Public Goods with Congestion: a Mechanism Design Approach

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Abstract

I consider the problem of efficient provision of the public good with congestion in the setting with asymmetric information. I show, in particular, that when congestion is taken into account, in a wide class of economies it is *possible* to construct an incentive compatible mechanism that always produces the good at the efficient level, balances the budget and satisfies each consumer's voluntary participation constraint. This result is in contrast with the corresponding *impossibility* result for pure public goods due to Rob (1989) and Mailath and Postlewiate (1990).

1 Introduction

Economic literature devoted to the problem of public good provision is enormous. This problem had been studied in a huge variety of settings and from different perspectives. In the present paper we concentrate on a very particular sub-problem. The society has to undertake a single public project and this project, if undertaken, produces a public good at a single possible level. The society, however, wants to undertake the project if and only if it is efficient to do so, that is the project has to be implemented if and only if it increases the joint welfare of the society.

The good is, traditionally, called public if it is both non-rival and non-excludable.[1] This definition implies, firstly, that no consumer cares about how many others she shares the good with, and secondly, nobody can be excluded from consumption. As Buchanan (1965) pointed out in his seminal work on *club* goods, many of the public goods are excludable, and in many cases consumers *do* care about how many others consume the good. These goods are rival to some extent, or in our terminology –

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[1]In this paper we further call such good *pure public* good.

"congested". Examples of such goods are abundant. A swimming pool that charges membership fees is an excludable public good and when it becomes crowded, customers, obviously, adversely affect each other's welfare. Toll road is an excludable public good, but traffic jams happen even there. University library is restricted to the students and the faculty, but the desired book is quite often checked out. In fact, the majority of the public goods are characterized by some congestion.

This paper concentrates on the problem of financing the provision of excludable public goods with congestion. We show, in particular, that when congestion is taken into account, in a wide class of economies it is *possible* to construct an incentive compatible mechanism that always produces the good at the efficient level, always balances the budget (satisfies BB) and satisfies each consumer's voluntary participation constraint (satisfies IR). Our possibility result holds even if the information about the efficient level of the public good is dispersed among the potential consumers, and every consumer just knows her private valuation for the good and the distribution of others' valuations. This result is in contrast with the corresponding *impossibility* result for pure public goods.[2] Thus, the incentive problem is relaxed, when the consumers create congestion, decreasing each others' utility from consumption. It is noteworthy, that we do not impose any congestion artificially, instead, we pick up a realistic feature of the environment and incorporate it in our model.

Intuitively, our possibility result should not come as a huge surprise. When the good is completely rival, that is *pure private*, it is possible to construct an efficient, incentive compatible mechanism that satisfies BB and IR. We show, however, that our existence result is non-trivial. The budget can be balanced in the situations where the good is still "quite public". Our results suggest that to balance the budget, less that half of the consumers should be excluded. It is worth clarifying, that the number of consumers, that are excluded, is not a policy variable of the designer. The good is always produced at the efficient level, and if from the efficiency perspective it is beneficial to include exactly m consumers, exactly m consumers are included.

At this point our approach contrasts with the literature on *excludable* (and non-rival) public goods. There, after the good is produced, it is inefficient to exclude *any* consumer. If some of the consumers are not excluded, though, the mechanism runs the budget deficit, and the project is not undertaken.[3] Thus, to achieve efficiency the planner has to create an inefficiency. To take the case further, note that to balance the budget the planner has to exclude some of the consumers and then commit to keep them excluded. As soon as it is efficient to add those consumers after the project is constructed, there is an extra surplus to be shared between the excluded consumers and the planner.[4] Therefore, to balance the budget with excludable and non-rival public good, the model has to incorporate some commitment devices. Our

[2] We will discuss this result in the Literature Review subsection.

[3] Hence, at most second-best level of efficiency can be achieved, versus first-best level in my paper.

[4] Here, as a digression, we treat the planner as a producer of the good, who is motivated by profit maximization.

formulation is obviously free from these defects.

1.1 Literature Review

We first discuss the literature on the problem of provision of the *pure* public good. If the information about the impact of the project on the society's welfare is publicly available, the problem of achieving efficiency via incentive compatible, BB and IR mechanism has a solution. Indeed, in the setting with complete information a number of authors offered successful mechanisms, see Hurvicz (1979), Varian (1994) and Walker (1981), to mention just a few. Complete information seems to be quite strong an assumption, but it can be relaxed substantially. Jackson and Moulin (1992) offer a remarkable mechanism that satisfies BB and IR and implements the project efficiently whenever every consumers knows her private valuation of the project, and at least two consumers know the sum of all the private valuations.

If we restrict the information, that each consumer possess, to his private valuation and the distribution of others' valuations, satisfying both BB and IR becomes a problem. Green and Laffont (1977) demonstrate that every efficient, IR, dominant strategy incentive compatible mechanism runs a deficit in some realizations.[5] Arrow (1979) and d'Aspremont and Gérard-Varet (1979) present an efficient mechanism, which is Bayesian incentive compatible and BB, however the IR constraint may be violated. Rob (1989) and Mailath and Postlewaite (1990) show that requiring IR and BB along with Bayesian incentive compatibility precludes efficiency.[6] Moreover, they demonstrate that the probability of ever undertaking the project goes to zero when the number of consumers goes to infinity and the cost of the project is increasing with the size of the economy. Hellwig (2003), in contrast, shows that when the cost of the project is fixed, and the number of the consumers increases, the provision level converges in distribution to the efficient level.

Several authors consider the possibility of constructing efficient, BB and IR mechanisms in the setting where the public good, still completely non-rival, is excludable. Dearden (1997) demonstrates that BB, IR and efficiency can then be achieved in the limit, when the number of consumers goes to infinity.[7] Hellwig (2003) and Norman (2003) show that efficient, BB and IR mechanism can be approximated by a simple fixed entry fee mechanism.

There is extensive literature on club goods, starting from Buchanan (1965). For a nice survey of the results with complete information see Cornes and Sandler (1986).

[5] Laffont and Maskin (1979) show that efficient, BB, mechanism can be constructed if interim individual rationality is replaced with ex-ante individual rationality. See also Laffont and Maskin (1982) for a survey of the results with dominant strategy implementable mechanisms.

[6] Independence of "types" and an assumption about the continuum of possible private valuations is important here.

[7] Dearden (1997) and Ledyard and Palfrey (1999), in more general setting, provide a characterization of ex-ante efficient, BB and IR mechanisms. Cornelli (1996) studies the problem of provision of excludable public good by profit maximizing monopoly and obtains very similar characterization.

Very little is done in incomplete information setup. This paper is the only one, to my best knowledge, that treats public goods with congestion in the setting with quasilinear utility.

Jackson and Nicolò (2003) examines the strategy-proof provision of club goods. Consumers in their setting have single-peaked preferences over the public good level and also care about how many other consumers have an access to the good. Jackson and Nicolò (2003) show that strategy-proofness and efficiency are in general incompatible with individual stability, a notion that requires that consumers, who are given an access to the good, weakly prefer to consume, and those who are excluded, weakly prefer not to consume.[8] Bogomolnaia and Nicolò (2002) in similar setting (preferences are single-peaked over the level of the good and negatively affected by congestion) study the problem of locating two public facilities. When each of the consumers has to be assigned to exactly one of these facilities, strategy-proof and efficient allocation rule exists and satisfies ex-post stability, that is ex-post no consumer wants to be assigned to the facility, other than the one she is assigned to by the rule.

Moulin (1994) and further Dearden (1998) and Olszewski (1999) examine the properties of an appealingly simple and robust *serial* cost-sharing mechanism for provision of excludable, non-rival public goods. They show that serial mechanism Pareto dominates any IR, BB, coalition strategy-proof mechanism that satisfies anonymity.[9] Deb and Razzolini (1999a) further demonstrate that serial mechanism Pareto dominates any strategy-proof, BB, anonymous mechanism that satisfies individual stability in the sense of Jackson and Nicolò (2003). Serial mechanism excludes customers and therefore achieves only second-best efficiency.

2 The Model

Economy consists of n potential consumers and a social planner. Set N denotes the set of all potential consumers. Each of the consumers has an endowment of a numeraire good, which is convenient to think of as money. The social planner is endowed with a technology, that can transform the numeraire into the public good. The technology can produce one unit of indivisible public good and requires an amount C of the numeraire as an input. The public good is excludable and, to some extent, rival.

Each of the consumers i draws a real valued *value* v_i, distributed according to a smooth distribution function F with the density $f(v) > 0$ on the interval $[0, 1]$.[10] The draws are independent. The distribution F is common knowledge, but v_i is consumer i's private information. It is convenient to think of these values as ranked in descending order. From now on, v_i denotes the i-th highest value among n draws. This v_i is the value, that consumer i attaches to the public good, if she consumes it

[8] Strategy-proofness and efficiency can also be incompatible with IR.

[9] Serial mechanism by construction is budget balanced.

[10] Our proof relies on the fact that the left end of the support is 0. The right end of the support being set at 1 is without loss of generality.

alone as a private good. As usual, $\mathbf{v}$ denotes the "state of the world" – the profile of the realized values $(v_1, v_2, ..., v_n)$. Where necessary we will write $(v_i, \mathbf{v}_{-i})$, where $\mathbf{v}_{-i}$ stands for the ranked values of all the consumers except i. The set of all possible realization of the values is denoted with $\mathbf{V}$.

Utility that consumer i derives from consumption of the public good depends on the private value v_i and on "congestion" – the number of the consumers, who are given an access to the public good. The extent to which the consumers affect each others utility is determined by the *congestion schedule.*

Definition 1 *Mapping* $\boldsymbol{\alpha} : \mathbb{N} \to [0,1]$ *is a* congestion schedule *if* $\alpha_1 = 1$ *and* $\alpha_k \geq \alpha_{k+1}$ *for all* k.

The congestion schedule for the *pure* public good is denoted by $\boldsymbol{\rho}$ with $\rho_k = 1$ for all k.

If m consumers are given an access to the public good, then the utility that any one of them, say consumer i, derives from consumption, is $\alpha_m v_i$. The fact that congestion schedule is decreasing implies that whenever an extra consumer gets an access to the public good, the utility that all the former users derive from the good decreases.[11] The congestion schedule is assumed to be commonly known among the consumers and the social planner.

The prerogative of the social planner is to make a decision concerning the production of the public good. Firstly, the decision is made about the *allocation* M, that is whether to produce the public good or not and if produce, which consumers to give an access to the good. Formally, the set $M \subseteq N$ denotes the set of the consumers who are given an access to the public good. If the good is not produced, $M = \varnothing$. Secondly, the decision is made about the *payments* $\mathbf{t}$, that is how much of the numeraire each of the consumers has to contribute (or receive), under the provision that the sum of the contributions covers C – the amount of the input necessary. As a result of the planners decision consumer i experiences a *gain in utility*

$$U_i(M, \boldsymbol{\alpha}, v_i) = \begin{cases} \alpha_m v_i - t_i & \text{if } i \in M, \\ -t_i & \text{if } i \notin M, \end{cases}$$

where $m = \#M$ and t_i stands for the contribution of the endowment that a consumer i is asked to make as a part of the input into the public good. After the decision is made and the set M is determined, the *social welfare* in the economy is defined as

$$SW(M, \boldsymbol{\alpha}, \mathbf{v}) \equiv \begin{cases} \alpha_m \sum\limits_{j \in M} v_j - C & \text{if } M \neq \varnothing, \\ 0 & \text{if } M = \varnothing. \end{cases}$$

The primary objective of the social planner is efficiency, that is maximizing the social welfare in the economy, but the efficient decision depends on consumers' private

[11] Vickrey (1969) distinguishes between six types of traffic congestion. Our formulation is equivalent to the most general of his – "general density congestion".

information. The planner may ask the consumers to reveal their private information and then make the decision based on their reports $\widetilde{\mathbf{v}}$.[12] For this purpose, the planner designs a set of rules or a mechanism that maps the reports into the decision and announces the mechanism to the consumers. It is assumed that the planner can commit to the mechanism she announces. Denote the set of all possible allocations with $\mathcal{M}$ and the set of all possible payments with $\mathcal{T}$.

Definition 2 *A mechanism* $(M, \mathbf{t}) : \mathbf{V} \rightarrow \mathcal{M} \times \mathcal{T}$ *is* (ex-post) efficient *if at every* $\mathbf{v} \in \mathbf{V}$ *the allocation* $M(\boldsymbol{\alpha}, \mathbf{r})$ *maximizes social welfare* $SW(M, \boldsymbol{\alpha}, \mathbf{v})$*).*

Definition 3 *A mechanism* $(M, \mathbf{t}) : \mathbf{V} \rightarrow \mathcal{M} \times \mathcal{T}$ *is* (ex-post) budget balanced *if at every* $\mathbf{v} \in \mathbf{V}$ *the payment* $\mathbf{t}(\boldsymbol{\alpha}, \mathbf{r})$ *satisfies*

$$\sum\nolimits_{i \in N} t_i(\boldsymbol{\alpha}, \mathbf{r}) = C.$$

Clearly, if the decision about the efficient allocation is non-trivial and depends on consumers' private information, the planner has to ask the consumers about their values. A consumer, however, need not tell the truth and may "play strategically". Therefore, there is a need in designing a game, where each of the consumers has an incentive to, firstly, participate and secondly, tell the truth about his value. The decision on whether to participate and whether to tell the truth is made at the *interim* stage, that is when the consumer has already drawn his private valuation, but before the outcome is determined. As we have already assumed, the future outcome, from consumer i's point of view, depends on the information held by the other consumers. Thus, each consumer, when making participation/non-participation and truth-telling/not truth-telling decisions has to take an expectation over the possible realizations of the values of other consumers. To summarize, the consumer is given "Bayesian incentives."

Definition 4 *A mechanism* $(M, \mathbf{t}) : \mathbf{V} \rightarrow \mathcal{M} \times \mathcal{T}$ *is* (Bayesian) incentive compatible *if for all* $i \in N$*, at every* $\mathbf{v} \in \mathbf{V}$*, "truth-telling" constitutes a Bayesian-Nash equilibrium of the resulting game.*

Introduce $U_i(v_i) = E_{\mathbf{v}_{-i}} \left[U_i(M, \boldsymbol{\alpha}, v_i) \right].$

Definition 5 *A mechanism* $(M, \mathbf{t}) : \mathbf{V} \rightarrow \mathcal{M} \times \mathcal{T}$ *is* (interim) individually rational *if for all* $i \in N$*, at every* $\mathbf{v} \in \mathbf{V}$*,*

$$U_i(v_i) \geq 0,$$

where 0 *denotes consumer* i*'s* individual rationality *level.*[13]

[12] As usual, without loss of generality, we restrict our attention to direct revelation mechanisms.

[13] The individual rationality level represents consumer i's outside option. In public goods literature it is typically assumed to be equal to zero. This is not without loss of generality, but we follow the convention as soon as our main focus is on the role of congestion.

The assumption that IR is satisfied only interim (not ex-post) is not fully innocuous. It implies that the social planner is endowed with some coercive power, so that (ex-post) she can force the consumers to make the payments, that they have agreed to pay (interim). Alternatively, one can think of a situation where consumers are asked to make the payments and only after that the outcome is announced. The existence of some power on the planner's side is already assumed, however, when we assume that the planner is able to exclude some agents from consumption.

Existence of an ex-post efficient and ex-post budget balanced mechanism that satisfies Bayesian incentive compatibility and interim individual rationality is known to be problematic in many settings. The following Theorem provides a decisive answer to this existence question.

Theorem 1 (Krishna and Perry (2000)) *An efficient, incentive compatible, BB and IR mechanism exists* if and only if *the Vickrey-Clarke-Groves (VCG) mechanism results in an expected surplus.*[14]

2.1 The Vickrey-Clarke-Groves Mechanism

In this Section we introduce the *VCG mechanism*. Every consumer is asked to report her value to the social planner, who then determines the allocation of the public good and the payment from each consumer.

Definition 6 $K(\boldsymbol{\alpha}, \mathbf{v})$ *is an* allocation rule *of the VCG mechanism if at every* $\mathbf{v} \in \mathbf{V}$*:*

$$SW\left(K(\boldsymbol{\alpha}, \mathbf{v}), \mathbf{v}\right) = \max_{M} SW(M, \boldsymbol{\alpha}, \mathbf{v}), \tag{1}$$

and every $\arg\max_M SW(M, \boldsymbol{\alpha}, \mathbf{v}) \subseteq K(\boldsymbol{\alpha}, \mathbf{v})$.

We need a few extra notations. Similarly to $SW(M, \boldsymbol{\alpha}, \mathbf{v})$ introduce

$$SW_{-i}(M, \boldsymbol{\alpha}, \mathbf{v}) \equiv \begin{cases} \alpha_m \sum\limits_{i \neq j \in M} v_j - C & \text{if } M \neq \varnothing, \\ 0 & \text{if } M = \varnothing. \end{cases}$$

Then introduce the social welfare in the economy where v_i is set at 0,

$$SW(M, \boldsymbol{\alpha}, \mathbf{v}_{-i}) \equiv SW(M, \boldsymbol{\alpha}, (0, \mathbf{v}_{-i})).$$

Now similarly introduce $K(\boldsymbol{\alpha}, \mathbf{v}_{-i})$ with a meaning of an allocation in the economy where v_i is set at 0.

[14] Makowski and Mezzetti (1994) show that efficient, incentive compatible, BB and IR mechanism exists if and only if there exists efficient, dominant strategy incentive compatible, IR mechanism that balances the budget ex-ante.

Definition 7 $\mathbf{t}(\boldsymbol{\alpha}, \mathbf{v}) \equiv (t_i(\boldsymbol{\alpha}, \mathbf{v}), \mathbf{t}_{-i}(\boldsymbol{\alpha}, \mathbf{v}))$ *is a payment rule of the VCG mechanism if for every* $i \in N$, *at every* $\mathbf{v}$*:*

$$t_i(\boldsymbol{\alpha}, \mathbf{v}) = SW(K(\boldsymbol{\alpha}, \mathbf{v}_{-i}), \mathbf{v}_{-i}) - SW_{-i}(K(\boldsymbol{\alpha}, \mathbf{v}), \mathbf{v})$$

The following lemma is standard and is given here for completeness.

Lemma 1 *The generalized VCG mechanism is efficient and for every consumer i truth-telling constitutes a weakly dominant strategy in the generalized VCG mechanism.*

Proof. Efficiency follows from the definition of the allocation rule. Now we show weak dominance.

Indeed, suppose that for some consumer i, for some profile of the values of the others $\widehat{\mathbf{v}}_{-i}$ consumer i strictly prefers to report $\widehat{v}_i \neq v_i$. Then, i's ex-post payoff is

$$\alpha_k v_i + SW_{-i}(K(\boldsymbol{\alpha}, \widehat{\mathbf{v}}), \widehat{\mathbf{v}}) - SW(K(\boldsymbol{\alpha}, \widehat{\mathbf{v}}_{-i}), \widehat{\mathbf{v}}_{-i}))$$

versus

$$\alpha_{k'} v_i + SW_{-i}(K(\boldsymbol{\alpha}, (v_i, \widehat{\mathbf{v}}_{-i})), (v_i, \widehat{\mathbf{v}}_{-i})) - SW(K(\boldsymbol{\alpha}, \widehat{\mathbf{v}}_{-i}), \widehat{\mathbf{v}}_{-i})),$$

where $k = \#K(\boldsymbol{\alpha}, \widehat{\mathbf{v}})$, $k' = \#K(\boldsymbol{\alpha}, (v_i, \widehat{\mathbf{v}}_{-i}))$. The last two terms cancel out and the first two in each line by the supposition imply

$$SW(K(\boldsymbol{\alpha}, \widehat{\mathbf{v}}), (v_i, \widehat{\mathbf{v}}_{-i})) > SW(K(\boldsymbol{\alpha}, (v_i, \widehat{\mathbf{v}}_{-i})), (v_i, \widehat{\mathbf{v}}_{-i})).$$

But this contradicts the fact $K(\boldsymbol{\alpha}, (v_i, \widehat{\mathbf{v}}_{-i}))$ is an efficient allocation at the profile $(v_i, \widehat{\mathbf{v}}_{-i})$. ■

Individual rationality for the generalized VCG mechanism in the more abstract setup is proven in Krishna and Perry (2000).

3 Congested Public Goods and Budget Deficit

We following definition is borrowed from the theory of the pure public goods.

Definition 8 *Consumer i is said to be* pivotal *under schedule $\boldsymbol{\alpha}$ at the realization* $\mathbf{v}$, *if* $K(\boldsymbol{\alpha}, \mathbf{v}) \neq \varnothing$, *while* $K(\boldsymbol{\alpha}, \mathbf{v}_{-i}) = \varnothing$.

Clearly, more than one pivotal consumer may exist. In addition, suppose $v_i > v_j$ and j is pivotal. Then i is obviously pivotal. As the following Lemma illustrates, the presence of pivotal consumers has importnant implications for the budget surpus in the VCG mechanism.

Lemma 2 *Suppose at* $\mathbf{v}$ *such that* $\#K(\boldsymbol{\alpha}, \mathbf{v}) = k \geq 2$ *there exists a pivotal consumer. Then the VCG mechanism runs the budget deficit at* $\mathbf{v}$.

Proof. We show that $\sum t_i(\boldsymbol{\alpha}, \mathbf{v}) \leq C$. Suppose, without loss of generality, that there is just one pivotal consumer, namely consumer 1. (If there is more than one pivotal consumer, consumer 1 is pivotal). Then, by the definition of the payments in the VCG,

$$\sum_{i \in K(\boldsymbol{\alpha}, \mathbf{v})} t_i(\boldsymbol{\alpha}, \mathbf{v}) = C + \sum_{i \in K(\boldsymbol{\alpha}, \mathbf{v}_{-1})} SW(K(\boldsymbol{\alpha}, \mathbf{v}_{-i}), \mathbf{v}_{-i}) - (k-1) SW(K(\boldsymbol{\alpha}, \mathbf{v}), \mathbf{v}) \leq C.$$

The equality follows form the fact that,

$$\sum_{i \in K(\boldsymbol{\alpha}, \mathbf{v})} \sum_{i \in K(\boldsymbol{\alpha}, \mathbf{v}_{-i})} SW_{-i}(K(\boldsymbol{\alpha}, \mathbf{v}), \mathbf{v}) = (k-1) SW(K(\boldsymbol{\alpha}, \mathbf{v}), \mathbf{v}),$$

and the inequality follows from the fact that $SW(K(\boldsymbol{\alpha}, \mathbf{v}_{-i}), \mathbf{v}_{-i})) \leq SW(K(\boldsymbol{\alpha}, \mathbf{v}), \mathbf{v})$ for every $i \in K(\boldsymbol{\alpha}, \mathbf{v})$. ■

Corollary 1 *With pure public good the VCG mechanism runs an expected budget deficit.*

Proof. Indeed, if at $\mathbf{v}$ the pure public good is not produced, the budget is exactly balanced. If at $\mathbf{v}$ the good is produced and there is a pivotal consumer, by Lemma 2 the VCG mechanism runs a deficit. If at $\mathbf{v}$ the good is produced and there is no pivotal consumer, then the total payment at $\mathbf{v}$ is 0 and the budget runs the deficit $-C$. ■

Corollary 1 together with Theorem 1 proves the Rob (1989) and Mailath and Postlewaite (1990) impossibility result for pure public goods.

With congested public goods every consumer, who is given an access the to the good, exerts an externality on the rest of the society, and every such consumer pays a positive amount. Under congestion, a consumer obviously pays when she is pivotal, but by Lemma 2 in these realizations the VCG mechanism runs a deficit. At the realizations, where no consumer is pivotal, the mechanism may run a deficit, but it may also run a surplus. When the latter realizations "contribute enough" to the budget, the VCG mechanism runs an expected budget surplus.

3.1 Congested versus Pure Public Goods

In this subsection we demonstrate that VCG mechanism raises higher surpluses in the economy with congested good as compared to the economy with pure public goods. To concentrate on the role of congestion, we maintain an assumption that the cost C is the same for the congested and for the pure public good. Recall that $\boldsymbol{\rho}$ denotes the congestion schedule for the pure public good.

Proposition 1 *Suppose $K(\boldsymbol{\alpha}, \mathbf{v}) \neq \varnothing$ for the congested good with congestion schedule $\boldsymbol{\alpha}$. Then the budget surplus in the VCG mechanism under congested good exceeds the corresponding surplus under the pure public good.*

Proof. Clearly if it is efficient to produce the good under some congestion, it is also efficient to produce the good as pure public good. Thus, $K(\boldsymbol{\rho}, \mathbf{v}) \neq \varnothing$, and we just need to show that

$$\sum t_i(\boldsymbol{\alpha}, \mathbf{v}) \geq \sum t_i(\boldsymbol{\rho}, \mathbf{v}).$$

Choose an arbitrary consumer $i \in K(\boldsymbol{\alpha}, \mathbf{v})$. Obviously $i \in K(\boldsymbol{\rho}, \mathbf{v})$. There are two possibilities – either consumer i is pivotal under $\boldsymbol{\rho}$ or she is not pivotal.

Suppose at $\mathbf{v}$, i is pivotal under $\boldsymbol{\rho}$. Obviously, if $i \in K(\boldsymbol{\alpha}, \mathbf{v})$ is pivotal under $\boldsymbol{\rho}$, she is pivotal under $\boldsymbol{\alpha}$ as well. Then,

$$t_i(\boldsymbol{\alpha}, \mathbf{v}) = C - SW_{-i}(K(\boldsymbol{\alpha}, \mathbf{v}), \mathbf{v}) \geq C - SW_{-i}(K(\boldsymbol{\rho}, \mathbf{v}), \mathbf{v}) = t_i(\boldsymbol{\rho}, \mathbf{v})$$

as

$$SW_{-i}(K(\boldsymbol{\rho}, \mathbf{v}), \mathbf{v}) \geq SW_{-i}(K(\boldsymbol{\alpha}, \mathbf{v}), \mathbf{v}).$$

Therefore, $t_i(\boldsymbol{\alpha}, \mathbf{v}) \geq t_i(\boldsymbol{\rho}, \mathbf{v})$.

Suppose i is not pivotal under $\boldsymbol{\rho}$. Then, $t_i(\boldsymbol{\rho}, \mathbf{v}) = 0$, and

$$t_i(\boldsymbol{\alpha}, \mathbf{v}) = SW_{-i}(K(\boldsymbol{\alpha}, \mathbf{v}_{-i}), \mathbf{v}_{-i}) - SW_{-i}(K(\boldsymbol{\alpha}, \mathbf{v}), \mathbf{v}) \geq 0 = t_i(\boldsymbol{\rho}, \mathbf{v}),$$

where the inequality follows from efficiency of the mechanism at the profile $(0, \mathbf{v}_{-i})$ under the schedule $\boldsymbol{\alpha}$.

Thus $t_i(\boldsymbol{\alpha}, v) \geq t_i(\boldsymbol{\rho}, v)$ for any $i \in K(\boldsymbol{\alpha}, v)$.

Notice finally that any $i \in N \backslash K(\boldsymbol{\alpha}, \mathbf{v})$ cannot be pivotal under $\boldsymbol{\rho}$ by the supposition that $K(\boldsymbol{\alpha}, \mathbf{v}) \neq \varnothing$ and hence $t_i(\boldsymbol{\alpha}, \mathbf{v}) - t_i(\boldsymbol{\rho}, \mathbf{v}) = 0$ for all such i. ■

Remark 1 *In the case $K(\boldsymbol{\alpha}, \mathbf{v}) = \varnothing$ the good is not produced under $\boldsymbol{\alpha}$, while under $\boldsymbol{\rho}$ by Corollary 1 the budget cannot be in the surplus.*

Thus we know that any congestion is "better" than pure public good in terms of the budget surplus. Proposition 1, however, does not answer the question of whether the VCG mechanism in the economy with congestion runs a surplus or a deficit. To answer this type of question, we have to get some measure on "how much" an economy with congested good improves (in terms of the surplus) relatively to the economy with pure public good.

4 Budget Deficit in Two-Agent Economy

The previous section compares congested goods with the uncongested in terms of the budget surplus in the VCG mechanism. This section compares the goods of different "degree of congestion" with each other.

Definition 9 *Congestion schedule* $\boldsymbol{\alpha}$ *is* more congested *than congestion schedule* $\boldsymbol{\beta}$*, further denoted as* $\boldsymbol{\alpha}\mathbb{C}\boldsymbol{\beta}$*, if*

$$\alpha_k \leq \beta_k \text{ for every } k.$$

The definition is given for the economy of an arbitrary size. We proceed with the analysis of the two-agent economy. Suppose the economy consists of two consumers, whose values are v_1 and v_2. Denote the congestion schedule $\boldsymbol{\beta}$ for this economy with $(1, b)$. $K(\boldsymbol{\beta}, \mathbf{v}) = \varnothing$ if $\max\{v_1, b(v_1 + v_2)\} < C$. $K(\boldsymbol{\beta}, \mathbf{v}) = \{1, 2\}$, that is the good is produced and both consumers are given an access, if $b(v_1 + v_2) \geq \max\{v_1, C\}$. The payments in the generalized VCG mechanism in this case are determined as

$$t_i(\boldsymbol{\beta}, \mathbf{v}) = \begin{cases} (1-b)\, v_{-i} & \text{if } v_{-i} \geq C \\ C - bv_{-i} & \text{if } v_{-i} < C \end{cases}.$$

Similarly, $K(\boldsymbol{\beta}, \mathbf{v}) = \{1\}$, that is the good is produced and consumer 1 has an exclusive rights to use it, if $v_1 > b(v_1 + v_2)$ and $v_1 \geq C$. The payments in this case

$$\begin{aligned} t_1(\boldsymbol{\beta}, \mathbf{v}) &= \begin{cases} v_2 & \text{if } v_2 \geq C \\ C & \text{if } v_2 < C \end{cases} \\ t_2(\boldsymbol{\beta}, \mathbf{v}) &= 0. \end{aligned}$$

Observe, that the economy at least breaks even when the good is not produced and when it is produced for consumer 1 only. When both consumers are given an access, by Lemma 2, the economy is in the deficit if $v_2 < C$. (Indeed, consumer 1 is pivotal then.) In addition, the economy is in the deficit when $v_2 \geq C$ and $(1-b)(v_1+v_2) < C$.

Now consider another congestion schedule $\boldsymbol{\alpha} = (1, a)$ with $a < b$, that is $\boldsymbol{\alpha}\mathbb{C}\boldsymbol{\beta}$. In the realizations $\mathbf{v}$ such that $K(\boldsymbol{\alpha}, \mathbf{v}) = \varnothing$, but $K(\boldsymbol{\beta}, \mathbf{v}) = \{1, 2\}$ the economy breaks even under the schedule $\boldsymbol{\alpha}$, but runs a deficit under $\boldsymbol{\beta}$. (This realization "transits" from region **ii** into region **i** in Figure 1.) Observe further, that in the realizations $\mathbf{v}$ such that $K(\boldsymbol{\beta}, \mathbf{v}) = \{1, 2\}$, but $K(\boldsymbol{\alpha}, \mathbf{v}) = \{1\}$ with $v_2 < C$, the economy breaks even under $\boldsymbol{\alpha}$, but runs a deficit under $\boldsymbol{\beta}$. This realization transits from region **ii** into region **iii** in the Figure.

In the realization $\mathbf{v}$ such that $K(\boldsymbol{\alpha}, \mathbf{v}) = K(\boldsymbol{\beta}, \mathbf{v}) = \{1, 2\}$ and $(1-a)(v_1+v_2) \geq C$ but $(1-b)(v_1+v_2) < C$, the economy runs a surplus under $\boldsymbol{\alpha}$, and a deficit under $\boldsymbol{\beta}$. This realization transits from region **ii** into region **iv** in the Figure. Observe now that in the realization $\mathbf{v}$ such that $K(\boldsymbol{\beta}, \mathbf{v}) = \{1, 2\}$, but $K(\boldsymbol{\alpha}, \mathbf{v}) = \{1\}$ with $v_2 \geq C$, the budget surplus under $\boldsymbol{\alpha}$ is higher than the one under $\boldsymbol{\beta}$. Indeed, $t_1(\boldsymbol{\beta}, \mathbf{v}) + t_2(\boldsymbol{\beta}, \mathbf{v}) = (1-b)(v_1+v_2) < v_2$, where the inequality follows from the fact that $b(v_1 + v_2) > v_1$. This realization transits from the region **iv** into the region **v** in the Figure. Observe finally, that every payment is decreasing with b.

To sum up, in every realization the budget surplus under $\boldsymbol{\alpha}$ exceeds the one under $\boldsymbol{\beta}$. Thus we have shown that, in the economy with two consumers, for any two congestion schedules $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, such that $\boldsymbol{\alpha}\mathbb{C}\boldsymbol{\beta}$, the ex-post budget surplus under $\boldsymbol{\alpha}$ is higher

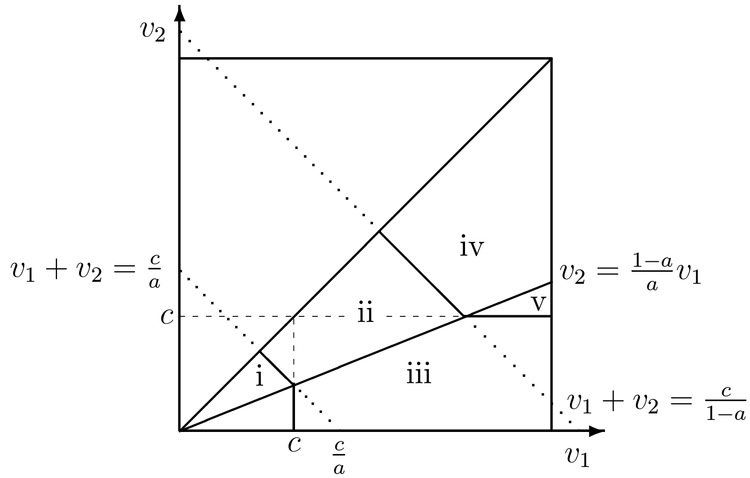

Figure 1: Budget deficit in two-agent economy

than the one under $\boldsymbol{\beta}$. To put it more extreme, the budget surplus under the pure *private* good regime is higher than the budget surplus under any "congested" public good regime. Therefore, we have established a new, quite pessimistic impossibility result.

Conjecture 1 *Suppose the cost C is such that the good is inefficient to produce in case it will be consumed as a pure private good. Then the generalized VCG mechanism runs budget deficit under any congestion.*

We have proven this Conjecture for the economy with two consumers. The following sections demonstrate that the two agent economy is, in fact, a non-generic case and Conjecture 1 is likely to be incorrect if $n > 2$.

5 Step-Like Congestion Schedules

In this section we consider the schedules that consist of a number of 1's followed by a number of 0's. Such step-like schedules are of some interest on their own. A roller coaster ride is an example of an excludable public good that can accommodate at most m consumers, (m is the number of seats). First m customers create no congestion to each other, but with the $m+1$'st customer included the ride simply cannot start, everybody has to be seated and tightly buckled up. Thus the $m+1$'st consumer creates extreme congestion on the first m. Any public good with "fixed capacity" is associated with step-like congestion schedule.

The existence proof for step-like schedules is more transparent than the one for the schedules of the general form. The reason is that with step-like schedules the number of the consumers, who are given an access to the good, is effectively exogenous. In the above example it is efficient to provide the good either for the m consumers with the highest values or for nobody. Denote the schedule $1, ..., 1_{m-1}, 1_m, 0_{m+1}, ..., 0_n$ with

m, this should create no confusion. The cost of the good, $C(m)$, is assumed to be increasing with m. We derive a *lower bound* on the expected budget surplus under the schedule m. If the lower bound is positive, the VCG mechanism runs an expected budget surplus and we have the desired existence result. In what follows $EBS(m)$ denotes the expected budget surplus in the VCG mechanism under congestion schedule m.

Lemma 3 *For $1 \leq m \leq n-1$ suppose $K(m, \mathbf{v}) \neq \varnothing$. Then,*

$$\sum t_i(m, \mathbf{v}) \geq m v_{m+1}. \tag{2}$$

and the lower bound is strict.

Proof. [15] Pick an arbitrary $i \in K(m, \mathbf{v})$.

Suppose i is not pivotal under m. By efficiency, $\#K(m, \mathbf{v}_{-i}) = \#K(m, \mathbf{v}) = m$. Then,

$$t_i(m, \mathbf{v}) = \sum_{j \in K(m, \mathbf{v}_{-i})} v_j - \sum_{i \neq j \in K(m, \mathbf{v})} v_j = \sum\nolimits_{i \neq j=1}^{m+1} v_j - \sum\nolimits_{i \neq j=1}^{m} v_j = v_{m+1}.$$

Suppose i is pivotal under m. Then $t_i(m, \mathbf{v}) = C(m) - \sum_{i \neq j=1}^{m} v_j$. Suppose that $C(m) - \sum_{i \neq j=1}^{m} v_j < v_{m+1}$. Then $K(m, \mathbf{v}_{-i}) \neq \varnothing$ which contradicts the supposition that i is pivotal. Thus, $t_i(m, \mathbf{v}) \geq v_{m+1}$.

For the consumer i that is not given an access to the good, $t_i(m, v) = 0$. ■

Thus, the total revenue at the realizations, where the good is efficient to produce, is bounded below by $m v_{m+1}$ hence the budget surplus is bounded below by $m v_{m+1} - C$. At the realizations $\mathbf{v}$, such that $K(m, \mathbf{v}) = \varnothing$, the good is not produced and the budget is exactly balanced. Therefore, the lower bound on the expected budget surplus in the VCG mechanism under the schedule m,

$$EBS_F(m) \geq E_F(\{m v_{m+1} - C\} \cdot I\{v_1 + ... + v_m \geq C\}), \tag{3}$$

where $I\{v_1 + ... + v_m \geq C\}$ is an indicator function which takes a value of 1 when $v_1 + ... + v_m \geq C$, and 0 otherwise.

Now we use the derived lower bound to show the existence of the economies where the congested public good is financed via budget balanced mechanisms. We proceed in a sequence of Lemmas.

Lemma 4 *Suppose distribution F stochastically dominates the uniform distribution. Then the VCG mechanism runs an expected budget surplus in the economy with n consumers, under congestion schedule m, if $\frac{m(n-m)}{n+1} \geq C$.*

[15] In the proofs in this section we use explicit notations for the relevant social welfares.

Proof. We start with the uniform distribution, U. We have to show that (3) is non-negative.

$$EBS_U(m) \geq E_U\left(mv_{m+1} \cdot I\left\{v_1 + ... + v_m \geq C\right\}\right) - C \cdot E_U\left(I\left\{v_1 + ... + v_m \geq C\right\}\right)$$

It is easy to see that $E_U\left(mv_{m+1} \cdot I\left\{v_1 + ... + v_m \geq C\right\}\right)$ is equivalent to

$$\frac{m(n-m)}{n+1} J_{n+1}(C,m), \text{ where } J_n(C,m) = E_U\left(I\left\{v_1 + ... + v_m \geq C\right\}\right).$$

The joint distribution of n order statistics of n independent draws from distribution F can be written as $n! \cdot \prod_{i=1}^{n} f(v_i)$, where f is the density of F, see e.g. Arnold et.al. (1992). For the uniform "parent" distribution, the joint distribution of the n order statistics is just $n!$. Consider two integrals, one that corresponds to $E_U\left(v_{m+1} \cdot I\left\{v_1 + ... + v_m \geq C\right\}\right)$ and the other – to $E_U\left(I\left\{v_1 + ... + v_m \geq C\right\}\right)$. For both integrals $\int_0^{v_{m+1}} ... \int_0^{v_{n-1}} dv_n...dv_{m+2} = \frac{1}{(n-m-1)!}\left(v_{m+1}\right)^{n-m-1}$. Observe further that $\int_0^{v_m} v_{m+1}^{n-m} dv_{m+1} = \frac{1}{n-m+1}\left(v_m\right)^{n-m+1}$ and $\int_0^{v_m} v_{m+1}^{n-m-1} dv_{m+1} = \frac{1}{n-m}\left(v_m\right)^{n-m}$. Thus, the presence of v_{m+1} in the first integral just adds 1 to the power at the $n-m$'th step of integration, which increases the power in the final expression by 1 and in combination with $n!$ produces the coefficient $\frac{n-m}{n+1}$.

Now, we argue for the distribution that stochastically dominates the uniform. Random variable v' *stochastically dominates* random variable v, denoted $v' \geq_{st} v$, if

$$\Pr\left\{v > u\right\} \leq \Pr\left\{v' > u\right\} \text{ for every } u \in [0,1].$$

Equivalently, distribution F on $[0,1]$ stochastically dominates distribution G on $[0,1]$, $F \geq_{st} G$ if $F(v) \leq G(v)$ for every $v \in [0,1]$. Finally, $F \geq_{st} U$ if $F(v) \leq v$ for every $v \in [0,1]$. Then by stochastic dominance

$$E\left(mv_{m+1} \cdot I\left\{v_1 + ... + v_m \geq C\right\}\right) \geq E\left(mF\left(v_{m+1}\right) \cdot I\left\{v_1 + ... + v_m \geq C\right\}\right),$$

for any distribution $F \geq_{st} U$.

Similarly, to how this was done for the uniform distribution observe that $\int_0^{v_{m+1}} ... \int_0^{v_{n-1}} f(v_{m+2})...f(v_n)dv_n...dv_{m+2} = \frac{1}{(n-m-1)!}F^{n-m-1}(v_{m+1})$. Further observe that $\int_0^{v_m} F^{n-m}\left(v_{m+1}\right) dF\left(v_{m+1}\right) = \frac{1}{n-m+1}F^{n-m+1}\left(v_m\right)$ and $\int_0^{v_m} F^{n-m-1}\left(v_{m+1}\right) dF\left(v_{m+1}\right) = \frac{1}{n-m}F^{n-m}\left(v_m\right)$. Again, the presence of $F\left(v_{m+1}\right)$ in the first integral just adds 1 to the power at the $n-m$'th step of integration, which increases the power in the final expression by 1 and in combination with $n!$ produces the coefficient $\frac{n-m}{n+1}$.

Thus, for any $F \geq_{st} U$,

$$\begin{aligned} E_F\left(mv_{m+1} \cdot I\left\{v_1+...+v_m \geq C\right\}\right) &\geq \frac{m(n-m)}{n+1} J_{n+1}(C,m), \\ \text{where } J_n(C,m) &= E_F\left(I\left\{v_1+...+v_m \geq C\right\}\right). \end{aligned} \tag{4}$$

And

$$EBS_F\left(m\right) \geq \frac{m(n-m)}{n+1} J_{n+1}(C,m) - C \cdot J_n(C,m).$$

We have to establish a few properties of $J_n(C,m)$. Firstly, $J_n(C,m) \geq 0$ and $J_n(C,m) > 0$ for any $C < m$. Secondly, for a fixed m, $J_n(C,m)$ is increasing in n. To show this we need to extend the notion of stochastic dominance to random vectors. Notation $\mathbf{v}' \geqq \mathbf{v}$ implies $v_i' \geq v_i$ for every $i = 1,2,...,n$. A set $U \subseteq [0,1]^n$ is an *upper* set if $\mathbf{v}' \in U$, whenever $\mathbf{v}' \geqq \mathbf{v}$ and $\mathbf{v} \in U$. A function $\phi(\mathbf{v})$ is *increasing* if $\phi(\mathbf{v}) \leq \phi(\mathbf{v}')$ for every $\mathbf{v}$ and $\mathbf{v}' \geqq \mathbf{v}$. Let $\mathbf{v}$ and $\mathbf{v}'$ be two random vectors such that

$$\Pr\left\{\mathbf{v} \in U\right\} \leq \Pr\left\{\mathbf{v}' \in U\right\}, \text{ for all upper sets } U \subseteq [0,1]^n.$$

Then, $\mathbf{v}'$ *stochastically dominates* $\mathbf{v}$, denoted $\mathbf{v}' \geq_{st} \mathbf{v}$. It is known that $\mathbf{v}' \geq_{st} \mathbf{v}$ if and only if

$$E\left(\phi(\mathbf{v}')\right) \geq E\left(\phi(\mathbf{v})\right)$$

for every increasing function $\phi(\mathbf{v})$, such that an expectation exists (see e.g. Shaked and Shanthikumar (1994), pp. 113-114).

Suppose we make n independent draws from the uniform distribution. These form random vector $\mathbf{v}_n = (v_1, v_2, ..., v_n)$ (v's are not ordered yet and hence independent here). Order these and collect the m highest statistics, these form random vector $\mathbf{v}_n^m$. Next make $n+1$ independent draws from the same distribution and form random vectors $\mathbf{v}_{n+1} = (v_1, v_2, ..., v_{n+1})$ and $\mathbf{v}_{n+1}^m$. Clearly, $\mathbf{v}_{n+1}^m \geqq \mathbf{v}_n^m$ and therefore $\mathbf{v}_{n+1}^m \geq_{st} \mathbf{v}_n^m$.

Obviously, $\phi = I\left\{v_1+...+v_m \geq C\right\}$ is an increasing function, hence $E\left(\phi\left(\mathbf{v}_{n+1}^m\right)\right) \geq E\left(\phi\left(\mathbf{v}_n^m\right)\right)$. Therefore, for a fixed m, $J_n(C,m)$ is increasing in n.

Thus, $J_{n+1}(C,m) \geq J_n(C,m)$, which implies that

$$EBS_F\left(m\right) \geq \left(\frac{m(n-m)}{n+1} - C\right) \cdot J_n(C,m). \tag{5}$$

We have argued that $J_n(C,m) \geq 0$. Hence $EBS_F\left(m\right) \geq 0$ if $\frac{m(n-m)}{n+1} - C \geq 0$. Moreover, suppose $C < \frac{m(n-m)}{n+1}$, $C < m$. Then $J_n(C,m) > 0$ and $EBS_F\left(m\right) > 0$. ■

Introduce the per capita cost of the good, $\frac{C}{m}$ and the share of the population, who is given an access to the good, $\frac{m}{n}$. Then $\frac{m(n-m)}{n+1} \geq C$ for any n is approximately equivalent to $1 - \frac{m}{n} \geq \frac{C}{m}$. The latter inequality is quite illustrative. In particular, it implies that when the good is "almost private," $\frac{m}{n}$ is close to zero, the VCG mechanism runs a surplus no matter what the cost is. When, the good is "almost public," $\frac{m}{n}$ close to one, only very small per capita cost can be covered. If $n, m \to \infty$, for the VCG to run a surplus, the per capita cost should go to zero, which is coherent with the findings of Rob (1989), Mailath and Postlewaite (1990) and Hellwig (2003).

In the case of the *pure* public good, Rob (1989) and Mailath and Postlewaite (1990) show that when the cost of the good grows proportionally to the size of the economy, the probability of the efficient allocation via budget balanced mechanism goes to zero. Hellwig (2003) shows that if the cost stays bounded, while the economy grows, an allocation arbitrary close to the efficient one can be supported when the number of consumers is sufficiently large. For the case of *congested* public goods we show that the cost need not be bounded, and may increase with the size of the economy. Equivalently, for the efficient mechanism to balance the budget, the per capita cost need not go to zero. When a fixed proportion of the population is given an access to the good, $1 - \frac{m}{n} \geq \frac{C}{m}$ offers a transparent upper bound on the per capita cost, that can be covered by the VCG mechanism. To reiterate, the VCG mechanism runs expected budget surpluses even when the economy grows, and the number of the consumers, who are given an access to the good and the total cost of the project grow proportionally to the size of the economy.

Lemma 4 implies that the VCG mechanism runs a surplus in the economy with $n = 3$ consumers, when the congested good is allocated to only two of them, if the cost is no higher than $\frac{1}{2}$. Our characterization of the "admissible" cost is derived from the lower bound on the expected surplus, therefore for C such that $\frac{1}{2} < C < 2$, the VCG mechanism may still run an expected surplus. Numerical simulations suggest that for $n = 3$ and $m = 2$ the VCG runs a surplus if the cost is no higher than 0.58.

Our characterization is tight, though, when $m \to \infty$. Indeed, when m and hence n are large, there are many consumers with virtually identical values, therefore almost none of them is pivotal. Hellwig (2003) estimates that the probability that a given consumer is pivotal decreases with n at the rate $1/\sqrt{n}$. Thus, asymptotically we can think of the situation as if the good is provided for m consumes for the payment of v_{m+1}. Then, in the limit, with $n, m \to \infty$ the VCG runs a surplus if $C \leq \frac{m(n-m)}{n+1}$ and runs a deficit if $C > \frac{m(n-m)}{n+1}$.

Corollary 2 *For the uniform distribution, for any m, for any C, there exists N such that for any $n \geq N$ the VCG mechanism runs an expected budget surplus in the economy with n consumers, under congestion schedule m.*

Proof. If $C > m$, the VCG mechanism trivially runs a surplus, since the good is never efficient to produce. Further we consider $C \leq m$. Inequality (4) in the proof of Lemma 4 implies that the expected *revenue* in the VCG mechanism under congestion schedule m is bounded below by $\frac{m(n-m)}{n+1} E\left(I\left\{v_1 + ... + v_m \geq C\right\}\right)$. Observe that,

$$\lim_{n \to \infty} \frac{m(n-m)}{n+1} E\left(I\left\{v_1 + ... + v_m \geq C\right\}\right) = m.$$

Therefore, for a fixed m, for any $C \leq m$ there exists N such that for any $n \geq N$, $\frac{m(n-m)}{n+1} E\left(I\left\{v_1 + ... + v_m \geq C\right\}\right) \geq C$ is satisfied. Moreover, if $C < m$, $EBS_U\left(m\right) > 0$. ■

Further, we extend the result to other distributions on $[0, 1]$. We first deal with the distributions of the sort $F(v) = v^a$ with $0 < a < 1$. It is possible to compute the expectation (3) and obtain closed form solution, but this solution is difficult to analyze. I am unable to obtain a characterization similar to the one in Lemma 4, that is one can increase n and m at particular rates and maintain expected surpluses in the VCG mechanism "along the way". Corollary 2 easily extends to any distribution $F(v) = v^a$.

Lemma 5 *For distribution $F(v) = v^a$ with $a > 0$, for any m, for any C, there exists N such that for every $n \geq N$, the VCG mechanism runs an expected budget surplus in the economy with n consumers, under congestion schedule m.*

Proof. We use the result Corollary 2 for the uniform distribution to obtain similar result for distribution v^a.

Consider the following thought experiment. Suppose we have two distributions, one is the uniform on $[0, 1]$ and the other is $F = v^a$ with $a > 0$ (we are particularly interested in small a's). Suppose we make n independent draws from the uniform distribution, order the draws and collect $m+1$ highest statistics. These form random vector $\left(v_1, v_2, ..., v_{m+1}\right)_U$. Next we make n' independent draws from distribution $F = v^a$, order the draws and collect $m+1$ highest statistics. These form random vector $\left(v_1, v_2, ..., v_{m+1}\right)_F$. We further show

Claim 1. *For any $a > 0$, for any m, for any $n \geq m$ there exists N, possibly $N = N(a, m, n)$ such that for any $n' \geq N$,*

$$\left(v_1, v_2, ..., v_{m+1}\right)_F \geq_{st} \left(v_1, v_2, ..., v_{m+1}\right)_U.$$

Suppose we make n independent draws from the uniform distribution. These form random vector $\mathbf{v}_U = \left(v_1, v_2, ..., v_n\right)_U$ (v's are not ordered yet and hence independent here). Next make $k \times n$ independent draws from the distribution F and form random vector $\mathbf{v}_F = \left(\left(v_1, v_2, ..., v_k\right)_1, \left(v_1, v_2, ..., v_k\right)_2, ..., \left(v_1, v_2, ..., v_k\right)_n\right)_F$. Clearly, $\left(v_1, v_2, ..., v_k\right)_i$ is independent of $\left(v_1, v_2, ..., v_k\right)_j$ for any $i \neq j$. Now consider $\left(v_1, v_2, ..., v_k\right)_i$ and v_i, the i-th bracket in vector $\mathbf{v}_F$ and the i-th component in vector $\mathbf{v}_U$. The distribution of the highest element in $\left(v_1, v_2, ..., v_k\right)_i$ is $F^k = \left(v^a\right)^k$. The distribution of v_i is v. For any $a > 0$ there exists k, for example $k = \left[\frac{1}{a}\right] + 1$, such that $\left(v^a\right)^k \leq v$ for every $v \in [0, 1]$, and this inequality holds for all higher k. Now consider vector $\mathbf{v}_F^1$ such that $\left\{\mathbf{v}_F^1\right\}_i = \max\left(v_1, v_2, ..., v_k\right)_i$. Components of $\mathbf{v}_F^1$ are independent. Thus, we have argued that $\mathbf{v}_F^1 \geq_{st} \mathbf{v}_U$.

By Theorem 4.B.10 in Shaked and Shanthikumar (1994) if $\mathbf{v}_F^1 \geq_{st} \mathbf{v}_U$ and $\boldsymbol{\gamma} : \mathbb{R}^n \rightarrow \mathbb{R}^l$ is any l-dimensional increasing function (vector function is increasing iff every its component is increasing), then the l-dimensional vectors $\boldsymbol{\gamma}(\mathbf{v}_F^1)$ and $\boldsymbol{\gamma}(\mathbf{v}_U)$ satisfy $\boldsymbol{\gamma}(\mathbf{v}_F^1) \geq_{st} \boldsymbol{\gamma}(\mathbf{v}_U)$. Consider function $\boldsymbol{\gamma}$ that maps vectors of independent random variables into vectors of $m+1$ highest order statistics. Obviously, $\boldsymbol{\gamma}$ is an increasing function. Therefore, Claim 1 is established.[16]

Consider function $\phi \equiv mv_{m+1} \cdot I\{v_1 + ... + v_m \geq C\}$. It is easy to verify that ϕ is an increasing function. Claim 1 then implies that for every m and every $n \geq m$, for any $a > 0$,

$$E_F\left(mv_{m+1} \cdot I\{v_1 + ... + v_m \geq C\}\right) \geq E_U\left(mv_{m+1} \cdot I\{v_1 + ... + v_m \geq C\}\right),$$

if we make $n' \geq n \cdot \left(\left[\frac{1}{a}\right] + 1\right)$ draws from distribution $F = v^a$ and n draws from the uniform. Since for any m and any C, for large enough n, the VCG mechanism runs a surplus in the economy with n consumers, the schedule m and the uniform distribution, we can find n' large enough, so that the VCG mechanism runs a surplus in the economy with n' consumers, the schedule m and distribution $F = v^a$. ■

Take some distribution F on $[0,1]$. If one can find $a > 0$ such that F stochastically dominates distribution v^a, the existence result of Lemma 5 extends to the distribution F. It is possible, however, to come up with an example of a distribution with a smooth c.d.f. and $f(v) > 0$, which is not stochastically dominated by the distribution v^a for any $a > 0$. Then consider an interval $[\varepsilon, 1-\varepsilon]$ for a small $\varepsilon > 0$. Clearly, $0 < F(\varepsilon) \leq F(v) \leq F(1-\varepsilon) < 1$ for any $v \in [\varepsilon, 1-\varepsilon]$. Therefore, for any $\varepsilon > 0$ there exists $a > 0$ such that $F(v) \leq v^a$ for any $v \in [\varepsilon, 1-\varepsilon]$. If $C > m$, the VCG trivially runs a surplus. If $C = m$, it is easy to see that the budget of the VCG mechanism is exactly balanced. Suppose $C < m$. Then by Corollary 2, $EBS_U(m) > 0$. By continuity of the expected revenue in the VCG mechanism and stochastic dominance we can extend the existence result of Lemma 5 to any smooth distribution with positive density.

This limit result is less satisfactory than the characterization of Lemma 4. Indeed. one may say that to get the surpluses in the VCG we have to increase the population in the economy to infinity, restricting the good for almost private consumption. This is clearly the consequence of the generality of the result. If the distribution is very skewed towards zero, the value that each and every consumer derives from the good is very low and we need a continuum of potential consumers to cover the cost of the project. When some additional information about the distribution is available, our results are more specific. Lemma 5 implies, in particular, the following. Suppose with the uniform distribution the VCG mechanism runs a surplus under congestion schedule m and cost C in the economy with n potential consumers. If some estimates imply that the actual distribution is close to, say, the square root distribution, then, other things equal, the VCG runs a surplus in the economy with $2n$ potential consumers.

[16] Strictly speaking for distribution F, $\boldsymbol{\gamma}$ maps from the big $k \times n$ vector $\mathbf{v}_F$, not from the vector $\mathbf{v}_F^1$. But, obviously the vector $\boldsymbol{\gamma}(\mathbf{v}_F) \geqq \boldsymbol{\gamma}\left(\mathbf{v}_F^1\right)$ and therefore $\boldsymbol{\gamma}(\mathbf{v}_F) \geq_{st} \boldsymbol{\gamma}\left(\mathbf{v}_F^1\right) \geq_{st} \boldsymbol{\gamma}(\mathbf{v}_U)$.

If F stochastically dominates the uniform distribution, the tighter characterization of Lemma 4 extends to distribution F. For given m and C, n that guarantees the surpluses with the uniform distribution also suffices for any F that stochastically dominates the uniform. We summarize the findings of this Section in,

Proposition 2 *For any F, for any m, for any C, there exists N, such that for every $n \geq N$ the VCG mechanism runs an expected budget surplus in the economy with n consumers, under congestion schedule m. If F stochastically dominates the uniform distribution, the VCG mechanism runs an expected budget surplus in the economy with n consumers, under congestion schedule m, if $\frac{m(n-m)}{n+1} \geq C$.*

Suppose the VCG mechanism runs an expected surplus for a given cost and congestion. If we increase the population of the economy, and keep the cost and the congestion fixed, by Lemma 5 the VCG mechanism still runs a surplus. Then one could conjecture the following monotonicity result. Suppose the VCG runs a surplus for some congested good, then the VCG also runs a surplus for any good which is more congested. The conjecture is quite subtle, though. One can model this situation in a more straightforward way – increase the congestion (decrease m) maintaining n and C fixed. Numerical simulations suggest that monotonicity does not hold here. One can run surpluses with less congested good and deficits with more congested good.

Proposition 2 suggests a key to the puzzle. In both exercises the cost of the project is assumed to be fixed. In the former, though, the cost is shared among the fixed number of consumes and the increase of the population does not directly increase anybody's cost share. In the latter exercise the fixed cost is shared among fewer and fewer consumers. Intuitively, it provides extra incentives to "shade" the valuation, as soon as the consumer, who is given an access to the good, expects to pay the "higher price".

It is instructive to compare the generalized VCG mechanism in our setting with the mechanisms proposed in the literature for provision of excludable, non-rival public goods

mechanism	this paper[17]	Moulin (1994) serial	Deb and Razzolini (1999b)
produce	iff $\sum_{i=1}^{m} v_i \geq C$	for $\max m$ s.t. $v_m \geq \frac{C}{m}$	for $\max m$ s.t. $v_m \geq \frac{C}{m}$
include	$1, 2, ..., m$	$1, 2, ..., m$	$1, 2, ..., m$
each pays	v_{m+1}	$\frac{C}{m}$	$\max\{v_{m+1}, \frac{C}{m}\}$

Both Moulin's serial mechanism and Deb and Razzolini's auction-like mechanism balance the budget by construction, satisfy the IR, but both are inefficient. Therefore, these mechanisms are not helpful in the quest for an efficient, BB and IR mechanism.

[17] This column of the Table represents our lower bound estimates on the payments in the VCG. For the sake of comparison with other mechanisms assume that we describe the fictitious mechanism that provides the good for the consumers with m highest values if their sum is above C and asks for the payment of v_{m+1} from each of the m.

6 Congestion Schedules of the General Form

Observe that, whenever at a given realization $\mathbf{v}$ the good is produced under the schedule $\boldsymbol{\alpha}$, it is also produced under any schedule $\boldsymbol{\beta}$ such that $\boldsymbol{\alpha}\mathbb{C}\boldsymbol{\beta}$. In general, it is not true, though, that $K(\boldsymbol{\alpha},\mathbf{v}) \subseteq K(\boldsymbol{\beta},\mathbf{v})$.[18]

Definition 10 *Schedule $\boldsymbol{\alpha}$ is a* rotation *of the schedule $\boldsymbol{\beta}$, if $\boldsymbol{\alpha}\mathbb{C}\boldsymbol{\beta}$ and*

$$\frac{\alpha_{k+1}}{\alpha_k} \le \frac{\beta_{k+1}}{\beta_k} \text{ for every } k \text{ such that } \alpha_k > 0.$$

Lemma 6 *Suppose $\boldsymbol{\alpha}$ is a rotation of $\boldsymbol{\beta}$. Then at every $\mathbf{v} \in \mathbf{V}$,*

$$K(\boldsymbol{\alpha},\mathbf{v}) \subseteq K(\boldsymbol{\beta},\mathbf{v}).$$

Proof. If $K(\boldsymbol{\beta},\mathbf{v}) = \varnothing$, then $K(\boldsymbol{\alpha},\mathbf{v}) = \varnothing$ and the inclusion follows. If $K(\boldsymbol{\alpha},\mathbf{v}) = \varnothing$, but $K(\boldsymbol{\beta},\mathbf{v}) \neq \varnothing$, the Lemma is again obvious. Suppose $K(\boldsymbol{\alpha},\mathbf{v}) \neq \varnothing$, then $K(\boldsymbol{\beta},\mathbf{v}) \neq \varnothing$. Now suppose the Lemma is violated at some $\mathbf{v}$. This implies that there exist m and $m' > m$ such that,

$$\begin{aligned} \beta_m \sum_{i=1}^{m} v_i &\ge \beta_k \sum_{i=1}^{k} v_i, \text{ for every } k, \text{ and} \\ \alpha_{m'} \sum_{i=1}^{m'} v_i &\ge \alpha_k \sum_{i=1}^{k} v_i, \text{ for every } k. \end{aligned}$$

In particular,

$$\begin{aligned} \beta_m \sum_{i=1}^{m} v_i &> \beta_{m'} \sum_{i=1}^{m'} v_i \text{ and} \\ \alpha_{m'} \sum_{i=1}^{m'} v_i &\ge \alpha_m \sum_{i=1}^{m} v_i. \end{aligned}$$

The strict inequality follows from the fact that the allocation rule of the generalized VCG "chooses" the largest set of consumers, whenever indifferent between a few sets. The above inequalities imply

$$\frac{\beta_m}{\alpha_m} > \frac{\beta_{m'}}{\alpha_{m'}} \Leftrightarrow \frac{\alpha_{m'}}{\alpha_m} > \frac{\beta_{m'}}{\beta_m},$$

which violates the supposition that $\boldsymbol{\alpha}$ is a rotation of $\boldsymbol{\beta}$. ■

Lemma 7 *Suppose $K(\boldsymbol{\alpha},\mathbf{v}) = \{1,2,...,m\}$. Then*

$$\sum t_i(\boldsymbol{\alpha},\mathbf{v}) \ge m\alpha_m v_{m+1}.$$

[18] An assumption, called "rotation", that will be introduced in Section 6, guarantees the inclusion of these sets.

Proof. Consider a consumer $i \in K(\boldsymbol{\alpha}, \mathbf{v})$. We show that $t_i(\boldsymbol{\alpha}, \mathbf{v}) \geq \alpha_m v_{m+1}$. By definition of the payments in the VCG mechanism,

$$t_i(\boldsymbol{\alpha}, \mathbf{v}) = SW(K(\boldsymbol{\alpha}, \mathbf{v}_{-i}), \mathbf{v}_{-i}) - SW_{-i}(K(\boldsymbol{\alpha}, \mathbf{v}), \mathbf{v}),$$

where $SW_{-i}(K(\boldsymbol{\alpha}, \mathbf{v}), \mathbf{v}) = \sum_{i \neq j=1}^{m} \alpha_m v_j$. Then $SW_{-i}(K(\boldsymbol{\alpha}, \mathbf{v}), \mathbf{v}) + \alpha_m v_{m+1} = \sum_{i \neq j=1}^{m+1} \alpha_m v_j$. Thus, $SW_{-i}(K(\boldsymbol{\alpha}, \mathbf{v}), \mathbf{v}) + \alpha_m v_{m+1}$ can be written as $SW(K(\boldsymbol{\alpha}, \mathbf{v}), \mathbf{v}_{-i})$, the social welfare that the economy without i can reach if the good is still produced for the m consumers with the highest values. By efficiency of the VCG mechanism, $SW(K(\boldsymbol{\alpha}, \mathbf{v}_{-i}), \mathbf{v}_{-i}) \geq SW(K(\boldsymbol{\alpha}, \mathbf{v}), \mathbf{v}_{-i})$, hence $t_i(\boldsymbol{\alpha}, \mathbf{v}) \geq \alpha_m v_{m+1}$ for every $i \in K(\boldsymbol{\alpha}, \mathbf{v})$. ■

Lemma 8 *For any C, for any $\boldsymbol{\alpha}$, there exists N such that for any $n \geq N$, the VCG mechanism runs a surplus in the economy with n consumers, under congestion schedule $\boldsymbol{\alpha}$.*

Proof. First argue for the uniform distribution.

Conditional on the allocation being $\{1, 2, ..., m\}$ the surplus in the VCG mechanism is

$$E\left(m\alpha_m v_{m+1} \cdot I\left\{\alpha_m\left(v_1 + ... + v_m\right) \geq C\right\}\right).$$

With the uniform distribution the above can be rewritten as

$$m\alpha_m \frac{n-m}{n+1} J_{n+1}(C, m, \alpha_m), \text{ where } J_n(C, m, \alpha_m) = E\left(I\left\{\alpha_m\left(v_1 + ... + v_m\right) \geq C\right\}\right).$$

If $C \geq m\alpha_m$ the good is never produced for the set $\{1, 2, ..., m\}$ therefore the budget is balanced. Suppose $C < m\alpha_m$. As before

$$\lim_{n \to \infty} m\alpha_m \frac{n-m}{n+1} J_{n+1}(C, m, \alpha_m) = m\alpha_m.$$

Thus, there exists some N_m such that the VCG runs a surplus conditional on the fact that the allocation is $\{1, 2, ..., m\}$.

Consider all m such that the corresponding $m\alpha_m > C$. Find $N = \max N_m$. Because the relevant function is increasing we are done for any distribution that stochastically dominates the uniform and for all distibutions v^a with $0 < a < 1$ we can keep sampling as in Lemma 5. Then we complete the ardument for any distribution on $[0, 1]$ by continuity. ■

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