Long–term Equilibria of Repeated Consistently Competitive Games

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June 20, 2003

Discussion Paper
Europa–Universität Viadrina, Frankfurt (Oder)

Abstract

The class of consistently competitive games canonically unifies Prisoner's Dilemmas, contests, auctions, and Bertrand competitions. If those games are repeated infinitely, the players have to negotiate about the strategies that are to be repeated infinitely. These negotiations, however, are perturbed by the possibility that players make defective proposals (defective proposals are sensibly not maintained in the long term). The opponents' defections have to be detected and retaliated. In this study, these aspects (negotiations and defections) are analyzed jointly, and (thus) a refinement concept for Folk theorem equilibria is introduced.

JEL Classification: C73 (Stochastic and Dynamic Games), C78 (Bargaining Theory), D43 (Oligopoly), D44 (Auctions)

Keywords: repeated game, equilibrium refinement, folk theorem, negotiation, auction, Bertrand competition

*I would like to thank Friedel Bolle for numerous discussions and the Deutsche Forschungsgemeinschaft for fincancial support.

1 Introduction

A number of social interactions share two attributes: the same agents interact repeatedly, not caring about the number of remaining rounds (i.e. infinitely), and their interactions are (in some sense) competitive. Roughly, games shall be called competitive, if the best–reply dynamics converge *monotonically* in terms of competitiveness (which is defined implicitly), and if they converge to socially *decreasingly efficient* outcomes (i.e. to outcomes with decreasing aggregate payoffs). More precisely, for each strategy combination some of the players are better off deviating to more competitive strategies, which harms the respective opponents, and implies that some opponents are better off deviating to more competitive strategies in turn. Moreover, *all deviations* to more competitive strategies are socially inefficient, and the players are generally best off to reply increasingly competitive moves of their opponents with increasingly competitive moves of their own (*consistency*). Instances of these *consistently competitive games* are Prisoner's Dilemmas, rent–seeking contests, auctions, and Bertrand (price) competitions. To avoid inconveniences, note that there are other games, that have been called competitive, as exchange economies (Aumann, 1966; Shapley and Shubik, 1969) and generalizations of zero–sum games (Kats and Thisse, 1992; Ewerhart, 2003). These are not addressed in the following.

We will argue that players of infinitely repeated consistently competitive games (IRCGs) refine their moves with respect to a kind of negotiation rationality, as a result of which they seem to restrict themselves to increasingly competitive moves. The basic idea behind that appears to be rather intuitive, and, applied to repeated Prisoner's Dilemmas, was sketched out by authors as early as Rapoport (1966, Chapter 9):

> "... it seems plausible to play C [cooperation] as a way of communicating to the opposing player that one is ready to cooperate if he will. If he goes along, then a *tacit* agreement might be reached to continue to cooperate (i.e., choose C). ... Each player is prevented from defecting to D by the knowledge that the other, in order to save himself from the worst outcome associated with unilateral cooperation, will be forced to retaliate by also playing D on successive plays. But the choice (D,D) is punishing to both. It therefore seems sensible to stick to (C,C) ..."

According to that, the reply of D to D needs not be intended to retaliate the opponent for a couple of rounds, but as the profit–maximizing reply of a player, who assumes that his opponent's defection applies infinitely. An interpretation of such long–term considerations is, that the players negotiate about the strategy combination that is to be played in the long–term, i.e. that they look for the best strategy combination that is stable in the long term (as in "If I deviate from (C,C) to (D,C), hoping to realize that in the long term, I will see that the opponent replies with deviating to (D,D); that is, (D,C) is unstable, and the best thing I can do is sticking to (C,C)."). In a real play, a player might nonetheless deviate from (C,C) to (D,C), and could say afterwards: "That was mistake, sorry. Now I am going to play C." Along these lines, we shall define a defective deviation as *a deviation (from some strategy combination) that is sensibly not maintained in the long term by the deviating player* (note, that there are no undefective deviations in a repeated PD).

Let us assume, that we have found a strategy combination, that perfectly reflects what is sensibly maintained in the long term (the above negotiation process will provide us with that). Then, we might assume that the players adopt the respective outcome (and defend it against defections) without further reflections (as it is done in Folk theorems, see Rubinstein, 1979, and Fudenberg and Maskin, 1986). Alternatively, we might assume that the players adopt the reasoning behind long–term negotiations, but that the negotiations as such take place *while* they are playing (and while they are retaliating defections). The former amounts to selecting specific instances of Folk theorem equilibria, differing from older approaches only, in that the negotiation process appears to be modeled more naturally here (since it is leaned precisely on what is sensibly not maintained in the long term, a rather appealing definition of defections). The latter approach, however, amounts to refining Folk theorem equilibria, we will be able to proceed along that one as well (apparently, we are able to capitalize upon a distinctively naturale definition of the negotiation process). Eventually, when comparing these approaches, we will observe a paradigm shift in constructing repeated game equilibria: away from retaliating deviations from an exogenously injected strategy combination, towards simultaneously negotiating about the ultimately played strategy combination and retaliating deviations from the path of plausible negotiation proposals.

In a preliminary step towards such a refinement concept, we will characterize the general negotiation outcome, and to do so, we will analyze a game, where the negotiations are carried out efficiently (i.e. they are carried out such that their results indeed reflect what the players should sensibly maintain in the long term). In this game, the players can explicitly propose maintainable strategy combinations, and their actual moves are formulated in relation to their standing proposals. Rather crucially, the proposals are required to be non–decreasingly competitive. On the one hand, this restriction implies that the players would never make a defective proposal (i.e. a proposal that is more competitive than is sensible in the long term), since defective moves are *per se* profitable only if the players can return to undefective moves afterwards (and get trusted soon again). On the other hand, we will see that (thanks to the consistency of the payoff structure of consistenly competitive games) this restriction is not biassing the moves of sincerely negotiating players, i.e. they are best off moving increasingly competitive when their opponents are doing so. As a result of that, in these games (called *games of long–term concerns*) the players are indeed able to negotiate efficiently about the benchmark of cooperativeness. The subgame–perfect equilibria (Selten, 1965, 1975) of these games turn out to be a subset of the sequential equilibria (Kreps and Wilson, 1982) of a game, that is called multiple–round game: the moves are increasingly competitive, the game proceeds unless all players stick to their moves of the previous round, and only the moves of the final round are payoff–relevant (for an analysis of multiple–round auctions, see e.g. Breitmoser, 2002). That is, we can induce the negotiation–rational moves (i.e. the path of efficient negotiations) backwardly in a multiple–round game.

Contrary to the game of long–term concerns, in a plain repeated game, maintainable strategy combinations are not proposed explicitly. Instead, these have to be deduced from the actual moves, which are conglomerates of negotiation proposals and (possibly defective) actual moves. We will see, however, that for players who expect their opponents to implicitly negotiate through increasingly competitive proposals (which those are doing in turn), a subgame–perfect equilibrium of IRCGs can be constructed, which has it that the players never defect nor deviate from the "negotiation–rational" path derived above. As mentioned already, this equilibrium construction rests on abstractedly defined components, and can therefore be interpreted as an equilibrium refinement concept. Thus, we are able to refine the continuum of Folk theorem equilibria with respect to "negotation–rational" equilibrium paths. Actually, the continuum of Folk theorem equilibria is resolved, even though the players are allowed to move more freely than in Folk theorems (in that they may negotiate).

Besides these introductory (and some conclusive) remarks, this paper has two sections. In Section 2, the class of consistently competitive games is defined, several implications of this definition are put forward, and the major instances are introduced. In Section 3, the distinction of defective and "negiotative" concerns is discussed, "defection–proof" equilibria are derived (based on the analyses of Rubinstein, 1979, and Fudenberg and Maskin, 1986), the long–term negotiations are analyzed, and eventually, these ingredients are combined in the analysis of IRCGs. A number of proofs are gathered in the appendix.

2 The Consistently Competitive Basis Game

2.1 Definition

Infinitely repeated games consist of infinite repetitions of basis games. The class of basis games, that we shall restrict our attention to, is defined in two steps. In the first step, a couple of restrictions concerning the basis game as a whole are presented, and in the second step, characteristics are presented, that have to hold in each of the game's "dimensions" (of which each game has at least one; but often, the strategy space has to be split into several of those, in order to pass the restrictions of the second step, e.g. in multiple–object auctions). Concerning the first step, the definition of the basis game, its strategy set has to be finite, and, if multiple–dimensions are to be distinguished, then the overall payoff has to be linear in the dimensions.

Definition 1 (Basis Game) *The game extends into M dimensions $D = \{D_i\}_{i=1\ldots M}$ and involves N players $B = \{B_j\}_{j=1\ldots N}$. The strategy set for player $b \in B$ in dimension $d \in D$ of the game is $\overline{S}_{b,d} \subset \mathbf{N}$, with $\mathbf{N}$ as the set of natural numbers and $\left|\overline{S}_{b,d}\right| < \infty$. Besides, the following sets be defined.*

$$\overline{S}_b = \bigtimes_d \overline{S}_{b,d}, \qquad \overline{S}_d = \bigtimes_b \overline{S}_{b,d}, \qquad \overline{S} = \bigtimes_b \overline{S}_b,$$

$$\overline{S}_{-b,d} = \bigtimes_{c \neq b} \overline{S}_{c,d}, \qquad \overline{S}_{-bc,d} = \bigtimes_{b' \neq b,c} \overline{S}_{b',d}, \qquad \overline{S}_{-b} = \bigtimes_{c \neq b} \overline{S}_c, \qquad \overline{S}_{-b,-d} = \bigtimes_{i \neq d} \overline{S}_{-b,i},$$

Similarly, projections are defined, e.g. for $S \in \overline{S}$, the projection onto $\overline{S}_{-b,d}$ is denoted by $S_{-b,d}$. The payoff function for player b in dimension d, $p_{b,d}(S_d) : \overline{S}_d \to \mathbf{R}$, is independent of the strategies in the other dimensions, i.e. the overall payoff $p_b(S) : \overline{S} \to \mathbf{R}$ is, for all $S \in \overline{S}$, $p_b(S) = \sum_{d \in D} p_{b,d}(S_d)$.

This class of games contains a subclass, which instances shall be called consistently competitive. Basically, in consistently competitive games, the strategies can be ranked in a way, that allows to call one strategy to be *more competitive* than another one (which comes hand in hand with a couple of characteristics that are outlined below), if and only if the first strategy is ranked higher than the second one. We shall refer to the *competitiveness* of a strategy as the number of strategies that are less competitive than the strategy in question, and we will assume, that the competitiveness is increasing in the natural numbers $S_{b,d} \in \overline{S}_{b,d}$ representing the strategies (i.e. that the strategies are already ranked properly).

Basically, the idea of competition, as it is understood here, is captured in a game, where the strategies $S_{b,d} \in \overline{S}_{b,d}$ can be decomposed into elementary components α_i, $S_{b,d} = \cup_{i \in \overline{A}(S_{b,d})} \alpha_i$, that have certain characteristics. Basically, each component has a price and provides some competitive power. The competitive power of an aggregated strategy is the sum of the single components' contributions. The gross–payoff of each player is a function of the players' *relative* competitive powers, and the net–payoff is the difference of gross–payoff and costs of the employed components. Finally, the components α_i are assumed to be applied in decreasing order with respect to their efficiency (i.e. with respect to the ratio of "competitive power" to price). That is, the components α_i can be arranged, such that each strategy in dimension d, $S_{b,d} \in \overline{S}_{b,d}$, can be described through a number $n \in \mathbf{N}$ with $S_{b,d} = \cup_{i \leq n} \alpha_i$.

This game has a couple of distinctive characteristics. First, the own payoff roughly has a concave structure in the own competitiveness (first increasing, then constant, and eventually decreasing), which stems from the decreasing efficiency of the components. Secondly, the own payoff is decreasing in the opponents' competitiveness, which stems from the decreasing relative competitive power. Thirdly, the social income (the aggregated net–payoffs of the players) is decreasing in the players' competitivenesses, which stems from the gross payoff's reliance on the players' *relative* competitive powers (i.e. competition merely helps to redistribute the social gross income, at rising costs; by the way, inefficiency is a commonly recognized side effect of competition). Additionally, we will require that the game fulfills the following consistency requirement: if a player is better off adding a component to his strategy (facing some set of opposing components), then this addition is profitable against any superset of these opposing components. This is not a general characteristic of the above game, but it will be crucial in our analysis of repeated games. Finally, it will be assumed that there is generally a player, who is better off deviating to his most competitive strategy (which is the one, that includes all of the components in his arsenal); thus, essentially, redundant strategies are ruled out. Usually, the last restriction is equivalent to focusing on a subset of the actual strategy space, to which one's concerns are restricted in any case.

Before the assumptions of consistently competitive games can be formalized, some functions need to be introduced. The most and least competitive strategies of player b in dimension d are highlighted by stars, i.e.

$$S^*_{b,d} \in \arg\max_{S' \in \overline{S}_{b,d}} S', \qquad S_{*(b,d)} \in \arg\min_{S' \in \overline{S}_{b,d}} S'. \tag{1}$$

Similarly, we have a (set–valued) best–reply function, and most or least competitive best replies, i.e.

$$BR_{b,d}\left(S_{-b,d}\right) := \arg\max_{S' \in \overline{S}_{b,d}} p_{b,d}\left(S', S_{-b,d}\right), \qquad BR_b\left(S_{-b}\right) := \bigtimes_{d \in D} BR_{b,d}\left(S_{-b,d}\right), \tag{2}$$

$$BR^*_{b,d}\left(S_{-b,d}\right) \in \arg\max_{S' \in BR_{b,d}\left(S_{-b,d}\right)} S', \qquad BR_{*(b,d)}\left(S_{-b,d}\right) \in \arg\min_{S' \in BR_{b,d}\left(S_{-b,d}\right)} S'. \tag{3}$$

The Nash equilibria are accumulated in the set

$$\overline{EQ}_d = \left\{S_d \in \overline{S}_d : S_{b,d} \in BR_{b,d}\left(S_{-b,d}\right) \forall b \in B\right\}. \tag{4}$$

The predecessor of a strategy is the strategy, that is the most competitive of all strategies, that are less competitive than the strategy in question (or, in case there is no less competitive strategy, the predecessor is the strategy itself), i.e.

$$PD_{b,d}\left(S_{b,d}\right) \in \begin{cases} \{S_{b,d}\} & \text{, if } S_{b,d} = S_{*(b,d)}, \\ \arg\max_{S' < S_{b,d}} S' & \text{, otherwise.} \end{cases} \tag{5}$$

The first–order difference of the payoffs is defined as b's gain, if c deviates from his (c's) predecessor strategy to the strategy in question, i.e.

$$\Delta_c p_{b,d}(S_d) = p_{b,d}(S_d) - p_{b,d}\left[S_{-c,d}, PD_{c,d}\left(S_{c,d}\right)\right], \tag{6}$$

and the more general difference function (which concerns b's payoff when c deviates) is defined as

$$D^b_{c,d}\left(S_d, S'_{c,d}\right) = p_{b,d}\left(S'_{c,d}, S_{-c,d}\right) - p_{b,d}\left(S_d\right), \tag{7}$$

$$D^b_{c,d}\left(S_{-c,d}, S'_{c,d}, S''_{c,d}\right) = D^b_{c,d}\left(\left\{S_{-c,d}, S'_{c,d}\right\}, S''_{c,d}\right). \tag{8}$$

Finally, if a relation of two vectors V^1, V^2 from the same space is referred to (or, equivalently, a relation of matrices), then this relation is understood in one of the following ways.

$$V^1 = V^2 \Leftrightarrow V^1_i = V^2_i \,\forall i \qquad\qquad V^1 \leq V^2 \Leftrightarrow V^1_i \leq V^2_i \,\forall i \tag{9}$$

$$V^1 < V^2 \Leftrightarrow V^1 \leq V^2,\ V^1 \neq V^2 \qquad\qquad V^1 \ll V^2 \Leftrightarrow V^1_i < V^2_i \,\forall i \tag{10}$$

Definition 2 (Consistently Competitive Game, CG) *In consistently competitive games (CGs), the payoff function satisfies the characteristics* (C1,C2,C3,C4,C5)*, as defined in the following, for all players $b \in B$ and dimensions $d \in D$. The varieties, that are distinguished, are called strict (SCG), weak (WCG), and irreducible weak (ICG). In WCGs and ICGs, characteristic* $(C2*)$ *is fulfilled additionally.*

(C1) The deviation incentives are monotonically decreasing in the own competitiveness Roughly, it is required that the payoff is concave in the own competitiveness. This is weakened, however, for we do not require that the first difference function is monotonically decreasing, but that the signs of the first differences are non–increasing (in this context, the sign is understood as the function that evaluates to $\{1, 0, -1\}$). In WCGs, it is required that, if one is worse off deviating from some strategy to the successor strategy, then one is never better off deviating from any more competitive strategy to the respective successor (given the opponents' strategies are held fixed). That is, for all $S_{-b,d} \in \overline{S}_{-b,d}$, for all $S^1_{b,d} < S^2_{b,d} \in \overline{S}_{b,d} \setminus \{S_{*(b,d)}\}$, and with $s_i := \text{sign}\left[\Delta_b p_{b,d}\left(S^i_{b,d}, S_{-b,d}\right)\right]$,

$$\text{SCG:} \quad s_1 \geq s_2, \qquad\qquad \text{WCG, ICG:} \quad s_1 < 0 \Rightarrow s_2 \leq 0. \tag{11}$$

(C2) Competition is harmful to the opponents The payoffs of b's opponents are decreasing in b's competitiveness, i.e. for all $S_d \in \overline{S}_d$, for all $S'_{b,d} \in \overline{S}_{b,d} : S'_{b,d} > S_{b,d}$, and for all $c \neq b$

$$\text{SCG:} \quad D^b_{b,d}\left(S_d, S'_{b,d}\right) > 0, \qquad\qquad \text{WCG, ICG:} \quad D^b_{b,d}\left(S_d, S'_{b,d}\right) \geq 0. \tag{12}$$

(C2$*$) No relevance for the own payoff implies general irrelevance This is required only in WCGs and ICGs, in order to fix the otherwise very loosely defined payoff structure (by the way, this characteristic implies, that SCGs are not generally WCGs). If b's payoff from $(S_{b,d}, S_{-b,d})$ is equal to that from $\left(S'_{b,d}, S_{-b,d}\right)$, with $S'_{b,d} > S_{b,d}$, then $S'_{b,d}$ and $S_{b,d}$ are generally equivalent in reply to strategies $S'_{-b,d} \geq S_{-b,d}$, or, if that is not fulfilled, it is sufficient that there is an opponent who would gain through deviating from $(S_{b,d}, S_{-b,d})$ to a more competitive strategy. That is, for all $S_d \in \overline{S}_d$, all $S'_{b,d} \in \overline{S}_{b,d} : S'_{b,d} > S_{b,d}$, and all $S'_{-b,d} \in \overline{S}_{-b,d} : S'_{-b,d} \geq S_{-b,d}$,

$$D^b_{b,d}\left(S_d, S'_{b,d}\right) = 0 \tag{13}$$

$$\Rightarrow \quad D^c_{b,d}\left(S'_{-b,d}, S_{b,d}, S'_{b,d}\right) = 0 \quad \forall c \neq b, \tag{14}$$

$$\text{or} \quad \exists c \neq b,\ \exists S'_{c,d} \in \overline{S}_{c,d} : S'_{c,d} > S_{c,d}, \quad D^c_{c,d}\left(S_d, S'_{c,d}\right) > 0.$$

(C3) The deviation incentives are monotonically increasing in the opponents' competitiveness In particular, if one is better off deviating (from a strategy to some more competitive strategy) in reply to some combination of the opponents' strategies, then one is better off deviating (in the same way) in reply to any more competitive combination of opponents' strategies, i.e. for all $T^1_{-b,d} < T^2_{-b,d} \in \overline{S}_{-b,d}$, and for all $S^1_{b,d} < S^2_{b,d} \in \overline{S}_{b,d}$, and with $s_i := \text{sign}\left[D^b_{b,d}\left(T^i_{-b,d}, S^1_{b,d}, S^2_{b,d}\right)\right]$,

$$\text{SCG:} \quad s_1 \leq s_2, \qquad\qquad \text{WCG, ICG:} \quad s_1 \geq 0 \ \Rightarrow\ s_2 \geq 0. \tag{15}$$

(C4) Competition is socially inefficient From the social point of view (assuming, the society is equivalent to the set of players), competition is inefficient, i.e. any gain of b from deviating to a more competitive strategy is outweighed by the losses of b's opponents. In WCGs, we simply require that there are losses of some opponent if b is gaining, i.e. for all $S_d < T_d \in \overline{S}_d$ and all $S'_{b,d} \in \overline{S}_{b,d} : S'_{b,d} > S_{b,d}$,

$$\text{SCG:} \quad \sum_{b \in B} p_{b,d}(S_d) > \sum_{b \in B} p_{b,d}(T_d), \tag{16}$$

$$\text{WCG, ICG:} \quad D^b_{b,d}\left(S_d, S'_{b,d}\right) > 0 \quad \Rightarrow \quad \exists c \in B,\ D^c_{b,d}\left(S_d, S'_{b,d}\right) < 0. \tag{17}$$

(C5) Generally, one player is better off deviating to his most competitive strategy For all $S_d \in \overline{S}_d \setminus \{S^*\}$

$$\text{SCG:} \quad \exists b \in B,\ D^b_{b,d}\left(S_d, S^*_{b,d}\right) > 0, \qquad \text{WCG:} \quad \exists b \in B,\ D^b_{b,d}\left(S_d, S^*_{b,d}\right) \geq 0, \tag{18}$$

$$\text{ICG:} \quad \exists b \in B,\ \Delta_b p_{b,d}\left(S_d\right) \geq 0. \tag{19}$$

2.2 Implications of the Characteristics of Consistently Competitive Games

There are several implications, that result immediately from the above characteristics. Some of the implications are presented in the following, as later arguments rest on those, and as these may be helpful to illustrate the above definition of CGs. First, let us consider the rather technical issue of examining the effects of eliminations of strategies with respect to the above characteristics in SCGs and WCGs. We can show, that (weak or strict) consistent competitiveness is robust against eliminations of strategies, if it is not a player's most competitive strategy that is to be eliminated (the most competitive strategies constitute the main equilibria of CGs, and are therefore called relevant in the following). For most of the characteristics, this robustness is obvious, since those are required to hold for each pair of strategy combinations, independently of irrelevant strategies. Only the robustness of *(C1)* is slightly more involved, but the payoffs' concavity (as defined above) is contained in any case. A weaker kind of robustness holds for ICGs; there, only the respective least competitive strategies may be eliminated (any other strategy may be relevant for *C5* to be fulfilled).

Implication 1 (Robustness against eliminations of irrelevant strategies) *An SCG (WCG) keeps an SCG (WCG, respectively), whenever a strategy $S_{b,d} \neq S^*_{b,d}$ of any $b \in B$ in any $d \in D$ is eliminated. An ICG keeps an ICG, whenever a strategy $S_{b,d} = S_{*(b,d)}$ of any $b \in B$ in any $d \in D$ is eliminated.*

Next let us look at an implication, that concerns a facet of the consistency of the deviation incentives: the competitiveness of a player's best reply is weakly increasing in the competitiveness of his opponents' strategies, which results straightforwardly from *(C3)*.

Implication 2 (Self–enforcing competition) *In any CG, the competitivenesses of one's most competitive best replies is weakly increasing in the opponents' competitiveness, i.e. for all $S_{-b,d} < T_{-b,d} \in \overline{S}_{-b,d}$, we have $BR^*_{b,d}(S_{-b,d}) \leq BR^*_{b,d}(T_{-b,d})$, and in SCGs, this holds similarly for the least competitive best replies, i.e. $BR_{*(b,d)}(S_{-b,d}) \leq BR_{*(b,d)}(T_{-b,d})$.*

A further implication concerns a rather illustrative "personal inefficiency" of competition, which is a weak kind of Pareto inefficiency and holds along each player's best replies: the more competitive b's opponents are, the less profitable b's best reply is.

Implication 3 (Shrinking payoffs of best replies) *In CGs, the payoff from b's best replies is decreasing in the opponents' competitiveness, i.e. for all $S_{-b,d} < T_{-b,d} \in \overline{S}_{-b,d}$, all $S^{BR}_{b,d} \in BR_{b,d}(S_{-b,d})$, and all $T^{BR}_{b,d} \in BR_{b,d}(T_{-b,d})$,*

$$\text{SCG:} \quad p_{b,d}\left(S^{BR}_{b,d}, S_{-b,d}\right) > p_{b,d}\left(T^{BR}_{b,d}, T_{-b,d}\right), \tag{20}$$

$$\text{WCG, ICG:} \quad p_{b,d}\left(S^{BR}_{b,d}, S_{-b,d}\right) \geq p_{b,d}\left(T^{BR}_{b,d}, T_{-b,d}\right). \tag{21}$$

Next, a simple implication concerning the players' incentives structure has to be mentioned.

Implication 4 (Incentives to deviate) *In a CG, and for any strategy combination $S_d \in \overline{S}_d$, any player, who is better off deviating to a more competitive strategy $T_{b,d} > S_{b,d}$, is also (strictly/weakly) better off deviating to any strategy $S'_{b,d} \in (S_{b,d}, T_{b,d})$, i.e.*

$$\text{SCG:} \quad D^b_{b,d}(S_d, T_{b,d}) > 0 \quad \Rightarrow \quad D^b_{b,d}\left(S_d, S'_{b,d}\right) > 0, \tag{22}$$

$$\text{WCG ,ICG:} \quad D^b_{b,d}(S_d, T_{b,d}) \geq 0 \quad \Rightarrow \quad D^b_{b,d}\left(S_d, S'_{b,d}\right) \geq 0. \tag{23}$$

A further pair of implications concerns the Nash equilibria of CGs (which shall be called competitive equilibria). This is straightforward in SCGs, since there is generally at least one player, who is strictly better off deviating to his most competitive strategy. Hence, on the one hand, there can be no equilibrium in pure strategies other than the combination of the most competitive strategies. On the other hand, there can be no other equilibrium in mixed strategies. Take any combination of mixed strategies, and consider the combination of the least competitive pure strategies, that are played with positive probabilities. There is a player, who is better off playing his most competitive strategy against this combination of opponents' strategies (due to *C5*), and therefore against any combination of more competitive strategies, including any mixed one. Hence, that player is better off assigning the probability, that he assigned to that strategy, to his most competitive strategy, and generally, there must always be a player deviating to a more competitive mixed strategy. Finally, there is also a positive way to show, that the combination of the most competitive strategies is an equilibrium (the unique one, as we know). Consider the following iterative (pure–strategy) process: starting in S_{*d}, each player, who is better off deviating to his most competitive strategy, is deviating to any of his best replies to the opponents' strategies (any of those must be more competitive than his current strategy). Since there is always at least one player better off deviating, this (monotonically increasing) process must reach the combination of the most competitive strategies (in SCGs). Therefore, each b's most competitive strategy $S^*_{b,d}$ must have been the best reply to some $S_{-b,d}$ in that process, and hence must be the best reply to $S^*_{-b,d}$ (for $S^*_{-b,d} \geq S_{-b,d}$, see also *C3*). Because of that, the combination of the most competitive strategies must be a combination of best replies to each other.

Implication 5 (Strict competitive equilibrium) *In SCGs, there is a unique Nash equilibrium, which is the combination of the most competitive strategies.*

The characterization of the set of Nash equilibria of WCGs and ICGs is more difficult (the discussion of which will be restricted to pure strategy equilibria; based on that, and on the previous arguments, the case of mixed strategies is straightforward). The set of Nash equilibria can be decomposed into several subsets of equivalent and interchangeable equilibria, for each of which a "connectedness" to the combination of the most competitive strategies can be derived. Precisely, we can show, that for each equilibrium strategy combination $EQ_d \in \overline{EQ}_d$, any more competitive strategy combination $S_d \in \overline{S}_d : S_d > EQ_d$ is also an equilibrium combination $S_d \in \overline{EQ}_d$, and that it is payoff–equivalent to and interchangeable with EQ_d. Hence, the following implication results.

Implication 6 (Weak competitive equilibria) *In a WCG/ICG, the set of equilibria $\overline{EQ}_d$ is the conjunction of sets $\overline{EQ}_d = \cup_i \overline{EQ}^i_d$, that consists of interchangeable equilibria, and for each $S'_d \in \overline{EQ}^i_d$ we also have that any $S''_d > S'_d$ is $S''_d \in \overline{EQ}^i_d$, i.e.*

$$\overline{EQ}^i_d \equiv \bigtimes_{b \in B} \overline{S}^{Eq,i}_{b,d}, \qquad \textit{with} \quad \overline{S}^{Eq,i}_{b,d} = \left\{ S_{b,d} \in \overline{S}_{b,d} : \exists EQ^i_d \in \overline{EQ}^i_d,\ S_{b,d} \geq EQ^i_{b,d} \right\}. \tag{24}$$

Moreover, all of the equilibria are payoff equivalent to the most competitive strategy combination and (therefore) to each other, i.e. $p_{b,d}\left(E'_d\right) = p_{b,d}\left(E''_d\right)$ for all $b \in B$ and all $E'_d, E''_d \in \overline{EQ}_d$.

Another implication, one that is rather illustrative with respect to the structure of the deviation incentives, is that CGs are generally iteratively dominance solvable (in the weak or strict sense, respectively). Hence, the competitive equilibrium is the only rationalizable strategy combination (given the players' rationality is common knowledge, see e.g. Pearce, 1984). First, let us define dominance solvability.

Definition 3 (Iterative dominance solvability) *A game is considered iteratively dominance solvable (in the weak/strict sense), if the iterative elimination of (weakly/strictly) dominated strategies implies a set of strategies, all combinations of which are payoff–equivalent.*

Let $\overline{S}^{El,t}_{b,d}$ denote the set of strategies of $b \in B$ after t steps of eliminations $(0 \leq t < \infty)$. Assume, that the combination of the least competitive strategies $S^{El,t}_{*d}$ of these sets is not an equilibrium. Then, there must be a player $b \in B$, whose least competitive strategy is less competitive than any of his best replies to the opponents' least competitive strategies $S^{El,t}_{*(b,d)} \notin BR_{b,d}\left(S^{El,t}_{*(-b,d)}\right)$, i.e. for all $S^{BR,t}_{b,d} \in BR_{b,d}\left(S^{El,t}_{*(-b,d)}\right)$ we have $S^{BR,t}_{b,d} > S^{El,t}_{*(b,d)}$, and thus for all $T_{-b,d} \in \bigtimes_{c \neq b} \overline{S}^{El,t}_{c,d}$, of which we know $T_{-b,d} \geq S^{El,t}_{*(-b,d)}$, we have (due to $C3$) in SCGs

$$p_{b,d}\left(S^{El,t}_{*d}\right) < p_{b,d}\left(S^{BR,t}_{b,d}, S^{El,t}_{*(-b,d)}\right) \qquad \Rightarrow \qquad p_{b,d}\left(S^{El,t}_{*(b,d)}, T_{-b,d}\right) < p_{b,d}\left(S^{BR,t}_{b,d}, T_{-b,d}\right) \tag{25}$$

and in WCGs/ICGs, the implied relation is fulfilled weakly. Therefore, any $S^{BR,t}_{b,d}$ dominates weakly or strictly (respectively) the least competitive strategy $S^{El,t}_{*(b,d)}$ of player b, and the latter can therefore be eliminated (note that the order of eliminations of weakly dominated strategies is irrelevant here). Now assume, that the combination of the least competitive strategies is an equilibrium. In an SCG, all but the initially most competitive strategies must have been eliminated and a unique combination is left. This one is equivalent to the competitive equilibrium. In WCGs/ICGs, we know from Implication 6, that any combination of strategies, that are more competitive than $S^{El,t}_{*d}$, must be payoff–equivalent to and interchangeable with $S^{El,t}_{*d}$. Hence, the game as a whole is dominance solvable.

Implication 7 (Dominance solvability) *Be $\overline{S}^{El,\infty}_{d}$ the set of combinations of all iteratively undominated strategies (in the weak or strict sense) of a (weak or strict, respectively) CG, then $\overline{S}^{El,\infty}_{d} \subseteq \overline{EQ}_d$, and the equilibria in $\overline{S}^{El,\infty}_{d}$ are payoff–equivalent and interchangeable.*

In each iteration of the elimination process, the set of rationalizable strategies declines (unless all of the strategies are rationalizable), i.e. based on previous eliminations of irrational moves, further moves are declared irrational, and eliminated. The opposite approach is to determine rational (most profitable) moves based on previously attained beliefs about rational (most likely) moves. One of its formalizations is called best–reply dynamics (see, e.g., Bernheim, 1984). Generally, the best–reply dynamics do not converge to any equilibrium (not to speak of a global convergence to the same equilibrium), but in the case of CGs, the competitive equilibrium results, whatever the initial beliefs are. To examine this in detail, let us define the interior of the strategy space, $\overline{I}_d \subseteq \overline{S}_d$. The interior is defined to comprise all strategy combinations, where all players are playing the best reply to the respective strategies of their opponents, or a less competitive strategy.

Definition 4 (Interior of the strategy set) *A strategy combination S_d is in the interior $\overline{I}_d \subseteq \overline{S}_d$, if for all players $b \in B$, the respective strategy $S_{b,d}$ is not more competitive than the most competitive best reply to the respective opponents' strategies $S_{-b,d}$, i.e. $\overline{I}_d := \left\{S_d \in \overline{S}_d : \; S_{b,d} \leq BR^*_{b,d}(S_{-b,d}) \;\forall b \in B\right\}$.*

Based on the mapping $BR^*_d : \overline{S}_d \to \overline{S}_d$, which gives for any strategy combination the combination of each player's most competitive best reply to the respective opponents' strategies, we can define the best–reply dynamics as the iterative process $\breve{S}^t_d$

$$\breve{S}^{t+1}_d = BR^*_d\left(\breve{S}^t_d\right), \qquad \text{with} \qquad BR^*_d(S_d) := \left\{BR^*_{b,d}(S_{-b,d})\right\}_{b \in B}. \tag{26}$$

Table 1: A Prisoner's Dilemma

	Player 2	
Player 1	C	D
C	$(2,2)$	$(0,3)$
D	$(3,0)$	$(1,1)$

Now, it is straightforward to show that, inside the interior, the process is monotically increasing in terms of competitiveness, and generally stays inside the interior. For each strategy combination outside the interior, moreover, there is a less competitive strategy combination inside it (e.g. S_{*d}), which obviously keeps (weakly) less competitive in all iterations of the best–reply dynamics, but converges to S_d^*. Hence, best–reply dynamics starting outside the interior converge to the equilibrium as well (but not monotonically).

Implication 8 (Convergence of the Best–Reply Dynamics) *The best–reply dynamics according to* (26) *in a CG globally converge to the combination of the most competitive strategies,*

$$\forall \breve{S}_d^0 \in \overline{S}_d, \quad \exists t^* : \breve{S}_d^t = S_d^* \; \forall t > t^*, \tag{27}$$

and when this process has entered the interior of the strategy set (Definition 4), then it will stay inside the interior in all of the following iterations,

$$\exists t, \, \breve{S}_d^t \in \overline{I}_d \quad \Rightarrow \quad \breve{S}_d^{t'} \in \overline{I}_d \quad \forall t' > t. \tag{28}$$

Eventually, these illustrations to the definition of CGs are concluded with introducing some (already well known) examples. The first instance is a relative of the Prisoner's Dilemma (see Table 1, and Straffin, 1980), which shall be called PD_0 in the following. The relative is the *Prisoners' Dilemma* (PD), which shall be defined as a game that is iteratively dominance solvable with socially inefficient defections, instead of a game that is non–iteratively dominance solvable with a Pareto inefficient equilibrium.

Definition 5 (Prisoners' Dilemma, PD) *An* $N \times 2$*–game is called Prisoners' Dilemma, if it is iteratively dominance solvable in the strict sense, and if any player's defection is socially inefficient (i.e. the sum of all players' payoffs is decreasing), regardless of the opponents' strategies.*

Thus, a PD is a dilemma of all prisoners, but not of each one anymore (hence, the varied spelling). The respective variations, however, allow us to state the following.

Implication 9 (PD–like Competitiveness) *Any strict consistently competitive* $N \times 2$ *game is a Prisoners' Dilemma.*

A related implication concerns the game, that results when a (strict consistently competitive) game is restricted to two strategies per player (one of which is arbitrary, the other one is his initially most competitive one). This game must be still strict consistently competitive, and therefore a PD. That is, the relation of any uncompetitive strategy combination to the combination of most competitive strategies is equivalent to the relation of the strategies in a PD.

Implication 10 (Restrictions to $N \times 2$ games) *Any SCG, that is restricted to exactly two strategies per player (one of which is his most competitive strategy), is a PD.*

As a result of this, in any region of the strategy space, the players are better off defecting to the most competitive strategy combination, and the class of CGs turns out to be a proper generalization of the structure of PDs to more than two strategies. Another game, that is considered to be a generalized PD (in fact, to be a generalized PD_0), is the tragedy of the commons (Hardin, 1968), with the payoff function $p^b = \delta \sum_c s_c - s_b$ with $\frac{1}{N} < \delta < 1$. It is a game with an arbitrary number of players and strategies, yet is still non–iteratively dominance solvable, and there are allocations that are Pareto superior to the allocation of the Nash equilibrium. Obviously, this game is consistently competitive (in decreasing contributions s_b), and therefore, consistent competition can be considered to comprise this kind of interactions as well.

Let us now turn to games, that are structurally more complex than those above, i.e. to games that comprise more than two strategies per player, and are (at the same time) not non–iteratively dominance solvable (i.e. where the elimination process requires more than one iteration). The general prototype for an SCG is the rent–seeking contest (Tullock, 1980) as defined in the following. Besides, note also that the contest is also a prototype for several all–pay games (e.g. patent and innovation races, see Baye and Hoppe, 2002).

Definition 6 (Contest) *The strategy space of player $b \in B$ is $\overline{S}_b \subset \mathbf{N} \setminus \{0\}$, the payoff is $p_b(S_b, S_b^\Sigma) = \frac{s_b}{S_b + S_b^\Sigma} - \delta S_b$, with $S_b^\Sigma = \sum_{c \neq b} S_c$ and $\delta \in \boldsymbol{R} : \delta > 0$. Moreover, for all $b \in B$ we have $\max_{s \in \overline{S}_b} s \leq \frac{n-1}{\delta n^2}$ with $n = |B|$.*

Implication 11 (Contest is strict consistently competitive) *A two–players contest (according to Definition 6) is strict consistently competitive.*

The general prototype for WCGs is the (first–price, sealed–bid) auction. In auctions (contrary to contests), only the winners have payoff–relevant transactions. Thus, if a player is not the high–bidder, and neither becomes it from increasing his bid by a given amount, then there are no consequences for his payoff resulting from this bid increment. Basically because of that, auctions are considered merely weak consistently competitive. Besides, in order to determine the high–bidder, the auctioneer may have to resolve ties, which we rule out by requiring that different bidders can never bid equal amounts. This simplifies the notation and the game structure a little bit, but does not imply notable side–effects with respect to the usual characteristics of auctions. There are two further restrictions in the following definition of auctions, that concern the maximal bids which the players can make.

Definition 7 (Auction) *The strategy space of any player $b \in B$ is $\overline{S}_b \subseteq \mathbf{N}$, and the payoff is (depending on some $S_b \in \overline{S}_b$, and $S_{-b} \in \overline{S}_{-b}$)*

$$P_b(S_b, S_{-b}) = \begin{cases} W_b - \delta S_b - \varepsilon_b & \text{, if } \delta S_b + \varepsilon_b > \max_{c \neq b} \delta S_c + \varepsilon_c \\ 0 & \text{, otherwise,} \end{cases} \tag{29}$$

with $0 \leq \varepsilon_b < \delta$, $\varepsilon_b \neq \varepsilon_c$ $\forall c$, and the valuations W_b, with $|W_b - W_c| > 2\delta$ for all $b \neq c \in B$. Moreover, for all players $b \in B$ and all strategies $S_b \in \overline{S}_b$, we have $W_b > S_b * \delta + \varepsilon_b$, and the highest–bidding *player* $b^* \in \arg\max_{b\in B} \delta s_b + \varepsilon_b$ *has only one strategy that beats all opponents' strategies, i.e.*

$$\arg\max_{S'_{b^*}\in\overline{S}_{b^*}} S'_{b^*} \equiv \left\{ S''_{b^*} \in \overline{S}_{b^*} : \ \delta S''_{b^*} + \varepsilon_{b^*} > \max_{S_c\in\overline{S}_c} \delta S_c + \varepsilon_c \ \ \forall c \neq b^* \right\}. \tag{30}$$

Implication 12 (Auctions are weak consistently competitive) *Auctions (according to Definition 7) are weak consistently competitive.*

Finally, let us consider the prototype for ICGs, Bertrand competition of n competitors (Bertrand, 1883). By the way, the following discusion applies similarly to auctions, where tied bids can not be excluded; with the difference, that in Betrand competition the payoff is *sublinearly* decreasing (beyond the best reply) in increasingly competitive strategies (decreasing prices), for the consumers' demand is increasing. In order to simplify the model, let us assume that the demand is linear, $D(p) = 1 - p$, the marginal costs are constant, $C_b(x) = cx$, and equivalent for all players. The equilibrium price in a monopoly is $\frac{c+1}{2}$, and through their strategies $S_b \in \overline{S}$, the players describe the (absolute) discount they give with respect to the monopoly price. That is, strategy S_b coincides with setting the price to $P(S_b) = \frac{c+1}{2} - S_b\delta$, with $\delta \in R : \delta > 0$. In order to satisfy the characteristics for ICGs, the strategies need to be restricted through

$$S_b < \sqrt{\frac{(1-c)^2}{4\delta^2} + \frac{n}{(n-1)^2}} - \frac{n}{n-1} \qquad \underset{\delta\to 0}{\Leftrightarrow} \qquad \delta S_b < \frac{1-c}{2}. \tag{31}$$

Thus, for sufficiently small δ, the most competitive strategies imply prices, that are arbitrarily near the marginal costs c, and easily beyond the price implied by the Cournot–equilibrium of n competitors, $\frac{n+c}{n+1}$ (Cournot, 1839).

Definition 8 (Bertrand competition) *The strategy sets are for all* $b \in B$

$$\overline{S}_b = \left\{ S \in \mathbf{N} : S < \sqrt{\frac{(1-c)^2}{4\delta^2} + \frac{n}{(n-1)^2}} - \frac{n}{n-1} \right\}, \tag{32}$$

with $n = |B| > 1$, $c \in \mathbf{R} : 0 \leq c < 1$, $\delta \in \mathbf{R} : 0 < \delta < 1$. *The payoff for b, depending on* $S \in \overline{S}$, *is*

$$p^b(S) = \begin{cases} 0 & , S_b \neq \max_c S_c, \\ \Pi_b & , \textit{otherwise}, \end{cases} \qquad \textit{with} \quad \Pi_b = \frac{1}{k}\left(\frac{(1-c)^2}{4} - S_b^2\delta^2\right), \tag{33}$$

$$k = \left| \arg\max_b S_b \right|. \tag{34}$$

Implication 13 (Bertrand competition is irreducible weak consistently competitive) *Bertrand competition (according to Definition 8) is an ICG.*

3 The Relation of Repeated Games to Multiple–Round Games

3.1 General Remarks

Historically, multiple–round games evolved out of one–round games (in economic experiments), when trial repetitions were introduced to increase the probability of observing Nash equilibrium play in specific (uncompetitive) games (Smith, 1977, 1979). If the basis game is consistently competitive, however, the relationship of repeated and multiple–round games is much closer than that of one–round and multiple–round games. Multiple–round games are derived from repeated games through, on the one hand, converting the first n rounds into payoff–irrelevant trial repetitions (n is defined as the number of the first round, where the moves of the previous round are repeated by all players), and restricting the moves in the subsequent (payoff–relevant) rounds to those, that had been the repeated ones. Note, that these restrictions would be naturally satisfied, if the players' strategies (in repeated games) could be described as mappings of the previous round's moves (as a result of which moves, that are repeated once, are repeated forever), and if the players would not discount future payoffs too much (then the pre–fixed point payoffs are irrelevant besides the infinitely often paid fixed point payoffs). On the other hand, the moves in multiple–round games are required to be non–decreasing with respect to a given ranking of the strategies (first proposed by Banks et al., 1989). For these two modifications, multiple–round and repeated games are substantially different. But we will see that, given the basis game is consistently competitive and the players' behavior is refined in a certain way, the initial moves of any repeated game equilibrium (until a strategy combination has been repeated twice in a row) also combine to a multiple–round game equilibrium; and the remaining moves in the repeated game are equivalent to those that had been repeated. As a result of that, the analysis of infinitely repeated CGs simplifies enormously (to an analysis of multiple–round games). This is shown and discussed in the following.

In an infinitely repeated game, the current payoffs are independent of the previous–round moves, and so seem to be the current strategies (which would imply, in our case, that the competitive equilibrium is to be played in each round). Intuitively, however, human players construct intertemporal strategical dependencies, in that they evaluate the probability that their opponents cooperate (based on their previous moves), and they cooperate (move uncompetitively) when that probability is high. Early game–theoretic approaches, that were aimed to construct uncompetitive repeated–game equilibria, seemed to exploit that intuition directly: cooperate, when the opponent is trustable, and the opponent is trustable, when he cooperated in the previous round (*tit–for–tat*). Given the opponents start trustful, this is an equilibrium that is not only constructed, but also constructive (as competition is avoided), and so seems to be the underlying idea of trust–evaluations. This approach has been generalized by Rubinstein (1979) and Fudenberg and Maskin (1986), who slightly shifted the focus in the construction of the strategical interdependencies: from evaluating the opponents' trustabilities ex post, towards enforcing cooperativeness (trustable moves) ex ante by threatening to retaliate deviations. More precisely, based on an arbitrary division of the strategy space into cooperative and uncooperative moves, they showed that it is generally possible to construct retaliation programs, such that it is equilibrial that all players move cooperatively in all rounds (provided the cooperative payoffs exceed the minimax payoffs). It remains questionable, however, how the opponents reach an agreement about what is to be considered cooperative, and how they resolve disagreements. Usually, some cooperative concept of equilibrium selection is assumed to be applied.

We will examine a non–cooperative approach. In order to do so, we have to define abstractedly, what it means to face uncooperative opponents. As a first working definition, let us say that a strategy $S_{b,d} \in \overline{S}_{b,d}$ is uncooperative, *if the respective player is worse off sticking to* $S_{b,d}$ *(or to some* $S'_{b,d} > S_{b,d}$*) than to some* $T_{b,d} < S_{b,d}$ *in all of the remaining rounds, when he accounts for the (possibly differing)* *replies of his opponents to these moves*. This definition is very near to the initially cited argument of Rapoport (1966) and implies, essentially, that moving uncooperatively is unprofitable unless the deviating player aims to exploit the opponents' trustfulness for a single round (after that, the player would try to convince his opponents to cooperate again). We will restrict the relevant opponents' replies to those that are cooperative conditional on what is considered cooperative in the subgame reached (as a result of which our definition is recursive). Put less abstractly, with his deviation to $S_{b,d}$, b claims he would earn more from $S_{b,d}$ (or some $S'_{b,d} > S_{b,d}$) than he did before (from $T_{b,d}$), even if his opponents would reply (but not retaliate) this deviation; or, put from another point of view, with his deviation, b is claiming that the current outcome (based on $T_{b,d}$) would be unfair to him in the long term, as it awards him with less payoff than the negotiation outcome of a game where his strategy set is restricted to $\left\{S'_{b,d} : S'_{b,d} \geq S_{b,d} > T_{b,d}\right\}$. If that claim is justified, $S_{b,d}$ shall be considered cooperative, otherwise the deviation to $S_{b,d}$ is considered defective.

The justifiability of b's claim is evaluated in a stand–alone negotiation, where b's strategy set is restricted to what he claims would be sufficient, and where his opponents' strategy sets are restricted according to the claims they made before. The latter is not a unique way to proceed, but its consequences will turn out rather favorable with respect to implementing a refinement concept for Folk theorem equilibria (by the way, these consequences include that the evaluating players actually solve for the equilibrium payoffs in a multiple–round game). Before we can turn to that, however, we shall note that the requirement of increasingly competitive proposals is not biassing the negotiation process (and is therefore a feasible basis of a refinement concept). Precisely, we can show that the players are best off moving increasingly competitively when their opponents are moving so (i.e. that this assumption is equilibrial), as the consistency of the payoff structure (see characteristic *C3* and Implication 2) implies that, for all $S \in \bar{I}$ and all $b \in B$,

$$p_b(S_b, S'_{-b}) = \max_{S'_b \leq S_b} p_b\left(S'_{b,d}, S'_{-b,d}\right) \qquad \forall S'_{-b} \geq S_{-b} \tag{35}$$

S_b is the most profitable of all strategies, that are at most as competitive as S_b, and that holds in reply to any strategy that is at least as competitive as S_{-b}. That is, whatever it is that the opponents come up with in the following rounds, when one's current move is not excessively competitive (i.e. inside the interior $\bar{I}$, differing behavior would not be individually rational) and the opponents move increasingly competitive, then one is never better off moving less competitively than currently. Moreover, even off the equilibrium path, i.e. if the opponents move less competitively than is equilibrial (as a result of which, a player might find himself outside the interior $\bar{I}$), then the initially aspired negotiation outcome would still be equilibrial, and this player would not be better off moving backwards either. Thus, it is secured that our definition of the negotiation process is not corrupting the structure of IRCGs.

Summing up these considerations, let us put forward a second working definition of uncooperative strategies: *a strategy is uncooperative, if the player would not play that way in the respective subgame of a multiple–round game*, and the actual set of cooperative (i.e. undefective) strategies can be calculated through non–cooperatively analyzing a multiple–round game. As indicated already, we shall call any game, that we will (or players implicitly do) use to define the set of "cooperative moves" as a *negotiation*. However, we shall not stick simply to the multiple–round game, as negotiations (within repeated games) do not take place without the possibility of interferences (temporary defections of some players). Hence, it is more appropriate to reserve the term "negotiation outcome" to the equilibrium of a game, where negotiations and defections take place simultaneously. However, in order to avoid interferences due to informational imperfections concerning the distinction of negotiation proposals and defections (which are not part of the negotiations as such), we shall secure that these components are perfectly distinguishable in our negotiation game.

Therefore, in the game that we shall use to define cooperative behavior, the players are asked in each round to reveal (besides their actual, possibly defective moves) which levels of cooperativeness the propose in reply to their opponents' levels. These proposals shall be referred to as the *long–term* (or, second–order) components of the strategies. The actual moves of the repeated game shall be referred to as the *short–term* (or, first–order) components of the players' strategies. It is required that the second–order moves are increasingly competitive (which is called *long–term consistency* in the following). Thus, we are very close to our first working definition of cooperative/defective behavior, but it is accounted for all essential characteristics of repeated games. The consequences of using such a definition of cooperative/defective behavior are evaluated in the remainder of this paper.

3.2 Short–term Equilibria

As mentioned above, we will distinguish (explicitly or implicitly) first–order and second–order components. First, we will analyze the game of short–term concerns—a game with an explicit distinction of these components. Its subgame–perfect equilibria are called short–term equilibria. In that game, the strategy sets are restricted to adaptations of the short–term components (the payoff–relevant moves), while the negotiation proposals $P^{b,t'} \in \overline{S}$ are assigned exogenously (such that $P^{b,t'} \equiv P^{c,t'} =: P^{t'} \; \forall b, c, t'$). The short–term moves are based on (i.e. derived from) sets $\overline{C}_{b'}^{b,t'} = \{0,1\}$, that comprise the possible evaluations of b''s cooperativeness, out of the eyes of b in round t'. Player b' is evaluated cooperative in round t, $C_{b'}^{b,t'} = 1$, if his move $S_{b'}^{t}$ is less competitive than or equal to $P_{b'}^{b,t}$. The set $\overline{C}^{b,t'} = \bigtimes_{b' \in B} \overline{C}_{b'}^{b,t'}$ comprises the possible evaluations of all players out of the eyes of b in round t'. In this subsection, the evaluations out of the eyes of b are equivalent to those of any other player c, $C^{b,t'} \equiv C^{c,t'}$, as those are based on the same proposals $P^{b,t'} \equiv P^{c,t'}$, but in the general repeated game (which is analyzed further below), this needs not be the case. In order to keep the notation consistent, the evaluating player (b in $C^{b,t}$, $P^{b,t}$) is indexed already in this subsection. The exogenously set proposals will be allowed to vary in a predetermined way (i.e. it is a path instead of a strategy), which is a slight generalization of the usual Folk theorems, but not of substantial impact. In the following, remember that S^* denotes the combination of the most competitive strategies, and that S_* denotes the combination of the least competitive ones. Moreover, we adopt the preference relation of Fudenberg and Maskin (1986), which basically supposes, that the players discount future payoffs by $(1-\delta)$.

Definition 9 (Game of short–term concerns) *The time–varying benchmark P^t is defined as*

$$P^0 \in \left\{ S \in \overline{I} : p_b(S) > p_b(S^*) \; \forall b \in B \right\} =: \tilde{I}, \qquad P^t = \begin{cases} PU\left(P^{t-1}\right) & \text{, if } S^{t-1} \leq P^{t-1}, \\ P^{t-1} & \text{, otherwise,} \end{cases} \tag{36}$$

where it is required that $\tilde{I} \neq \emptyset$. $S_{b,d}^t$ as the move of player b in dimension d and round t; $PU : \overline{S} \to \overline{S}$ is *the proposal–updating function. It is required with respect to PU, that*

$$p_b(P) > p_b(S^*)\ \forall b \in B \quad \Rightarrow \quad p_b[PU(P)] > p_b(S^*)\ \forall b \in B \qquad \forall P \in \overline{S}, \tag{37}$$

$$\text{and} \qquad P \in \overline{I} \quad \Rightarrow \quad PU(P) \in \overline{I} \qquad \forall P \in \overline{S}. \tag{38}$$

Based on that, the individual proposals are defined as $P^{b,t} \equiv P^t$*, and each player* $b \in B$ *has to devise a system of functions* $\forall t$*, that describe b's moves depending on the previous moves in relation to* $P^{b,t}$

$$S_b^t : \bigtimes_{t'<t} \overline{C}^{b,t'} \to \overline{S}_b^t \equiv \overline{S}_b \qquad \text{with} \quad \overline{C}_{b'}^{b,t'} \ni C_{b'}^{b,t'} = \begin{cases} 1 & , \text{if } S_{b'}^{t'} \leq P_{b'}^{b,t'}, \\ 0 & , \text{otherwise.} \end{cases} \tag{39}$$

The payoffs in round t are $p_b(S^t)$*, and the payoff function complies with the characteristics of CGs. Each subgame–perfect equilibrium in the game of short–term concerns, that implies* $S^t = P^t\ \forall t$ *and for all discount rates* $(1-\delta) : \delta \in (\underline{\delta}, 1)$ *with* $\underline{\delta} < 1$*, is called short–term equilibrium.*

Note, that the above definition of time–varying proposals includes the standard case of constant uncompetitive proposals, by setting $PU(S) = S\ \ \forall S \neq P^0$, and setting $PU\left(P^0\right)$ respectively.

Leaned on the analysis of Fudenberg and Maskin (1986), and on the notation of Rubinstein (1979), we will describe a system of retaliation programs, which is a short–term equilibrium in *any* game of short–term concerns, regardless of the benchmark P^t (given it complies with the above assumptions). Let us begin with showing, that the payoffs of the benchmark strategies are indeed higher than the minimax payoffs, i.e. for all $b \in B$ and for all t, we see by using *(C1)* and *(C2)*, respectively,

$$p_b\left(P^{b,t}\right) \underset{\text{Assum}}{>} p_b(S^*) \underset{C1}{=} \max_{S_b \in \overline{S}_b} p_b\left(S_b, S^*_{-b}\right) \underset{C2}{=} \min_{S_{-b} \in \overline{S}_{-b}} \max_{S_b \in \overline{S}_b} p_b(S_{-b}, S_b). \tag{40}$$

Moreover, S^*_{-b} is an effective way of punishing *any* $b \in B$, i.e. the payoff of any b's best reply to the retaliation strategy is indeed less than his payoff from the uncompetitive proposals, as

$$p_b\left(P^{b,t}\right) > p_b(S^*) \equiv P_b\left(S^*_{-b}, \left\{BR_{b,d}\left(S^*_{-b,d}\right)\right\}_{d \in D}\right), \qquad \forall\, b \in B,\ \forall t, \tag{41}$$

and S^*_b is the optimal defense (reply) to S^*_{-b}. Hence, we need only consider one way of punishments, that fits for all players, and we can even ignore deviations in the punishment phase (since those are neither profitable for the deviating player, nor harmful to the opponents). The minimal retaliation, that $b \in B$ experiences in any round of the punishment phase is L_b, and the maximal surplus (with respect to the competitive equilibrium) that b might realize from defecting is R_b, as defined in the following (again, these values will be noted as being out of the eyes of b in round t, concerning b')

$$L_{b'}^{b,t} := p_{b'}\left(PM^{b,t,b'}\right) - p_{b'}(S^*) > 0, \qquad R_{b'}^{b,t} := p_{b'}\left(P_{-b'}^{b,t}, BR^*_{b'}\left(P_{-b'}^{b,t}\right)\right) - p_b(S^*), \tag{42}$$

$$\text{with} \quad PM^{b,t,b'} \in \arg \min_{P \in \left\{\tilde{P}^{b,t'}\right\}_{t'>t}} p_{b'}(P), \qquad \tilde{P}^{b,t} = P^{b,t}, \quad \tilde{P}^{b,t'} = PU\left(\tilde{P}^{b,t'-1}\right)\ \forall t' > t, \tag{43}$$

i.e. $PM^{b,t,b'}$ is the strategy combination, that induces the minimal payoff from cooperation to b' (of all strategies that are to be played in the rounds following round t, out of the eyes of b). Note that it is sufficient to know $P^{b,t}$, in order to calculate $L_{b'}^{b,t}$ and $R_{b'}^{b,t}$. Regarding the discount rate δ, we will generally require, that (for all b, b', t)

$$\delta > 1 - \frac{L_{b'}^{b,t}}{R_{b'}^{b,t}} \quad \Leftrightarrow \quad R_{b'}^{b,t} < \frac{L_{b'}^{b,t}}{1-\delta} \quad \Leftrightarrow \quad 1 - (1-\delta)\frac{R_{b'}^{b,t}}{L_{b'}^{b,t}} > 0, \tag{44}$$

which implies that it is generally possible to retaliate any player effectively, i.e. that, at least after a long run of retaliations, any deviating player regrets his defection.

Based on that, we can define functions $m_{b'}^{b,t} : \bigtimes_{t'<t} \overline{C}^{b,t'} \times \overline{P}^{b,t} \to \overline{M}_{b'}^{b,t} \equiv \mathbf{N}$ in a recursive way, such that $m_{b'}^{b,t}$ describes the number of rounds, that b' is still to be punished (out of the eyes of b in round t),

$$m_{b'}^{b,1} = 0, \qquad m_{b'}^{b,t} \underset{t>1}{=} \begin{cases} \left\lceil \log_\delta \left(1 - (1-\delta) * \frac{R_{b'}^{b,t-1}}{L_{b'}^{b,t-1}} \right) - 1 \right\rceil & , \text{if } C_{b'}^{b,t-1} \neq 1,\ m^{b,t-1} = \mathbf{0}, \\ m_{b'}^{b,t-1} - 1 & , \text{if } m_{b'}^{b,t-1} > 0, \\ 0 & , \text{otherwise.} \end{cases} \tag{45}$$

Thus, eventually, we can define a function $S_b^t : \overline{M}^{b,t} \to \overline{S}_b$

$$S_b^t \left(m^{b,t} \right) = \begin{cases} S_b^* & , \text{if } \exists c \in B : m_c^{b,t} > 0, \\ P_b^{b,t} & , \text{otherwise,} \end{cases} \tag{46}$$

and the retaliation program is completed. Defecting from retaliating an opponent is generally unprofitable, as the retaliation strategies are best replies to the opponents' retaliation and defense strategies. Defecting from S_b^t towards a less competitive strategy, when no one is to be retaliated, is unprofitable, because b's payoff is increasing in his competitiveness (as $P^{b,t} \in \overline{I}$), and the opponents' strategies would be unaffected. Defecting from S_b^t towards a more competitive strategy, when no one is to be retaliated, implies a one–round gain of $R_{b'}^{b,t} - \left(p_{b'}\left(P^{b,t}\right) - p_{b'}\left(S^*\right) \right)$, at most, but some $\underline{v}$ rounds of retaliations with losses of at least $L_{b'}^{b,t}$ (with respect to the cooperatively realized payoff). An implication of the definition of $\underline{v}$ in Equation (45) is, that that such a defection does not pay, i.e. with $\Sigma_{b'}^{b,t}$ as the aggregated surplus that b' realizes, when all players (including him) are cooperating in all of the rounds $t' \geq t$ (out of the eyes of b),

$$\Sigma_{b'}^{b,t} = \sum_{t' \geq t} \delta^{(t'-t)} \left[p_{b'} \left(\tilde{P}^{b,t'} \right) - p_{b'}\left(S^*\right) \right] \geq \frac{1}{1-\delta} L_{b'}^{b,t}, \tag{47}$$

we see that

$$\Sigma_{b'}^{b,t} \geq R_{b'}^{b,t} + \delta^{\underline{v}+1} \Sigma_{b'}^{b,t} \qquad \Leftrightarrow \qquad \left(1 - \delta^{\underline{v}+1}\right) \Sigma_{b'}^{b,t} \geq R_{b'}^{b,t} \tag{48}$$

is generally secured through setting $\underline{v}$ such that

$$\delta^{\underline{v}+1} \leq 1 - (1-\delta) \frac{R_{b'}^{b,t}}{L_{b'}^{b,t}}, \qquad \text{which implies} \qquad R_{b'}^{b,t} \leq \frac{1-\delta^{\underline{v}+1}}{1-\delta} L_{b'}^{b,t} \leq \left(1 - \delta^{\underline{v}+1}\right) \Sigma_{b'}^{b,t}. \tag{49}$$

Proposition 1 *The retaliation programs from Equations* (42)–(46) *are generally short–term equilibrial as defined in Definition 9.*

3.3 Long–term Equilibria

In this subsection, the game of long–term concerns is analyzed, where both, the long–term component and the short–term component of the repeated game strategy may be adapted; but still, these components are to be announced explicitly. This game shall provide us with an idea of how the players would negotiate about the definition of *cooperativeness*, when they could do so efficiently (as argued above).

In each round, the players announce proposals $PR_b \in \overline{S}\ \forall b$ for that benchmark. Those comprise **demands** $PR_{b_c} = \left\{PR_{b_{(c,d)}}\right\}_{d \in D}$ for all $c \neq b$, which describe the level of cooperativeness that b requires c to comply with, and an **offer** $PR_{b_b} = \left\{PR_{b_{(b,d)}}\right\}_{d \in D}$, that b is about to play if all opponents do comply with his demands. Similarly to the game of short–term concerns, the proposal of b' is not simply noted as $PR_{b'}$, but as the proposal $PR^{b,t}_{b'}$, that describes what b believes in round t. In the plain game of long–term concerns, the proposals are announced explicitly, and hence, all players have equivalent beliefs, $PR^{b,t} \equiv PR^{c,t}$ for all b,c,t. In the general repeated game (which is analyzed in the next subsection), this needs not be the case. In order to keep the notation consistent, the more general notation is used already in this subsection.

In each round, each player b updates and announces his proposal $PR^{b,t}_b$. This is described through a function, that maps any history of proposals to an updated proposal of b,

$$PR^{b,t}_b : \bigtimes_{t'<t} \bigtimes_{b' \in B} \overline{PR}^{b,t'}_{b'} \to \overline{PR}^{b,t}_b \equiv \overline{S}. \tag{50}$$

Using the updated proposal, and using the previous cooperativenesses $C^{b,t'}$ out of b's eyes, the actual (payoff–relevant) move of the repeated game is calculated,

$$S^t_b : \bigtimes_{t'<t} \overline{C}^{b,t'} \times \overline{PR}^{b,t}_b \to \overline{S}^t_b \equiv \overline{S}_b. \tag{51}$$

The $C^{b,t'}$ describe the relations of the moves in t' to b's (updated) proposals, $PR^{b,t'+1}_b$ in $t'+1$, and are defined precisely as those of Equation (39), with $P^{b,t'}_{-b} := PR^{b,t'+1}_{b_{-b}}$. We will generally assume that a player b', who considers himself defecting (or, retaliating), is also considered to be so by his opponents, or formally, that his opponents' demands in $t+1$ are not more competitive than the offer of b' in t,

$$PR^{b,t+1}_{b_{b'}} \leq PR^{b',t}_{b'_{b'}} \equiv PR^{b,t}_{b'_{b'}} \qquad \forall b' \neq b \in B. \tag{52}$$

Moreover, remember that we assumed that the players would move long–term consistently, i.e. the competitiveness of the players' offers is not decreasing in time, $PR^{b,t}_c \geq PR^{b,t-1}_c$ for all b,c,t.

With that in mind, and based on the retaliation programs from Equations (42)–(46), with $P^{b,t}_c := PR^{b,t}_{c_c}$, let us consider the following: all players are cooperating out of their eyes, i.e. $S^t_c \leq PR^{b,t}_{c_c}\ \forall c$, and all players, but b, consider also their opponents cooperating, i.e. $S^t \leq PR^{c,t}_c\ \forall c \neq b$ and $S^t \nleq PR^{b,t}_b$. This player (b) is therefore retaliating some of his opponents by playing S^*_b, which, in turn, the opponents consider to be a defection that calls for retaliation. In these circumstances, b suffers from implementing his retaliation, and (considering that the "defective" opponent can not reduce the competitiveness of his proposal offer anymore) b is better off adapting either his retaliation programs (to rule out retaliations of opponents, that consider themselves cooperating) or his proposal demand, such that his demands are generally not more demanding than the opponents' offers. In any case, b's proposal demand turns out irrelevant: either explicitly (in the first option), or implicitly (in the second option). To illustrate the latter, proposal demands, that are never more cooperative (i.e. never more demanding) than the opponents' offers, imply (in addition to Equation 52) that the current demands generally equal the opponents' previous offers, i.e. $PR^{b,t+1}_{b_{b'}} \equiv PR^{b,t}_{b'_{b'}}$ for all $b \neq b', t$. Hence, if the opponents' demands are in line with all offers, then the player in question is best off not to care about his demands; overall, therefore, it is equilibrial that the players neglect their demands. In the following, we will assume that they do. That is, the adaptations of the proposals will be restricted to adaptations of the offers, and the offers shall be referred to as $P_b^{b,t} := PR_{b_b}^{b,t}$.

Definition 10 (Game of long–term concerns) *Each player $b \in B$ has to devise a system of proposal functions $P_b^{b,t}$, with $P_b^{c,t} := P_b^{b,t}\ \forall c \neq b$.*

$$P_b^{b,t} : \bigtimes_{t'<t} \overline{P}^{b,t'} \to \overline{P}_b^{b,t} \equiv \overline{S}_b, \qquad \text{with} \quad P_b^{b,t} \geq P_b^{b,t-1} \quad \forall b,t>0, \quad \text{and} \quad P^{b,0} = S_*, \tag{53}$$

and a retaliation program according to (39)*. The payoffs in round t are $p_b(S^t)$, and the payoff function complies with the characteristics of CGs. Any strategy combination* (53)*, that combines with a short–term equilibrial* (39) *to a subgame–perfect equilibrium in the game of long–term concerns, for all discount rates $(1-\delta) : \delta \in (\underline{\delta}, 1)$ for some $\underline{\delta} < 1$, is called long–term equilibrium.*

Let us consider a player, who induces that he would gain from altering (i.e. from increasing the competitiveness of) his proposal offer $P_b^{b,t}$. Furthermore, he is discounting future payoffs by $(1-\delta)$ with $\delta < 1$. Obviously, this player would adapt the proposal as soon as possible. Moreover, since the players are completely informed and rational (which is common knowledge), there is no information to be deduced from the history of the proposals (except for their current realizations, which restrict the players' strategy sets). This implies that the following moves depend only on the current proposals, and (as the players do not delay adaptations of the proposals) the players move equivalently in subgames with equivalent initial proposals. Therefore, any combination of long–term moves, that is played for two consecutive rounds, must be a fixed point in the long–term moves, and it keeps played for the remainder of the infinite repetitions. We can easily verify, that any such long–term fixed point is in the interior of the strategy space (since anything else is not individually rational) and that it implies an allocation that is Pareto–superior or equivalent to the competitive allocation. Hence, infinite repetitions of any long–term fixed point combine with the retaliation program from Equations (42)–(46), or any other short–term equilibrial instance of (39), to a subgame–perfect equilibrium in the game of long–term concerns, and this combination would be long–term equilibrial according to Definition 10 (provided the players are required not to move less competitively than their fixed point strategies, as they are once this fixed point is reached).

Moreover, there generally exists some $\underline{\underline{\delta}} < 1$, such that any player with a discount rate $(1-\delta)$: $\delta \in (\underline{\underline{\delta}}, 1)$ considers any change in the pre–fixed point payoffs irrelevant in comparison to any change in the fixed point payoffs. If r is the number of pre–fixed point rounds, $\Delta_{pre} > 0$ is the suitably averaged gain in the pre–fixed point payoff (due to some move), and $\Delta_{fix} > 0$ the loss in the fixed point payoff, then we can calculate this set of feasible discount rates in the following way.

$$\frac{1-\delta^r}{1-\delta}\Delta_{pre} < \frac{\delta^r}{1-\delta}\Delta_{fix}, \qquad \Leftrightarrow \qquad \delta \in \left(\left(\frac{\Delta_{pre}}{\Delta_{pre}+\Delta_{fix}}\right)^{\frac{1}{r}}, 1\right) \neq \emptyset. \tag{54}$$

Hence, for suitable $\delta < 1$, the payment of pre–fixed point moves is only of secondary relevance with respect to the moves of the players in a game of long–term concerns (primarily, the fixed point payoff has to be maximized). Moreover, as we know from the short–term analysis, nobody would defect if the retaliation programs are short–term equilibrial. Hence, any long–term equilibrium path must also be a subgame–perfect equilibrium path in a game, where defections are impossible, where the moves are required to be (weakly) increasingly competitive, and where only the moves, that are the first ones to be repeated in two consecutive rounds, are payoff–relevant (precisely, the pre–fixed point segments of these paths are equivalent). This kind of game shall be introduced formally now.

Definition 11 (Multiple–round game) *In a multiple–round game, each player $b \in B$ has to devise a system of functions*

$$S_b^t : \bigtimes_{t'<t} \overline{S}^{t'} \to \overline{S}_b^t \equiv \overline{S}_b, \qquad \textit{with} \qquad S^0 = S_* \quad \textit{and} \quad S^t \geq S^{t-1}\ \forall t > 0. \tag{55}$$

The payoff is derived from the moves of round t^*, that is the first round where the played strategies equal that of the previous round, i.e. where $S^{t^*} = S^{t^*-1}$. The payoff is $p_b(S^{t^*})$, and the payoff function complies with the characteristics of CGs.

The opposite direction, however, does not hold generally, i.e. there are multiple–round game equilibria, that do not induce long–term equilibria. For, if a player might deviate from a multiple–round game equilibrium equilibrium path without consequences for his fixed point payoff, then the pre–fixed point point payoff becomes relevant. Thus, in a game of long–term concerns, a player might gain (marginally) through deviating from a multiple–round game equilibrium path. This can be ruled out when we require that the multiple–round game equilibria be refined in a certain way, towards (say) *path–optimized* multiple–round game equilibria. Essentially, in these refined equilibria, any unilateral pre–fixed point deviation induces a loss to the deviating player (in the fixed point payoff, or in the pre–fixed point ones). There are (at least) two ways to model such a refinement concept, which are, however, related rather closely. According to the first approach, we might rest on a concept as trembling–hand perfectness Selten (1975), according to which the pre–fixed point strategies in the multiple–round game would be perturbed (when the failure to adapt one's strategy has probability ε, and actual maladaptations have probability ε^{N+1}, with N as the number of players). Then, given the fixed point payoffs are fixed, it is of highest priority to optimize the pre–fixed point paths, as it is the most likely of (payoff–relevant) mistakes, that all players fail to adapt their moves and let the multiple–round game end under the current allocation. According to the second possible approach, the players' payoffs are perturbed, such that fixed point payoff equivalences are eliminated. This will be discussed further below (see also Breitmoser, 2002).

Regardless of the actually chosen approach, we see that any (and only a) path–optimized, subgame–perfect multiple–round game equilibrium induces the pre–fixed point segment of proposals, that combine with a short–term equilibrial retaliation program (39) and discount rates arbitrarily close to zero to a subgame–perfect equilibrium of a game of long–term concerns. That is, only such proposals are long–term equilibrial.

Proposition 2 (Equivalence of long–term and multiple–round game equilibria) *The proposal updating function of a game of long–term concerns is long–term equilibrial (see Definition 10), if and only if it is induced by a path–optimzed subgame–perfect equilibrium of the respective multiple–round game.*

3.4 Equilibria of infinitely repeated games

Eventually, we are examining plain infinitely repeated CGs as defined in the following.

Definition 12 (Infinitely repeated consistently competitive game) *Each player $b \in B$ has to devise a set of functions*

$$S_b^t : \bigtimes_{t'<t} \overline{S}^{t'} \rightarrow \overline{S}_b^t \equiv \overline{S}_b, \tag{56}$$

with $\overline{S}^{t'} = \bigtimes_{b' \in B} \overline{S}_{b'}^{t'}$; his payoff in round t is $p_b(S^t)$, and the payoff function is in accordance with the characteristics of CGs.

Most crucially, the players are not told, how their opponents' moves are to be decomposed into the short–term components and the long–term ones. This decomposition is of substantial relevance, as the intended signals differ. Moreover, short–term replies are derived in forward inductions (and the respective equilibria are forward induction equilibria), whereas long–term replies are derived in backward inductions. Let us assume that the players know the different ideas behind long–term and short–term moves, and, for their part, separate those in their own considerations. To do that efficiently, the players try to deduce (generally imperfectly and recursively) their opponents' long–term moves (proposals) from their plain repeated–game moves, some initial proposals $P^{b,0}$, and some information about the retaliations,

$$P_{-b}^{b,t} : \overline{S}^t \times \overline{M}^{b,t} \times \overline{P}_b^{b,t-1} \rightarrow \overline{P}_{-b}^{b,t} \equiv \overline{S}. \tag{57}$$

$$\tag{58}$$

Based on these hypothesized proposals, they update their own proposals (which, however, is only preliminary and will be corrected, if the opponents must have misunderstood these, see below)

$$P_b^{b,t+1} : \bigtimes_{t' \leq t} \overline{P}^{b,t'} \rightarrow \overline{P}_b^{b,t+1} \equiv \overline{S}_b. \tag{59}$$

By carrying out the recursion, this function combines with (57) to a function that deduces (generally imperfectly) the history of the opponents' proposals from the history of the moves, and updates the own proposals,

$$P^{b,*} : \bigtimes_{t' \leq t} \overline{S}^{t'} \times \bigtimes_{t' \leq t} \overline{M}^{b,t'} \times \overline{P}^{b,0} \rightarrow \bigtimes_{t' \leq t} \overline{P}^{b,t'} \times \overline{P}_b^{b,t+1}. \tag{60}$$

When we combine this with a function according to (39)

$$S_b^t : \bigtimes_{t'<t} \overline{C}^{b,t'} \rightarrow \overline{S}_b^t \equiv \overline{S}_b, \tag{61}$$

considering that

$$C^{b,t'} : \overline{S}^{t'} \times \overline{P}^{b,t'} \rightarrow \overline{C}^{b,t'} \qquad \text{and} \qquad m^{b,t} : \bigtimes_{t'<t} \overline{C}^{b,t'} \times \overline{P}^{b,t} \rightarrow \overline{M}^{b,t}, \tag{62}$$

then we have completed a repeated–game strategy.

The updating of the hypothesized opponents' proposals will be formulated (in a first step) as depending on a set $\tilde{P}^{b,t}_{b'}$ of *plausible long–term moves* (which in turn will be calculated using the information described in Equation 57). Moreover, the players will extract information about the proposals, if and only if the game had not been in the retaliation phase, and if the player in question played a move in $\tilde{P}^{b,t}_{b'}$ or a move that is less competitive than some move in $\tilde{P}^{b,t}_{b'}$. Besides, we will assume, that the players assume that their opponents would move long–term consistently (this assumption is common knowledge), i.e. that the implied long–term moves are non–decreasingly competitive, $P^{b,t} \geq P^{b,t-1}$ for all b,t. This does not imply that the players have to move long–term consistently, but we will see that they are best off doing so (thus seen, this assumption is equilibrial, and, more importantly, the resulting moves are equilibria). Moreover, this assumption is a substantial relaxation of the assumption, that is employed in Folk theorem equilibria (where the proposals are assumed to be fixed). Finally (and as indicated already), we will assume that the players correct their own proposals, if the opponents must have deducted those incorrectly (in order to circumvent unresolvable misunderstandings). That is, the own proposal is *rededucted*, and the preliminary one, derived in (59) is "overwritten." We will not analyze that point formally, as it merely simplifies to show that the moves of the game of long–term concerns constitute equilibria in the second–order moves of repeated games; apparently, the players are unable to improve upon that by provocing misunderstandings (thus, rededucting the own proposals is optimal). These characteristics give rise to the following proposal–deduction function, with $P^{b,0}_{b'} = S_{*b'}$

$$P^{b,t}_{b'} = \begin{cases} \left\{ \max \left\{ S^t_{b',d}, P^{b,t-1}_{b',d} \right\} \right\}_d & \text{, if } \exists P_{b'} \in \tilde{P}^{b,t}_{b'},\ S^t_{b'} \leq P_{b'},\ \text{and } m^{b,t} = \mathbf{0}, \\ P^{b,t-1}_{b'} & \text{, otherwise.} \end{cases} \qquad \forall b' \tag{63}$$

Basically, there are two different approaches to construct sets $\tilde{P}^{b,t}$ of plausible proposals: ex–ante evaluations of the moves' plausibilities, and ex–post ones. The ex–ante way is to deduce the set of plausible proposals before the actual moves are made (basically, from the previous hypothesized proposals, and usually in iterative processes), and the ex–post way is to deduce the set of plausible proposals from the previous proposal hypotheses and the actual moves. Moreover, there are two varieties of the ex–ante way (besides the possible requirement of equilibrium play, which appears not to be less restrictive than Folk theorems, however, and is therefore skipped). The first variety, the "positive" way, is in the spirit of the best–reply dynamics: in each iteration further moves are added to the set of plausible moves, based on the moves that have been considered plausible in the previous iterations. Secondly, the "negative" way (which is in the spirit of rationalizability): more and more moves are considered implausible (or, irrational), based on the moves that had been considered implausible before. Both of these varieties of ex–ante ways, however, are inapplicable in our case. For, the positive way may be too restrictive to consider resistance–dominant equilibrium proposals to be plausible (for resistance dominance see Güth and Kalkofen, 1989, and instances of such cases can be found in Breitmoser, 2002), and the negative way can be too unrestrictive to be useful (as there may be an abundance of long–term equilibria, and therefore hardly implausible moves, see Breitmoser, 2002, again). For these reasons, we will apply a variant of an ex–post evaluation. In order to do so, let us introduce b's *long–term payoff* $LP_b(P^t)$ of proposals P^t, which shall refer to the backward–induction payoff resulting in the respective subgame of a multiple–round game. Thus, a proposal of $b' \in B$ shall be considered plausible, if the long–term payoff $LP_{b'}$ of it in combination with the opponents' actual moves is not less than the long–term payoff of the previous hypothesized proposal of b' in combination with the opponents' actual moves. Formally, the set of plausible moves is

$$\tilde{P}^{b,t}_{b'} = \left\{ P'_{b'} \in \overline{S}_{b'} : LP_{b'}\left(P'_{b'}, S^t_{-b'}\right) \geq LP_{b'}\left(P^{b,t-1}_{b'}, S^t_{-b'}\right) \right\}. \tag{64}$$

Now, let us consider the combination of our proposal updating function (63, 64), a short–term equilibrial (39), and a long–term equilibrial (59), which is a complete strategy in a repeated game according to Definition 12. This strategy combination shall be called *negotiation–proof* and is generally subgame–perfect in a repeated game (as is shown in the following).

If no player is defecting and the players make (implicitly) long–term equilibrial proposals, then their proposals are deducted correctly from the repeated game moves, since equilibrium proposals are best replies to each other, and therefore not worse than the respective previous–round proposals. Moreover, short–term defections, that the opponents identify as such, are retaliated explicitly, and therefore dominated to be carried out in the first place. Contrary to that, short–term defections that the opponents mistake to be a long–term proposal, are replied through adaptations of their proposals. Thus, a different long–term equilibrium results, which must imply a payoff vector, that is weakly Pareto inferior or equivalent to the initial allocation (otherwise the first equilibrium would not be an equilibrium of our game). Hence, it is equilibrial not to defect, and likewise it is equilibrial not to deviate to excessively competitive long–term proposals. Explicitly or implicitly, any deviation from the long–term equilibrium path towards more competitive strategies would be retaliated.

Moreover, for the following reasons, the players are worse off deviating unilaterally from some standing proposals to less competitive ones (given the equilibrium path had not been left before). On the one hand, the deviating player's payoff would not increase thanks to favorable replies of his opponents, since they would not reply this adaptation differently than they replied his previous proposal (as they do not recognize the return to less competitive proposal). On the other hand, the deviating player's payoff would not increase thanks to unilaterally reaching more profitable areas of the strategy space, because the long–term equilibrium is path–optimized (in case the fixed point had not been reached yet) and because the fixed point proposals are in the interior of the strategy space (which applies otherwise, see also Equation 35). All in all, based on the constructed strategy combination, it is not profitable to deviate to a strategy above or below the long–term equilibrium path, and that holds regardless of whether the deviations are meant short–term of long–term. Hence, any player is best off moving short–term and long–term equilibrially, when the opponents do so. By the way, note that it is neither profitable to deviate substantially from the hypothesis updating function of (63, 64), since any possibly resulting deviation from the long– or short–term equilibrium path puts the respective player worse off.

Proposition 3 *There generally exists a* $\underline{\delta}$*, such that for all discount rates* $(1-\delta) : \delta \in (\underline{\delta}, 1)$ *any combination of (63, 64), a short–term equilibrial* (39)*, and a long–term equilibrial* (59) *is a subgame–perfect equilibrium of an infinitely repeated game according to Definition 12.*

Finally, let us inspect *defection–proof* strategies, which shall refer to combinations of (63, 64), a short–term equilibrial (39), and any instance of (59). We already know that, if the (59) is long–term equilibrial, then the resulting strategy combination is a subgame–perfect equilibrium of the repeated game for all discount rates with δ in some $(\underline{\delta}, 1)$. Now, let us assume that some proposal function can be inserted into the defection–proof strategy, such that the resulting function is a subgame–perfect equilibrium in the repeated game, even though the proposal function is not long–term equilibrial. Then, there is a player who is better off adapting his proposal in the game of long–term concerns (and therefore better off deviating in a repeated game, if his adaptation would be deducted correctly). Now, since he is better off adapting the proposal in a game of long–term concerns, he can not be worse off doing so in the respective multiple–round game; and hence, his adaptation would be deducted correctly. Thus, he can profitably deviate from the initial strategy combination, which therefore is disequilibrial. All in all, we can propose the following.

Proposition 4 (Long–term equilibria in repeated consistently competitive games) *In any infinitely repeated CG (Definition 12), there exists some* $\underline{\delta} < 1$*, such that the combination of (63, 64), short–term equilibrial* (39)*, and an instance of* (59) *is a subgame–perfect equilibrium for arbitrary discount rates* $(1-\delta) : \delta \in (\underline{\delta}, 1)$*, if and only if the* (59) *component is long–term equilibrial.*

4 Conclusive Remarks

In order to model an infinitely repeated consistenly competitive game (IRCG) comprehensively and efficiently, both, negotiations and defections, have to be considered simultaneously. To accomplish that, the players need to deduct their opponents' negotiative proposals from their actual moves. With respect to that, (63, 64) seems to be appropriate, since it works properly along the long–term equilibrium path, and (off the equilibrium path) cascades of misunderstandings are avoided and proposals are ignored, when those are implausible in light of the opponents' actual proposals (even if the ignored proposals should be rationalizable). If the players employ our (or a similar) deduction function, then there remain two aspects in repeated games: short–term equilibriality (i.e. to suppress defections by threatening to retaliate those) and long–term equilibriality (i.e. to negotiate optimally). Of those aspects, short–term equilibriality is of secondary relevance, as it lends support only off the long–term equilibrium path (there, it defends the equilibrium against defections), but it is (beyond that) not payoff relevant. Thus seen, both, the deduction function and the retaliation program, appear to be rather technical features of strategies in IRCGs, that rational players might simply be expected to handle properly. Moreover, since these features are implemented equivalently in all IRCGs, we can neglect the details of their implementations, and simply require that they are have set in the way we set them (or similarly). Above, the resulting behavior has been called defection–proof.

As a result of assuming defection–proof strategies, the repeated game degenerates (essentially) into a multiple–round game, and the equilibrium strategies can be induced backwardly. The resulting equilibrium might be called *negotiated*, as, after ruling out defections, the players concentrate on negotiating. Basically, the negotiated equilibrium (including its technical components) is constructed, as it rests on the (invented) notion of *trust* (which splits the repeated game moves into first–order and second–order ones). However, within the set of all equilibria that rest on trust (e.g. the Folk theorem equilibria), negotiated equilibria result from refining towards defection–proofness. As a result of that refinement, the main obstacle of Folk theorem equilibria (the arbitrariness with respect to the supported strategy combination) is resolved, even though the players are less restricted here. Secondly, the game, that the refinedly moving players apparently play, is more illustrative and less complex than the original game (which are characteristics, that refinement concepts usually do not share). Thirdly, the long–term equilibrium is found in the IRCG itself, and therefore appears to be more naturale a choice than those proposed by other (cooperative, say) concepts.

Above, it was mentioned, that there would be two manifestations of refinement concepts to path–optimize multiple–round game equilibria. On the one hand, strategy perturbations (trembling–hand perfectness, Selten, 1975), which (implicitly) induces incentives to care about the pre–fixed point paths, and on the other hand, payoff perturbations (mood–perfectness, Breitmoser, 2002), such that different paths generally imply different fixed point payoffs. Above, it was suggested that it would be optimal to employ any measure, that would increase the pre–fixed point payoff. Roughly, this corresponds with both, trembling–hand perfect equilibria in multiple–round games (when the players forget to move with probability ε and fail otherwise with probability ε^{N+1}), and mood–perfect equilibria, when the players are envying (i.e. when the players marginally prefer to harm their opponents). Let us suppose, however, that the interacting players have a reputation to maintain, which concerns their aggressivenesses in the path optimization. That is, the way that their moves are replied depends on the way, that they previously path–optimized multiple–round game equilibria. In these circumstances, it is likely to be inappropriate to be generally aggressive (see Breitmoser, 2002, for a similar result concerning multiple–round auctions).

Finally, let us look at implications for Bertrand competition. In our case, the competitors have equivalent marginal costs, and in the one–round equilibrium, the market price equals those (approximately). As is frequently argued (for example by Bertrand), however, prices can be adapted easily, and thus, Bertrand competitors interact (infinitely) repeatedly. The corresponding Folk theorem equilibria suggest, that uncompetitive prices may be equilibrial. The long–term equilibrium is (naturally) more precise about this: the competitors share the market at the (Pareto efficient) monopoly price (given the resulting market shares, in cases of tied prices, are independent of the market price; this equilibrium can be induced easily in the respective multiple–round game). By the way, the equilibriality of the monopoly price holds equivalently, when the competitors face capacity constraints; provided, no competitor's best reply to his opponents maximal quantities is more profitable to him than his payoff from the shared monopoly outcome. Notably, the latter condition is fulfilled, when the competitors set their capacities according to their respective oligopoly equilibrium quantities (which was proposed by Kreps and Scheinkman, 1983). Hence, the primary determinant of the equilibrium outcome in a combined model of capacity constraints and *infinitely repeated* Bertrand competition is not the fact, that the capacities have to be planned, but that the price setting is repeated infinitely.

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A Proofs

Proof of Implication 1 *(C2, C2*, C4)* are required to hold for each pair of strategy combinations, *(C3)* for each rectangle of strategy combinations, and *(C5)* for each strategy combination in relation to the most competitive strategies. Hence, these characteristics are untouched by eliminations. To evaluate the impact of eliminations on *(C1)*, let us consider $S'_{b,d}, S''_{b,d}, S'''_{b,d} \in \overline{S}_{b,d}$, with $S'_{b,d} = PD_{b,d}\left(S''_{b,d}\right) = PD^2_{b,d}\left(S'''_{b,d}\right)$, where $S''_{b,d}$ is the strategy to be eliminated. In case $S''_{b,d}$ has no predecessor or no successor, *(C1)* is not touched. The difference operator after the eliminations shall be denoted with Δ^A_c, and it is

$$\begin{aligned} & S^A_1 := \Delta^A_c p_{b,d}\left(S'_{b,d}\right) \equiv \Delta_c p_{b,d}\left(S'_{b,d}\right), \\ \text{and} \quad & S^A_2 := \Delta^A_c p_{b,d}\left(S'''_{b,d}\right) \equiv \Delta_c p_{b,d}\left(S'''_{b,d}\right) + \Delta_c p_{b,d}\left(S''_{b,d}\right). \end{aligned}$$

Now, there are two cases to be distinguished. First, consider $S_{-b,d} \in \overline{S}_{-b,d}$, such that $S''_{b,d} \notin BR_{b,d}\left(S_{-b,d}\right)$. Here, the triangular (i.e. concave) payoff structure is preserved, as (in SCGs) both summands in S^A_2 have equal signs, and thus, $\text{sign}\left[S^A_2\right] \equiv \text{sign}\left[\Delta_c p_{b,d}\left(S''_d\right)\right]$. This implies, that $\text{sign}\left[S^A_1\right] \geq \text{sign}\left[S^A_2\right]$, as we know that $\text{sign}\left[S'_{b,d}\right] \geq \text{sign}\left[S''_{b,d}\right]$. Secondly, consider the strategy combinations $S_{-b,d} \in \overline{S}_{-b,d}$ with $S''_{b,d} \in BR_{b,d}\left(S_{-b,d}\right)$. Then $S'_{b,d}$ or $S'''_{b,d}$ become best replies (if these had not been so already), and we see that in SCGs

$$\begin{aligned} p_{b,d}\left(S'_d, S_{-b,d}\right) > p_{b,d}\left(S'''_d, S_{-b,d}\right) \quad &\Rightarrow \quad \text{sign}\left[S^A_1\right] \geq 0 \geq -1 = \text{sign}\left[S^A_2\right], \\ p_{b,d}\left(S'_d, S_{-b,d}\right) = p_{b,d}\left(S'''_d, S_{-b,d}\right) \quad &\Rightarrow \quad \text{sign}\left[S^A_1\right] \geq 0 \geq 0 = \text{sign}\left[S^A_2\right], \\ p_{b,d}\left(S'_d, S_{-b,d}\right) < p_{b,d}\left(S'''_d, S_{-b,d}\right) \quad &\Rightarrow \quad \text{sign}\left[S^A_1\right] = 1 \geq 1 = \text{sign}\left[S^A_2\right], \end{aligned}$$

the requirements are fulfilled (and similarly it is in WCGs). ■

Proof of Implication 2 On the one hand, in SCGs, we see that no one of the best replies to $T_{-b,d} > S_{-b,d} \in \overline{S}_{-b,d}$ can be less competitive than $S^{BR}_{*(b,d)} = BR_{*(b,d)}\left(S_{-b,d}\right)$, since for all $S'_{b,d} < S^{BR}_{*(b,d)}$ we have (as a result of *C3*)

$$D^b_{b,d}\left(S_{-b,d}, S'_{b,d}, S^{BR}_{*(b,d)}\right) > 0 \quad \Rightarrow \quad D^b_{b,d}\left(T_{-b,d}, S'_{b,d}, S^{BR}_{*(b,d)}\right) > 0. \tag{65}$$

On the other hand, in both, strict and weak consistently competitive games, we have for any strategy $S''_{b,d} < S^{*BR}_{b,d} = BR^*_{b,d}\left(S_{-b,d}\right)$ that (again due to *C3*)

$$D^b_{b,d}\left(S_{-b,d}, S''_{b,d}, S^{*BR}_{b,d}\right) \geq 0 \quad \Rightarrow \quad D^b_{b,d}\left(T_{-b,d}, S''_{b,d}, S^{*BR}_{b,d}\right) \geq 0, \tag{66}$$

and hence, if any $S''_{b,d} \in BR_{b,d}\left(T_{-b,d}\right)$, then also $S^{*BR}_{b,d} \in BR_{b,d}\left(T_{-b,d}\right)$. ■

Proof of Implication 3 We know for any two opponents' strategy combinations $S_{-b,d} < T_{-b,d} \in \overline{S}_{-b,d}$, that the best replies' relation is $S^{*BR}_{b,d} \leq T^{*BR}_{b,d}$ with $S^{*BR}_{b,d} := BR^{*}_{b,d}\left(S_{-b,d}\right)$ and $T^{*BR}_{b,d} := BR^{*}_{b,d}\left(T_{-b,d}\right)$. First, consider $S^{*BR}_{b,d} = T^{*BR}_{b,d}$. We know from *(C2)* that in SCGs

$$p_{b,d}\left(S^{*BR}_{b,d}, S_{-b,d}\right) > p_{b,d}\left(S^{*BR}_{b,d}, T_{-b,d}\right) \equiv p_{b,d}\left(T^{*BR}_{b,d}, T_{-b,d}\right) \tag{67}$$

(and similarly in WCGS). Moreover, if $S^{*BR}_{b,d} < T^{*BR}_{b,d}$, we know from the combination of *(C1)* and *(C2)*, that (in both SCGs and WCGs)

$$p_{b,d}\left(S^{*BR}_{b,d}, S_{-b,d}\right) > p_{b,d}\left(T^{*BR}_{b,d}, S_{-b,d}\right) \geq p_{b,d}\left(T^{*BR}_{b,d}, T_{-b,d}\right). \tag{68}$$

Since the payoffs to all other best replies to $S_{-b,d}$ are equivalent to that of $S^{*BR}_{b,d}$, i.e. $p_{b,d}\left(S^{*BR}_{b,d}, S_{-b,d}\right) = p_{b,d}\left(S^{BR}_{b,d}, S_{-b,d}\right)\ \forall S^{BR}_{b,d} \in BR_{b,d}\left(S_{-b,d}\right)$, and similarly with $T^{*BR}_{b,d}$ and all $T^{BR}_{b,d} \in BR_{b,d}\left(T_{-b,d}\right)$, the following implication results. ■

Proof of Implication 4 Consider a player b, who is better off deviating from some strategy $S'_{b,d}$ to a more competitive strategy $S''_{b,d} > S'_{b,d}$ (while the opponents' strategies $S_{-b,d}$ are held fixed). If $S''_{b,d}$ is b's best reply to $S_{-b,d}$, then his payoff is (strictly/weakly) increasing in his competitiveness over all strategies in the set $\left[S'_{b,d}, BR_{*(b,d)}\left(S_{-b,d}\right)\right]$, and constant over the strategies in $\left[BR_{*(b,d)}\left(S_{-b,d}\right), S''_{b,d}\right]$. Therefore, he is (strictly/weakly) better off deviating from $S'_{b,d}$ towards any strategy in this set $\left(S'_{b,d}, S''_{b,d}\right)$. If $S''_{b,d}$ is not the best reply, then there are two possible cases for intermediate strategies $T'_{b,d}, T''_{b,d} \in \overline{S}_{b,d}$, as these may be more or less competitive than the best–reply strategies, i.e.

$$S'_{b,d} < T'_{b,d} \leq BR_{*(b,d)}\left(S_{-b,d}\right) \leq BR^{*}_{b,d}\left(S_{-b,d}\right) \leq T''_{b,d} \leq S''_{b,d}. \tag{69}$$

In both cases, and because of $p_{b,d}\left(S''_{b,d}, S_{-b,d}\right) > p_{b,d}\left(S'_{b,d}, S_{-b,d}\right)$, we see (due to *C1*) that in SCGs, the intermediate strategies offer a higher payoff than the initial $S'_{b,d}$, i.e.

$$\begin{aligned} &p_{b,d}\left(S'_{b,d}, S_{-b,d}\right) < p_{b,d}\left(T'_{b,d}, S_{-b,d}\right) \\ &\qquad \text{and}\ \ p_{b,d}\left(T''_{b,d}, S_{-b,d}\right) > p_{b,d}\left(S''_{b,d}, S_{-b,d}\right) > p_{b,d}\left(S'_{b,d}, S_{-b,d}\right), \end{aligned}$$

and similarly in WCGs. Hence, the player is (strictly/weakly) better off deviating to any $T_{b,d} \in \left(S'_{b,d}, S''_{b,d}\right)$ in this second case as well. ■

Proof of Implication 6 For any $EQ_d \in \overline{EQ_d} : EQ_d \neq S^{*}_{d}$, there exists a $b \in B$, who can deviate to his successor strategy $S'_{b,d} = PD^{-1}_{b,d}(S_{b,d})$, without losing (due to *C5* and Implication 4) and without gaining (because the initial strategy combination is an equilibrium). Hence (and since no opponent can be better off deviating from EQ_d), *(S2∗)* assures us, that b's deviation is irrelevant out of his opponents' eyes.

$$\begin{aligned} && p_b\left(S_d\right) = p_b\left(S_{-b,d}, S'_{b,d}\right) && \quad (70) \\ \Rightarrow && p_c\left(S_d\right) = p_c\left(S_{-b,d}, S'_{b,d}\right) && \forall c \neq b \quad (71) \\ \Rightarrow && D^{c}_{c,d}\left(S_d, S'_{c,d}\right) = D^{c}_{c,d}\left(\left\{S_{-b,d}, S'_{b,d}\right\}, S'_{c,d}\right) && \forall S'_{c,d} > S_{c,d},\ \forall c \neq b. \quad (72) \end{aligned}$$

From $\left\{S_{-b,d}, S'_{b,d}\right\}$, there is again a player who is not worse off deviating. On the one hand, this might be b again. In this case, his deviation from $S'_{b,d}$ to $''_{b,d} = PD_{b,d}^{-1}\left(S'_{b,d}\right)$ can not be profitable, otherwise S_d would not be equilibrial. Hence, this deviation is again irrelevant out of his opponents' eyes, and it is likewise irrelevant whether he plays $S_{b,d}$ or $S''_{b,d}$. On the other hand, a player $c \neq b$ might be not worse off deviating. As we know, c's deviations from $\left(S_{-b,d}, S'_{b,d}\right)$ are equivalent to those from S_d, i.e. unprofitable (since S_d is an equilibrium). Hence, these strategies of c are irrelevant (out of his opponents' eyes), and interchangeable. In this way, we can induce, that for each b' all strategies $S'_{b',d} > S_{b',d}$ are interchangeable and payoff–equivalent. This implies, that all of their combinations are equilibrial. ■

Proof of Implication 8 First, let us consider processes, that start in the interior of the strategy space. Consider any $S_d \in \bar{I}_d$, with $S_d \neq S_{*d}$, and calculate $T_d = BR_d(S_d)$, then it must be fulfilled that (for it is in the interior, and for Implication 2)

$$T_d = BR^*_d(S_d) \quad \Rightarrow \quad T_d \geq S_d \quad \Rightarrow \quad BR^*_d(T_d) \geq BR^*_d(S_d), \tag{73}$$

and therefore also $BR^*_d(T_d) \geq T_d$, which implies that T_d is again in the interior, i.e. $T_d \in \bar{I}_d$. Since we took the most competitive best replies to get T_d, it must be $T_d \geq S_d$. Moreover, if the initial strategy combination S_d has not been equilibrial, then there must exist a $b \in B$, suhc that $T_{b,d} > S_{b,d}$, and hence $T_d > S_d$, and if it has been equilibrial, then we know that $T_d = S^*_d$ (regardless of whether a weak or strict CG is considered). Thus, and since the numbers of strategies are finite, the processes of best–reply dynamics generally reaches S^*_d, and it does so in a finite number of steps.

Secondly, let us consider any strategy combination $S_d \in \bar{S}_d$, with $S_d \notin \bar{I}_d$, and take any $T_d \in \bar{I}_d$, such that $T_d < S_d$ (for instance the combination of the least competitive strategies). Now consider the sequences $\check{S}_d$ and $\check{T}_d$, starting in $\check{S}^0_d = S_d$ and $\check{T}^0_d = T_d$, respectively. Because of the increasing competitiveness of the best replies in the opponents' strategies (Implication 2), we find that at any stage t of these processes $\check{S}^t_{b,d} \geq \check{T}^t_{b,d}\ \forall b \in B$, i.e. $\check{S}_d$ keeps weakly more competitive than $\check{T}_d$. Hence, and since $\check{T}_d$ is converging to the competitive equilibrium, $\check{S}_d$ does converge as well. ■

Proof of Implication 11 Obviously, the opponent's payoff is decreasing in the own competitiveness *(C2)*, as the opponent's cake share is decreasing in one's competitiveness, whereas his stake is constant, and competition is socially inefficient *(C4)*, as the cake size is constant and any stake increase is merely aimed at redistributing it (and therefore lost for the society of the players). The remaining characteristics are slightly more involved. The Nash equilibrium of a two–player contest in general is $s = \frac{n-1}{\delta n^2}$; above it is assumed that there are no strategies that are more competitive than these. First, $p_1(a+s_1, b) \leq p_1(a,b)$, for some $a, b, s_1 > 0$, is equivalent to

$$b \leq \delta(a+b+s_1)(a+b) \tag{74}$$

$$\text{which implies} \quad b \leq \delta(a+b+k+s_2)(a+b+k) \qquad \text{for } k+s_2 \geq s_1, k \geq 0, s_2 \geq 0, \tag{75}$$

The latter is equivalent to $p_1(a+k+s_2, b) \leq p_1(a+k, b)$. Similarly, we can show that $p_1(a+k+s2, b) \geq p_1(a+k, b)$ implies $p_1(a+s_1, b) \geq p_1(a,b)$, and *(C1)* is (essentially) established. Secondly,

$p_1(a+s,b) \geq p_1(a,b)$ is equivalent to (for $a+s \leq \frac{n-1}{\delta n^2}$, $s \geq 0$)

$$
\begin{aligned}
& b \geq \delta(a+b+s)(a+b) \quad \Leftrightarrow \quad a+\frac{s}{2} \leq \sqrt{\frac{b}{\delta}+\frac{s^2}{4}}-b \\
& \underset{b+k \leq \frac{1}{4\delta}-\delta\frac{s^2}{4}}{\Rightarrow} \qquad \sqrt{\frac{b+k}{\delta}+\frac{s^2}{4}}-(b+k) \geq \sqrt{\frac{b}{\delta}+\frac{s^2}{4}}-b \geq a+\frac{s}{2} \\
& \underset{\frac{1}{4\delta}-\delta\frac{s^2}{4} \leq b+k \leq \frac{(n-1)^2}{\delta n^2}}{\Rightarrow} \qquad \sqrt{\frac{b+k}{\delta}+\frac{s^2}{4}}-(b+k) \geq \frac{n-1}{\delta n^2} \geq a+\frac{s}{2} \\
& \qquad \Rightarrow \quad b+k \geq \delta(a+b+k+s)(a+b+k),
\end{aligned} \tag{76}
$$

which is equivalent to $p_1(a+s,b+k) \geq p_1(a,b+k)$ for $b+k \leq \frac{(n-1)^2}{\delta n^2}$. Again, the opposite direction is derived similarly, and *(C3)* can be established along these lines. Finally, note that

$$
\begin{aligned}
& b \leq \frac{(n-1)^2}{\delta n^2}, \qquad a \leq \frac{b}{n-1}, \qquad a+k \leq \frac{n-1}{\delta n^2} \\
& \Rightarrow \quad \frac{n-1}{n} \geq \frac{\delta nb}{n-1} \quad \Leftrightarrow \quad 1 \geq \frac{\delta nb}{n-1}+\frac{1}{n} \quad \Leftrightarrow \quad b \geq \frac{\delta nb^2}{n-1}+\frac{\delta nb}{\delta n^2} \\
& \Rightarrow \quad b \geq \delta\frac{nb}{n-1}\left(b+\frac{n-1}{\delta n^2}\right) \quad \Leftrightarrow \quad b \geq \delta\left(b+\frac{b}{n-1}\right)\left(b+\frac{n-1}{\delta n^2}\right) \\
& \qquad \Rightarrow \quad b \geq \delta(b+a)(b+a+k) \quad \Leftrightarrow \quad p_1(a,b) \leq p_1(a+k,b),
\end{aligned} \tag{77}
$$

and thus, the deviation to the most competitive strategy is profitable for any player, whose initial stake is not higher than the mean of his opponents' stakes. Hence, *(C5)* is established and the proof is completed. ■

Proof of Implication 12 The best reply to any opponents' strategy combination is the least competitive strategy, that outbids all of the opponents' bids (if there is no such strategy, then all payoffs are equivalent, and the case is straightforward). The payoff of any move, that is less competitive than the best reply, is zero, and the payoff is decreasing for strategies beyond the best reply. Hence, *(C1)* is fulfilled. *(C2), (C2*),* and *(C4)* are fulfilled as well, since, from increasing a bid, the opponents' payoffs are either unaffected (if the high–bidder stays the same), or some opponent's payoff is decreasing (if one becomes the high–bidder). *(C3)* is fulfilled, because the payoff increase due to moving from some strategy to a more competitive one is negative, only if the initial strategy is higher than the best reply. In the case of *(C3)*, however, we are secured that the initial strategy is less competitive than the best reply to some opponents' bids, and hence, it is less competitive than the best reply to higher bids. Finally, *(C5)* is fulfilled, as, if the highest–bidding player is not the high–bidder, he is better off deviating to his most competitive strategy, or, otherwise, any other player is not worse off deviating. Note, that the case, that the highest–bidding player is the high–bidder, but does not employ his most competitive strategy, and no one of his opponents has a more competitive strategy to play, is ruled out in the above definition. Thus, the proof is completed. ■

Proof of Implication 13 Let us define $M_{-b} = \max_{c \neq b} S_c$, the most competitive strategy of b's opponents. Thus, the set of potential best replies to S_{-b} can be confined to two strategies: M_{-b} and $1+M_{-b}$.

If all of b's opponents play M_{-b}, then b's payoff from $1+M_{-b}$ is higher than that from M_{-b}, which is secured through the definition of the strategy sets.

$$S < \sqrt{\frac{(1-c)^2}{4\delta^2} + \frac{n}{(n-1)^2}} - \frac{n}{n-1} \tag{78}$$

$$\Rightarrow \quad \left(S + \frac{n}{n-1}\right)^2 < \frac{(1-c)^2}{4\delta^2} + \left(\frac{n}{n-1} - 1\right)\left(\frac{n}{n-1}\right) \tag{79}$$

$$\Leftrightarrow \quad \frac{n}{n-1}S^2 + 2S + 1 < \frac{n-1}{n} * \frac{(1-c)^2}{4\delta^2} \tag{80}$$

$$\Leftrightarrow \quad \frac{1}{n}\left(\frac{(1-c)^2}{4} - S^2\delta^2\right) < \frac{(1-c)^2}{4} - (S+1)^2\delta^2 \tag{81}$$

Thus, the required characteristics are secured. Regardless of M_{-b} or $1+M_{-b}$ is more profitable, *(C1)* is fulfilled, as the payoff of strategies, that are less competitive than M_{-b} is zero, and the payoff is decreasing beyond $1+M_{-b}$. *(C2)* is fulfilled, since, when the own payoff is increasing from some move, then one becomes a tied or single winner, and in any case, some opponents (those, that were winners before) are worse off now. Concerning *(C2$*$)*, b's payoff might keep constant, if he moves from a strategy below M_{-b} to another one below M_{-b}, or if b moves from M_{-b} to some $i+M_{-b}$. In the first case, b's move is irrelevant (and keeps irrelevant, when the opponents move more competitively), and in any instance of the second case, one of the opponents can profitable deviate. Concerning *(C3)*, M_{-b} is increasing in the opponents competitiveness. Now, if b's payoff increases, then b moves from a strategy below M_{-b} to M_{-b} or to a more competitive strategy, or b moves from M_{-b} to some $i+M_{-b}$, and the payoff might decrease, if b moves from M_{-b} or beyond M_{-b} to a more competitive strategy. Let us assume, that there is a contradiction to *(C3)*. Hence, in the initial strategy combination (before the opponents move), b's initial strategy at most M_{-b} (since his payoff is increasing from his move), and in the second combination, b's initial strategy is at least M_{-b} (since b is now losing from his move). Hence, the opponents' moves have not affected M_{-b}, b's payoff from M_{-b} has not risen, but the payoffs of the more competitive strategies are still as they were in reply to the opponents' initial strategy. Hence, the payoff difference from the deviation from M_{-b} has not decreased from the opponents' moves, and there is no contradiction to *(C3)*. The aggregate payoff (of all players) is

$$\Pi_{agg} \equiv \frac{(1-c)^2}{4} - S_{\max}^2\delta^2 \qquad \text{with} \qquad S_{\max} = \max_b S_b \tag{82}$$

is generally decreasing in $S_{\max}$, and thus not increasing in increasingly competitive moving players *(C4)*. Finally, if all of b's opponents and b himself play M_{-b}, then b gains from moving to $1+M_{-b}$ (see above), otherwise, an opponent is not worse off deviating to a more strategy. ■