\right)\right).$$

Rearranging the order of summation, we have:

$$\sum_{j\in C_q\backslash\{i\}}\left(\sum_{S\subset C_q:i,j\in S,|S|=l}\lambda_i^N\Upsilon_i^{S\backslash\{j\}}\left(V_S\right)\right)+\binom{|C_q|-1}{l-1}v\left(N\right)$$
$$-\sum_{j\in N\backslash\{i\}}\left(\sum_{S\subset C_q:i\in S,j\notin S,|S|=l}\lambda_j^N\Upsilon_j^N\left(V\right)\right)-\sum_{j\in C_q\backslash\{i\}}\left(\sum_{S\subset C_q:i,j\in S,|S|=l}\lambda_j^N\Upsilon_j^{S\backslash\{i\}}\left(V_S\right)\right).$$

We now analyze the four terms separately:

1. First term is equal, by Lemma 1, to

$$\sum_{j\in C_q\backslash\{i\}}\lambda_i^N\left(\sum_{T\subset C_q\backslash\{j\}:i\in T,|T|=l-1}\Upsilon_i^T\left(V_T\right)\right)$$

   which coincides, by (7.1), with

$$\binom{|C_q|-2}{l-2}\sum_{j\in C_q\backslash\{i\}}\lambda_i^N\Upsilon_i^{N\backslash\{j\}}\left(V\right).$$

2. Since $v(N)=\lambda_i^N\Upsilon_i^N\left(V\right)+\sum_{j\in C_q\backslash\{i\}}\lambda_j^N\Upsilon_j^N\left(V\right)+\sum_{j\in N\backslash C_q}\lambda_j^N\Upsilon_j^N\left(V\right)$, second term is equal to

$$\binom{|C_q|-1}{l-1}\lambda_i^N\Upsilon_i^N\left(V\right)+\binom{|C_q|-1}{l-1}\sum_{j\in C_q\backslash\{i\}}\lambda_j^N\Upsilon_j^N\left(V\right)$$
$$+\binom{|C_q|-1}{l-1}\sum_{j\in N\backslash C_q}\lambda_j^N\Upsilon_j^N\left(V\right).$$

3. Third term is equal to

$$-\sum_{j\in C_q\backslash\{i\}}\left(\sum_{S\subset C_q:i\in S,j\notin S,|S|=l}\lambda_j^N\Upsilon_j^N\left(V\right)\right)-\sum_{j\in N\backslash C_q}\left(\sum_{S\subset C_q:i\in S,j\notin S,|S|=l}\lambda_j^N\Upsilon_j^N\left(V\right)\right)$$

since for each $j \in C_q \backslash \{i\}$, there are $\binom{|C_q|-2}{l-1}$ possible sets S, such that $S \subset C_q$, $i \in S$, $j \notin S$, and $|S| = l$; and for each $j \in N \backslash C_q$, there are $\binom{|C_q|-1}{l-1}$ possible sets S, such that $S \subset C_q$, $i \in S$, $j \notin S$, and $|S| = l$, last expression coincides with

$$-\binom{|C_q|-2}{l-1} \sum_{j \in C_q \backslash \{i\}} \lambda_j^N \Upsilon_j^N (V) - \binom{|C_q|-1}{l-1} \sum_{j \in N \backslash C_q} \lambda_j^N \Upsilon_j^N (V).$$

4. Fourth term is equal, by Lemma 1, to

$$-\sum_{j \in C_q \backslash \{i\}} \lambda_j^N \left(\sum_{T \subset C_q \backslash \{i\} : j \in T, |T| = l-1} \Upsilon_j^T (V_T) \right),$$

which coincides, by (7.1), with

$$-\binom{|C_q|-2}{l-2} \sum_{j \in C_q \backslash \{i\}} \lambda_j^N \Upsilon_j^{N \backslash \{i\}} (V).$$

Since $\binom{|C_q|-1}{l-1} = \binom{|C_q|-2}{l-1} + \binom{|C_q|-2}{l-2}$, adding terms 2 and 3 we obtain

$$\binom{|C_q|-1}{l-1} \lambda_i^N \Upsilon_i^N (V) + \binom{|C_q|-2}{l-2} \sum_{j \in C_q \backslash \{i\}} \lambda_j^N \Upsilon_j^N (V).$$

Then,

$$l \lambda_i^N \sum_{S \subset C_q : i \in S, |S| = l} \Upsilon_i^S (V_S) =$$

$$\binom{|C_q|-2}{l-2} \sum_{j \in C_q \backslash \{i\}} \lambda_i^N \Upsilon_i^{N \backslash \{j\}} (V) + \binom{|C_q|-1}{l-1} \lambda_i^N \Upsilon_i^N (V)$$
$$+ \binom{|C_q|-2}{l-2} \sum_{j \in C_q \backslash \{i\}} \lambda_j^N \left(\Upsilon_j^N (V) - \Upsilon_j^{N \backslash \{i\}} (V) \right).$$

In Theorem 2 we prove, without using this lemma, that Υ satisfies $ABCAP$ and hence,

$$= \binom{|C_q|-2}{l-2} \sum_{j \in C_q \backslash \{i\}} \lambda_i^N \Upsilon_i^{N \backslash \{j\}} (V) + \binom{|C_q|-1}{l-1} \lambda_i^N \Upsilon_i^N (V)$$
$$+ \binom{|C_q|-2}{l-2} \sum_{j \in C_q \backslash \{i\}} \lambda_i^N \left(\Upsilon_i^N (V) - \Upsilon_i^{N \backslash \{j\}} (V) \right)$$

$$= \binom{|C_q|-1}{l-1} \lambda_i^N \Upsilon_i^N (V) + \binom{|C_q|-2}{l-2} \left(\sum_{j \in C_q \setminus \{i\}} \lambda_i^N \Upsilon_i^N (V) \right).$$

Since $\binom{|C_q|-1}{l-1} + \binom{|C_q|-2}{l-2} (|C_q|-1) = l \binom{|C_q|-1}{l-1}$, the last expression coincides with

$$l \binom{|C_q|-1}{l-1} \lambda_i^N \Upsilon_i^N (V)$$

which is precisely the right side of (7.2). ■

Proof of Theorem 2. It is straightforward to prove that Υ satisfies EF and IS. By Proposition 1, we know that Υ satisfies $BCONS$.

Let $\left(\Upsilon^S\right)_{S \subset N}$ be a consistent coalitional payoff configuration. Then, there exists a supporting hyperplane game (N, V', C) such that $\Upsilon^S = \Upsilon^S (V', C)$. Moreover, $V'(S) = \left\{ y \in \mathbb{R}^S : \lambda^S * y \leq v(S) \right\}$, where $\lambda^S \in \mathbb{R}^S_{++}$ is an orthonormal vector to $\partial V(S)$ and $v(S) = \max \left\{ \lambda^S * x : x \in V(S) \right\}$.

We now prove that Υ satisfies $ABCAC$. In order to simplify the notation, we assume that $S = N$. By EF and Remark 2, $v(N) = \sum_{C_r \in C} \left(\sum_{j \in C_r} \lambda_j^N \Upsilon_j^N \right)$. Applying this to (3.1) we obtain that for all $C_q \in C$,

$$|C| \sum_{j \in C_q} \lambda_j^N \Upsilon_j^N =$$

$$= \sum_{C_r \in C \setminus C_q} \left(\sum_{j \in C_q} \lambda_j^N \Upsilon_j^{N \setminus C_r} \right) + \sum_{C_r \in C} \left(\sum_{j \in C_r} \lambda_j^N \Upsilon_j^N \right) - \sum_{C_r \in C \setminus C_q} \left(\sum_{j \in C_r} \lambda_j^N \Upsilon_j^{N \setminus C_q} \right)$$

$$= \sum_{C_r \in C \setminus C_q} \left(\sum_{j \in C_q} \lambda_j^N \Upsilon_j^{N \setminus C_r} \right) + \sum_{j \in C_q} \lambda_j^N \Upsilon_j^N + \sum_{C_r \in C \setminus C_q} \left(\sum_{j \in C_r} \lambda_j^N \left(\Upsilon_j^N - \Upsilon_j^{N \setminus C_q} \right) \right).$$

If we subtract $\sum_{C_r \in C \setminus C_q} \left(\sum_{j \in C_q} \lambda_j^N \Upsilon_j^N \right) = (|C|-1) \sum_{j \in C_q} \lambda_j^N \Upsilon_j^N$ in both sides, then

$$\sum_{j \in C_q} \lambda_j^N \Upsilon_j^N =$$

$$\sum_{C_r\in C\setminus C_q}\left(\sum_{j\in C_q}\lambda_j^N\left(\Upsilon_j^{N\setminus C_r}-\Upsilon_j^N\right)\right)+\sum_{j\in C_q}\lambda_j^N\Upsilon_j^N+\sum_{C_r\in C\setminus C_q}\left(\sum_{j\in C_r}\lambda_j^N\left(\Upsilon_j^N-\Upsilon_j^{N\setminus C_q}\right)\right).$$

Hence,

$$0=\sum_{C_r\in C\setminus C_q}\left(\sum_{j\in C_q}\lambda_j^N\left(\Upsilon_j^{N\setminus C_r}-\Upsilon_j^N\right)\right)+\sum_{C_r\in C\setminus C_q}\left(\sum_{j\in C_r}\lambda_j^N\left(\Upsilon_j^N-\Upsilon_j^{N\setminus C_q}\right)\right),$$

which means that Υ satisfies $ABCAC$.

We now prove that Υ satisfies $ABCAP$. In order to simplify the notation, we assume that $S=N$. Given $i\in C_q\in C$, by (3.2),

$$\begin{aligned}
|C_q|\lambda_i^N\Upsilon_i^N &= \sum_{j\in C_q\setminus\{i\}}\lambda_i^N\Upsilon_i^{N\setminus\{j\}}+\sum_{j\in C_q}\lambda_j^N\Upsilon_j^N-\sum_{j\in C_q\setminus\{i\}}\lambda_j^N\Upsilon_j^{N\setminus\{i\}}\\
&= \sum_{j\in C_q\setminus\{i\}}\lambda_i^N\Upsilon_i^{N\setminus\{j\}}+\lambda_i^N\Upsilon_i^N+\sum_{j\in C_q\setminus\{i\}}\lambda_j^N\left(\Upsilon_j^N-\Upsilon_j^{N\setminus\{i\}}\right)\\
&= \sum_{j\in C_q\setminus\{i\}}\lambda_i^N\left(\Upsilon_i^{N\setminus\{j\}}-\Upsilon_i^N\right)+|C_q|\lambda_i^N\Upsilon_i^N+\sum_{j\in C_q\setminus\{i\}}\lambda_j^N\left(\Upsilon_j^N-\Upsilon_j^{N\setminus\{i\}}\right).
\end{aligned}$$

Then,

$$0=\sum_{j\in C_q\setminus\{i\}}\lambda_i^N\left(\Upsilon_i^{N\setminus\{j\}}-\Upsilon_i^N\right)+\sum_{j\in C_q\setminus\{i\}}\lambda_j^N\left(\Upsilon_j^N-\Upsilon_j^{N\setminus\{i\}}\right),$$

which means that Υ satisfies $ABCAP$.

We now prove that Υ satisfies COV. Given $i\in C_q\in C$, let $\left(N,\widetilde{V},C\right)$ be obtained from (N,V,C) by a change in player i's utility. Let a and b be the corresponding constants. We proceed by induction over the number of coalitions of C.

If C has only one coalition ($C=\{N\}$) and since Ψ satisfies COV:

- $\Upsilon_i^N\left(\widetilde{V}\right)=\Psi_i^N\left(\widetilde{V}\right)=a\Psi_i^N(V)+b=a\Upsilon_i^N(V)+b$ and
- $\Upsilon_j^N\left(\widetilde{V}\right)=\Psi_j^N\left(\widetilde{V}\right)=\Psi_j^N(V)=\Upsilon_j^N(V)$ for each $j\in N\setminus\{i\}$.

Assume the result holds when $|C|$ has at most $m-1$ coalitions. We prove it when $|C|=m$.

By (3.1),

$$|C|\sum_{j\in C_q}\tilde{\lambda}_j^N\Upsilon_j^N\left(\widetilde{V}\right)=$$

$$\sum_{C_r\in C\setminus C_q}\left(\sum_{j\in C_q}\tilde{\lambda}_j^N\Upsilon_j^{N\setminus C_r}\left(\widetilde{V}\right)\right)+\tilde{v}\left(N\right)-\sum_{C_r\in C\setminus C_q}\left(\sum_{j\in C_r}\tilde{\lambda}_j^N\Upsilon_j^{N\setminus C_q}\left(\widetilde{V}\right)\right).$$

By induction hypothesis, $\Upsilon_i^{N\setminus C_r}\left(\widetilde{V}\right)=a\Upsilon_i^{N\setminus C_r}\left(V\right)+b$ when $C_r\neq C_q$; moreover, $\Upsilon_j^{N\setminus C_r}\left(\widetilde{V}\right)=\Upsilon_j^{N\setminus C_r}\left(V\right)$ when $j\neq i$. Since $\tilde{\lambda}_i^N=\dfrac{\lambda_i^N}{a}$ and $\tilde{\lambda}_j^N=\lambda_j^N$ when $j\neq i$,

$$\begin{aligned}
|C|\sum_{j\in C_q}\tilde{\lambda}_j^N\Upsilon_j^N\left(\widetilde{V}\right) &= \sum_{C_r\in C\setminus C_q}\left(\sum_{j\in C_q\setminus\{i\}}\lambda_j^N\Upsilon_j^{N\setminus C_r}\left(V\right)+\lambda_i^N\Upsilon_i^{N\setminus C_r}\left(V\right)+\frac{b\lambda_i^N}{a}\right)\\
&\quad +v\left(N\right)+\frac{b\lambda_i^N}{a}-\sum_{C_r\in C\setminus C_q}\left(\sum_{j\in C_r}\lambda_j^N\Upsilon_j^{N\setminus C_q}\left(V\right)\right)\\
&= \sum_{C_r\in C\setminus C_q}\left(\sum_{j\in C_q}\lambda_j^N\Upsilon_j^{N\setminus C_r}\left(V\right)\right)+v\left(N\right)\\
&\quad -\sum_{C_r\in C\setminus C_q}\left(\sum_{j\in C_r}\lambda_j^N\Upsilon_j^{N\setminus C_q}\left(V\right)\right)+|C|\frac{b\lambda_i^N}{a}\\
&= |C|\sum_{j\in C_q}\lambda_j^N\Upsilon_j^N\left(V\right)+|C|\frac{b\lambda_i^N}{a},
\end{aligned}$$

where the last equality comes because Υ satisfies (3.1).

Given $k\in C_q$, by (3.2),

$$|C_q|\tilde{\lambda}_k^N\Upsilon_k^N\left(\widetilde{V}\right)=\sum_{j\in C_q\setminus\{k\}}\tilde{\lambda}_k^N\Upsilon_k^{N\setminus\{j\}}\left(\widetilde{V}\right)+\sum_{j\in C_q}\tilde{\lambda}_j^N\Upsilon_j^N\left(\widetilde{V}\right)-\sum_{j\in C_q\setminus\{k\}}\tilde{\lambda}_j^N\Upsilon_j^{N\setminus\{k\}}\left(\widetilde{V}\right).$$

By the induction hypothesis and the previous result, if $k=i$ then,

$$|C_q|\tilde{\lambda}_i^N\Upsilon_i^N\left(\widetilde{V}\right) = \sum_{j\in C_q\setminus\{i\}}\left(\lambda_i^N\Upsilon_i^{N\setminus\{j\}}\left(V\right)+\frac{b\lambda_i^N}{a}\right)$$

$$
\begin{aligned}
& + \sum_{j\in C_q} \lambda_j^N \Upsilon_j^N(V) + \frac{b\lambda_i^N}{a} - \sum_{j\in C_q\setminus\{i\}} \lambda_j^N \Upsilon_j^{N\setminus\{i\}}(V) \\
= & \sum_{j\in C_q\setminus\{i\}} \lambda_i^N \Upsilon_i^{N\setminus\{j\}}(V) + \sum_{j\in C_q} \lambda_j^N \Upsilon_j^N(V) \\
& - \sum_{j\in C_q\setminus\{i\}} \lambda_j^N \Upsilon_j^{N\setminus\{i\}}(V) + \frac{b\lambda_i^N}{a}|C_q| \\
= & |C_q|\lambda_i^N \Upsilon_i^N(V) + \frac{b\lambda_i^N}{a}|C_q|,
\end{aligned}
$$

where the last equality comes because Υ satisfies (3.2).

Therefore,

$$\Upsilon_i^N\left(\widetilde{V}\right) = a\Upsilon_i^N(V) + b.$$

By the induction hypothesis and the previous result, if $k \neq i$ then,

$$
\begin{aligned}
|C_q|\lambda_k^N \Upsilon_k^N\left(\widetilde{V}\right) = & \sum_{j\in C_q\setminus\{k\}} \lambda_k^N \Upsilon_k^{N\setminus\{j\}}(V) + \sum_{j\in C_q} \lambda_j^N \Upsilon_j^N(V) + \frac{b\lambda_i^N}{a} \\
& - \sum_{j\in C_q\setminus\{k,i\}} \lambda_j^N \Upsilon_j^{N\setminus\{k\}}(V) - \lambda_i^N \Upsilon_i^{N\setminus\{k\}}(V) - \frac{b\lambda_i^N}{a} \\
= & \sum_{j\in C_q\setminus\{k\}} \lambda_k^N \Upsilon_k^{N\setminus\{j\}}(V) + \sum_{j\in C_q} \lambda_j^N \Upsilon_j^N(V) - \sum_{j\in C_q\setminus\{k\}} \lambda_j^N \Upsilon_j^{N\setminus\{k\}}(V) \\
= & |C_q|\lambda_k^N \Upsilon_k^N(V).
\end{aligned}
$$

Therefore, $\Upsilon_k^N\left(\widetilde{V}\right) = \Upsilon_k^N(V)$.

Given $C_r \in C \setminus C_q$, using arguments similar to those used for C_q we can conclude that

$$\sum_{j\in C_r} \tilde{\lambda}_j^N \Upsilon_j^N\left(\widetilde{V}\right) = \sum_{j\in C_r} \lambda_j^N \Upsilon_j^N(V).$$

Now using (3.2) it is not difficult to conclude that for each $j \in C_r$, $\Upsilon_j^N\left(\widetilde{V}\right) = \Upsilon_j^N(V)$.

Thus, Υ satisfies COV. ■

Proof of Theorem 3. In Theorem 2 we proved that the consistent coalitional value satisfies these five properties in the class of hyperplane games.

We now prove the reciprocal. Let $\widetilde{\Upsilon}$ be a single value satisfying these five properties. We will show that $\widetilde{\Upsilon} = \Upsilon$. We proceed by induction on the number of players. If there is only one player, then, by EF, $\widetilde{\Upsilon} = \max\{x : x \in V(\{i\})\} = \Upsilon$.

When $|N| = 2$ we can assume without loss of generality that $\lambda_i^{\{i\}} = \lambda_j^{\{j\}} = 1$. There are two possible coalition structures, $C^1 = \{i, j\}$ or $C^2 = \{\{i\}, \{j\}\}$.

Given $a \in \mathbb{R}$, let (N, v^a) be the TU game given by $v^a(\{i\}) = v^a(\{j\}) = a$ and $v^a(N) = 1$.

Since $\widetilde{\Upsilon}$ satisfies EF and IS, we conclude that

$$\widetilde{\Upsilon}_i^N\left(v^a, C^1\right) = \widetilde{\Upsilon}_j^N\left(v^a, C^1\right) = \frac{1}{2}.$$

Since $\widetilde{\Upsilon}$ satisfies EF and $ABCAC$, we conclude that

$$\widetilde{\Upsilon}_i^N\left(v^a, C^2\right) = \widetilde{\Upsilon}_j^N\left(v^a, C^2\right) = \frac{1}{2}.$$

A similar result can be obtained for Υ.

Since any hyperplane game with two players (N, V, C) can be obtained from v^a (for some a) by linear transformation of utilities of players, and Υ and $\widetilde{\Upsilon}$ satisfy COV it is straightforward to prove that for each $i \in N$,

$$\widetilde{\Upsilon}_i^N = \frac{v(N) + \lambda_i^N v(\{i\}) - \lambda_j^N v(\{j\})}{2\lambda_i^N} = \Upsilon_i^N.$$

Moreover,

$$\lambda_i^N \Upsilon_i^N - \lambda_j^N \Upsilon_j^N = \lambda_i^N \widetilde{\Upsilon}_i^N - \lambda_j^N \widetilde{\Upsilon}_j^N = \lambda_i^N v(\{i\}) - \lambda_j^N v(\{j\}). \tag{7.3}$$

Assume that $\widetilde{\Upsilon} = \Upsilon$ for hyperplane games with at most $n-1$ players with $n \geq 3$. We prove $\widetilde{\Upsilon} = \Upsilon$ when (N, V, C) is a hyperplane game with n players.

We first prove that for each $C_q \in C$,

$$\sum_{j \in C_q} \lambda_j^N \Upsilon_j^N(V) = \sum_{j \in C_q} \lambda_j^N \widetilde{\Upsilon}_j^N(V). \tag{7.4}$$

By induction hypothesis we know that, for each $S \subsetneq N$, $\widetilde{\Upsilon}^S(V) = \Upsilon^S(V)$. Given $C_q \in C$, by (3.1),

$$\sum_{j \in C_q} \lambda_j^N \Upsilon_j^N(V) =$$

$$
\begin{aligned}
&= \frac{1}{|C|}\left[\sum_{C_r \in C \backslash C_q}\left(\sum_{j \in C_q} \lambda_j^N \Upsilon_j^{N \backslash C_r}(V)\right) + v(N) - \sum_{C_r \in C \backslash C_q}\left(\sum_{j \in C_r} \lambda_j^N \Upsilon_j^{N \backslash C_q}(V)\right)\right] \\
&= \frac{1}{|C|}\left[\sum_{C_r \in C \backslash C_q}\left(\sum_{j \in C_q} \lambda_j^N \widetilde{\Upsilon}_j^{N \backslash C_r}(V)\right) + v(N) - \sum_{C_r \in C \backslash C_q}\left(\sum_{j \in C_r} \lambda_j^N \widetilde{\Upsilon}_j^{N \backslash C_q}(V)\right)\right].
\end{aligned}
$$

Since $\widetilde{\Upsilon}$ satisfies EF, $v(N) = \sum_{C_r \in C}\left(\sum_{j \in C_r} \lambda_j^N \widetilde{\Upsilon}_j^N(V)\right)$. Then,

$$
\begin{aligned}
|C| \sum_{j \in C_q} \lambda_j^N \Upsilon_j^N(V) &= \sum_{C_r \in C \backslash C_q}\left(\sum_{j \in C_q} \lambda_j^N \widetilde{\Upsilon}_j^{N \backslash C_r}(V)\right) + \sum_{j \in C_q} \lambda_j^N \widetilde{\Upsilon}_j^N(V) \\
&\quad + \sum_{C_r \in C \backslash C_q}\left(\sum_{j \in C_r} \lambda_j^N \left(\widetilde{\Upsilon}_j^N(V) - \widetilde{\Upsilon}_j^{N \backslash C_q}(V)\right)\right).
\end{aligned}
$$

We add and subtract $\sum_{C_r \in C \backslash C_q}\left(\sum_{j \in C_q} \lambda_j^N \widetilde{\Upsilon}_j^N(V)\right) = (|C| - 1) \sum_{j \in C_q} \lambda_j^N \widetilde{\Upsilon}_j^N(V)$,

$$
\begin{aligned}
&= \sum_{C_r \in C \backslash C_q}\left(\sum_{j \in C_q} \lambda_j^N \left(\widetilde{\Upsilon}_j^{N \backslash C_r}(V) - \widetilde{\Upsilon}_j^N(V)\right)\right) + |C| \sum_{j \in C_q} \lambda_j^N \widetilde{\Upsilon}_j^N(V) \\
&\quad + \sum_{C_r \in C \backslash C_q}\left(\sum_{j \in C_r} \lambda_j^N \left(\widetilde{\Upsilon}_j^N(V) - \widetilde{\Upsilon}_j^{N \backslash C_q}(V)\right)\right)
\end{aligned}
$$

So,

$$|C|\left(\sum_{j \in C_q} \lambda_j^N \Upsilon_j^N(V) - \sum_{j \in C_q} \lambda_j^N \widetilde{\Upsilon}_j^N(V)\right) =$$

$$= \sum_{C_r \in C \setminus C_q} \left(\sum_{j \in C_r} \lambda_j^N \left(\widetilde{\Upsilon}_j^N (V) - \widetilde{\Upsilon}_j^{N \setminus C_q} (V) \right) \right)$$
$$- \sum_{C_r \in C \setminus C_q} \left(\sum_{j \in C_q} \lambda_j^N \left(\widetilde{\Upsilon}_j^N (V) - \widetilde{\Upsilon}_j^{N \setminus C_r} (V) \right) \right).$$

Since $\widetilde{\Upsilon}$ satisfies $ABCAC$, we conclude that the last expression is equal to 0. Then,

$$\sum_{j \in C_q} \lambda_j^N \Upsilon_j^N (V) = \sum_{j \in C_q} \lambda_j^N \widetilde{\Upsilon}_j^N (V).$$

We now prove that $\widetilde{\Upsilon}_i^N = \Upsilon_i^N$ for each $i \in C_q \subset N$. We denote by V_S and $\widetilde{V}_S$ the reduced games associated to Υ and $\widetilde{\Upsilon}$ respectively.

By (7.4), if $C_q = \{i\}$ we conclude that $\widetilde{\Upsilon}_i^N = \Upsilon_i^N$.

Assume that $C_q \neq \{i\}$. For each $j \in C_q \setminus \{i\}$ we consider $S = \{i, j\}$. We know that V_S and $\widetilde{V}_S$ are hyperplane games. We denote by v_S and $\tilde{v}_S$ the associated functions to V_S and $\widetilde{V}_S$. By the definition of reduced game and the induction hypothesis,

$$\widetilde{V}_S (\{i\}) = V_S (\{i\}) \text{ and } \widetilde{V}_S (\{j\}) = V_S (\{j\}).$$

Hence, $v_S (\{i\}) = \tilde{v}_S (\{i\})$ and $v_S (\{j\}) = \tilde{v}_S (\{j\})$.

Since $\widetilde{\Upsilon}$ satisfies EF we conclude that $v(N) = \sum_{k \in N} \lambda_k^N \widetilde{\Upsilon}_k^N (V)$. Then,

$$\widetilde{V}_S (S) = \left\{ (x_i, x_j) \in \mathbb{R}^{\{i,j\}} : \lambda_i^N x_i + \lambda_j^N x_j \leq \lambda_i^N \widetilde{\Upsilon}_i^N (V) + \lambda_j^N \widetilde{\Upsilon}_j^N (V) \right\}.$$

By the efficiency of $\widetilde{\Upsilon}$ and (7.3),

$$\lambda_i^N \widetilde{\Upsilon}_i^S \left(\tilde{V}_S \right) + \lambda_j^N \widetilde{\Upsilon}_j^S \left(\tilde{V}_S \right) = \lambda_i^N \widetilde{\Upsilon}_i^N (V) + \lambda_j^N \widetilde{\Upsilon}_j^N (V)$$

$$\lambda_i^N \widetilde{\Upsilon}_i^S \left(\tilde{V}_S \right) - \lambda_j^N \widetilde{\Upsilon}_j^S \left(\tilde{V}_S \right) = \lambda_i^N \tilde{v}_S (\{i\}) - \lambda_j^N \tilde{v}_S (\{j\}).$$

If we sum on $C_q \setminus \{i\}$ both expressions,

$$\lambda_i^N \sum_{j \in C_q \setminus \{i\}} \widetilde{\Upsilon}_i^S \left(\tilde{V}_S \right) + \sum_{j \in C_q \setminus \{i\}} \lambda_j^N \widetilde{\Upsilon}_j^S \left(\tilde{V}_S \right) = \lambda_i^N \left(|C_q| - 1 \right) \widetilde{\Upsilon}_i^N (V) + \sum_{j \in C_q \setminus \{i\}} \lambda_j^N \widetilde{\Upsilon}_j^N (V)$$

$$\lambda_i^N \sum_{j\in C_q\setminus\{i\}} \widetilde{\Upsilon}_i^S\left(\tilde{V}_S\right) - \sum_{j\in C_q\setminus\{i\}} \lambda_j^N \widetilde{\Upsilon}_j^S\left(\tilde{V}_S\right) = \lambda_i^N\left(|C_q|-1\right)\tilde{v}_S(\{i\}) + \sum_{j\in C_q\setminus\{i\}} \lambda_j^N \tilde{v}_S(\{j\}).$$

Since $\widetilde{\Upsilon}$ satisfies $BCONS$, $\sum_{j\in C_q\setminus\{i\}} \widetilde{\Upsilon}_i^S\left(\tilde{V}_S\right) = \left(|C_q|-1\right)\widetilde{\Upsilon}_i^N(V)$. Hence,

$$\lambda_i^N \widetilde{\Upsilon}_i^N(V) + \sum_{j\in C_q\setminus\{i\}} \lambda_j^N \widetilde{\Upsilon}_j^S\left(\tilde{V}_S\right) = \sum_{j\in C_q} \lambda_j^N \widetilde{\Upsilon}_j^N(V)$$

$$\left(|C_q|-1\right)\lambda_i^N \widetilde{\Upsilon}_i^N(V) - \sum_{j\in C_q\setminus\{i\}} \lambda_j^N \widetilde{\Upsilon}_j^S\left(\tilde{V}_S\right) = \lambda_i^N\left(|C_q|-1\right)\tilde{v}_S(\{i\}) + \sum_{j\in C_q\setminus\{i\}} \lambda_j^N \tilde{v}_S(\{j\}).$$

A similar analysis for Υ yields,

$$\lambda_i^N \Upsilon_i^N(V) + \sum_{j\in C_q\setminus\{i\}} \lambda_j^N \Upsilon_j^S(V_S) = \sum_{j\in C_q} \lambda_j^N \Upsilon_j^N(V)$$

$$\left(|C_q|-1\right)\lambda_i^N \Upsilon_i^N(V) - \sum_{j\in C_q\setminus\{i\}} \lambda_j^N \Upsilon_j^S(V_S) = \lambda_i^N\left(|C_q|-1\right) v_S(\{i\}) + \sum_{j\in C_q\setminus\{i\}} \lambda_j^N v_S(\{j\}).$$

By (7.4),

$$\begin{aligned} &\lambda_i^N \widetilde{\Upsilon}_i^N(V) + \sum_{j\in C_q\setminus\{i\}} \lambda_j^N \widetilde{\Upsilon}_j^S\left(\tilde{V}_S\right) \\ &= \lambda_i^N \Upsilon_i^N(V) + \sum_{j\in C_q\setminus\{i\}} \lambda_j^N \Upsilon_j^S(V_S). \end{aligned} \tag{7.5}$$

Since $\tilde{v}_S(\{i\}) = v_S(\{i\})$ and $\tilde{v}_S(\{j\}) = v_S(\{j\})$,

$$\begin{aligned} &\left(|C_q|-1\right)\lambda_i^N \widetilde{\Upsilon}_i^N(V) - \sum_{j\in C_q\setminus\{i\}} \lambda_j^N \widetilde{\Upsilon}_j^S\left(\tilde{V}_S\right) \\ &= \left(|C_q|-1\right)\lambda_i^N \Upsilon_i^N(V) - \sum_{j\in C_q\setminus\{i\}} \lambda_j^N \Upsilon_j^S(V_S). \end{aligned} \tag{7.6}$$

Adding (7.5) and (7.6),

$$|C_q|\lambda_i^N \Upsilon_i^N(V) = |C_q|\lambda_i^N \widetilde{\Upsilon}_i^N(V)$$

which means that $\widetilde{\Upsilon}_i^N(V) = \Upsilon_i^N(V)$. ■

Proof of Remark 3. $ABCAC$ is independent of the rest of properties because the consistent value satisfies EF, IS, COV, and $BCONS$ but not $ABCAC$.

Using arguments similar to those used by Winter (1992), we can conclude that the rest of properties are independent. ■

Proof of Theorem 4. By Theorem 2, we know that Υ satisfies these properties.

We now prove the reciprocal. We proceed by induction on the number of players. The result is trivially true for $n = 1$. Assume the result holds for each $S \subsetneq N$.

Assume now $\left(\tilde{\Upsilon}^S\right)_{S \subset N}$ is a payoff configuration associated to a value $\tilde{\Upsilon}$ satisfying these properties. By Remark 2 and EF, for each $S \subset N$ there exists $\lambda^S \in \mathbb{R}^S_{++}$ satisfying $\lambda^S * \tilde{\Upsilon}^S = v(S)$, where $v(S) = \max\left\{\lambda^S * x : x \in V(S)\right\}$. Let (N, V', C) be the corresponding hyperplane game, *i.e.* for each $S \subset N$,

$$V'(S) = \left\{y \in \mathbb{R}^S : \lambda^S * y \leq v(S)\right\}.$$

It is enough to prove that $\tilde{\Upsilon}^S = \Upsilon^S(V')$ for all $S \subset N$. By induction hypothesis, for each $S \subsetneq N$, $\tilde{\Upsilon}^S = \Upsilon^S(V')$. We will show that $\tilde{\Upsilon}^N = \Upsilon^N(V')$. By simplicity we take $\Upsilon^N = \Upsilon^N(V')$. Assume that $i \in C_q \in C$.

Since $\tilde{\Upsilon}$ satisfies EF and $ABCAC$, using arguments similar to those used in the proof of Theorem 3, we can conclude that for each $C_q \in C$,

$$\sum_{j \in C_q} \lambda_j^N \Upsilon_j^N = \sum_{j \in C_q} \lambda_j^N \tilde{\Upsilon}_j^N.$$

By (3.2),

$$|C_q| \lambda_i^N \Upsilon_i^N = \sum_{j \in C_q \setminus \{i\}} \lambda_i^N \Upsilon_i^{N \setminus \{j\}} + \sum_{j \in C_q} \lambda_j^N \Upsilon_j^N - \sum_{j \in C_q \setminus \{i\}} \lambda_j^N \Upsilon_j^{N \setminus \{i\}}.$$

Since $\sum_{j \in C_q} \lambda_j^N \Upsilon_j^N = \sum_{j \in C_q} \lambda_j^N \tilde{\Upsilon}_j^N$ and the induction hypothesis,

$$\begin{aligned}
|C_q| \lambda_i^N \Upsilon_i^N &= \sum_{j \in C_q \setminus \{i\}} \lambda_i^N \tilde{\Upsilon}_i^{N \setminus \{j\}} + \sum_{j \in C_q} \lambda_j^N \tilde{\Upsilon}_j^N - \sum_{j \in C_q \setminus \{i\}} \lambda_j^N \tilde{\Upsilon}_j^{N \setminus \{i\}} \\
&= \sum_{j \in C_q \setminus \{i\}} \lambda_i^N \tilde{\Upsilon}_i^{N \setminus \{j\}} + \lambda_i^N \tilde{\Upsilon}_i^N + \sum_{j \in C_q \setminus \{i\}} \lambda_j^N \left(\tilde{\Upsilon}_j^N - \tilde{\Upsilon}_j^{N \setminus \{i\}}\right)
\end{aligned}$$

if we add and substract $\sum\limits_{j\in C_q\backslash\{i\}} \lambda_i^N \tilde{\Upsilon}_i^N = (|C_q|-1)\,\lambda_i^N \tilde{\Upsilon}_i^N$, we obtain:

$$= \sum_{j\in C_q\backslash\{i\}} \lambda_i^N \left(\tilde{\Upsilon}_i^{N\backslash\{j\}} - \tilde{\Upsilon}_i^N\right) + |C_q|\lambda_i^N \tilde{\Upsilon}_i^N + \sum_{j\in C_q\backslash\{i\}} \lambda_j^N \left(\tilde{\Upsilon}_j^N - \tilde{\Upsilon}_j^{N\backslash\{i\}}\right).$$

Then,

$$|C_q|\lambda_i^N \left(\Upsilon_i^N - \tilde{\Upsilon}_i^N\right) = \sum_{j\in C_q\backslash\{i\}} \lambda_j^N \left(\tilde{\Upsilon}_j^N - \tilde{\Upsilon}_j^{N\backslash\{i\}}\right) - \sum_{j\in C_q\backslash\{i\}} \lambda_i^N \left(\tilde{\Upsilon}_i^N - \tilde{\Upsilon}_i^{N\backslash\{j\}}\right).$$

Since $\tilde{\Upsilon}$ satisfies $ABCAP$, we conclude that the last expression is equal to 0. Then, $\tilde{\Upsilon}_i^N = \Upsilon_i^N$. ■

Proof of Remark 4. EF is independent of the rest of properties. The value

$$\Gamma_i^N = 0 \text{ for each } NTU \text{ game } (N,V,C) \text{ and } i \in N$$

satisfies $ABCAC$ and $ABCAP$ but not EF.

$ABCAP$ is independent of the rest of properties. The random order coalitional value satisfies EF and $ABCAC$ but not $ABCAP$.

$ABCAC$ is independent of the rest of properties.

Given a hyperplane game (N,V,C) we define, for each $i \in N$,

$$\Omega_i^N = \frac{v(N)}{|N|\,\lambda_i^N}.$$

Let $\pi \in \Pi_q$ be a permutation of players in C_q. We consider $f(\pi) \in \mathbb{R}^{C_q}$ such that for each $i \in C_q$,

$$f_i(\pi) = \max\left\{x_i : \left(\left(\Omega_j^S\right)_{j\in\overline{C_q}}, (f_j(\pi))_{j\in P(\pi,i)}, x_i\right) \in V(S)\right\},$$

where $S = \overline{C_q} \cup P(\pi,i) \cup \{i\}$.

It is straightforward to prove that

$$f_i(\pi) = \frac{v(S) - \sum\limits_{j\in\overline{C_q}} \lambda_j^S \Omega_j^S - \sum\limits_{j\in P(\pi,i)} \lambda_j^S f_j(\pi)}{\lambda_i^S}.$$

Then, given $i \in C_q \in C$, we define Γ as follows:

$$\Gamma_i^N = \frac{1}{|\Pi_q|} \sum_{\pi \in \Pi_q} f_i(\pi).$$

For each $C_q \in C$ and $\pi \in \Pi_q$, since Ω satisfies EF, $\sum_{j \in C_q} \lambda_j^N \Omega_j^N = \sum_{j \in C_q} \lambda_j^N f_j(\pi)$. Hence, $\sum_{j \in C_q} \lambda_j^N \Omega_j^N = \sum_{j \in C_q} \lambda_j^N \Gamma_j^N$. Then, it is trivial to see that Γ satisfies EF in the class of hyperplane games.

We now prove that Γ satisfies $ABCAP$.

For each $j \in C_q$ we denote the set of permutations of Π_q where j is the last player by $\Pi_q(j)$. If $j \neq i$, then player i's expected marginal contribution conditioned to j being last, is the same as in the game $(N \setminus \{j\}, V, C_{-j})$, which is $\Gamma_i^{N\setminus\{j\}}$, *i.e.*

$$\frac{1}{|\Pi_q(j)|} \sum_{\pi \in \Pi_q(j)} f_i(\pi) = \frac{1}{|\Pi_q^{C_{-j}}|} \sum_{\pi \in \Pi_q^{C_{-j}}} f_i(\pi) = \Gamma_i^{N\setminus\{j\}}.$$

Given $\pi \in \Pi_q(i)$,

$$\begin{aligned} f_i(\pi) &= \frac{v(N) - \sum_{j \in \overline{C_q}} \lambda_j^N \Omega_j^N - \sum_{j \in C_q \setminus \{i\}} \lambda_j^N f_j(\pi)}{\lambda_i^N} \\ &= \frac{\sum_{j \in C_q} \lambda_j^N \Omega_j^N - \sum_{j \in C_q \setminus \{i\}} \lambda_j^N f_j(\pi)}{\lambda_i^N} \\ &= \frac{\sum_{j \in C_q} \lambda_j^N \Gamma_j^N - \sum_{j \in C_q \setminus \{i\}} \lambda_j^N f_j(\pi)}{\lambda_i^N}. \end{aligned}$$

Then,

$$\begin{aligned} \frac{1}{|\Pi_q(i)|} \sum_{\pi \in \Pi_q(i)} f_i(\pi) &= \frac{\sum_{j \in C_q} \lambda_j^N \Gamma_j^N}{\lambda_i^N} - \frac{1}{|\Pi_q(i)|} \sum_{\pi \in \Pi_q(i)} \frac{\sum_{j \in C_q \setminus \{i\}} \lambda_j^N f_j(\pi)}{\lambda_i^N} \\ &= \frac{\sum_{j \in C_q} \lambda_j^N \Gamma_j^N}{\lambda_i^N} - \frac{1}{\lambda_i^N} \sum_{j \in C_q \setminus \{i\}} \lambda_j^N \frac{1}{|\Pi_q(i)|} \sum_{\pi \in \Pi_q(i)} f_j(\pi) \end{aligned}$$

$$= \frac{\sum\limits_{j \in C_q} \lambda_j^N \Gamma_j^N}{\lambda_i^N} - \frac{1}{\lambda_i^N} \sum_{j \in C_q \backslash \{i\}} \lambda_j^N \Gamma_j^{N \backslash \{i\}}$$

$$= \frac{\sum\limits_{j \in C_q} \lambda_j^N \Gamma_j^N - \sum\limits_{j \in C_q \backslash \{i\}} \lambda_j^N \Gamma_j^{N \backslash \{i\}}}{\lambda_i^N}.$$

Thus, for each $i \in C_q \in C$,

$$\begin{aligned} \Gamma_i^N &= \frac{1}{|\Pi_q|} \sum_{\pi \in \Pi_q} f_i(\pi) \\ &= \frac{1}{|\Pi_q|} \sum_{j \in C_q \backslash \{i\}} \sum_{\pi \in \Pi_q(j)} f_i(\pi) + \frac{1}{|\Pi_q|} \sum_{\pi \in \Pi_q(i)} f_i(\pi). \end{aligned}$$

Since $|\Pi_q| = |C_q| \, |\Pi_q(j)|$ for each $j \in C_q$, the last expression can be rewritten as

$$\begin{aligned} & \frac{1}{|C_q|} \left[\sum_{j \in C_q \backslash \{i\}} \frac{1}{|\Pi_q(j)|} \sum_{\pi \in \Pi_q(j)} f_i(\pi) + \frac{1}{|\Pi_q(i)|} \sum_{\pi \in \Pi_q(i)} f_i(\pi) \right] \\ = & \frac{1}{|C_q|} \left[\sum_{j \in C_q \backslash \{i\}} \Gamma_i^{N \backslash \{j\}} + \frac{\sum\limits_{j \in C_q} \lambda_j^N \Gamma_j^N - \sum\limits_{j \in C_q \backslash \{i\}} \lambda_j^N \Gamma_j^{N \backslash \{i\}}}{\lambda_i^N} \right]. \end{aligned}$$

Then,

$$|C_q| \lambda_i^N \Gamma_i^N = \sum_{j \in C_q \backslash \{i\}} \lambda_i^N \Gamma_i^{N \backslash \{j\}} + \sum_{j \in C_q} \lambda_j^N \Gamma_j^N - \sum_{j \in C_q \backslash \{i\}} \lambda_j^N \Gamma_j^{N \backslash \{i\}}.$$

Since $|C_q| \lambda_i^N \Gamma_i^N = \sum\limits_{j \in C_q} \lambda_i^N \Gamma_i^N$, it is easy to conclude that Γ satisfies $ABCAP$.

If we proceed with Γ in the same way that we did with Υ we can extend Γ to the set of NTU games and prove that Γ also satisfies EF and $ABCAP$ in the class of NTU games. Of course, $\Gamma \neq \Upsilon$. ■

Proof of Corollary 1: Since each TU game is a hyperplane game, we conclude that the consistent coalitional value is a single value. Repeating the same arguments that in the proof of Theorem 4 for TU games we can obtain that there is at most a value (on the set of TU games) satisfying EF, $ABCAC$, and $ABCAP$. Then, we only need to prove that the Owen value ϕ satisfies these properties.

We know that ϕ satisfies EF. We now prove that ϕ satisfies $ABCAC$ and $ABCAP$. By simplicity we assume that $S = N$.

Since ϕ satisfies $BCAC$, for each $C_q, C_r \in C$

$$\sum_{j \in C_q} \left(\phi_j^N - \phi_j^{N \backslash C_r} \right) = \sum_{j \in C_r} \left(\phi_j^N - \phi_j^{N \backslash C_q} \right).$$

Then,

$$\sum_{C_r \in C \backslash C_q} \left(\sum_{j \in C_q} \left(\phi_j^N - \phi_j^{N \backslash C_r} \right) \right) = \sum_{C_r \in C \backslash C_q} \left(\sum_{j \in C_r} \left(\phi_j^N - \phi_j^{N \backslash C_q} \right) \right)$$

which means that ϕ satisfies $ABCAC$ in TU games.

Since ϕ satisfies $BCAP$, for each $C_q \in C$ and $i, j \in C_q$

$$\phi_i^N - \phi_i^{N \backslash \{j\}} = \phi_j^N - \phi_j^{N \backslash \{i\}}.$$

Then,

$$\sum_{j \in C_q \backslash \{i\}} \left(\phi_i^N - \phi_i^{N \backslash \{j\}} \right) = \sum_{j \in C_q \backslash \{i\}} \left(\phi_j^N - \phi_j^{N \backslash \{i\}} \right)$$

which means that ϕ satisfies $ABCAP$ in TU games. ■

Proof of Lemma 2: Let $F = \left(F^S \right)_{S \subset N}$ be the random order coalitional payoff configuration for (N, V, C). By definition, F_j^N is the expected marginal contribution of player j over all the $|\Pi^C|$ admissible permutations of players with respect to C. We classify these permutations in $|C|$ groups according the last coalition C_r in such permutations.

Let $\Pi^C(C_r)$ be the set of admissible permutations with respect to C in which players of coalition C_r are in the last position. Notice that $|\Pi^C| = |C||\Pi^C(C_r)|$ for each $C_r \in C$.

If $C_r \neq C_p$, then the expected marginal contribution for each player $j \in C_p$ in the permutations of $\Pi^C(C_r)$ coincides with the expected marginal contribution of player j in the game $(N \backslash C_r, V, C \backslash C_r)$, which is $F_j^{N \backslash C_r}$, *i.e.*

$$\frac{1}{|\Pi^C(C_r)|}\sum_{\pi\in\Pi^C(C_r)} d_j(\pi) = \frac{1}{|\Pi^{C\setminus C_r}|}\sum_{\pi\in\Pi^{C\setminus C_r}} d_j(\pi) = F_j^{N\setminus C_r}. \tag{7.7}$$

Moreover, for each $\pi \in \Pi^C(C_q)$,

$$\sum_{j\in C_q}\lambda_j^N d_j(\pi) = v(N) - \sum_{C_r\in C\setminus C_q}\left(\sum_{j\in C_r}\lambda_j^N d_j(\pi)\right).$$

Then,

$$\begin{aligned}
&\frac{1}{|\Pi^C(C_q)|}\sum_{\pi\in\Pi^C(C_q)}\left[\sum_{j\in C_q}\lambda_j^N d_j(\pi)\right]\\
=\ &\frac{1}{|\Pi^C(C_q)|}\sum_{\pi\in\Pi^C(C_q)}\left(v(N) - \sum_{C_r\in C\setminus C_q}\sum_{j\in C_r}\lambda_j^N d_j(\pi)\right)\\
=\ &\frac{1}{|\Pi^C(C_q)|}\sum_{\pi\in\Pi^C(C_q)} v(N) - \sum_{C_r\in C\setminus C_q}\left(\sum_{j\in C_r}\lambda_j^N\left(\frac{1}{|\Pi^C(C_q)|}\sum_{\pi\in\Pi^C(C_q)} d_j(\pi)\right)\right)\\
=\ &v(N) - \sum_{C_r\in C\setminus C_q}\left(\sum_{j\in C_r}\lambda_j^N F_j^{N\setminus C_q}\right).
\end{aligned} \tag{7.8}$$

We have then:

$$\begin{aligned}
\sum_{j\in C_q}\lambda_j^N F_j^N &= \sum_{j\in C_q}\lambda_j^N\frac{1}{|\Pi^C|}\sum_{\pi\in\Pi^C} d_j(\pi)\\
&= \sum_{j\in C_q}\lambda_j^N\left(\sum_{C_r\in C}\frac{1}{|\Pi^C|}\sum_{\pi\in\Pi^C(C_r)} d_j(\pi)\right)
\end{aligned}$$

since $\left|\Pi^C\right| = |C|\left|\Pi^C(C_r)\right|$, the last expression can be rewritten as

$$\sum_{j\in C_q}\lambda_j^N\frac{1}{|C|}\sum_{C_r\in C}\frac{1}{|\Pi^C(C_r)|}\sum_{\pi\in\Pi^C(C_r)} d_j(\pi) =$$

$$\frac{1}{|C|}\left[\sum_{j\in C_q}\lambda_j^N\sum_{C_r\in C\setminus C_q}\underbrace{\frac{1}{|\Pi^C(C_r)|}\sum_{\pi\in\Pi^C(C_r)} d_j(\pi)} + \underbrace{\frac{1}{|\Pi^C(C_q)|}\sum_{\pi\in\Pi^C(C_q)}\left(\sum_{j\in C_q}\lambda_j^N d_j(\pi)\right)}\right]$$

the terms above brackets are those given in (7.7) and (7.8), so:

$$= \frac{1}{|C|}\left[\sum_{j\in C_q}\lambda_j^N \sum_{C_r\in C\backslash C_q} F_j^{N\backslash C_r} + v(N) - \sum_{C_r\in C\backslash C_q}\left(\sum_{j\in C_r}\lambda_j^N F_j^{N\backslash C_q}\right)\right]$$

$$= \frac{1}{|C|}\left[\sum_{C_r\in C\backslash C_q}\left(\sum_{j\in C_q}\lambda_j^N F_j^{N\backslash C_r}\right) + v(N) - \sum_{C_r\in C\backslash C_q}\left(\sum_{j\in C_r}\lambda_j^N F_j^{N\backslash C_q}\right)\right]$$

which is precisely the statement of this lemma. ■

8. References

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