# Coase and Aumann-Myerson's solution for Network Formation Games: Misleading Policy Implications.

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**Abstract**

The Coase theorem is evaluated when there is *sequential* strategic coalitional behavior and more than two agents are involved in the externality. Links or networks, needed for gains of cooperation, are proposed and payoffs are evaluated according to "who is more connected". We use the Aumann-Myerson's bargaining solution (1988). Two firms pollute a third one. Without liabilities, once the first link is *agreed,* the two players *choose* not to form extra links because they *foresee* a decrease in their bargaining power and hence lower payoffs; so only two-firm coalitions form even with strict superadditivity and costless links. If the latter is interpreted as zero transaction costs, then the latter result is inconsistent with the Coase theorem as with liability the grand coalition forms (thus efficiency results). We give sufficient and necessary conditions for related results in general three-person games. It is pointed out that for policy purposes it is crucial to distinguish inefficiency induced by factors others than transaction costs. Finally we give a counterexample in which allowing for externalities in coalition induces the grand coalition to form. As a solution concept, we use the extended Shapley value in Partition function form (Myerson 1977) and the link formation game of Aumann and Myerson.

*Ricardo Nieva is Visiting Faculty at Rochester Institute of Technology. Thanks again to Leonid Hurwicz as this paper was originated when he was wondering in a related project about enforcer games(Nieva October 2002) if its core was empty. The bad news for the implementation theory is that it was empty as in the game in this present paper.

# 1 Introduction

A version of the Coase theorem-that, with zero transaction costs and zero income effects, the identical Pareto-efficient allocation of resources will emerge regardless of the initial assignment of property rights- is evaluated when there is strategic coalitional behavior and more than two agents are involved in a production externality as in Aviazian, Callen and Lipnowski (1987). In less technical terms and to give intuition, we pose the following informal question, if there is costless bargaining among three firms, in which two of them generate pollution that affects negatively the production of the third firm: will the three firms come up with an agreement that for example leads the polluters to shut down and be compensated by the third firm *independently* if the polluters are liable or not, and the society as a whole gains, i.e. the outcome is the same and efficient?

We use the Aumann-Myerson (1988) bargaining solution concept for games of endogenous networks with sequential proposals. We claim that the Aumann-Myerson bargaining solution cannot be used in a proper formulation of the Coase theorem. In an informal way again, we claim that the Coase theorem does not hold if the latter bargaining solution is a realistic one, even though there is perfect information. Misleading policy implications are discussed.[1]

To illustrate environments where Aumann-Myerson's solution concept is relevant, in our opinion, for production externalities, we have an example due to Maschler (cited in Aumann and Myerson (1988)). Consider the coalitional game $v$ with identical players 1, 2 and 3 summarized as: $v(B) = 0$ if $|B| = 1$, $v(B) = 60$ if $|B| = 2$, $v(B) = 72$ if $|B| = 3$, where $|B|$ stands for number of players in coalition $B$ and $v(B)$ denotes its worth, the maximum payoff that the respective coalition can achieve. If every player acts alone they get zero each. Any pair colluded gets instead 60. Three players in a coalition yield still 72. The standard question in cooperative game theory is which coalitions form and how payoffs are distributed.

Most cooperative solutions argue that the three-player coalition results (or grand coalition) and 72 would be divided in an appropriate way. But suppose players 1 and 2 meet *in absence* of 3. Looking ahead, they would be happy with 30 each and would not invite 3 to join negotiations and instead, only a two-person coalition would be the outcome despite of its inefficiency. They fear that once 3 joins negotiations, the three players would be in symmetric roles and 72 may be divided equally, say (24,24,24) (for example according to the cooperative solution given by the Shapley value). Informally and to give intuition in our model, we could say that if the latter situation arises every player would end up with the same bargaining power.

Following the same example as in Aviazian, Callen and Lipnowski (1987 and 1981), that use instead the bargaining set as a solution concept, and a reply by Coase

[1] For a treatment of different interpretations of the Coase theorem and their implications, and for an adequate formulation of the mentioned theorem in terms of an adequate solution concept (say, analyzing bargaining outcomes with the core solution concept) see Hurwicz (1995).

(1981) we have two polluters that can affect a third agent. Two factories (firms 1 and 2) pollute a neighborhood laundry (firm 3). Let the characteristic function $v$ denote profits per day where $v(1) = 1 \quad v(2) = 2 \quad v(3) = 3 \quad v(1,2) = 8 \quad v(1,3) = 9 \quad v(2,3) = 10 \quad v(123) = 12$. It is reasonable to motivate the latter coalitions' profits as assuming that collusion can be thought as implying more worth because the polluters shut down and let the other firms produce and thus generate more profit. Firms do not discount the future. This characteristic function, as in Maschler's example, is strictly superadditive or equivalently colluding always increases total gains. Hence only the grand coalition can maximize joint profits.

It seems straightforward to think that with no transaction costs, or equivalently costless bargaining, firms have an incentive to negotiate and internalize the externality. In particular the grand coalition internalizes the externality completely and the joint gain would be one more than for any two-agent coalition. We explain that the latter is not the case when we use a different bargaining solution concept than the core because for this case the core is *empty* (as it is also the case in Maschler's example).

More formally and following Aumann and Myerson (1988), it is assumed that we have a coalitional game with side payments and perfect information. The procedure for bilateral links (necessary for gains of cooperation) or, equivalently, bilateral negotiations proposals by pairs of players follow a rule of order (thus, two players can meet in the *absence* of the third). Proposers evaluate their link proposals according to Myerson values (1977), that are predictions for individual payoffs or Shapley values that take into account the link structure, i.e. who has more "connections" matters. Evaluations depend additionally on other links that might be induced by them accepting or rejecting a proposed link (proposers are non-myopic). We assume also that the proposal procedure leads to a finite game. Finally, after the last link or, equivalently, bilateral negotiation has been agreed upon, every pair has a last chance to propose again. Given that this is a finite game of perfect information we have subgame perfect equilibriums in pure strategies in which no more links are accepted.

Let us assume in our example, as in the Coase theorem, that there is no liability. For the characteristic function above, once a link between any of the pair of agents called upon to propose is *agreed* and their payoffs are evaluated according to the Myerson value of the prospective new link structure, thinking ahead, each firm would not find it optimal to propose any other link. This happens because the second link induces the next pair of players, not linked yet, to propose and accept the third last link. Intuitively it should be clear (in contrast with the players in the first link thinking about forming a second link) that more connections are better in the latter case, thus the grand coalition forms. The players who formed the first link would see their payoffs or Myerson values decrease even if the game is strictly superadditive. In other words the first pair to form does not want to *individually* "communicate" with the third agent because even though total gains increase, their decreased bargaining power leads the first pair to lower payoffs. The latter happens despite of the inefficiency of the outcome.

In this kind of games, for the Coase's theorem to hold the following is necessary: Assuming that the game is *strictly* superadditive the complete graph (or everyone linked with everyone) that implies the grand coalition must be the unique subgame perfect equilibrium outcome. Otherwise a Pareto-optimal allocation need not obtain.

It is not clear from Coase if the bargaining procedure, when leading to an inefficient outcome, can be thought as creating a transaction cost. If not then we claim that the Aumann-Myerson bargaining solution cannot be used in a proper formulation of the Coase theorem[2] because of two reasons. First as we explained above, if there are no liabilities only non-efficient allocations are stable or equivalently the grand coalition doesn't form. Second, as Aivazian and Callen show, if the polluters are liable, then the grand coalition would result, the polluters would discontinue operations, and a Pareto-optimal allocation of resources would result.

Note that the cost of forming links is zero or we could say we have zero transactions costs. In other words it is not that negotiations are not possible. Instead two players *decide voluntarily and it is in their best self interest not* to choose to form additional links or coalitions because of the induced consequences. Ex-ante, before any pair is called to propose, we could argue instead that the rule of order of pairs called upon to propose induces a transaction cost. Therefore we have *ambiguity.* The policy implications are important because in case of inefficiency, and if this is a reasonable or realistic bargaining solution (see Nieva October 2002 for an argument in this sense in enforcer games), we would be looking to design institutions to deal with transaction costs that *don't even exist.*

One might think that the example above is patological, but note that in it we could say that production of externalities exhibits decreasing marginal productivity (see a precise definition of the term in section 2). Thus the gains in adding more polluters to a coalition decreases. It is also implied that adding a second polluter to say the coalition formed by players 1 and 3 decreases the percapita worth as is the case in the example above. Both implications in turn are not uncommon at all.

Also we give sufficient and necessary conditions in claim 2a for similar results for any three-person game that is superadditive. Only one-link graphs form (implying that the grand coalition does not form), iff for *at least* one-link graph structure (two players linked by a line represent a one-link graph for example) the Myerson values for the respective two players linked is strictly greater than their Shapley values or equivalently Myerson values in the complete graph. It is intuitively clear that if adding one more agent to a two agent coalition (ceteris paribus or all other thinks being constant) adds enough worth then the assumptions of claim 2a do not hold anymore and the complete graph that implies the grand coalition forms. In other words if the gains of adding one more player to the two person coalition are big

[2] Hurwicz (1996) points out that solution concepts with nonempty sets of solutions (at least in common real world examples ) that are both efficient and individually rational (PO-IR) should be used.

enough there is no way two agents would gain by not inviting the third one to the negotiations table. In our environment, if production of externalities exhibits high enough marginal productivity in production of pollution (see definition in section 2), even though the latter condition (adding one more player adds enough worth when a new coalition forms) might be satisfied the assumption of ceteris paribus may be violated as we have simultaneous changes in the worths of different coalitions. Thus the complete graph might not form.

It is important to see that in the characteristic function given above the core is empty. Nieva (October 2002) uses also the Aumann-Myerson solution concept and show that enforcer games can lead also to empty cores and find examples in a superadditive (not strictly superadditive) game in which the core is empty but the grand coalition forms. So emptiness of the core is not a sufficient condition for the grand coalition not to form at least in superadditive games. This result might look counterintuitive but it is important to realize that the key here is that 2 players block 1 imputation, i.e. the one given by the Shapley values of the complete graph, with the Myerson values implied by the first link they agreed to form. Even if this blocking is not possible and the complete graph forms by claim 2, the core can still be empty. Just think of the Shapley value imputation in question be blocked by a coalition of 2 that doesn't use the Myerson values.

In a related paper Dixit (2000) also points out to a sequential game where noncooperative behavior induces inefficient outcomes in a context of public goods. In our set up we have instead externalities in production when coalition formation depends on the voluntary "communication" structure (links) and agents can collude *even* in the first stage. In contrast, in the mentioned paper, Dixit assumes that the game in its first stage is non-cooperative and in the second one cooperative. According to the way we understood his model, we think that his assumption is ad-hoc in the sense that non-coalition formation in the first stage is not an equilibrium outcome according to the Nash program, but an assumption.

Finally in contrast to Aviazan and Callen (1987), allowing for spillovers in coalition formation when we have unilateral externalities of 2 producers again a third one may induce the grand coalition to form as the unique equilibrium outcome. The latter will happen if the collution of two polluter firms decreases enough the payoff of the third firm in comparison to the case where all firms operate separately. As solution concepts, we use the extended Shapley value in Partition function form (Myerson 1977) and the link formation game of Aumann and Myerson (1988).

In the next section we describe the environment more in detail. A review of the Myerson Value and Aumann-Myerson solution for sequential network formation games follow together with the equilibrium analysis and results in section number 3. Section 4 includes conclusions. The appendix includes the environment allowing for coalition formation with the counterexample to Avazan and Callen.

## 2 Environment

Using the same example as in Aivazian and Callen (1981), we consider 2 factories (firm 1 and 2) polluting a neighborhood laundry (firm 3). Firms do not discount the future. The laundry does not emit pollutants. Let the characteristic function $v$ denote profits per day where it is assumed that in absence of negotiations they get:

$v(1) = 1 \qquad v(2) = 2 \qquad v(3) = 3$

If negotiations are possible and merges brings about the reduction of pollution we will have the following coalitional values:

$v(1,2) = 8 \qquad v(1,3) = 9 \qquad v(2,3) = 10 \qquad v(123) = 12$

The characteristic function is strictly superadditive or in other words the formation of coalition always gives a net positive gain. Hence only the grand coalition can maximize joint profits. If players 1 and 3 would collude their net gain would be of 5. Even if the two factories collude the net gain is positive. One way to motivate the latter would be that one of them pollutes the other one or maybe some economies to scale or some fixed cost sharing kick in. For simplicity we assume that there are not spillovers in coalition formation (i.e. the value say of firm 3 is the same if the firms 1 and two collude or not).Also, for assessing the importance of marginal productivity in production of pollution for our results, it will be useful to assume that firms 1 and 2 cannot pollute each.

Only if the grand coalition forms the externality is completely internalized (we can imagine that the polluters close down) and total gains are maximized. We point out that this example is not patological because we could say that production of externalities exhibits decreasing marginal productivity. More precisely:

Definition:

The *marginal productivity in production of pollution* of polluter 1 is defined as the extra amount of pollution of firm 1 imposed on the laundry, player 3, and is directly proportional to the difference $v(123) - v(23)$ or $v(23) - v(3)$. The lower the marginal productivity the lower is the gain of forming the coalitions with the polluters given our assumption that the polluters shut down. We define for player 2 the respective term in analogous way$\blacksquare$.

Also in this example the core is empty, where the core is the set of allocations $x$ that satisfy the following conditions:

$x_1 \geqq v(1) \qquad x_2 \geqq v(2) \qquad x_3 \geqq v(3)$

$x_1 + x_2 \geqq v(12) \qquad x_1 + x_3 \geqq v(13) \qquad x_2 + x_3 \geqq v(23)$

$x_1 + x_2 + x_3 \geq v(123)$

Note that if the core is non-empty then $v(123) \geq \frac{v(12)+v(13)+v(23)}{2}$. Clearly this does not hold in the example above.

Before we review the idea of networks and the related negotiations or communication environment let $N$ be a finite set of players. Given $N$, let $CL$ be the set of all coalitions (non-empty subsets) of $N$,

$CL = \{B \subseteq N, B \neq \varnothing\}$

Let $PT$ be the set of partitions or *coalition structures* of $N$, so
$\{B_1, ..., B_l\} \in PT$ iff:
$U^l_{i=1} B_i = N, \forall j\ B_j \neq \varnothing, \forall k B_i \cap B_k = \varnothing$ if $k \neq j$.

A richer negotiations framework (allowing for network formation) will help us formulate our arguments precisely. For modelling this richer kind of setups Myerson (1977) defined a cooperation structure (or cooperation graph) in a coalitional game. This graph is one whose vertices are the players .As in Aumann and Myerson (1987) a link between two players (and edge of the graph) exists if it is possible for them to carry on meaningful direct negotiations with each other.

In particular, ordinary partitions or coalition structures$\{B_1, ..., B_l\}$ may be modeled within this framework by defining two players to be linked if and only if they belong to the same $B_j$.

In our case for three players we have the empty graphs without links. The complete graph $g^N$ in contrast is represented as:

```
  3
 / \
1——2
graph N
```

The one link graphs representation is:

```
  3           3          3
 /             \
1   2       1   2      1——2
graph 1     graph 2    graph 3
```

The two links graphs:

```
      3                 3        3
     / \                 \      /
    1   2              1— 2    1—2
   graph 4            graph 5  graph 6
```

As is explained in the next section the bargaining environment can be modelled sequentially in a game with finite number of stages.

With respect to the information structure, after any stage, everyone observes past moves including mixed actions.

# 3 Decision rules and solution concepts in a sequential model of endogenous networks

## 3.1 The Myerson value and the Aumann-Myerson solution

This section follows Aumann and Myerson (1988) very closely.

### 3.1.1 The Myerson value

The Shapley value evaluates the players' prospects when there is full and free communication among all of the players–when the cooperation structure is 'full', when any two players are linked. Otherwise a player say i who is totally isolated(no links with anyone) can expect to get nothing beyond his own worth $v(\{i\})$; in general the more links a player has with others, the better the expected prospects.

Myerson (1977) defined an extension of the Shapley value of a coalitional game $v$ to the case of an arbitrary cooperation structure g for games with transferable utility as is our case. In particular, if $g$ is the complete graph on the all-player set $N$ (any two players are directly linked) as in $g^N$, then Myerson's value coincides with Shapley's. Moreover, if the cooperation graph $g$ corresponds to the coalition structure $(B_1, ..., B_l)$ in the sense indicated here, then the Myerson value of a member $i$ of $B_j$ is the Shapley value of $i$ as a player of the game $v|B_j$($v$ restricted to $B_j$). The key difference in this restricted game is that the worth of the union of say $B_j \cup B_k$equals the worth of the sum $v(B_j)+v(B_k)$ according to the original characteristic function $v$. Given that there are no links between some player in $j$ and some player in $k$ they don't get the benefits derived from the union of the coalitions in question according to $v$. If the game is strictly super-additive then for all $k$ and $j$, where $j \neq k$, $v\,(B_j \cup B_k) > v(B_j)+v(B_k)$.

Let be given a coalitional game $v$ with $N$ as player set and $g$ a graph whose vertices are the players. For each player $i$ and given the graph $g$ and the characteristic function $v$, the Myerson value $\phi_i^g = \phi_i^g(v)$ is determined by the following axioms:

Axiom 1. If a graph $g$ is obtained from another graph $h$ by adding a single link, namely the one between players $i$ and $j$, then $i$ and $j$ gain (or lose) equally by the change; that is,

$\phi_i^g - \phi_i^h = \phi_i^g - \phi_i^h$

Axiom 2. If $S$ is a connected component of $g$, then the sum of the values of the players in $S$ is the worth of $S$; that is,

$\sum \phi_i^g(v) = v(S)$,

where a connected component of a graph is a maximal set of vertices of which any two may be joined by a chain of linked vertices.

The Myerson value is unique and if $v$ is superadditive, as our game is, then two players who form a new link never lose by it: Note that the two sides of the equation in Axiom 1 are nonnegative. Myerson also established the following practical method: Given $v$ and $g$, define a coalitional game $v^g$ by

$v^g(S) := \sum v^g(S_j)$,

where the sum ranges over the connected components $S_j^g$ of the graph $g|S$ ($g$ restricted to $S$). Then

$\phi_i^g(v) = \phi_i(v^g)$

where $\phi_i$ denotes the ordinary Shapley value for player $i$.

### 3.1.2 The sequential link formation game

Given a coalitional game $v$ with $n$ players with no links, the game consists of pairs of players being offered to form links according to some rule of order. Links are formed when both parties agree; once it is formed, a link cannot be destroyed and all previous proposals and rejections are known history (this game is of perfect information). The crucial detail of the proposal procedure is that it leads to a finite game, and that after the last link has been formed, each of the $\frac{n(n-1)}{2}$ pairs must be given a final opportunity to form an additional link (as in bridge). At this point some cooperation graph $g$ has formed and the payoff to player $i$ is then the Myerson Value and is denoted by $\phi_i^g(v)$.

Given that the game is of perfect information it has subgame perfect equilibriums in pure strategies. Each such equilibrium is associated with a unique cooperation graph $g$, namely the graph reached at the end of play. Any such $g$ (for any choice of the order on pairs) is called a natural structure for $v$ (or a natural outcome of the linking game). If all subgame perfect equilibriums of subgames (for any choice of order) dictate that no additional links form, then $g$ is called stable.

## 3.2 Equilibrium Analysis

### 3.2.1 Equilibrium without liabilities and "zero" transaction costs

In this subsection we compute the Myerson values for different graphs and find the subgameperfect equilibriums or Aumann-Myerson solutions (for an illustration see last subsubsection).

The Myerson value for graph 1 is $(3.5, 2, 5.5)$
The Myerson value for graph 2 is $(1, 4.5, 5.5)$
The Myerson value for graph 3 is $(3.5, 4.5, 4)$
The Myerson value for graph 4 is $(3, 4, 5)$
The Myerson value for graph 5 is $(2\frac{1}{6}, 5\frac{4}{6}, 4\frac{1}{6})$
The Myerson value for graph 6 is $(4\frac{4}{6}, 3\frac{1}{6}, 4\frac{1}{6})$
The Myerson value for the complete graph is $(3, 4, 5)$

Claim 1: The endogenous game of link formation with the Myerson value as fixed valuation has 3 natural subgame perfect equilibria where *either* coalition 13, 12 or 23 form but never the two or three-link graph (for a general proof see claim 2 below).

Proof: Suppose we are at a stage where there are 2 links. The players that have not formed a link yet have an incentive to form one and the complete graph forms. Therefore we can assume that from a 2 links graph the 3 links graph will form inevitably.

Beginning with one link, say graph 1, (where the link is between players 1 and 3), player 3 would not like any other link because 5.5 is the maximum she can get; as both players that propose a link next have to agree to get connected, it doesn't matter what player 2's prospects are. However players 1 and 2 have an incentive to form one, though thinking ahead player 1 won't because she would get less after the complete graph inevitably forms.

Imagine now we begin with graph 2; looking ahead players 2 and 3 would not like to form a link with 1 because their payoffs would decrease after the complete graph forms. We have an analogous result if we begin with graph 3 instead.

Beginning with no links, if we have the rule of order 12 13 23 (pairs of player $(1, 2)$, $(1, 3)$, $(2, 3)$ propose bilateral links in that order), provided that the first two pairs refused, the last pair would get linked (otherwise they would get their individual worths $v(2)$ and $v(3)$). One stage backward if players in 12 have rejected agents in 13 get to propose. Player 3 is indifferent between accepting or rejecting because in either case she gets 5.5. The other member of the proposed link, player 1, has as *a* weakly dominant stage action to accept other wise she may be left out with only 1.in payoff.

Thus at the first stage the optimal individual action will depend on what player 3 does in the second stage:

a)Suppose link 13 forms, then 1 will be indifferent between accepting or rejecting the link with player 2 because in either case she gets 3.5. Player 2, has as a weakly dominant stage action to accept other wise she may be left out with only a payoff of 2. Hence there are two possible subgame perfect equilibriums depending on what 1 decides. Either 12 or 13 forms.

b)If 3 rejects 1 will have as dominant action to accept link with 2. Player 2 will always reject because then 23 would finally form.

Hence depending on what 3 does in the second proposal there are possibility of 3 outcomes. In any case the complete network doesn't form!.

In an analogous way, neither the two link or three-link graph form if we instead have the rules of order 12 23 13.or 23 12 13. We can apply alternatively claim 2 below directly■.

We state a result from Nieva (October 2002): An empty core is not sufficient for the complete graph not to form if the game is superadditive. A counterexample is given in the latter paper though the game is not strictly superadditive.

## Claim 2:

Assume we have 3 person normalized superadditive games that are non-trivial (i.e. value of 2 person coalitions are strictly positive) or where $v(i) + v(j) < v(i + j)$ for $i \neq j$ and $i, j \in (1, 2, 3)$.

a) Only one link graphs form, iff for a*t least* 1 link graph structure (graphs 1, 2 and 3 in our model) the Myerson values for the respective two players linked is strictly greater than their Shapley values in the complete graph. If the above conditions hold for graphs either $i$, or $i$ and $j$, or $i$, $j$ and $k$ then graphs $i$, $i$ or $j$, and $i$, or $j$ or $k$ form respectively in equilibrium for $i \neq j \neq k$ and $i, j, k \in (1, 2, 3)$ in our model.

b) In contrast the complete graph forms iff the conditions in (a) do not hold.

## Proof:

Claim 2a :Without loss of generality assume that the 13 link is associated to a graph with such characteristics. Suppose three links or the complete graph form. Then for some 2 link graph it is optimal for two players to propose the last link. Also there is a one link graph that finds it optimal to accept a second link. If all links would satisfy the assumptions of the claim then the complete graph would have a deviation as a second link would not be accepted by neither of the players in the first link to form. We focus then on the situation when at least there is one link among 23 and 12 that has not strictly higher Myerson values for both of their two members. We will show that at least one of the members in the first link that was formed on the path to the complete graph in the assumed equilibrium wants to deviate. Again without loss of generality let us assume that the latter first link is 12. Thus (and consistent with proposal procedure) the pair of players (1,3) are still out there to propose in case 12 is rejected and *maybe* also the pair (2,3).

1st case: Assume players in 23 have Myerson values that are not both strictly higher than their Shapley values in the complete graph. If player 1 accepts link 12 she gets her Shapley value in the complete graph by definition of equilibrium. If p1 rejects she gets her Myerson value in 13. She expects in one case p3 to reject a link with p2 and accept a link with p3 if the rule of order mandates first pair (2,3) and second (1,3). In the other case, pair (1,3) is next and p1 would reject link 12 knowing that player 3 would accept 13. If only 13 is left to propose the result is obviously the same. Thus player 1 deviates.

2nd case: Assume instead that 23 has Myerson values that are both strictly higher than their Shapley values in the complete graph. if p3's Myerson value in 23 is strictly lower than her's in 13 then the same argument used in the 1st case holds. If the latter Myerson value is strictly higher instead then p2 deviates. If p3's Myerson value in 23 is equal to the one she gets in 13 then, depending on what p3 does, there would be associated deviations. Note that in this 2nd case for sure both13 and 23 are still out there to propose otherwise the game would have ended already.

Finally we need to check if the two link graph does not form (otherwise the grand coalition forms *though* with an incomplete graph).By assumption $v(i)+v(j) < v(i+j)$ for $i \neq j$ and $i, j \in (1, 2, 3)$ So by definition of the Myerson value (see end of this subsubsection for a related illustration), the players not linked form the third link as their payoffs increase; thus there is a deviation.

Conversely without loss of generality assume that only one-link graph forms and for no one-link graph the Myerson values for both associated two players linked is strictly greater than their Shapley values in the complete graph. Let us take one-link graphs. The players in this first link anticipate that a two link graph always lead to the complete graph and hence it is optimal at least for one player to accept a second link independently of the rule of order. This is a contradiction.

Given that this is a finite game of perfect information there is at least one subgameperfect equilibrium in pure strategies. Thus the rest of claim 2a follows.

Claim 2b: Given that $v(i) + v(j) < v(i + j)$ for $i \neq j$ and $i, j \in (1, 2, 3)$ a 2 link graph never forms in any equilibrium thus only the complete graph forms$\blacksquare$.

The reader can check that in the example with production externalities given above the assumptions of claim 2 hold.

### 3.2.2 The Role of Marginal Productivity in Pollution Production: An Illustration of The Myerson Value

It is intuitively clear that if adding one more agent to a two agent coalition adds enough worth then the assumptions of claim 2a do not hold anymore and the complete graph that implies the grand coalition forms. In an environment with production externalities, if production of externalities exhibits high enough marginal productivity in production of pollution (see definition in section 2) then the assumptions of claim 2a might be satisfied even though adding one more agent to a two agent coalition adds enough worth. We explain the latter assertion by illustrating the use of the Myerson value in what follows:

We focus on the Shapley value and the Myerson value of player 1 when the only link corresponds to the one between players 1 and 3.

By definition, the Shapley value for player 1 for a 3-person game is:

$\phi_1 = \frac{1}{6}(123 + 132 + 213 + 231 + 312 + 321)$, where
$123 = v(1)$
$132 = v(1)$
$213 = v(21) - v(1)$
$231 = v(123) - v(23)$
$312 = v(13) - v(3)$
$321 = v(123) - v(23)$

The triplets stand for the extra addition to the total worth if player 1 "gets in" depending on different orderings. The Myerson value for player 1 if only graph 1 forms or there is only one link, i.e. the link corresponds to the one between players 1 and 3, can be computed as follows:

$\phi_1^{g=1} = \frac{1}{6}(123 + 132 + 213 + 231 + 312 + 321)$, where
$123 = v(1)$
$132 = v(1)$
$213 = [v(2) + v(1)] - v(1) = v(2)$
$231 = [v(13) + v(2)] - [v(2) + v(3)] = v(13) - v(3)$
$312 = v(13) - v(3)$
$321 = [v(13) + v(2)] - [v(2) + v(3)] = v(13) - v(3)$

We are interested in:
$\phi_1 - \phi_1^{g=1} = [v(21) - v(1)] + 2[v(123) - v(23)] + [v(13) - v(3)] - v(2) - 3[v(13) - v(3)]$
$= [v(21) - v(1)] + 2[v(123) - v(23)] - v(2) - 2[v(13) - v(3)]$,
after rearranging,
$= [v(21) - v(1)] - v(2) + 2[v(123) - v(23)] - 2[v(13) - v(3)]$

If the $v(123)$ is high enough then ceteris paribus $\phi_1 > \phi_1^{g=1}$. After picking $v(123)^*$ such that the conditions hold for claim 2b, then the complete graph will form. That means $v(123)^*$ (ceteris paribus) has to be high enough so that at least for all players in one-link graphs it is the case that their Myerson value is lower or equal than her Shapley value in the complete graph. An increase in marginal productivity in production of pollution will produce an increase in $v(123)$. but it also increases $[v(13) - v(3)]$, thus the effect is undetermined. It would be worth while to have explicit functional forms for the referred marginal productivity as for the following argument. Suppose firms 1 and 2 are identical and don't pollute each other and that there is a "parallel shift" up in the marginal productivity in production of pollution. Then the last two terms in the previous equation cancel each other out and we have a constant term:

$\phi_1 - \phi_1^{g=1} = [v(21) - v(1)] - v(2) > 0$,

by strict superadditivity. Hence we have no effect. It would be also interesting to relax the assumption that the 2 firms do not pollute each other. In this the latter result does not need to hold.

We could also check under which conditions an increase in the referred marginal productivity increases the difference $\phi_1 - \phi_1^{g=1}$. The same exercise should be done for all possible one-link graphs and their respective two members so that the conditions for claim 2a do not hold . Only then we could analyze properly the role of the marginal productivity in production of pollution in the formation of the grand coalition. Summarizing, we can claim that marginal pollution productivity of pollution is not necessarily a determinant in the formation of the complete graph.

### 3.2.3 Equilibrium with liabilities

We state Aviazian and Callen's (1981) result. If players 1 and 2 are liable for the pollution damages to player 3 it is obvious that the grand coalition solution must obtain and 1 nor 2 will be allowed to produce. Here it is assumed that this is the most efficient outcome. Otherwise we would have to specify each firm's production function, or equivalently, a level of output for each level of pollution; Aviazian and Callen claim that the results would not change in this latter case.

# 4 Conclusions

The complete graph that implies the grand coalition will form if firms are liable. In the case they are not, neither the complete graph and nor the incomplete two-link graph (both implying the grand coalition) will form. Hence the Aumann and Myerson solution has to be excluded for a proper formulation of the Coase theorem. For example as Hurwicz (1996) argues, solution concepts with nonempty sets of solutions (at least in common real world examples ) that are both efficient and individually rational (PO-IR) should be used.

Our conclusion could be questioned though if the bargaining procedure, as long as it leads to an inefficient equilibrium outcome, is considered a transaction cost. Note however that the cost of forming links is zero or we could say that we have zero transactions costs. The important point is that it is not that negotiations are not possible. Instead two players *decide voluntarily and it is in their best self interest not* to choose to form additional links or coalitions because of the induced consequences. In the latter sense this richer negotiation framework (endogenous networks) *allows* in a rigorous way the distinction between impossibility to negotiate (in which case we could properly say there is indeed a transaction cost; for example if links were costly) and not willingness to cooperate. It is our opinion that the latter case could not be properly called a transaction cost. The punchline for short would be in a very informal way that the Coase theorem doesn't hold even with perfect information.

Finally ex-ante, before any pair is called to propose, we could argue instead that the rule of order of pairs called upon to propose induces a transaction cost. We think that a better word would be historical path dependence as a way of illustrating North's argument (1991).

Thus we have *ambiguity* and the importance of this paper for *policy* issues in terms of identifying the determinants of inefficient outcomes other than transaction costs provided that the Aumann-Myerson solution is a reasonable ways of predicting negotiations in the real world[3].

With respect to the conditions for the grand coalition not to form we have two claims:

1. Only 1 link graphs form, iff for *at least* 1 link graph structures (1, 2 and 3 in our model) the Myerson values for the respective two players linked is strictly greater than their Shapley values in the complete graph. Thus the grand coalition never forms. For related results see Nieva (October 2002).

2. In an environment with production externalities, even if the production of externalities exhibits high enough marginal pollution productivity (see definition in section 2) the complete graph does not form necessarily.

# 5 Appendix: Externalities in Production and in Coalition Formation

In contrast to Aviazan and Callen (1987), allowing for spillovers in coalition formation when we have unilateral externalities of 2 producers again a third one may induce the grand coalition to form as the unique equilibrium outcome. The latter will happen if the collution of two polluter firms decreases enough the payoff of the third firm in comparison to the case where all firms operate separately. As solution concepts, we use the extended Shapley value in Partition function form (Myerson 1977) and the link formation game of Aumann and Myerson (1988).

#### 5.0.4 Environment

Let us assume that what coalitions can achieve depends on sets of links (or graphs as in Aumann and Myerson(1988) or Jackson and Wolinsky (1996)) among players and on the coalition structure following the partition function approach in which spillovers in coalition formation are possible(see Bloch (1996) Ray and Vohra (1996) and Myerson(1977b)). Loosely speaking, the extended Myerson value that allows externalities in coalition formation is a extended Shapley value or weighted average of contributions of players to coalitions taking into account also the corresponding contributions in different coalition structures. We assume that utility is transferable.

Before we describe the relationship between coalitions worth and links and coalitions structures, let $N$ be a finite set of players. Given $N$, let $CL$ be the set of all coalitions(nonempty subsets) of $N$,

$CL = \{S|S \subseteq N, S \neq \varnothing\}$

[3] In Nieva (October 2002) it is shown that the latter solution concept may be very useful to explain the existence of property rights and the difficulty of institutional change in enforcer games with empty cores for the case of Latin America.

Let $PT$ the set of partitions of $N$, so

$\{S^1, ..., S^l\} \in PT$ iff:

$U_{i=1}^{l} S^i = N$, $\forall j\ S^j \neq \varnothing, \forall k S^j \cap S^k = \varnothing$ if $k \neq j$.

Let $ECL$ be the set of embedded coalitions, that is the set of coalitions with specifications as to how the other player are aligned. Formally:

$ECL = \{(S, Q) | S \in Q \in PT\}$

For any finite set $L$, let $\mathbb{R}^L$ denote the set of real vectors indexed on the members of $L$.

In our case for three players we will begin with the complete graph $\mathrm{g}^N$ or, equivalently, with the original game where everyone is linked:

3<br>
/ \<br>
1——2<br>
graph N

For this graph we will have a game in partition function form that would correspond to a vector in $w^{g=N} \in \mathbb{R}^{ECL}$. For any such $w^N \in \mathbb{R}^{ECL}$ and any embedded coalition $(S, Q) \in ECL$, $w^N_{S,Q}$, the $(S, Q)$ component of $w^N$ is interpreted as the wealth, measured in units of transferable utility, which the coalition $S$ would have to divide among its members if all the players were aligned into the coalitions of partition $Q$.

We will have, for the example, as in Aviazian and Callen (1987)for the partition that consists of $\{\{1\}, \{2\}, \{3\}\}$:

$w^N_{\{1\},\{\{1\},\{2\},\{3\}\}} = 1$

$w^N_{\{2\},\{\{1\},\{2\},\{3\}\}} = 2$

$w^N_{\{3\},\{\{1\},\{2\},\{3\}\}} = 3$

If polluters collude then we have in contrast to Aviazan and Callen:

$w^N_{\{1,2\},\{\{1,2\},\{3\}\}} = 8$

$w^N_{\{3\},\{\{1,2\},\{3\}\}} = 0$

Thus, we have externalities in coalition formation as $0 = w^N_{\{3\},\{\{12\},\{3\}\}} \neq w^N_{\{3\},\{\{1\},\{2\},\{3\}\}} = 3$, i.e. the value of player 3 acting alone is dependent on the coalition structure.

If polluter 1 colludes with pollutee 3 we have

$w^N_{\{1,3\},\{\{1,3\},\{2\}\}} = 9$

$w^N_{\{2\},\{\{1,3\},\{2\}\}} = 2$

If polluter 2 colludes with pollutee 3 we have

$w^N_{\{2,3\},\{\{2,3\},\{1\}\}} = 10$

$w^N_{\{1\},\{\{2,3\},\{1\}\}} = 1$

The last partition, the grand coalition has one element, itself that is worth 12.

$w^N_{\{1,2,3\},\{\{1,2,3\}\}} = 12$

Let $\Phi_1(w^N)$ be the extended Shapley value for games with externalities in coalition formation for player 1 for the complete graph $N$. Following Myerson (1977b) we have:

$\Phi_1(w^N) = \frac{1}{3} w^N_{\{1,2,3\},\{\{1,2,3\}\}}$

$+\frac{1}{6}w^N_{\{1,2\},\{\{1,2\},\{3\}\}} - \frac{1}{3}w^N_{\{3\},\{\{1,2\},\{3\}\}}$
$+\frac{1}{6}w^N_{\{1,3\},\{\{1,3\},\{2\}\}} - \frac{1}{3}w^N_{\{2\},\{\{1,3\},\{2\}\}}$
$+\frac{2}{3}w^N_{\{1\},\{\{2,3\},\{1\}\}} - \frac{1}{3}w^N_{\{2,3\},\{\{2,3\},\{1\}\}}$
$+\frac{1}{6}w^N_{\{2\},\{\{1\},\{2\},\{3\}\}} + \frac{1}{6}w^N_{\{3\},\{\{1\},\{2\},\{3\}\}}$
$-\frac{1}{3}w^N_{\{1\},\{\{1\},\{2\},\{3\}\}}$

Plugging in values we get:

$\Phi_1(w^N) = \frac{1}{3}12 + \frac{1}{6}8 - \frac{1}{3}0 + \frac{1}{6}9 - \frac{1}{3}2 + \frac{2}{3}1 - \frac{1}{3}10 + \frac{1}{6}2 + \frac{1}{6}3 - \frac{1}{3}1 = 4$

Similarly for $\Phi_2(w^N)$, we get:

$\Phi_2(w^N) = \frac{1}{3}w^N_{\{1,2,3\},\{\{1,2,3\}\}}$
$+\frac{1}{6}w^N_{\{1,2\},\{\{1,2\},\{3\}\}} - \frac{1}{3}w^N_{\{3\},\{\{1,2\},\{3\}\}}$
$+\frac{1}{6}w^N_{\{2,3\},\{\{2,3\},\{1\}\}} - \frac{1}{3}w^N_{\{1\},\{\{2,3\},\{1\}\}}$
$+\frac{2}{3}w^N_{\{2\},\{\{1,3\},\{2\}\}} - \frac{1}{3}w^N_{\{1,3\},\{\{1,3\},\{2\}\}}$
$+\frac{1}{6}w^N_{\{1\},\{\{1\},\{2\},\{3\}\}} + \frac{1}{6}w^N_{\{3\},\{\{1\},\{2\},\{3\}\}}$
$-\frac{1}{3}w^N_{\{2\},\{\{1\},\{2\},\{3\}\}}$

$\Phi_2(w^N) = \frac{1}{3}12 + \frac{1}{6}8 - \frac{1}{3}0 + \frac{1}{6}10 - \frac{1}{3}1 + \frac{2}{3}2 - \frac{1}{3}9 + \frac{1}{6}1 + \frac{1}{6}3 - \frac{1}{3}2 = 5$

Finally for $\Phi_3(w^N)$ we have:

$\Phi_3(w^N) = \frac{1}{3}w^N_{\{1,2,3\},\{\{1,2,3\}\}}$
$+\frac{1}{6}w^N_{\{1,3\},\{\{1,3\},\{2\}\}} - \frac{1}{3}w^N_{\{2\},\{\{1,3\},\{2\}\}}$
$+\frac{1}{6}w^N_{\{2,3\},\{\{2,3\},\{1\}\}} - \frac{1}{3}w^N_{\{1\},\{\{2,3\},\{1\}\}}$
$+\frac{2}{3}w^N_{\{3\},\{\{1,2\},\{3\}\}} - \frac{1}{3}w^N_{\{1,2\},\{\{1,2\},\{3\}\}}$
$+\frac{1}{6}w^N_{\{1\},\{\{1\},\{2\},\{3\}\}} + \frac{1}{6}w^N_{\{2\},\{\{1\},\{2\},\{3\}\}}$
$-\frac{1}{3}w^N_{\{3\},\{\{1\},\{2\},\{3\}\}}$

Thus:

$\Phi_3(w^N) = \frac{1}{3}12 + \frac{1}{6}9 - \frac{1}{3}2 + \frac{1}{6}10 - \frac{1}{3}1 + \frac{2}{3}0 - \frac{1}{3}8 + \frac{1}{6}1 + \frac{1}{6}2 - \frac{1}{3}3 = 3$

# Summarizing: $\Phi^N = \Phi(w^N) = (4, 5, 3)$

We want to calculate the corresponding Shapley values for different graph structures. First, for one link graphs we have:

```
       3                       3                3
      /                         \
    1     2                 1     2         1——2
    graph 1                 graph 2         graph 3
```

For graph 1 we will have a game in partition function form that would correspond to a vector in $w^{g=1} \in \mathbb{R}^{ECL}$.

We will have in our example for the partition that consists of $\{\{1\},\{2\},\{3\}\}$:

$w^1_{\{1\},\{\{1\},\{2\},\{3\}\}} = 1$
$w^1_{\{2\},\{\{1\},\{2\},\{3\}\}} = 2$
$w^1_{\{3\},\{\{1\},\{2\},\{3\}\}} = 3$

If polluters collude:

$w^1_{\{1,2\},\{\{1,2\},\{3\}\}} = 3$

$w^1_{\{3\},\{\{12\},\{3\}\}} = 3$

We don't have externalities in coalition formation as ($3 = w^1_{\{3\},\{\{12\},\{3\}\}} = w^1_{\{3\},\{\{1\},\{2\},\{3\}\}} = 3$) players 1 and 2 are not linked.

If polluter 1 colludes with pollutee 3 we have

$w^1_{\{1,3\},\{\{1,3\},\{2\}\}} = 9$

$w^1_{\{2\},\{\{1,3\},\{2\}\}} = 2$

If polluter 2 colludes with pollutee 3 we have

$w^1_{\{2,3\},\{\{2,3\},\{1\}\}} = 5$

$w^1_{\{1\},\{\{2,3\},\{1\}\}} = 1$

The last partition, the grand coalition has one element, itself that is worth 12.

$w^1_{\{1,2,3\},\{\{1,2,3\}\}} = 11$

Now we compute the extended Myerson values for graph 1:

$\Phi_1(w^1) = \frac{1}{3}w^1_{\{1,2,3\},\{\{1,2,3\}\}}$
$+\frac{1}{6}w^1_{\{1,2\},\{\{1,2\},\{3\}\}} - \frac{1}{3}w^1_{\{3\},\{\{1,2\},\{3\}\}}$
$+\frac{1}{6}w^1_{\{1,3\},\{\{1,3\},\{2\}\}} - \frac{1}{3}w^1_{\{2\},\{\{1,3\},\{2\}\}}$
$+\frac{2}{3}w^1_{\{1\},\{\{2,3\},\{1\}\}} - \frac{1}{3}w^1_{\{2,3\},\{\{2,3\},\{1\}\}}$
$+\frac{1}{6}w^1_{\{2\},\{\{1\},\{2\},\{3\}\}} + \frac{1}{6}w^1_{\{3\},\{\{1\},\{2\},\{3\}\}}$
$-\frac{1}{3}w^1_{\{1\},\{\{1\},\{2\},\{3\}\}}$

$\Phi_1(w^1) = \frac{1}{3}11 + \frac{1}{6}3 - \frac{1}{3}3 + \frac{1}{6}9 - \frac{1}{3}2 + \frac{2}{3}1 - \frac{1}{3}5 + \frac{1}{6}2 + \frac{1}{6}3 - \frac{1}{3}1 = \frac{7}{2}$

Similarly for $\Phi_2(w^1)$, we get:

$\Phi_2(w^1) = \frac{1}{3}w^1_{\{1,2,3\},\{\{1,2,3\}\}}$
$+\frac{1}{6}w^1_{\{1,2\},\{\{1,2\},\{3\}\}} - \frac{1}{3}w^1_{\{3\},\{\{1,2\},\{3\}\}}$
$+\frac{1}{6}w^1_{\{2,3\},\{\{2,3\},\{1\}\}} - \frac{1}{3}w^1_{\{1\},\{\{2,3\},\{1\}\}}$
$+\frac{2}{3}w^1_{\{2\},\{\{1,3\},\{2\}\}} - \frac{1}{3}w^1_{\{1,3\},\{\{1,3\},\{2\}\}}$
$+\frac{1}{6}w^1_{\{1\},\{\{1\},\{2\},\{3\}\}} + \frac{1}{6}w^1_{\{3\},\{\{1\},\{2\},\{3\}\}}$
$-\frac{1}{3}w^1_{\{2\},\{\{1\},\{2\},\{3\}\}}$

$\Phi_2(w^1) = \frac{1}{3}11 + \frac{1}{6}3 - \frac{1}{3}3 + \frac{1}{6}5 - \frac{1}{3}1 + \frac{2}{3}2 - \frac{1}{3}9 + \frac{1}{6}1 + \frac{1}{6}3 - \frac{1}{3}2 = 2$

Finally for $\Phi_3(w^1)$ we have:

$\Phi_3(w^1) = \frac{1}{3}w^1_{\{1,2,3\},\{\{1,2,3\}\}}$
$+\frac{1}{6}w^1_{\{1,3\},\{\{1,3\},\{2\}\}} - \frac{1}{3}w^1_{\{2\},\{\{1,3\},\{2\}\}}$
$+\frac{1}{6}w^1_{\{2,3\},\{\{2,3\},\{1\}\}} - \frac{1}{3}w^1_{\{1\},\{\{2,3\},\{1\}\}}$
$+\frac{2}{3}w^1_{\{3\},\{\{1,2\},\{3\}\}} - \frac{1}{3}w^1_{\{1,2\},\{\{1,2\},\{3\}\}}$
$+\frac{1}{6}w^1_{\{1\},\{\{1\},\{2\},\{3\}\}} + \frac{1}{6}w^1_{\{2\},\{\{1\},\{2\},\{3\}\}}$
$-\frac{1}{3}w^1_{\{3\},\{\{1\},\{2\},\{3\}\}}$

Thus:

$\Phi_3(w^1) = \frac{1}{3}11 + \frac{1}{6}9 - \frac{1}{3}2 + \frac{1}{6}5 - \frac{1}{3}1 + \frac{2}{3}3 - \frac{1}{3}3 + \frac{1}{6}1 + \frac{1}{6}2 - \frac{1}{3}3 = \frac{11}{2}$

# Summarizing: $\Phi^1 = \Phi(w^1) = (3\frac{1}{2}, 2, 5\frac{1}{2})$

For graph 2 we will have a game in partition function form that would correspond to a vector in $w^{g=2} \in \mathbb{R}^{ECL}$.

We will have in our example for the partition that consists of $\{\{1\},\{2\},\{3\}\}$:

$w^2_{\{1\},\{\{1\},\{2\},\{3\}\}} = 1$

$w^2_{\{2\},\{\{1\},\{2\},\{3\}\}} = 2$

$w^2_{\{3\},\{\{1\},\{2\},\{3\}\}} = 3$

If polluters collude:

$w^2_{\{1,2\},\{\{1,2\},\{3\}\}} = 3$

$w^2_{\{3\},\{\{12\},\{3\}\}} = 3$

We don't have externalities in coalition formation as $(3 = w^N_{\{3\},\{\{12\},\{3\}\}} = w^N_{\{3\},\{\{1\},\{2\},\{3\}\}} = 3)$ players 1 and 2 are not linked.

If polluter 1 colludes with pollutee 3 we have

$w^2_{\{1,3\},\{\{1,3\},\{2\}\}} = 4$

$w^2_{\{2\},\{\{1,3\},\{2\}\}} = 2$

If polluter 2 colludes with pollutee 3 we have

$w^2_{\{2,3\},\{\{2,3\},\{1\}\}} = 10$

$w^2_{\{1\},\{\{2,3\},\{1\}\}} = 1$

The last partition, the grand coalition has one element, itself that is worth 12.

$w^2_{\{1,2,3\},\{\{1,2,3\}\}} = 11$

Now we compute the extended Myerson values for graph 2:

$\Phi_1(w^2) = \frac{1}{3}w^2_{\{1,2,3\},\{\{1,2,3\}\}}$
$+\frac{1}{6}w^2_{\{1,2\},\{\{1,2\},\{3\}\}} - \frac{1}{3}w^2_{\{3\},\{\{1,2\},\{3\}\}}$
$+\frac{1}{6}w^2_{\{1,3\},\{\{1,3\},\{2\}\}} - \frac{1}{3}w^2_{\{2\},\{\{1,3\},\{2\}\}}$
$+\frac{2}{3}w^2_{\{1\},\{\{2,3\},\{1\}\}} - \frac{1}{3}w^2_{\{2,3\},\{\{2,3\},\{1\}\}}$
$+\frac{1}{6}w^2_{\{2\},\{\{1\},\{2\},\{3\}\}} + \frac{1}{6}w^2_{\{3\},\{\{1\},\{2\},\{3\}\}}$
$-\frac{1}{3}w^2_{\{1\},\{\{1\},\{2\},\{3\}\}}$

$\Phi_1(w^2) = \frac{1}{3}11 + \frac{1}{6}3 - \frac{1}{3}3 + \frac{1}{6}4 - \frac{1}{3}2 + \frac{2}{3}1 - \frac{1}{3}10 + \frac{1}{6}2 + \frac{1}{6}3 - \frac{1}{3}1 = 1$

Similarly for $\Phi_2(w^2)$, we get:

$\Phi_2(w^2) = \frac{1}{3}w^2_{\{1,2,3\},\{\{1,2,3\}\}}$
$+\frac{1}{6}w^2_{\{1,2\},\{\{1,2\},\{3\}\}} - \frac{1}{3}w^2_{\{3\},\{\{1,2\},\{3\}\}}$
$+\frac{1}{6}w^2_{\{2,3\},\{\{2,3\},\{1\}\}} - \frac{1}{3}w^2_{\{1\},\{\{2,3\},\{1\}\}}$
$+\frac{2}{3}w^2_{\{2\},\{\{1,3\},\{2\}\}} - \frac{1}{3}w^2_{\{1,3\},\{\{1,3\},\{2\}\}}$
$+\frac{1}{6}w^2_{\{1\},\{\{1\},\{2\},\{3\}\}} + \frac{1}{6}w^2_{\{3\},\{\{1\},\{2\},\{3\}\}}$
$-\frac{1}{3}w^2_{\{2\},\{\{1\},\{2\},\{3\}\}}$

$\Phi_2(w^2) = \frac{1}{3}11 + \frac{1}{6}3 - \frac{1}{3}3 + \frac{1}{6}10 - \frac{1}{3}1 + \frac{2}{3}2 - \frac{1}{3}4 + \frac{1}{6}1 + \frac{1}{6}3 - \frac{1}{3}2 = \frac{9}{2}$

Finally for $\Phi_3(w^2)$ we have:

$\Phi_3(w^2) = \frac{1}{3}w^2_{\{1,2,3\},\{\{1,2,3\}\}}$
$+\frac{1}{6}w^2_{\{1,3\},\{\{1,3\},\{2\}\}} - \frac{1}{3}w^2_{\{2\},\{\{1,3\},\{2\}\}}$
$+\frac{1}{6}w^2_{\{2,3\},\{\{2,3\},\{1\}\}} - \frac{1}{3}w^2_{\{1\},\{\{2,3\},\{1\}\}}$
$+\frac{2}{3}w^2_{\{3\},\{\{1,2\},\{3\}\}} - \frac{1}{3}w^2_{\{1,2\},\{\{1,2\},\{3\}\}}$
$+\frac{1}{6}w^2_{\{1\},\{\{1\},\{2\},\{3\}\}} + \frac{1}{6}w^2_{\{2\},\{\{1\},\{2\},\{3\}\}}$

$-\frac{1}{3}w^2_{\{3\},\{\{1\},\{2\},\{3\}\}}$
Thus:
$\Phi_3(w^2) = \frac{1}{3}11 + \frac{1}{6}4 - \frac{1}{3}2 + \frac{1}{6}10 - \frac{1}{3}1 + \frac{2}{3}3 - \frac{1}{3}3 + \frac{1}{6}1 + \frac{1}{6}2 - \frac{1}{3}3 = \frac{11}{2}$

# Summarizing: $\Phi^2 = \Phi(w^2) = (1, 4\frac{1}{2}, 5\frac{1}{2})$

For graph 3 we will have a game in partition function form that would correspond to a vector in $w^{g=3} \in \mathbb{R}^{ECL}$.

We will have in our example for the partition that consists of $\{\{1\},\{2\},\{3\}\}$:
$w^3_{\{1\},\{\{1\},\{2\},\{3\}\}} = 1$
$w^3_{\{2\},\{\{1\},\{2\},\{3\}\}} = 2$
$w^3_{\{3\},\{\{1\},\{2\},\{3\}\}} = 3$
If polluters collude:
$w^3_{\{1,2\},\{\{1,2\},\{3\}\}} = 8$
$w^3_{\{3\},\{\{12\},\{3\}\}} = 0$

Thus, we have externalities in coalition formation as $0 = w^3_{\{3\},\{\{12\},\{3\}\}} \neq w^3_{\{3\},\{\{1\},\{2\},\{3\}\}} = 3$, i.e. the value of player 3 acting alone is dependent on the coalition structure because 1 and 2 are linked.

If polluter 1 colludes with pollutee 3 we have
$w^3_{\{1,3\},\{\{1,3\},\{2\}\}} = 4$
$w^3_{\{2\},\{\{1,3\},\{2\}\}} = 2$
If polluter 2 colludes with pollutee 3 we have
$w^3_{\{2,3\},\{\{2,3\},\{1\}\}} = 5$
$w^3_{\{1\},\{\{2,3\},\{1\}\}} = 1$
The last partition, the grand coalition has one element, itself that is worth 12.
$w^3_{\{1,2,3\},\{\{1,2,3\}\}} = 8$
Now we compute the extended Myerson values for graph 3:
$\Phi_1(w^3) = \frac{1}{3}w^3_{\{1,2,3\},\{\{1,2,3\}\}}$
$+\frac{1}{6}w^3_{\{1,2\},\{\{1,2\},\{3\}\}} - \frac{1}{3}w^3_{\{3\},\{\{1,2\},\{3\}\}}$
$+\frac{1}{6}w^3_{\{1,3\},\{\{1,3\},\{2\}\}} - \frac{1}{3}w^3_{\{2\},\{\{1,3\},\{2\}\}}$
$+\frac{2}{3}w^3_{\{1\},\{\{2,3\},\{1\}\}} - \frac{1}{3}w^3_{\{2,3\},\{\{2,3\},\{1\}\}}$
$+\frac{1}{6}w^3_{\{2\},\{\{1\},\{2\},\{3\}\}} + \frac{1}{6}w^3_{\{3\},\{\{1\},\{2\},\{3\}\}}$
$-\frac{1}{3}w^3_{\{1\},\{\{1\},\{2\},\{3\}\}}$
$\Phi_1(w^3) = \frac{1}{3}8 + \frac{1}{6}8 - \frac{1}{3}0 + \frac{1}{6}4 - \frac{1}{3}2 + \frac{2}{3}1 - \frac{1}{3}5 + \frac{1}{6}2 + \frac{1}{6}3 - \frac{1}{3}1 = \frac{7}{2}$
Similarly for $\Phi_2(w^3)$, we get:
$\Phi_2(w^3) = \frac{1}{3}w^3_{\{1,2,3\},\{\{1,2,3\}\}}$
$+\frac{1}{6}w^3_{\{1,2\},\{\{1,2\},\{3\}\}} - \frac{1}{3}w^3_{\{3\},\{\{1,2\},\{3\}\}}$
$+\frac{1}{6}w^3_{\{2,3\},\{\{2,3\},\{1\}\}} - \frac{1}{3}w^3_{\{1\},\{\{2,3\},\{1\}\}}$
$+\frac{2}{3}w^3_{\{2\},\{\{1,3\},\{2\}\}} - \frac{1}{3}w^3_{\{1,3\},\{\{1,3\},\{2\}\}}$
$+\frac{1}{6}w^3_{\{1\},\{\{1\},\{2\},\{3\}\}} + \frac{1}{6}w^3_{\{3\},\{\{1\},\{2\},\{3\}\}}$
$-\frac{1}{3}w^3_{\{2\},\{\{1\},\{2\},\{3\}\}}$

$\Phi_2(w^3) = \frac{1}{3}8 + \frac{1}{6}8 - \frac{1}{3}0 + \frac{1}{6}5 - \frac{1}{3}1 + \frac{2}{3}2 - \frac{1}{3}4 + \frac{1}{6}1 + \frac{1}{6}3 - \frac{1}{3}2 = \frac{9}{2}$

Finally for $\Phi_3(w^2)$ we have:

$\Phi_3(w^3) = \frac{1}{3}w^3_{\{1,2,3\},\{\{1,2,3\}\}}$
$+\frac{1}{6}w^3_{\{1,3\},\{\{1,3\},\{2\}\}} - \frac{1}{3}w^2_{\{2\},\{\{1,3\},\{2\}\}}$
$+\frac{1}{6}w^3_{\{2,3\},\{\{2,3\},\{1\}\}} - \frac{1}{3}w^2_{\{1\},\{\{2,3\},\{1\}\}}$
$+\frac{2}{3}w^3_{\{3\},\{\{1,2\},\{3\}\}} - \frac{1}{3}w^2_{\{1,2\},\{\{1,2\},\{3\}\}}$
$+\frac{1}{6}w^3_{\{1\},\{\{1\},\{2\},\{3\}\}} + \frac{1}{6}w^3_{\{2\},\{\{1\},\{2\},\{3\}\}}$
$-\frac{1}{3}w^3_{\{3\},\{\{1\},\{2\},\{3\}\}}$

Thus:

$\Phi_3(w^3) = \frac{1}{3}8 + \frac{1}{6}4 - \frac{1}{3}2 + \frac{1}{6}5 - \frac{1}{3}1 + \frac{2}{3}0 - \frac{1}{3}8 + \frac{1}{6}1 + \frac{1}{6}2 - \frac{1}{3}3 = 0$

# Summarizing: $\Phi^3 = \Phi(w^3) = (3\frac{1}{2}, 4\frac{1}{2}, 0)$

Analogously we could compute the extended Myerson value for the next three two-link types of graphs (for our results in next section we don't need to compute the latter ones)

```
     3                      3            3
   /   \                      \         /
  1     2                1——2         1——2
  graph 4                graph 5      graph 6
```

For graph 4 we will have a game in partition function form that would correspond to a vector in $w^{g=4} \in \mathbb{R}^{ECL}$.

We will have in our example for the partition that consists of $\{\{1\},\{2\},\{3\}\}$:

$w^4_{\{1\},\{\{1\},\{2\},\{3\}\}} = 1$
$w^4_{\{2\},\{\{1\},\{2\},\{3\}\}} = 2$
$w^4_{\{3\},\{\{1\},\{2\},\{3\}\}} = 3$

If polluters collude:

$w^4_{\{1,2\},\{\{1,2\},\{3\}\}} = 3$
$w^4_{\{3\},\{\{12\},\{3\}\}} = 3$

We don't have externalities in coalition formation as $(3 = w^4_{\{3\},\{\{12\},\{3\}\}} = w^4_{\{3\},\{\{1\},\{2\},\{3\}\}} = 3)$ players 1 and 2 are not linked.

If polluter 1 colludes with pollutee 3 we have

$w^4_{\{1,3\},\{\{1,3\},\{2\}\}} = 9$
$w^4_{\{2\},\{\{1,3\},\{2\}\}} = 2$

If polluter 2 colludes with pollutee 3 we have

$w^4_{\{2,3\},\{\{2,3\},\{1\}\}} = 10$
$w^4_{\{1\},\{\{2,3\},\{1\}\}} = 1$

The last partition, the grand coalition has one element, itself that is worth 12 because now everyone is directly or indirectly linked

$w^4_{\{1,2,3\},\{\{1,2,3\}\}} = 12$

For graph 5 we will have a game in partition function form that would correspond to a vector in $w^{g=5} \in \mathbb{R}^{ECL}$.

We will have in our example for the partition that consists of $\{\{1\},\{2\},\{3\}\}$:
$w^5_{\{1\},\{\{1\},\{2\},\{3\}\}} = 1$
$w^5_{\{2\},\{\{1\},\{2\},\{3\}\}} = 2$
$w^5_{\{3\},\{\{1\},\{2\},\{3\}\}} = 3$
If polluters collude:
$w^5_{\{1,2\},\{\{1,2\},\{3\}\}} = 8$
$w^5_{\{3\},\{\{12\},\{3\}\}} = 0$
Thus, we have externalities in coalition formation as $0 = w^5_{\{3\},\{\{12\},\{3\}\}} \neq w^5_{\{3\},\{\{1\},\{2\},\{3\}\}} = 3$, i.e. the value of player 3 acting alone is dependent on the coalition structure because 1 and 2 are linked.
If polluter 1 colludes with pollutee 3 we have
$w^5_{\{1,3\},\{\{1,3\},\{2\}\}} = 4$
$w^5_{\{2\},\{\{1,3\},\{2\}\}} = 2$
If polluter 2 colludes with pollutee 3 we have
$w^5_{\{2,3\},\{\{2,3\},\{1\}\}} = 10$
$w^5_{\{1\},\{\{2,3\},\{1\}\}} = 1$
The last partition, the grand coalition has one element, itself that is worth 12.
$w^5_{\{1,2,3\},\{\{1,2,3\}\}} = 12$
For graph 6 we will have a game in partition function form that would correspond to a vector in $w^{g=6} \in \mathbb{R}^{ECL}$.
We will have in our example for the partition that consists of $\{\{1\},\{2\},\{3\}\}$:
$w^6_{\{1\},\{\{1\},\{2\},\{3\}\}} = 1$
$w^6_{\{2\},\{\{1\},\{2\},\{3\}\}} = 2$
$w^6_{\{3\},\{\{1\},\{2\},\{3\}\}} = 3$
If polluters collude:
$w^6_{\{1,2\},\{\{1,2\},\{3\}\}} = 8$
$w^6_{\{3\},\{\{12\},\{3\}\}} = 0$
Thus, we have externalities in coalition formation as $0 = w^6_{\{3\},\{\{12\},\{3\}\}} \neq w^6_{\{3\},\{\{1\},\{2\},\{3\}\}} = 3$, i.e. the value of player 3 acting alone is dependent on the coalition structure because 1 and 2 are linked.
If polluter 1 colludes with pollutee 3 we have
$w^6_{\{1,3\},\{\{1,3\},\{2\}\}} = 9$
$w^6_{\{2\},\{\{1,3\},\{2\}\}} = 2$
If polluter 2 colludes with pollutee 3 we have
$w^6_{\{2,3\},\{\{2,3\},\{1\}\}} = 5$
$w^6_{\{1\},\{\{2,3\},\{1\}\}} = 1$
The last partition, the grand coalition has one element, itself that is worth 12.
$w^6_{\{1,2,3\},\{\{1,2,3\}\}} = 12$

## 5.1 A formal model of endogenous networks with Spillovers

Allowing for externalities in coalition formation doesn't affect the link formation game

### 5.1.1 Equilibrium Analysis

**A simple application with the Extended Myerson Value(1977b)** If we would have the Myerson value for all graphs, we could check for the subgame perfect equilibria. But there is a shorter way. Given that from 2 links graphs the players not linked find it optimal to link (see that the Myerson values goes up for both of them if they accept to form a new link (if externalities would be positive that would not be necessarily true) the theorem from the companion paper holds, i.e. it is necessary and sufficient for the grand coalition or complete graph to form if the Myerson values of the three types of one-link graphs are strictly lower than the corresponding Myerson values of the complete graph. For this example the latter holds as:

The myerson value for graph 1 is $(3\frac{1}{2}, 2, 5\frac{1}{2})$

The Myerson value for graph 2 is $(1, 4\frac{1}{2}, 5\frac{1}{2})$

The Myerson value for graph 3 is$(3\frac{1}{2}, 4\frac{1}{2}, 0)$

The Myerson value for graph 4 is

The Myerson value for graph 5 is

The Myerson value for graph 6 is

The Myerson value for the complete graph is $(4, 5, 3)$

In contrast we have for Aviazan and Callen's example:

The Myerson value for graph 1 is $(3.5, 2, 5.5)$

The Myerson value for graph 2 is $(1, 4.5, 5.5)$

The Myerson value for graph 3 is $(3.5, 4.5, 4)$

The Myerson value for graph 4 is $(3, 4, 5)$

The Myerson value for graph 5 is $(2\frac{1}{6}, 5\frac{4}{6}, 4\frac{1}{6})$

The Myerson value for graph 6 is $(4\frac{4}{6}, 3\frac{1}{6}, 4\frac{1}{6})$

The Myerson value for the complete graph is $(3, 4, 5)$

Conclusion:

The assumption of not externalities in coalition formation in Aviazan and Callen (1987) is not innocuous.

The intuition is that the two polluters colluded have more outside option in the complete graph because of the negative externality by joining them together and thus can extract more of the efficient outcome in the grand coalition. Recall that this was not possible in the original Aviazan and Callen's (1987) example because of the induced lower bargaining power in the grand coalition

# References


[1] Aviazian, Callen and Lipnowski(1987)"The Coase Theorem and Coalitional Stability" Economica 54, 517-520

[2] Aviazian, Callen and Lipnowski(1981)"The Coase Theorem and the empty core" Journal of Law and Economics 24 175-81

[3] Aumann R. Myerson " An endongenous formation of links between players and coalitions: an application of the shapley Value" in Physical-Verlag,Vienna

[4] Belleflamme Paul (2000)"Stable Coalition Structures with Open Membership and Asymmetric Firms"Games and Economic Behavior Vol. 30, No. 1,pp. 1-21 January 1,

[5] Bloch F. Coalitions and Networks in Industrial organization August 2001

[6] Bloch F. "Sequential Formation of Coalitions in Games with Externalities and Fixed Payoff Division" Games and Economic Behavior Vol. 14, No. 1, May 1, 1996 pp. 90-123

[7] Chatterjee, K., B. dutta, D. Ray, and K. sengupta(1993) " A Noncooperative theory of Coalitional Bargaining" Rev. Econ. Stud. 60, 463-477

[8] Coase R H (1981)"The Coase Theorem and the empty core: A reply" Journal of Law and Economics 24 183-87

[9] Coase R H (1981)"The Problem of Social Cost" Journal of Law and Economics 2,1-40

[10] Currarini and Massimo Morelli 2000 "Network Formation with sequential Demands Review of Economic Design 229-249 Vol 5(3) pp 229 249

[11] Dixit A. Olson M.(2000) Does voluntary participation undermine the Coase Theorem? Journal of Public Economics 76(2000) 309-335.

[12] Dutta, B. , a. van den Nouweland and s. Tijs, "Link Formation in Cooperative Situations" International Journal of Game Theory 27(1998) 245-256

[13] Hurwicz, Leonid(1995) "What is the Coase Theorem" Japan and the World Economy 7,49-74

[14] Hurwicz, Leonid(1999)"Revisiting externalities" Journal of Public Economic Theory.

[15] Jackson , M and A. Wolinksky (1996)" A strategic Model of social and economic Networks" Journal of economic Theory 71,44-71

[16] Montero M. and Okada A "Riskless vs Risky Bargaining procedure. The Aumann-Roth Controversy revisited (2002)

[17] Myerson Roger (1977b)"Values of games in partition function form" Int. Journal of Game Theory Vol 6 Issue 1 page 23-31 Physica-Verlag,Vienna

[18] Myerson Roger "(1980) Conference structures and fair allocation rules" Int. Journal of Game Theory Vol 9 Issue 3 page 169-82 Physica-Verlag,Vienna

[19] Myerson Roger (1977)"Graphs and Cooperation in Games" Mathematics of Operation Research 2, 225-9

[20] Nieva R.(June 2002)" Might makes right, the other Folk Theorem?. "Property rights as unique equilibria in repeated games with enforcement shocks" Downloadable at www.rit.edu/~rxngla

[21] Nieva R(October 2002) Third Party Enforcers or spontaneous Institutional Change? An extension of the Myerson Value for sequential games with endogenous networks.Downloadable at www.rit.edu/~rxngla

[22] Ray, D. and R. Vohra,(1999)" A Theory of Endogenous Coalition Structures" Games and Economic Behaviour 26, 286,336

[23] Sanchez-Mier L. (2002) Political Influence and Economic Change" abstract. University of Minnesota

[24] Santiago Sanchez-Pages "Rivalry, Exclusion and Coalitions" 2002

[25] Slikker and van den Nouweland (2001) Social and Economic Networks in Cooperative Game Theory" Kluwer Academic Publishers

[26] Watts, E.(2001) A dynamic model of network formation. "Games and Economic Behavior"