Egalitarian distributions in coalitional models: The Lorenz criterion

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Abstract

The paper presents a framework where the most important single-valued solutions in the literature of TU games are jointly analyzed. The paper also suggests that similar frameworks may be useful for other coalitional models.

Keywords: Coalitional games, egalitarian criteria, prenucleolus, Shapley value.

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1 Introduction

Traditionally, one of the objects of cooperative game theory is to define mechanisms that select for each game a set of vectors (the allocations), which set on many cases contains a unique element. The mechanisms proposed are then justified by some system of axioms used to claim that the resulting selection is fair. Therefore if society identifies fairness with the set of axioms it will accept the distributions selected by the mechanisms that satisfy the axioms. But as Young says (page 3 in his book "*Equity in Theory and Practice*" (1994)):

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> These sharing rules, which in some cases are elaborately defined, express a notion of equity in the division of jointly produced goods. By equitable I do not necessarily mean ethical or moral, but that which a given society considers to be appropriate to the need, status and contributions of its various members.

Under the same ideal of equity different societies could originate different sharing rules for the same sharing problems.

Two person games

Again quoting Young in "*Equity in Theory and Practice*" (page 15):

> A fruitful approach to this problem is to begin by asking what solution is most equitable when there are just two claimants. This case is usually simple to grasp intuitively. Once a standard of equity has been established for two-claimant situations, we may then solve a many-claimant problem according to the following principle: allocate the good so that every two claimants divide the amount allotted to them as they would if they were the only two claimants. This consistency principle turns out to be one of the most powerful unifying ideas in the theory of fair allocations.

Assume the following two-person TU game: $v(1) = 4, v(2) = 2$ and $v(12) = 8$. There are two *natural egalitarian* distributions for this game: one is the result of dividing the surplus of the game, i.e., the amount $v(12)-v(1)-v(2)$ equally between the players. In this egalitarian distribution the objective is to treat coalitions in an egalitarian way. And moreover there is a unique egalitarian distribution because in this case all the coalitions are treated identically, with each receiving the same amount of the surplus.

The other natural egalitarian distribution is the result of dividing the worth of the grand coalition as equally as possible. In this case the meaning of *as equally as possible* is that in the distribution player 1 cannot get less than 4 and player 2 cannot get less than 2. In this egalitarian distribution the objective is to give egalitarian treatment to the players in a restricted way.

This example illustrates that solutions may be egalitarian for the coalitions or for the players. The difference is that while selecting egalitarian distributions this egalitarianism is considered in the space of the vectors of satisfactions (a 2^n-dimensional space) or in the space of the vectors of payoffs (an n-dimensional space).

Solutions

The prenucleolus (Schmeidler, 1969) and the Shapley value (Shapley, 1953) are the best known single-valued solution concepts embodying the idea of egalitarianism over coalitions. These two solution concepts are identical in the class of two person games and in this class the solution is called the standard solution. The *egalitarian allocation* introduced by Dutta and Ray in 1989 is a known solution concept that is constructed on the notion of egalitarianism over the players. These authors argue that players could agree on egalitarian rules in situations where agents do not know which player are going to be in a game. They also argue that egalitarian rules should consider the selfish behavior of players and respect some stability criterion in order to avoid the formation of a blocking coalition to a given allocation. Therefore they are not claiming the extreme egalitarianism provided by an equal payoff to every player.

The main motivation of this paper is to present all those solutions and others in the same framework. Any normative selection is based on egalitarian considerations but this general motivation originates a lot of apparently different mechanisms of selecting distributions. The paper tries to give a more unified view of all those normative solution concepts. In fact, in the paper we claim that the idea of egalitarianism should be clarified answering the following two questions:

1. Which egalitarian criterion is used to make egalitarian comparisons between elements?

2. From which set are those elements taken?

The answers may help to understand the similarities and differences of different solution concepts.

2 Preliminaries

The Model

A *transferable utility n-person game* (in characteristic function form) is a pair (N, v) consisting of a finite set N and a real valued function, $v : 2^N \to R$, on the family 2^N of all subsets of N satisfying $v(\emptyset) = 0$. Elements of N are called *players* and the real valued function v is called the *characteristic function* of the game. Any subset S of the player set N is called a *coalition* and $v(S)$ is the *worth* of the coalition S in the game. The number of players in a coalition S is denoted by $|S|$. Usually we shall identify the game (N, v) with its characteristic function v.

A payoff to the players is represented by a real valued vector $x \in R^N$. The i-th coordinate of the vector x denotes the payoff given to player i. We denote $\sum_{i \in S} x_i$ by $y(S)$. The vector y is called *efficient* if $x(N) = v(N)$ and the set of all efficient vectors is called the *pre-imputation set* and is denoted by $PI(v)$. A subset of the pre-imputation set is the *imputation set* denoted by $I(v)$. An imputation is an efficient vector where $x_i \geq v(i)$ for all $i \in N$. The core is the set of preimputations for which each coalition receives at least its worth, i.e.,

$$C(v) = \{x \in PI(v); x(S) \geq v(S) \text{ for all } S \subset N\}.$$

Solutions and Properties

A solution concept ϕ on a set of games G^N is a mapping that associates a set $\phi(v) \subseteq PI(v)$ with every game v.

Some convenient and well-known properties of a solution concept ϕ on G^N are the following.

P1) ϕ satisfies **anonymity** if for each (N, v) in G^N and each bijective mapping $\tau : N \longrightarrow N'$ such that $(N', \tau v)$ in G^N it holds that $\phi(N', \tau v) = \tau(\phi(N, v))$ (where $(\tau v)(T) = v(\tau^{-1}(T))$, $\tau_j(x) = x_{\tau^{-1}(j)}$ $(x \in R^N, j \in N', T \subseteq N)$). In this case v and τv are equivalent games.

P2) ϕ satisfies the **equal treatment property** if for each (N, v) in G^N and for every $x \in \phi(N, v)$ interchangeable players i, j are treated equally, i.e., $x_i = x_j$. Here, i and j are interchangeable if $v(S \cup i) = v(S \cup j)$ for all $S \subseteq N \setminus \{i, j\}$.

P3) ϕ satisfies the **strong equal treatment property** if for each automorphism[1] π of the game (N, v) in G^N, and for all $x \in \phi(N, v)$ it holds that $x_{\pi(i)} = x_i$ for all $i \in N$.

[1]A permutation π of N is an automorphism of (N, v) if $v(\pi(S)) = v(S)$ for all $S \subseteq N$.

P4) ϕ satisfies **desirability** if for each (N,v) in G^N and for every $x \in \phi(N,v)$, $x_i \geq x_j$ if i is more desirable than j in v. We say that in a game v a player i is more desirable than a player j if $v(S \cup i) \geq v(S \cup j)$ for all $S \subset N \backslash \{i,j\}$.

P5) ϕ satisfies **pairwise reasonability** if x is pairwise reasonable for every $x \in \phi(N,v)$, for each (N,v) in G^N. A payoff vector is pairwise reasonable if for every $i,j \in N$,

$$x_i - x_j \geq m_{ij} = \min_{S \subseteq N \backslash \{i,j\}} [v(S \cup i) - v(S \cup j)] \text{ and}$$

Note that in a two-person game pairwise reasonability determines a unique payoff.

P6) ϕ satisfies **covariance** if $(N,v),(N,w) \in G^N$ with $w = \alpha v + \beta$ for some $\alpha > 0$ and some $\beta \in R^N$ implies that $\phi(N,w) = \alpha\phi(N,v) + \beta$ holds.

Egalitarian Criteria

For any vector $z \in R^d$ we denote by $\theta(z)$ the vector that results from z by permuting the coordinates in such a way that $\theta_1(z) \leq \theta_2(z) \leq ... \leq \theta_d(z)$. Let $x,y \in R^d$. We say that the vector x *Lorenz dominates* the vector y (denoted by $x \succ_L y$) if $\sum_{i=1}^{k} \theta_i(x) \geq \sum_{i=1}^{k} \theta_i(y)$ for all $k \in \{1,2,...,d\}$ and if at least one of these inequalities is strict. We say that the vector x *weakly Lorenz dominates* the vector y (denoted by $x \succeq_L y$) if $\sum_{i=1}^{k} \theta_i(x) \geq \sum_{i=1}^{k} \theta_i(y)$ for all $k \in \{1,2,...,d\}$.

Given a vector $x \in R^N$, and $w \in R_+^{n-1} \backslash \{0\}$ the *weighted satisfaction* $f^w(S,x)$ *of coalition* S *with* x in the game $v \in G^N$ is defined as

$$f^w(S,x) := w(|S|)\,(x(S) - v(S))\,.$$

Comment: A non-negative satisfaction of S with respect to y in the game v indicates the gain for coalition S if its members are paid according to x compared to the situation where coalition S operates on its own and receives the amount $v(S)$. We assume that coalitions with the same cardinality have the same weight and the grand coalition is not considered. We denote $w(|S|)$ by $w(s)$. Furthermore, $w(S) = 0$ for all s is a trivial case.

We denote by $f^w(x)$ the vector of weighted satisfactions (excluding the empty and the grand coalition) at x, that is,

$$f^w(x) = (f^w(S_1), f^w(S_2), ..., f^w(S_{2^n-2}))$$

where $(S_1, S_2, ..., S_{2^n-2})$ is an arbitrary fixed order[2] of coalitions. We denote by $F^w(PI(v))$ (or just F if there is no confusion) the set of weighted satisfactions associated to (N, v, w)

We say that x coalitionally Lorenz^w dominates y, denoted by $x \succ_{CL^w} y$, if $f^w(x) \succ_L f^w(y)$. If there is no confusion we omit the superscript w.

We say that the vector x *lexicographically dominates* the vector y (denoted by $x \succ_{lex} y$) if there exists k such that $\theta_i(x) = \theta_i(y)$ for all $i \in \{1, 2, ..., k-1\}$ and $\theta_k(x) > \theta_k(y)$. We say that the vector x *weakly lexicographically dominates* the vector y (denoted by $x \succeq_{lex} y$) if either there exists k such that $\theta_i(x) = \theta_i(y)$ for all $i \in \{1, 2, ..., k-1\}$ and $\theta_k(x) > \theta_k(y)$ or $\theta_i(x) = \theta_i(y)$ for all i.

3 Egalitarianism over coalitions

The most important single-valued solutions in the theory of coalitional games are probably the (pre)nucleolus and the Shapley value. We present those two solutions in the following way.

The w-prenucleolus is the solution resulting from the lexicographical maximization of the weighted satisfaction vector, i.e.,

$$\mathcal{PN}^w(v) = \{x \in PI(v); f^w(x) \succeq_{lex} f^w(y), \text{ for all } y \in PI(v)\}.$$

The prenucleolus is a single-valued solution where the weights for all the coalitions are identical.

Taking into account the homomorphism between $PI(v)$ and $F^w(PI(v))$ we can rewrite the above definition as

$$\mathcal{PN}(F^w(PI(v))) = \{x \in F^w(PI(v)); x \succeq_{lex} y, \text{ for all } y \in F^w(PI(v))\}.$$

According to this definition the prenucleolus[3] selects a vector of satisfactions and indirectly a preimputation. In this framework a solution concept ϕ on a set of spaces $\sum$ induced by the set of games G^N and the set of feasible vectors of weights, W, is a mapping that associates a set $\phi(v) \subseteq F^w(A(v))$ where $A(v) \subseteq PI(v)$ with every $F^w(A(v))$ in $\sum$.

[2]Fixing the order of coalitions is a technical trick, in order to obtain a homeomorphism between the topological vector spaces $PI(v)$ and $F^w(PI(v))$ (the space of preimputations and the space of weighted satisfactions of preimputations).

[3]We use the same name for both vectors.

Analogously we can define the w-nucleolus as

$$\mathcal{N}^w(v) = \{x \in I(v); f^w(x) \succeq_{lex} f^w(y), \text{ for all } y \in I(v)\}.$$

or in the new notation as

$$\mathcal{N}(F^w(I(v))) = \{x \in F^w(I(v)); x \succeq_{lex} y, \text{ for all } y \in (F^w(I(v))\}.$$

If all weights are equal, we get the ordinary nucleolus. Note that this set is non empty if the set of imputations is non empty.

The Shapley value is the unique optimal solution to the following problem (Keane, 1969)

$$\min \sum_{\emptyset \neq S \subset N} w^2(|S|)(v(S) - x(S))^2$$

$$s.t. \sum_{i \in N} x_i = v(N)$$

with weights $w(s) = \begin{pmatrix} n-2 \\ s-1 \end{pmatrix}^{-1/2}$. We will call this vector of weights *Shapley weights*. When the weights are $w(s) = 1$ for all s the optimal solution to the same problem is the least square prenucleolus (Ruiz, Valenciano and Zarzuelo, 1996).

We can rewrite the above program as

$$LS(F^w(PI(v)) = \arg \min_{.x \in F^w(PI(v)} \sum x_i^2$$

and also as

$$\mathcal{LS}(F^w(PI(v))) = \{x \in F^w(PI(v)); x \succeq_{LS} y, \text{ for all } y \in (F^w(PI(v))\}$$

where $\succeq_{LS}$ denotes the Least Square relationship.

Clearly, under those definitions the Shapley value and the prenucleolus are egalitarian selections on some sets. The space where the egalitarian distributions are selected is the space of weighted satisfactions. That means that we need a new element, a vector of weights, to define the space in which egalitarian selections are going to be made. Given a game and a vector of feasible weights (N, v, w) a space of weighted satisfactions is well-defined.

The prenucleolus uses the Rawls criterion as the egalitarian criterion in its definition, and the Shapley value uses the Least Square criterion. Under some conditions both criteria generate Lorenz undominated distributions.

In order to see that the Least Square criterion generates Lorenz undominated allocations note that for all $x, y \in PI(v)$ and a vector of weights w[4]

$$\sum_{\emptyset \neq S \subset N} w(s)(x(S) - v(S)) = \sum_{\emptyset \neq S \subset N} w(s)\ (y(S) - v(S)).$$

Now note that if $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$ and $x \succ_L y$ then

$$\sum_{i=1}^{n} x_i^2 < \sum_{i=1}^{n} y_i^2 \ .$$

The value $\sum_{\emptyset \neq S \subset N} w(s)(x(S) - v(S))$ could be considered as the surplus of the game for the coalitions once those coalitions have been weighted. The aim of any egalitarian distribution would be to divide this surplus equally among the coalitions. When this is possible the resulting element would be selected independently of the egalitarian criterion used. This is the case of the two-person games and other special cases[5].

Therefore it seems natural to ask for the set of Lorenz undominated distributions in a given space $F^w(PI(v))$. More precisely it seems natural to reject a Lorenz dominated distribution as a possible egalitarian distribution.

Given a game (N, v) and a vector of feasible weights w, the set of Lorenz undominated allocations[6] on $F^w(PI(v))$ is defined as

$$L(F^w(PI(v))) := \{x \in F^w(PI(v)) \mid \text{there is no } y \in F^w(PI(v)) \text{ such that } y \succ_L x\}\,.$$

[4]This result holds if we take weighted satisfactions whenever the coalitions of the same size are weighted equally. Throughout this work this is one of our assumptions. If coalitions with the same size have different weights then the sum over all weighted coalitions could be different for different preimputations and therefore it need not be true that Lorenz domination implies "least square domination".

[5]For example if only the singletons have a positive weight there is a unique egalitarian allocation. Also if only coalitions of cardinality $n - 1$ have positive weight there is only one egalitarian allocation.

[6]Arin and Feltkamp (2002) introduce those solutions concepts as subsets of $PI(v)$, with the name w-CoLoS. Sudhölter and Peleg (1998, 2000) introduce the set of maximal Lorenz satisfactions that coincide with the w-CoLoS where the weights are identical for all coalitions.

Again the homeomorphism between the topological vector spaces denoted by $PI(v)$ and $F^w(PI(v))$ (the space of preimputations and the space of weighted satisfactions of preimputations) implies that a selection on $F^w(PI(v))$ is a selection on $PI(N, v)$.

Comment: It is no clear a priori how the vector of weights should be selected. Concerning this fact there are three curious results (Arin and Feltkamp (2002)).

1. The prenucleolus is the only w-prenucleolus that satisfies dummy player property[7].

2. The Shapley value is the only weighted Least Square value that satisfies dummy player property.

3. There is no system of weights, w, for which the prenucleolus and the Shapley value belong always to the associated $L(F^w((PI(v))$.

In the definition of the Shapley value and the prenucleolus, there is no restriction on the space of $PI(v)$, nor therefore in its associated $F^w(PI(v))$. In particular both sets are unbounded. In the case of the nucleolus, the sets $I(v)$ and $F^w(I(v))$ are bounded.

Maschler, Potters and Tijs (1992) introduce the following two properties in the context of TU games:

P7) **Independence of Irrelevant Alternatives** (IIA): If F' is a subset of F, with $\phi(F) \subset F'$ then $\phi(F') = \phi(F)$.

P8) **Strong Independence of Irrelevant Alternatives** (sIIA): If F' is a subset of F, with $\phi(F) \cap F' \neq \emptyset$ then $\phi(F') = \phi(F) \cap F'$.

(In this notation we assume that ϕ is a well-defined correspondence that selects elements in the sets F' and F.)

(If ϕ selects a unique element in each set IIA and strong IIA are the same.)

Comment: This property is widely used in the context of bargaining situations. In bargaining models it is the objective of agents to reach a 'best'

[7] A solution concept ϕ satisfies the *dummy player property* if for every game $(N, v) \in \Gamma$, and every dummy player i, $x_i = v(i)$ for all $x \in \phi(N, v)$. A player i in a game (N, v) is a *dummy* if $v(S \cup \{i\}) - v(S) = v(i)$ for all $S \subset N \setminus \{i\}$.

agreement point given a non-empty set of possible agreement points. Informally, in bargaining situations the IIA property says the following. Suppose we have two bargaining situations, such that the set of possible agreement points of one problem is contained in the set of possible agreement points of the other problem. Furthermore, suppose that the bargaining solution of the larger problem is available in the smaller problem. Then this point should also be the solution of the smaller problem. The interpretation is that the solution was already the 'best' point of the larger set. Hence if no new alternatives are offered, and only irrelevant alternatives are canceled, then the solution should remain the same. Clearly in TU games one can see the solutions as best agreements and therefore solutions should satisfy some IIA requirements.

From the definitions of the Shapley value and the prenucleolus it is immediately apparent that both satisfy IIA whenever the sets under consideration are the sets of weighted satisfactions with the adequate weights.

The set of Lorenz undominated elements also satisfies IIA (the reason is that any element not in the Lorenz maximal set is Lorenz dominated by some element in the Lorenz maximal set) but not strong IIA as the following example shows:

Example 1 *Let (N, v) be a 3-person balanced game where* $v(1,2) = v(1,3) = v(1,2,3) = 10$ *and* $v(S) = 0$ *otherwise.*

Consider that all the coalitions are equally weighted and that the fixed order of coalitions (excluding the empty and the grand coalition) is

$$\{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}.$$

One can check that the element $(6, 2, 2, -2, -2, 4) \in L(F^w(PI(v)))$. (the preimputation associated to this vector of weighted satisfactions is $(6, 2, 2)$.)

Now consider $F' = \{x \in F : \ x_3 = 2 \text{ and } x_4 = -2\}$. The prenucleolus on F' is $z = (8, 0, 2, -2, 0, 2)$ that is by definition a Lorenz undominated element on F'. Note also that $z \notin L(F^w(PI(v))$ since $z_2 \neq z_3$ and because of ETP any element in $L(F^w(PI(v))$ satisfies $x_2 = x_3$. Therefore $L(F) \cap F' \neq \emptyset$ but $L(F') \neq L(F) \cap F'$.

Arin and Feltkamp (2002) show that given a game (N, v) and a vector of feasible weights, w, any set of Lorenz undominated allocations on $F^w(PI(v))$

satisfies properties P1-P6. That means that any solution that always belongs to some set of Lorenz undominated allocations on $F^w(PI(v))$ satisfies properties P2-P5. As we know, this is the case of the prenucleolus, the Shapley value and many other solutions.

4 Egalitarianism over players

When the objective is to find an egalitarian distribution directly for players we have a unique solution that is the result of sharing the worth of the grand coalition equally (this solution is called the equal share payoff). This solution does not take into account the different worth that different coalitions can have: it solution only depends on the value of the grand coalition. In general, this is not a desirable property. It seems necessary that the worth of the coalitions should influence the final allocation in some way. An interesting approach to solving this problem is to search for egalitarian allocations in sets of imputations that satisfy minimum requirements of stability. This stability will depend on the worth of the coalitions.

In this approach, the core appears to be the most natural set satisfying the requirements of stability. Therefore in the definitions of the previous section we replace the set $F^w(PI(v))$ by the set $C(v)$.

Now, using the Rawls criterion, the Least Square criterion and the Lorenz criterion we get solutions that are close to the prenucleolus, the Shapley value and the w-CoLoS. We denote them as the Leximin stable allocation (LSA), the Least Square value (LS) and the Lorenz stable set (LSS).

1. $LSA(v) = \{x \in C(v); x \succeq_{lex} y, \text{ for all } y \in C(v)\}$
2. $LS(v) = \arg \min_{.x \in C(v)} \sum x_i^2$
3. $LSS(C(v)) := \{x \in C(v) \mid \text{there is no } y \in C(v) \text{ such that } y \succ_{CL^w} x\}$

Comment[8]*:* Curiously, Dutta and Ray (1989) introduce the term Lorenz core to define a set that is not the set of Lorenz undominated allocations. When looking for stable and egalitarian solutions Dutta and Ray do not consider the core as the set of stable allocations. The notion of stability that we use is the core, whereas Dutta and Ray consider the *Lorenz core* as an alternative notion of stability that is justified on the following grounds.

[8]This comment was introduced in Arin and Iñarra (1996).

Consider a society formed by n members which has to select an egalitarian rule, and where the members of any coalition blocking this rule may form a subsociety subject to the same rule. In this setting, the fact that there exists a "blocking" coalition S whose worth exceeds the sum of payoffs assigned by the rule in the original game, does not guarantee that S is going to "deviate", since such deviation may not be credible. In fact, coalition S will only deviate, and form an alternative society, if none of its members receives an inferior payoff according to the given rule in the subsociety, which is defined by the subgame (S, v). (In this subgame the worth of any coalition is coincident with the worth in the original game). This argument questions the core as a notion of stability in this scenario, and calls for a new definition of stability. In this respect, Dutta and Ray (1989) define the Lorenz core, a set function that contains the core. Of course, the egalitarian allocation belongs to the Lorenz core.

However, in our opinion, the core may also be considered a satisfactory notion of stability in the setting above, since we consider that any coalition blocking to an allocation rule will really deviate and form a subsociety subject to the same rule. The argument runs as follows: Members of a blocking coalition when facing the possibility of deviating, will compare their own worth with the sum of payoffs offered by the rule in the entire society, choosing the maximal as their worth in the reduced game they may build. In this case, if we take into account individually rational rules (and the egalitarian allocation is certainly one) it is not difficult to see that the deviation of any blocking coalition will always be credible.[9].

The set of Lorenz undominated allocations in the core satisfies P1)-P5) and IIA and strong IIA. In fact, the properties can be renamed as IIC and sIIC where C means Core.

The egalitarian solutions defined in the core satisfy the Davis-Maschler reduced game property (consistency). This is a difference with respect to the egalitarian solutions defined in the space of vectors of weighted satisfactions. In this case only the prenucleolus satisfies the property[10].

In general, reduced game properties study the invariance of a solution when viewed by any subcoalition of players. We introduce now the Davis-Maschler

[9]See the discussion on the notion of stability of the core in Ray (1989)

[10]The Shapley value satisfies the Hart-Mas Colell reduced game property..

reduced game property[11].

Let Γ be a class of games. Let $v \in \Gamma$ be a game, $T \subset N$, and consider $T \neq N, \emptyset$ and a preimputation x. Then the *Davis-Maschler reduced game* (Davis and Maschler, 1965) with respect to $N \setminus T$ and x is the game $(N \setminus T, v_x)$[12] where

$$v_{\phi}^{N\setminus T}(S) := \begin{cases} 0 \quad \text{if } S = \emptyset \\ v(N) - \sum_{i \in T} x_i \quad if \quad S = N \setminus T \\ \max_{Q \subset T} \left\{ v(S \cup Q) - \sum_{i \in Q} x_i \right\} \qquad \text{for all } S \subset N \setminus T. \end{cases}$$

We also denote the game $(N \setminus T, v_x)$ by $v_x^{N\setminus T}$.
Let ϕ be a solution on Γ. We will say that

P9) ϕ satisfies the **Davis-Maschler reduced game property** on Γ if for every game $v \in \Gamma$, for all nonempty coalitions T and for all $x \in \phi(v)$ if $v_x^{N\setminus T} \in \Gamma$ then $x^{N\setminus T} \in \phi(v_x^{N\setminus T})$.

Comment: A criterion of egalitarianism is consistent if every coalition of players finds that the way they redistribute the amount allotted to them is egalitarian. The egalitarianism underlying the sharing rule is the same for all coalitions of players.

Arin and Iñarra (2001) show that $LSS(C(v))$ and $LSA(C(v))$ satisfy the Davis-Maschler reduced game property in the class of balanced games, the class where the solutions are well-defined[13]. Arin, Kuipers and Vermeulen (1998) show that $LS(C(v))$ satisfies also the property. Dutta (1990) shows that in the class of convex games the egalitarian allocation (Dutta-Ray) solution satisfies this property.

[11] This property states that if x is an element in the multi-valued function ϕ of a game $v \in \Gamma$, then for any non-trivial coalition T the projection of x into $N \setminus T$ belongs to the multi-valued function ϕ of the reduced game v for coalition $N \setminus T$ with respect to x whenever the reduced game belongs to the class.

[12] A Davis-Maschler reduced game is obtained from the original game by removing some players, who are assumed to be paid according to x. The total worth in the reduced game is equal to the total worth in the original game minus the payoff allocated to the removed players. The worth of a coalition in the reduced game is derived as the most profitable of the possibilities of cooperating with the players removed considering that these players are paid according to x.

[13] See also Hougaar, Peleg. and Thorlund-Petersen (2001).

Comment: Maschler, Potters and Tijs (1992) introduce a reduced game property for truncated games. A truncated game is a TU game with a set of permissible preimputations and a set of permissible coalitions. In this case a reduced game respect to a coalition S at x (where x is an element of the set of permissible imputations) is a new truncated game with a new set of permissible imputations where the payoffs for players in N/S are fixed according to x. This definition can be considered as a special case of the property of IIA.

The next table is a summary of results for TU games.

SUMMARY IN TU GAMES

Solution	Criterion	Domain	Properties
Prenucleolus	Rawls	$F(PI(v))$	P1-P9
Shapley value	Least Square	$F^{w_{Sh}}(PI(v))$	P1-P8
Nucleolus	Rawls	$F(I(v))$	P1-P9
w.CoLoS	Lorenz	$F^{w}(PI(v))$	P1-P7
Leximin	Rawls	$C(v)$	P1-P4 and P7-P9
Least Square value	Least Square	$C(v)$	P1-P4 and P7-P9
Lorenz stable set	Lorenz	$C(v)$	P1-P4 and P7-P9
Dutta-Ray solution	Lorenz Rawls	Lorenz core	P1-P4 and P7-P8
Equal share payoff	Least Square Lorenz	$PI(v)$	P1-P4 and P7-P9

In order to be precise we need to add the class of games in which the solution concept is well-defined. As we know the nucleolus is nonempty if the set of imputations is nonempty. Also it is important to take into account the class of games in which some properties have full meaning[14]. The Leximin, Least Square and Lorenz stable set are defined in the class of balanced games. The Dutta-Ray solution is well-defined in the class of convex games and in this case coincides with the Leximin solution. Finding a bigger class of games in which the Dutta-Ray solution is well-defined is an open question. The rest of the solutions are defined in the class of all TU games.

[14]In this sense, even when the nucleolus is well defined in a game could be empty for some reduced games. The nucleolus, in the class of all TU games, satisfies a slightly modified reduced game property (see Snijders, 1995).

5 Other coalitional models

The framework we have analyzed for TU games can be extended to other coalitional models. Again different solution concepts can be seen as particular answer to the two questions placed at the en of the *Introduction* of this paper.

For example, Aumann and Dreze (1975) introduce the model of cooperative games with coalitional structure. They add a third element to the set of players and the characteristic function, a coalitional structure that is a partition of the set of players. They define the classical solution concepts for this new case. The set of reference is not $PI(v)$ but a new set that depends on the specific coalitional structure and defined as follows:

$$X(N, v, B_k) = \left\{x \in R^N; x(S) = v(S) \text{ for all } S \in B_k\right\}$$

This is a new domain in which we can apply the different egalitarian criteria. Aumann and Dreze define the nucleolus in this way but they do not use the Least Square criterion to define a Shapley-like value in this context. They follow the axiomatic approach and define a value that does not coincide with the result of applying the Least Square criterion on the domain of weighted satisfactions generated by the set X.

Example 2 *Let* (N, v) *a 3-person balanced game where* $v(1,2) = v(1,3) = v(1,2,3) = 4$ *and* $v(S) = 0$ *otherwise.*

Consider that all the coalitions are equally weighted and that the fixed order of coalitions (excluded the empty and the grand coalition) is

$$\{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}.$$

Assume that the coalitional structure $\{3\}, \{1,2\}$ is formed. According to Aumann and Dreze the Shapley-like value in this game is $(2,2,0)$. If we apply the Least Square criterion to the set of vectors of satisfactions restricted to $x(1,2) = 4$ and $x(3) = 0$ we get the allocation $(3,1,0)$. We believe that this allocation is *as much a Shapley value as* the one proposed by Aumann and Dreze Note that in the case of 3-person games the Shapley weights are identical for all coalitions.

Lucas and Thrall (1963) introduce the model of games in partition function form where the characteristic function is a vector-valued function on the family of partitions of N. Different Shapley-like values have been analyzed in this model following axiomatic approaches. It is clear that a different approach can be considered to define Shapley-like concepts for games in partition function form.

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