On the application of an outside-option value to the gloves game

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Abstract

The paper presents a coalition-structure value that is meant to capture outside options of players in a cooperative game. It deviates from the Aumann-Drèze value by violating the null-player axiom. We apply this value to the gloves game.

Keywords: Aumann-Drèze value, Shapley value, core, null-player axiom, outside option, gloves game

1. Introduction

The gloves game is one of the most popular market games in cooperative game theory. It presupposes a player set $N = L \cup R$ where L and R are disjunct sets of players holding one left or one right glove, respectively. The coalition function (characteristic function) for the gloves game is given by

$$v_{L,R}(K) = \min(|L \cap K|, |R \cap K|).$$

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Thus, the worth of coalition K is equal to the number of pairs of gloves coalition K can assemble.

The core and the Shapley value for this game are well-known. The core represents the competitive solution where the holders of the scarce commodity (the right-glove owners in case of $|R| < |L|$) obtain a payoff of 1. This result holds for $|L| = 100$ and $|R| = 99$ as well as for $|L| = 100$ and $|R| = 1$. Shapley & Shubik (1969, p. 342) denounce the "violent discontinuity exhibited by ... the core". In contrast, the Shapley value is sensitive to the relative scarcity of the gloves. Facing 100 left gloves, the holder of the unique right glove achives a value of 0.99 while a right-glove holder out of 99 obtains about 0.54, only. This sensitivity is surely an attractive property of the Shapley value.

Note, however, that the Shapley value attributes a positive value to all players unless $|L| = 0$ or $|R| = 0$. However, we may suppose that some left-glove owners will not be able to strike a deal. They should then get a pay-off of zero. In our interpretation, the Shapley value is an ex-ante value, indicating the expected payoff to an agent in the gloves game before it is clear whether or not he will find a trading partner. Alternatively, one might be interested in an-post value that should give us an idea about the payoff for glove holders once they have or have not found a trading partner. We argue that the outside-option value presented in this paper might be a suitable candidate for that purpose.

Building on the Shapley value, several partitional values (or values for coalition structures) have been presented in the literature, most notably by Aumann & Drèze (1974) and Owen (1977). A coalition structure is a partition on the set of players; the sets making up the partition are called components. There is an important interpretational difference between the Aumann-Dreze (AD) value and the Owen value. For Aumann and Dreze, players are organized in (active) components in order to do business together. Then the players within each component should arguably get its worth, as in the Aumann-Dreze value (AD-value). This is the property of component efficiency. The idea of the Owen value is that players form bargaining components (unions etc.) that offer the service of all their members or no service at all. In this paper, we have the Aumann-Dreze interpretation in mind.

By component efficiency, the AD-value seems a good candidate for an ex-post value measuring the market power of agents. Of course, we have to specify a partition before we can apply the AD-value. Turning to the gloves game, we often assume maximal-pairs partitions. These are partitions that host $\min(|L|, |R|)$ components, each containing one left-glove holder and one right-glove owner. If

$|L| > |R|$, a maximal-pairs partition contains other components as well, with elements from L only. A left-glove and a right-glove owner who make up one component of the partition, receive an AD-value of $1/2$ each, irrespective of how many other left-hand or right-hand gloves are present. These payoffs do not accord well with our intuition about competition. The outside-option value obeys component efficiency, too. However, it produces results that are more sensitive to the relative scarcity of gloves.

Since the outside-option value applies to all coalition function, let us take a look at another game, an asymmetric version of an example used by Aumann & Myerson (1988). Assume player set $N = \{1, 2, 3\}$ and the characteristic, or coalition, function v on N which ascribes worths $v(\{1, 2\}) = v(\{2, 3\}) = 60$, $v(N) = 72$ and vanishes elsewhere. The Shapley value of this game is $(14, 44, 14)$. By efficiency, the whole of $v(N)$ is distributed among the players, but player 2 gets the lion's share. Now let $\mathcal{P} = \{\{1, 2\}, \{3\}\}$ be a coalition structure. For this coalition structure, the AD-value φ^{AD} and the outside-option value φ^{oo} can be computed to be

$$\begin{aligned} \varphi^{AD}(v, \mathcal{P}) &= (30, 30, 0), \\ \varphi^{oo}(v, \mathcal{P}) &= (20, 40, 0). \end{aligned}$$

Both values are component-efficient. The outside-option value attributes a higher payoff to player 2 than to player 1 thus reflecting the outside opportunities of player 2 ($v(2, 3) = 60 > 0 = v(1, 3)$).

In spirit, the bargaining set is close to our value. (In the above example, the bargaining set yields $(0, 60, 0)$, a somewhat "extreme" solution.) In fact, I find Maschler's (1992, pp. 595) introducing remarks pertinent to the value presented in this paper:

> During the course of negotiations there comes a moment when a certain coalition structure is "crystallized". The players will no longer listen to "outsiders", yet each [component] has still to adjust the final share of proceeds. (This decision may depend on options outside the [component], even though the chances of defection are slim).

The reader is also referred to the paper by Aumann & Drèze (1974) where outside options are modeled by redefining the characteristic function in a specific way. The result is a set of characteristic functions v_x^C, one for each component C of a coalition structure $\mathcal{P}$ (that also depends on a payoff vector $x \in \mathbb{R}^n$). The authors then go on to present coalition-structure spin-offs for the most widely used concepts within cooperative game theory. Interestingly, the coalition-structure Shapley value defined in that article (the AD-value introduced above) is the only one where the coalition functions v_x^C are not made use of.

What excuse could be offered for adding other outside-option values to those existent in the literature? To our knowledge, it is the only value close to the Shapley value that obeys component efficiency and takes outside options into account. Arguably, there are many economic and political situations where we need these properties. Apart from market games, one might think of the power within a government coalition. This power rests with the parties involved (component efficiency) but the power of each party within the government depends on other governments that might possibly form (outside options).

Close to the AD-approach, our outside-option value obeys component efficiency, symmetry and additivity. However, we argue that values φ^{oo} modelling component efficiency and outside options cannot possibly obey the null-player axiom. Consider $N = \{1, 2, 3\}$ and the unanimity game $u_{\{1,2\}}$ which maps the worth 1 to coalitions $\{1, 2\}$ and $\{1, 2, 3\}$ and the worth 0 to all other coalitions. We now look at the coalition structure $\mathcal{P}_1 = \{\{1, 3\}, \{2\}\}$. By component efficiency, we get $\varphi_1^{oo}\left(u_{\{1,2\}}, \mathcal{P}_1\right) + \varphi_3^{oo}\left(u_{\{1,2\}}, \mathcal{P}_1\right) = 0 = \varphi_2^{oo}\left(u_{\{1,2\}}, \mathcal{P}_1\right)$. Player 3 is a null player; his contribution to any coalition is zero. Yet, his payoff cannot be zero under φ^{oo}. The reason is this: Player 1 has outside options. By joining forces with player 2 (thus violating the existing coalition structure) he would have claim to a payoff of 1/2. Within the existing coalition structure, he will turn to player 3 to satisfy at least part of this claim. But then, player 3's payoff is negative.

It should also be clear that a component-efficient value that respects outside options cannot always coincide with the value for some "stable" partition. In our example, stable partitions might be given by $\mathcal{P}_2 = \{\{1, 2\}, \{3\}\}$ or $\mathcal{P}_3 = \{\{1, 2, 3\}\}$. By component efficiency the sum of payoffs for all three players is zero for $\mathcal{P}_1$ but 1 for $\mathcal{P}_2$ and $\mathcal{P}_3$.

Some readers might object to a negative payoff for player 3 by pointing to the possibility that player 3 departs from coalition $\{1, 3\}$ to obtain the zero payoff. However, for the purpose of determining the outside-option value, the coalition structure $\mathcal{P}$ is given. The stability of $\mathcal{P}$ is another -separate- issue that we will not deal with in this paper. Also, it is easy to show that negative payoffs need not bother us if we consider the gloves game and a maximal-pairs partitions.

Another objection points to the example of the game $-u_{\{1,2\}}$ together with the above partition $\mathcal{P}_1 = \{\{1, 3\}, \{2\}\}$. Here, player 1's outside options are negative.

If he were to join player 2, he would receive $-1/2$ and within the existing partition a component efficient value with outside options should attribute a negative payoff to player 1. A somewhat satisfactory interpretation could go as follows: Player 3 argues that player 1's payoff were negative if he would form a coalition with player 2. It is the existing component $\{1,3\}$ that prevents this negative payoff. Part of this gain should then go to player 3.

It has been noted that the outside-option value is close the AD-value and the Shapley value. Indeed, the outside-option value is a generalization of both these values. Since the Shapley value converges to the core for the gloves game, we will address the question (raised by Joachim Rosenmüller) whether the outside-option value converges to the core as well.

In our paper, the null-player axiom is substituted by the outside-option axiom. In the literature, different alternatives to the null-player axiom can be found. For example, Nowak & Radzik (1994) present a solidarity value where null players in unanimity games obtain a positive value. (Their value could easily be turned into a partitional one.) A very different approach is that by Napel & Widgren (2001). For the class of simple games they define so-called inferior players who form a superset of the set of null players. All inferior players get a payoff of zero according to their Strict Power Index, a close relative of the Banzhaf index.

The paper is organized as follows: In section 2 basic definitions are given. Section 3 presents axioms for the Shapley value and for the AD-value. We then apply these values and the core to the gloves game in section 4. The outside-option value is presented and axiomatized in section 5 where the reader will also find the application to the gloves game. Section 6 concludes the paper.

2. Definitions

Let $N=\{1,2,...,n\}$ be the player set. A game (in characteristic function form) is a pair (N,v) where v is a function $2^N\to\mathbb{R}$ such that $v(\emptyset)=0$. The set of all games on N is denoted G. A payoff vector x for N is an element of $\mathbb{R}^n$ or a function $N\to\mathbb{R}$. As usual, we abbreviate $\sum_{j\in S}x_j$ by $x(S)$ for all subsets S of N and let $x(\emptyset)=0$.

A game v is called monotonic if for any two coalitions K,K' fulfilling $\emptyset\subseteq K\subseteq K'\subseteq N$ we get $v(K)\leq v(K')$. v is a simple game if $v(K)\in\{0,1\}$ for all $K\subseteq N$. For simple games, any coalition K fulfilling $v(K)=1$ is called a winning coalition. The set of winning coalitions for v is denoted by $\mathbb{W}(v)$.

For any nonempty coalition $T\subseteq N$, $u_T(S)=1$, $S\supseteq T$; 0 otherwise, defines a game, called a unanimity game. It is well known that the set of those games (the cardinality of which is $2^n - 1$) is a basis of G in the sense of linear algebra.

A player $i \in N$ is a null player for $v \in G$ if $v(S \cup i) = v(S)$ for all $S \subseteq N$. (We sometimes abuse notation by omitting parentheses.)

Following Aumann & Drèze (1974), we define coalition structures: A coalition structure $\mathcal{P}$ on N (sometimes written as $(N, \mathcal{P})$) is a partition of N into components $C_1, ..., C_m$:

$$\mathcal{P} = \{C_1, ..., C_m\}.$$

The set of all partitions on N is denoted by $\mathfrak{P}$. For any player $i \in N$, $\mathcal{P}(i)$ denotes the component containing i. For any set $T \subseteq N$, the set of components containing any players from T is written $\mathcal{P}(T)$. These components are called T-components. (The reader will note $\mathcal{P}(i) \in \mathcal{P}$ and $\mathcal{P}(\{i\}) \subseteq \mathcal{P}$.)

The tuple $(N, v, \mathcal{P})$ is called a partition situation. Two players $i, j \in N$ are called symmetric with respect to $\mathcal{P}$, if $\mathcal{P}(i) = \mathcal{P}(j)$ and for all coalitions K obeying $i \notin K$ and $j \notin K$ we have

$$v(K \cup i) = v(K \cup j).$$

Rules of order σ on N are bijective functions $\sigma : N \to N$ where $\sigma(1)$ is to be understood as the first player in the order, $\sigma(2)$ as the second player etc. The set of all rules of order on N is denoted by Σ. The inverse $\sigma^{-1}(i)$ denotes player i's "position" in the rule of order σ. Then, we define $K_i(\sigma) := \{\sigma(1), ..., \sigma(\sigma^{-1}(i))\}$, i.e. $K_i(\sigma)$ is the set of players up to and including player i.

The Shapley value and other related values make heavy use of marginal contributions of players. For any coalition $S \subseteq N$ and any player $i \in N$ we define

$$MC_i^S(v) := v(S \cup i) - v(S \backslash i)$$

and, given some rule of order σ from Σ,

$$MC_i(v, \sigma) := v(K_i(\sigma)) - v(K_i(\sigma) \backslash i).$$

3. Axioms for the Shapley value and the AD-value

A value on $(N, \mathfrak{P})$ is a function $\psi : G \times \mathfrak{P} \to \mathbb{R}^n$. Values on $(N, \mathfrak{P})$ might obey one or several of the following axioms:

Axiom E (Efficiency):

$$\sum_{i \in N} \psi_i(v, \mathcal{P}) \equiv \psi(v, \mathcal{P})(N) = \nu(N)$$

Axiom CE (Component efficiency): For all $C \in \mathcal{P}$,

$$\sum_{i \in C} \psi_i (v, \mathcal{P}) \equiv \psi (v, \mathcal{P}) (C) = \nu (C) .$$

Axiom S (Symmetry): For all players i and j, symmetric with respect to partition $\mathcal{P}$

$$\psi_i (v, \mathcal{P}) = \psi_j (v, \mathcal{P}) .$$

Axiom N (Null player): For any null player $i \in N$,

$$\psi_i (v, \mathcal{P}) = 0.$$

Axiom N-S (Null player): For any nonempty set $T \subseteq N$,

$$\psi (u_T, \mathcal{P}) (T^c) = 0,$$

where $T^c := N \backslash T$.

Axiom N-AD (Null player): For any nonempty set $T \subseteq N$ and any $C \in \mathcal{P}$,

$$\psi (u_T, \mathcal{P}) (C \cap T^c) = 0.$$

Axiom A (Additivity): For any coalition functions $v_1, v_2 \in G$,

$$\psi (v_1 + v_2, \mathcal{P}) = \psi (v_1, \mathcal{P}) + \psi (v_2, \mathcal{P})$$

Axiom L (Linearity): For any coalition functions $v_1, v_2 \in G$ and any $\alpha \in \mathbb{R}$,

$$\psi (\alpha v_1 + v_2, \mathcal{P}) = \alpha \psi (v_1, \mathcal{P}) + \psi (v_2, \mathcal{P})$$

As is very well known, for $\mathcal{P} := \{N\}$, there exists a unique value on $(N, \mathfrak{P})$ satisfying the axioms E (or CE), S, N, and A (or L), the Shapley value, written $\varphi (v)$ for $v \in G$. It is given by

$$\varphi_i (v) = \frac{1}{n!} \sum_{\sigma \in \Sigma} MC_i (v, \sigma) , i \in N.$$

Alternatively, it is axiomatized by E (or CE), S, N-S, and L.

The AD-value is the Shapley value gained by restricting the coalition function to the components of a partition $\mathcal{P}$:

$$\varphi^{AD} (v, \mathcal{P}) := \left(\varphi_i \left(v|_{\mathcal{P}(i)} \right) \right)_{i \in I} .$$

The AD-value is uniquely determined by the axioms CE, S, N, and A. The same is true for the axioms CE, S, N-AD, and L.

4. Applying the core, the Shapley value and the AD-value to the gloves game

The core payoff to a right-glove owner is defined by some p ($0 \leq p \leq 1$) and by

$$\pi = \begin{cases} 1, & r < l \\ p, & r = l \\ 0, & r > l \end{cases}$$

We present this result also by the following matrix:

		no. of left-glove holders				
		0	1	2	3	4
no. of	1	0	p	1	1	1
right-	2	0	0	p	1	1
glove	3	0	0	0	p	1
holders	4	0	0	0	0	p

The insensitivity and discontinuity at $l = r$ is clearly visible. In contrast, the Shapley value is "less abruptly sensitive to the balance between supply and demand than ... the core, since it gives some credit for the bargaining position of the group in oversupply." (Shapley & Shubik (1969, p. 344))

We present the value for a right-glove owner within the following table (a small-scale reproduction of a table in Shapley & Shubik (1969, p. 344)):

		no. of left-glove holders				
		0	1	2	3	4
no. of	1	0	0.500	0.667	0.750	0.800
right-	2	0	0.167	0.500	0.650	0.733
glove	3	0	0.083	0.233	0.500	0.638
holders	4	0	0.050	0.133	0.271[1]	0.500

Shapley & Shubik (1969, p. 344) justify the following formulae for the gloves game:

$$\varphi_i\left(v_{L,R}\right) = \begin{cases} \frac{1}{2} + \frac{l-r}{2r} \sum_{k=1}^{r} \frac{l!r!}{(l+k)!(r-k)!}, & r \leq l \\ \frac{1}{2} - \frac{r-l}{2r} \sum_{k=0}^{l} \frac{l!r!}{(r+k)!(l-k)!}, & r > l \end{cases}, i \in R \tag{4.1}$$

[1] In case of 4 right-glove and 3 left-glove holders, the payoff to a right-glove holer is $19/70 \approx .27143$ while the authors put 0.272.

These formulae have been used to show that the Shapley value of the gloves game converges to the core: When replicating the game (i.e., increasing the number of left and right gloves by way of multiplication) the Shapley values converge toward 0 or 1 in case of $l \neq r$ (for $l = r$ we get a core payoff $\frac{1}{2}$).

As noted in the introduction, all players obtain a positive Shapley value although one might suspect that some left-glove holders will find no trading partner in case of $|L| > |R|$. The ex-post view can be reflected by the AD-value. To be precise, let $R = \{1, 2, ..., |R|\}$ and $L = \{|R| + 1, ..., |L| + |R|\}$ be the set of right-glove and left-glove owners, respectively. Also, let $N := L \cup R$, $l := |L|$, $r := |R|$ and $n := l + r$. Without loss of generality, assume $l \geq r$. We then consider maximal-pairs partitions such as

$$\mathcal{P} = \{\{1, r + 1\}, ..., \{r, 2r\}, \{2r + 1\} ..., \{l + r\}\}$$

or

$$\mathcal{P} = \{\{1, r + 1\}, ..., \{r, 2r\}, \{2r + 1..., l + r\}\}.$$

The AD-value does not take outside options into account. Hence, $\varphi_i^{AD}(v_{L,R}, \mathcal{P}) = \frac{1}{2}$ for every player $i = 1, ..., 2r$ and $\varphi_i^{AD}(v_{L,R}, \mathcal{P}) = 0$ for every player $i > 2r$. Thus, the AD-value takes the ex-post perspective while ignoring the relative scarcity of gloves.

5. The outside-option value

5.1. An outside-option axiom for unanimity games

In this section, we will present an axiom needed to axiomatize the outside-option (oo) value. Let $(N, \mathcal{P})$ be a coalition structure and $T \subseteq N$ a nonempty set.

Axiom N-oo (Outside options for unanimity games): If $\mathcal{P}$ contains a component C^T such that $T \subseteq C^T$, then for all $C \in \mathcal{P}$,

$$\psi(u_T, \mathcal{P}; \alpha)(C \cap T^c) = 0.$$

If $\mathcal{P}$ does not contain a component C such that $T \subseteq C$, then for all $C \in \mathcal{P}$,

$$\psi(u_T, \mathcal{P}; \alpha)(C \cap T^c) = -\alpha \frac{|C \cap T|}{|T|} \frac{|C \cap T^c|}{|T \cup C|}.$$

This axiom corresponds to the null-player axiom N. In fact, if there is a component C^T of $\mathcal{P}$ that contains T, all (symmetric!) null players receive a pay-off of 0.

This holds for all the players in $N \backslash C^T$ and for those in $C^T \backslash T$. If, however, such a component does not exist, the players from T find themselves in two or more components of $\mathcal{P}$. Then the worth of each component is zero. However, players from T might arguably not be content with a pay-off of zero. If they were not bound to the component they happen to find themselves in, they might possibly take part in a coalition promising to divide the worth of 1. By component efficiency, for every component C not containing T, positive payoffs for players from $T \cap C$ necessitate negative payoffs for players from $T^c \cap C$. Of course, players from T^c with negative payoffs could possibly better their lot by leaving their component. As noted in the introduction, the question of the stability of partitions will not concern us here.

Note that pay-offs to players in $C \cap T^c$ are zero if $C \cap T = \emptyset$ because in this case there are no $T-$players in C threatening to look for other coalitions.

Axiom N-oo makes use of $\frac{|C \cap T|}{|T|}$; this term can be interpreted as the probability that a player from $C \cap T$ (as opposed to a player from $C^c \cap T$) claims the unit pay-off. The second factor reflects the probability that players from $C \cap T^c$ actually do have to make up for the missed opportunities of T-players belonging to the same component.

Note that axioms CE, S and $N - oo$ imply

$$\psi_i (u_T, \mathcal{P}; \alpha) = \begin{cases} \frac{1}{|T|}, & i \in T \text{ and } \exists C \in \mathcal{P} : T \subseteq C, \\ 0, & i \notin T \text{ and } \exists C \in \mathcal{P} : T \subseteq C, \\ \alpha \frac{1}{|T|} \frac{|\mathcal{P}(i) \cap T^c|}{|\mathcal{P}(i) \cup T|}, & i \in T \text{ and } \nexists C \in \mathcal{P} : T \subseteq C, \\ -\alpha \frac{|\mathcal{P}(i) \cap T|}{|T|} \frac{1}{|T \cup \mathcal{P}(i)|}, & i \notin T \text{ and } \nexists C \in \mathcal{P} : T \subseteq C. \end{cases} \tag{5.1}$$

5.2. Axiomatizing the outside-option value

We will now define the outside-option value φ^{oo}. It is given by

$$\varphi_i^{oo} (v, \mathcal{P}, \alpha) = \frac{1}{n!} \sum_{\sigma \in \Sigma} \begin{cases} v (\mathcal{P} (i)) - \sum_{j \in \mathcal{P}(i) \backslash i} MC_j (\sigma, \mathcal{P}, \alpha), & \mathcal{P} (i) \subseteq K_i (\sigma), \\ MC_i (\sigma, \mathcal{P}, \alpha), & \text{otherwise,} \end{cases}$$

$i \in N$, where

$$MC_i (\sigma, \mathcal{P}, \alpha) = \alpha MC_i^{K_i(\sigma)} (v) + (1 - \alpha) MC_i^{K_i(\sigma) \cap P(i)} (v)$$

In looking at a rule of order σ and assuming $\alpha = 1$, player i gets her marginal contribution if she is not the last player in her component in σ, i.e., if $\mathcal{P} (i)$ is not included in $K_i(\sigma)$. If i is the last player in her component, she gets the worth of this component minus the pay-offs to the other players in her component.

In case of $\alpha > 0$, the oo-value reflects existing outside opportunities. For $\alpha = 1$, outside opportunities ($K_i(\sigma) \nsubseteq \mathcal{P}(i)$) count as much as inside opportunities ($K_i(\sigma) \subseteq \mathcal{P}(i)$), i.e. $\varphi_i^{oo}(v, \mathcal{P}, \alpha) = \varphi_i(v)$. Low values of α reflect some stability of the partition; individual i cannot seriously threaten to leave component $\mathcal{P}(i)$.

Theorem 5.1. *φ^{oo} is the unique value on $(N, \mathfrak{P})$ satisfying the axioms CE, S, N-oo, and L.*

The proof is given in the appendix.

5.3. Special cases and stability of coalition structures

The outside-option value is a generalization of both the Shapley and the AD-value in the following sense:

Lemma 5.2. *Let (v, N) be any game.*

- *For $\alpha = 0$, we get the AD-value, $\varphi_i^{oo}(v, \mathcal{P}, 0) = \varphi_i^{AD}(v, \mathcal{P})$.*
- *$\mathcal{P} = \{N\}$ yields the Shapley value, $\varphi_i^{oo}(v, \{N\}, \alpha) = \varphi_i(v)$ for all $\alpha \in [0, 1]$.*

Also, for simple monotonic games, a veto player obtains the Shapley value, as the following proposition shows. Its proof is relegated to the appendix.

Proposition 5.3. *Let (v, N) be a simple and monotonic game and $\mathbb{W}(v)$ its set of winning coalitions. Let there be a player $i_{veto} \in N$ who fulfills $i_{veto} \in W$ for all $W \in \mathbb{W}(v)$. Let $\mathcal{P}$ be a partition of N such that $\mathcal{P}(i_{veto}) \in \mathbb{W}(v)$. Then, $\varphi_{i_{veto}}^{oo}(v, \mathcal{P}, \alpha)$ is monotonously increasing in α and we have $\varphi_{i_{veto}}^{oo}(v, \mathcal{P}, 1) = \varphi_{i_{veto}}(v)$.*

In the gloves game $v_{L,R}$, if $R = \{1\}$ and $|L| \geq 1$, $v_{L,R}$ is a simple and monotonic game with 1 being its veto player. Then, if $|\mathcal{P}(r) \cap L| \geq 1$, player 1's payoff is monotonously increase in α and for $\alpha = 1$, 1 obtains the Shapley value.

The outside-option value can be negative as has been indicated in the introduction for the unanimity game. The same is true for the gloves game. Consider the partition $\mathcal{P} = \{\{1, 2, 3\}, \{4, 5, 6, 7\}\}$ where players 1 and 2 are from L and players 3 to 7 from R. Players 2 and 3 enjoy outside options. Indeed, these lead to a payoff of $\frac{3360-3456\alpha}{5040}$ for player 3 which is negative for sufficiently high α. However, negative payoffs are not possible for maximal-pairs partitions. Let $\mathcal{P}$ be a partition that puts a left-glove owner $\bar{l}$ and a right-glove owner $\bar{r}$ into one component: $\mathcal{P}(\bar{r}) = \mathcal{P}(\bar{l}) = \{\bar{l}, \bar{r}\}$. Let σ be a rule of order fulfilling $\mathcal{P}(\bar{r}) \subseteq K_{\bar{r}}(\sigma)$. Then player $\bar{r}$ is the last in $\mathcal{P}(\bar{r})$ and we have $v(\mathcal{P}(\bar{r})) = 1$. Then we have $\mathcal{P}(\bar{l}) \nsubseteq K_{\bar{l}}(\sigma)$ and player $\bar{l}$ obtains $MC_{\bar{l}}(\sigma, \mathcal{P}, \alpha) \leq 1$. This implies $v(\mathcal{P}(\bar{r})) - \sum_{j \in \mathcal{P}(\bar{r}) \setminus \bar{r}} MC_j(\sigma, \mathcal{P}, \alpha) = v(\mathcal{P}(\bar{r})) - MC_{\bar{l}}(\sigma, \mathcal{P}, \alpha) \geq 0$ and the following observation:

Remark 1. *Let $\bar{l} \in L, \bar{r} \in R$ and $\mathcal{P}$ be a partition obeying $\mathcal{P}(\bar{r}) = \mathcal{P}(\bar{l}) = \{\bar{l}, \bar{r}\}$. Then, $0 \leq \varphi^{oo}_{\bar{r}}(v_{L,R}, \mathcal{P}, \alpha) \leq 1$, $0 \leq \varphi^{oo}_{\bar{l}}(v_{L,R}, \mathcal{P}, \alpha) \leq 1$, and $\varphi^{oo}_{\bar{r}}(v, \mathcal{P}, \alpha) + \varphi^{oo}_{\bar{l}}(v, \mathcal{P}, \alpha) = 1$.*

Finally, the question if stability of partition structures is of interest. Adapting the definition proposed by Hart & Kurz (1983), we define stability in the following manner:

Definition 5.4. *A coalition structure $\mathcal{P}$ is stable for φ^{oo} if there is no coalition K such that all players from K profit from forming a component, i.e. if for all K we have*

$$\varphi^{oo}_i(v, \mathcal{P}; \alpha) \geq \varphi^{oo}_i(v, \{K, N \setminus K\}; \alpha) \text{ for some } i \in K.$$

We note without proof:

Proposition 5.5. *Stable coalition structures for φ^{oo} exist for all symmetric and convex games. Furthermore, the maximal-pairs partition is stable for φ^{oo} and the gloves game.*

5.4. Applying the outside-option value to the gloves game

Assuming $\alpha = 1$ and any maximal-pairs partition, we can calculate the outside-option value to a right-glove owner whose component also contains a left-glove owner. We obtain:

		no. of left-glove holders				
		0	1	2	3	4
no. of right-glove holders	1	0	0.500	0.667	0.750	0.800
	2	0	0.333	0.500	0.633	0.717
	3	0	0.250	0.367	0.500	0.614
	4	0	0.200	0.283	0.386	0.500

It seems clear that the value is an ex-post value while retaining the sensitivity to the relative scarcity. The reader may also note that in case of one right-glove owner, only, this agents obtains the Shapley value, in accordance with proposition 5.3.

In private communication, Joachim Rosenmüller conjectured that the outside-option value converges to the core for $\alpha = 1$. The following examples corroborate this conjecture:

replication factor	$n = 3, r = 1$	$n = 4, r = 1$
1	0.6666...	0.75
10	0.8531...	0.9278...
100	0.9734...	0.9904...

I suspect that the conjecture is correct but did not manage to provide a proof.

However, it can be shown that the outside-option value does not converge to the core for $\alpha < 1$. We show this by developing a formula corresponding to the Shapley formula 4.1. We start from any partition $\mathcal{P}$ and consider a right-glove player $\overline{r}$ for whom some player $\overline{l}$ exists such that $\mathcal{P}(\overline{r}) = \mathcal{P}(\overline{l}) = \{\overline{l}, \overline{r}\}$. (Indeed, the partition need not be maximal-pairs.) For any given rule of order σ, our right-glove player $\overline{r}$ achieves a payoff of $0, \alpha, 1 - \alpha$ or 1 :

$$\begin{aligned}
&1 - \alpha, \text{ if } \overline{l} \text{ occurs before } \overline{r} \text{ and if } \overline{l} \text{ increases the number of pairs in } K_{\overline{l}}(\sigma), \\
&1, \text{ if } \overline{l} \text{ occurs before } \overline{r} \text{ and if } \overline{l} \text{ does not increase the number of pairs in } K_{\overline{l}}(\sigma), \\
&\alpha, \text{ if } \overline{l} \text{ occurs after } \overline{r} \text{ and if } \overline{r} \text{ increases the number of pairs in } K_{\overline{r}}(\sigma), \text{ and} \\
&0, \text{ if } \overline{l} \text{ occurs after } \overline{r} \text{ and if } \overline{r} \text{ does not increase the number of pairs in } K_{\overline{r}}(\sigma).
\end{aligned}$$

These four cases can be put more succinctly as

cases	conditions		payoff
case I	$\sigma^{-1}(\overline{l}) < \sigma^{-1}(\overline{r})$	$\left\lvert K_{\overline{l}}(\sigma) \backslash \overline{l} \cap R\right\rvert > \left\lvert K_{\overline{l}}(\sigma) \backslash \overline{l} \cap L\right\rvert$	$1 - \alpha$
case II	$\sigma^{-1}(\overline{l}) < \sigma^{-1}(\overline{r})$	$\left\lvert K_{\overline{l}}(\sigma) \backslash \overline{l} \cap R\right\rvert \leq \left\lvert K_{\overline{l}}(\sigma) \backslash \overline{l} \cap L\right\rvert$	1
case III	$\sigma^{-1}(\overline{r}) < \sigma^{-1}(\overline{l})$	$\left\lvert K_{\overline{r}}(\sigma) \backslash \overline{r} \cap R\right\rvert < \left\lvert K_{\overline{r}}(\sigma) \backslash \overline{r} \cap L\right\rvert$	α
case IV	$\sigma^{-1}(\overline{r}) < \sigma^{-1}(\overline{l})$	$\left\lvert K_{\overline{r}}(\sigma) \backslash \overline{r} \cap R\right\rvert \geq \left\lvert K_{\overline{r}}(\sigma) \backslash \overline{r} \cap L\right\rvert$	0

Let $prob(I)$ stand for the probability of case I, $prob(II)$ for the probability of case II etc..Then $prob(I) + prob(II)$ is the probability of $\overline{l}$ occurring before $\overline{r}$ $(1/2)$ and the outside-option value for player $\overline{r}$ is given by

$$prob(I) \cdot (1 - \alpha) + prob(II) \cdot 1 + prob(III) \cdot \alpha$$

$$= \frac{1}{2} + \alpha \left[prob\left(III\right) - prob\left(I\right)\right],$$

By applying remark 1 to $\alpha = 1$, we learn that

$$\left| prob\left(III\right) - prob\left(I\right)\right| \leq \frac{1}{2}.$$

This implies

$$0 < \frac{1}{2} + \alpha \left[prob\left(III\right) - prob\left(I\right)\right] < 1 \text{ for all } \alpha < 1$$

and shows that convergence does not hold for $\alpha < 1$. The formulae for the probabilities are given in the appendix (and might be of some use for proving convergence for $\alpha = 1$.

6. Conclusions

In this paper, we develop a partitional value that is close the AD-value (in obeying comonent efficiency) and close the Shapley value (in being sensitive to relative scarcity and in special cases). We find that this value performs quite satisfactorily with respect to the gloves game. Of course, some empirical work would be needed to gain more confidence in this new value.

The oo-value makes use of axiom N-oo where the AD-value uses the null-player axiom N. Instead of axiom $N - oo$ one might postulate an alternative axiom which is like N-oo but attributes $\psi\left(u_T, \mathcal{P}\right)\left(C \cap T^c\right) = -\frac{|C\cap T|}{|T|}\frac{|C\cap T^c|}{|C|}$ if no component C of $\mathcal{P}$ can be found that fulfills $T \subseteq C$. The second factor in $-\frac{|C\cap T|}{|T|}\frac{|C\cap T^c|}{|C|}$ has the simple interpretation of letting all the players from C (rather than all the players from $C \cup T$ as in N-oo) have equal probability for paying a T-player who happens to complete T and to gain the unit payoff. Alas, a simple expression along the lines of φ^{oo} could not be found.

7. Appendix

A. Axiomatization of the outside-option value

Theorem A.1. *φ^{oo} is the unique value on $(N, \mathfrak{P})$ satisfying the axioms CE, S, N-oo, and L.*

Proof. As usual, the proof has to state two things. First, the value fulfills the axioms; second, there is no other value to do so. The second part is standard and makes heavy use of linearity and the fact that the unanimity games form a basis of G^N, e.g. Aumann (1989, pp. 30).

We will contend ourselves to comment on the interesting aspects of the first part. Axioms CE, S, and L are easily checked as is the first part of axiom N-oo. Turning to the second part of axiom N-oo, we fix a partition $\mathcal{P} = \{C_1, ..., C_m\}$ and a nonempty set $T \subseteq N$ and assume that T is not contained in any of $\mathcal{P}$'s components. Then, $u_T(C) = 0$ for all components C of $\mathcal{P}$ and all of the pay-offs carry the factor α because $u_T(\mathcal{P}(i)) = 0 = MC_i^{K_i(\sigma)\cap P(i)}(u_T)$ holds for all $i \in N$. Now, for a given $\sigma \in \Sigma_n$ and a given component C from $\mathcal{P}$, if a player i from $C \cap T^c$ is not the last of his component to occur in σ (i.e. $\mathcal{P}(i) \nsubseteq K_i(\sigma)$), he receives zero. If our player from $C \cap T^c$ happens to be the last C-player and to occur after all players from T, he has to pay -1 if, with probability $\frac{|C\cap T|}{|T|}$, a player from $C \cap T$ was the last of all the T-players to occur in σ. Therefore, players from $C \cap T^c$ obtain $-\alpha \frac{|C\cap T|}{|T|} \frac{|C\cap T^c|}{|T\cup C|}$. ■

B. The outside-option value for a veto player in a simple monotonic game

Proposition B.1. *Let (v, N) be a simple and monotonic game and $\mathbb{W}(v)$ its set of winning coalitions. Let there be a player $i_{veto} \in N$ who fulfills $i_{veto} \in W$ for all $W \in \mathbb{W}(v)$. Let $\mathcal{P}$ be a partition of N such that $\mathcal{P}(i_{veto}) \in \mathbb{W}(v)$. Then, $\varphi^{oo}_{i_{veto}}(v, \mathcal{P}, \alpha)$ is monotonously increasing in α and we have $\varphi^{oo}_{i_{veto}}(v, \mathcal{P}, 1) = \varphi_{i_{veto}}(v)$.*

Proof. If i_{veto} is the veto players, we have $MC_j^K(v) = 0$ for all players $j \neq i_{veto}$ and all coalitions $K \subseteq N$ and hence $MC_i(\sigma, \mathcal{P}, \alpha) = 0$ for all players $j \neq i_{veto}$ and all orderings $\sigma \in \Sigma$. For $\mathcal{P}(i_{veto}) \subseteq K_{i_{veto}}(\sigma)$, we obtain $v(\mathcal{P}(i_{veto})) - \sum_{j\in\mathcal{P}(i_{veto})\backslash i_{veto}} MC_j(\sigma, \mathcal{P}, \alpha) = 1$ by $\mathcal{P}(i_{veto}) \in \mathbb{W}(v)$. For $\mathcal{P}(i_{veto}) \nsubseteq K_{i_{veto}}(\sigma)$, whenver $MC_{i_{veto}}^{K_{i_{veto}}(\sigma)\cap P(i_{veto})} = 1$ we also have $MC_{i_{veto}}^{K_{i_{veto}}(\sigma)} = 1$ by monotonicity and by player i_{veto} 's veto power. Therefore, $MC_{i_{veto}}(\sigma, \mathcal{P}, \alpha)$ is monotonously increasing in α and so is $\varphi_{i_{veto}}^{oo}(v, \mathcal{P}, \alpha)$.

For $\alpha = 1$, let σ be any ordring from Σ. For $\mathcal{P}(i_{veto}) \subseteq K_{i_{veto}}(\sigma)$, we obtain $v(\mathcal{P}(i_{veto})) - \sum_{j\in\mathcal{P}(i_{veto})\backslash i_{veto}} MC_j(\sigma, \mathcal{P}, \alpha) = 1 = MC_{i_{veto}}^{K_{i_{veto}}(\sigma)}(v)$ by $\mathcal{P}(i_{veto}) \in \mathbb{W}(v)$ and by veto power. For $\mathcal{P}(i_{veto}) \nsubseteq K_{i_{veto}}(\sigma)$, we get $MC_{i_{veto}}(\sigma, \mathcal{P}, \alpha) = MC_{i_{veto}}^{K_{i_{veto}}(\sigma)}(v)$. Therefore, $\varphi_{i_{veto}}^{oo}(v, \mathcal{P}, \alpha) = \frac{1}{n!}\sum_{\sigma\in\Sigma} MC_{i_{veto}}^{K_{i_{veto}}(\sigma)}(v) = \varphi_{i_{veto}}(v)$. ■

C. The probabilities in the ouside-option formula for the gloves game

The probabilities for the first three cases are given by

$$prob(I) = \sum_{p_{\overline{l}}=2}^{n-1} \frac{1}{n}\frac{n-p_{\overline{l}}}{n-1} \cdot \begin{cases} \sum_{\rho=\max\left(\frac{p_{\overline{l}}-1}{2}+1, p_{\overline{l}}-l\right)}^{\min\left(p_{\overline{l}}-1, r-1\right)} \frac{\binom{r-1}{\rho}\binom{l-1}{p_{\overline{l}}-1-\rho}}{\binom{n-2}{p_{\overline{l}}-1}}, & p_{\overline{l}} \text{ odd} \\ \sum_{\rho=\max\left(\frac{p_{\overline{l}}}{2}, p_{\overline{l}}-l\right)}^{\min\left(p_{\overline{l}}-1, r-1\right)} \frac{\binom{r-1}{\rho}\binom{l-1}{p_{\overline{l}}-1-\rho}}{\binom{n-2}{p_{\overline{l}}-1}} & p_{\overline{l}} \text{ even} \end{cases}$$

$$prob(II) = \sum_{p_{\overline{l}}=2}^{n-1} \frac{1}{n}\frac{n-p_{\overline{l}}}{n-1} \cdot \begin{cases} \left(1 - \sum_{\rho=\max\left(\frac{p_{\overline{l}}-1}{2}+1, p_{\overline{l}}-l\right)}^{\min\left(p_{\overline{l}}-1, r-1\right)} \frac{\binom{r-1}{\rho}\binom{l-1}{p_{\overline{l}}-1-\rho}}{\binom{n-2}{p_{\overline{l}}-1}}\right), & p_{\overline{l}} \text{ odd} \\ \left(1 - \sum_{\rho=\max\left(\frac{p_{\overline{l}}}{2}, p_{\overline{l}}-l\right)}^{\min\left(p_{\overline{l}}-1, r-1\right)} \frac{\binom{r-1}{\rho}\binom{l-1}{p_{\overline{l}}-1-\rho}}{\binom{n-2}{p_{\overline{l}}-1}}\right) & p_{\overline{l}} \text{ even} \end{cases}$$

$$prob(III) = \sum_{p_{\overline{r}}=2}^{n-1} \frac{1}{n}\frac{n-p_{\overline{r}}}{n-1} \cdot \begin{cases} \sum_{\lambda=\max\left(\frac{p_{\overline{r}}-1}{2}+1, p_{\overline{r}}-r\right)}^{\min(p_{\overline{r}}-1, l-1)} \frac{\binom{l-1}{\lambda}\binom{r-1}{p_{\overline{r}}-1-\lambda}}{\binom{n-2}{p_{\overline{r}}-1}}, & p_{\overline{r}} \text{ odd} \\ \sum_{\lambda=\max\left(\frac{p_{\overline{r}}}{2}, p_{\overline{r}}-r\right)}^{\min(p_{\overline{r}}-1, l-1)} \frac{\binom{l-1}{\lambda}\binom{r-1}{p_{\overline{r}}-1-\lambda}}{\binom{n-2}{p_{\overline{r}}-1}} & p_{\overline{r}} \text{ even} \end{cases}$$

We will derive $prob(I)$, the other probabilities can be explained in a similar fashion. $prob(I)$ is the probability for $\overline{l}$ occurring before $\overline{r}$ and for $|K_{\overline{l}}(\sigma) \cap R| \geq |K_{\overline{l}}(\sigma) \cap L|$. The probability of $\overline{l}$ occurring before $\overline{r}$ is equal to

$$\underbrace{\sum_{p_{\bar{l}}=1}^{n}}_{\substack{\text{summing}\\ \text{over}\\ \text{all positions}\\ \text{player } \bar{l}\\ \text{may occupy}}} \underbrace{\frac{1}{n}}_{\substack{\text{probability}\\ \text{of } \bar{l} \text{ being at}\\ \text{any position}\\ 1,...,n}} \underbrace{\frac{n-p_{\bar{l}}}{n-1}}_{\substack{\text{probability}\\ \text{of } \bar{r} \text{ occurring}\\ \text{after position } p_{\bar{l}}}}$$

$$= \sum_{p_{\bar{l}}=1}^{n-1} \frac{1}{n}\frac{n-p_{\bar{l}}}{n-1} \text{ (no position after position } n\text{)},$$

i.e., we sum the probabilities for all positions $p_{\bar{l}} = 1, ..., n$ that player $\bar{l}$ may hold. Her probability for any of those positions is $\frac{1}{n}$. For position $p_{\bar{l}}$, the chance that player $\bar{r}$ appears later is $n - p_{\bar{l}}$ (the number of positions following $p_{\bar{l}}$) over $n - 1$ (the number of positions excepting $p_{\bar{l}}$). It will come to no surprise that $\sum_{p_{\bar{l}}=1}^{n-1} \frac{1}{n}\frac{n-p_{\bar{l}}}{n-1} = \frac{1}{2}$.

For any position $p_{\bar{l}} = 1, ..., n-1$ we consider the probability that a rule of order σ contains no less R-players than L-players if we restrict attention to positions 1 through $p_{\bar{l}} - 1$, i.e. the probability for $\left|K_{\bar{l}}(\sigma) \setminus \bar{l} \cap R\right| > \left|K_{\bar{l}}(\sigma) \setminus \bar{l} \cap L\right|$. In that case, $K_{\bar{l}}(\sigma)$ contains one more pair of gloves than $K_{\bar{l}}(\sigma) \setminus \bar{l}$. Our problem is to consider samples of $p_{\bar{l}} - 1$ gloves (glove holders) and determine whether they contain more right than left gloves. These samples contain neither $\bar{l}$ (we look at positions 1 through $p_{\bar{l}} - 1$) nor $\bar{r}$ (which occurs after $\bar{l}$). Therefore, we are dealing with $\binom{r-1+l-1}{p_{\bar{l}}-1} = \binom{n-2}{p_{\bar{l}}-1}$ samples. The probability of finding ρ right-hand gloves (and $p_{\bar{l}} - 1 - \rho$ left-hand gloves) is equal to $\frac{\binom{r-1}{\rho}\binom{l-1}{p_{\bar{l}}-1-\rho}}{\binom{n-2}{p_{\bar{l}}-1}}$ (the hypergeometrical distribution). We now have to restrict attention to those ρ (number of right gloves) that imply more right than left gloves in $K_{\bar{l}}(\sigma) \setminus \bar{l}$:

- If $p_{\bar{l}}$ is odd (even), this amounts to $\rho \geq \frac{p_{\bar{l}}-1}{2} + 1$ ($\rho \geq \frac{p_{\bar{l}}}{2}$).
- The maximal number of right gloves one may encounter in $K_{\bar{l}}(\sigma) \setminus \bar{l}$ is equal to $\left|K_{\bar{l}}(\sigma) \setminus \bar{l}\right| = p_{\bar{l}} - 1$. Therefore, $\rho \leq p_{\bar{l}} - 1$.

- Since the maximal number of left gloves in $K_{\bar{l}}(\sigma)\backslash\bar{l}$ is equal to $l-1$, the minimal number of right gloves is equal to $p_{\bar{l}}-1-(l-1)=p_{\bar{l}}-l$. Hence, $\rho\geq p_{\bar{l}}-l$.
- Finally, by $\sigma^{-1}\left(\bar{l}\right)<\sigma^{-1}\left(\bar{r}\right)$ we have $\rho\leq r-1$.

For $p_{\bar{l}}=1$, the first inequality implies $\rho\geq 1$ and the second, $\rho\leq 0$. Indeed, the left-glove holder at position 1 does not complete a pair and it suffices to start the summation at $p_{\bar{l}}=2$ as we do in the probability formulae.

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