Qualitative Voting*

Rafael Hortala-Vallve

London School of Economics and STICERD

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PRELIMINARY DRAFT

Abstract

As opposed to the *classical* voting system (one person - one decision - one vote) a *new* voting system is defined where agents are endowed with a given number of votes that can be distributed freely between a prearranged number of issues that have to be approved or dismissed. Its essence relies on allowing voters to express the intensity of their preferences in a simple and applicable manner. From a mechanism design perspective we first prove which allocations are implementable: for a social choice function to be implementable it should only care about the voter relative intensities between the issues at the *interim* stage. We also prove that this new voting system, Qualitative Voting, Pareto dominates Majority Rule in some general settings and, even more, it achieves the only *ex-ante* incentive compatible optimal allocation. Finally, an argument in favour of Majority Rule is presented showing that the optimal implementable allocation robust to any possible prior is the one achieved by Majority Rule.

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Contents

"*The majority principle is itself a product of agreement, and presupposes unanimity on at least one occasion*" Jean-Jacques Rousseau

1 Introduction

Voting is the paradigm of democracy. It reflects the will of taking everyone's opinion into account instead of imposing, by different means, the decision of a particular individual. At its root lies the fact that people should be allowed to freely cast their votes and, above all, they should be treated equally. Consequently, as opposed to many economic situations, voting is considered a situation where no side payments are allowed so that agents are treated in an ex-ante identical position and wealth effects play no role.

Voting is also viewed as a way to give legitimacy to public decisions. Legitimacy that is best achieved by unanimity but given the inefficiency that such rule would yield it is approached through Majority Rule (MR). MR is a compromise which we unanimously accepted -as stated in Rousseau's quote- given that it satisfies the three main properties any aggregating device should satisfy: equality, freedom and legitimacy.

Even though MR seems to be the most sensible voting rule, it has been repeatedly questioned. From an economist's perspective, and given that most of our work is built on the diverse behaviour of individuals with different marginal propensities to consume, produce, etc., the main worry should be the fact that it does not capture the intensity of the voters' preferences. Just as we contemplate the importance of the *willingness to pay* in the provision of public goods we should take into account the *willingness to influence* in a voting situation. As we know from the former case, an increase in the overall efficiency should also follow from doing so in the latter case.

The answer to this criticism has always been founded on the equality argument: if we were to treat differently a very enthusiastic voter from a very apathetic one, equality would no longer hold.[1] Nevertheless this reasoning is too narrow. In this paper we will show that we can build a very simple alternative that allows voters to express intensity and enhance the three criteria cited above.

[1]See Spitz (1984).

We will propose a voting system where the casted votes have an embedded quality which is associated somehow to the intensity of the voters' preferences. Accordingly, we will call such voting system Qualitative Voting (QV).

Let's be a bit more precise: in a setting with a closed agenda of N issues that have to be approved or dismissed, QV endows the voters with V votes that are simultaneously and freely distributed among the issues. Its essence relies on allowing voters to trade off voting power on one issue to strengthen it on a different one. Somehow, we are providing voters with a broader set of strategies than the classical "one person - one decision - one vote". Note that, the equality argument is satisfied given that all individuals are endowed with the same ex-ante voting power (say, V votes). Furthermore, freedom on the voter's strategies is broaden and the legitimacy on the resulting public decision is increased given that issues are now approved or dismissed on the grounds of overall intensity.

Note that allowing the intensity of the preferences to play a role in voting games goes hand-to-hand with considering a multiple issue situation. This argument is related to the one in Jackson and Sonnenschein (2003) where it is shown that linking decisions normally leads to Pareto improvements. In this paper we will show that any implementable allocation should only care about the voters' relative valuation between the issues. It follows that in the unidimensional case just the sign of the preference can be considered.

Very related to the mechanism at hand, Casella (2002) has proposed a system of *Storable Votes*: in a situation where voters have to decide over the same binary decision repeatedly over time, she proposes to allow votes to be stored and used in future meetings. Such voting system is proved to Pareto dominate MR in a general setting. Our framework is different in the sense that voters simultaneously cast all their votes and know their full preference profile at the time of voting (no time dimension). Besides, we undertake a mechanism design analysis which allows us not only to compare two particular voting rules but also to characterise all implementable allocations and, from them, identify the optimal ones.

From a mechanism design perspective the problem we are dealing with is one of multidimensional types with multilateral asymmetric information and no transfers. The main contribution lies on having no transfers but having cardinal utilities; most of the literature on mechanism design without transfers (as most of the literature on voting) is build on a setting of electing one alternative out of many and having ordinal utilities,[2] i.e. a setting of electing representatives. Within that literature, QV has a flavour of a *scoring rule*[3] though there is a crucial distinction: a *scoring rule* is used to elect one representative out of many, instead QV deals with a situation where N independent issues have to be approved or dismissed. Our setting is one of a repeated binary election.

The literature on alternatives to Majority Rule is related to our work in the sense that it provides mechanisms which capture the intensity of the voters preferences but their complexity undermines its applicability. On the one hand, Tullock (1976) develops an application of the *Clarke-Groves* mechanism to a voting framework. Needless to say, this requires monetary transfers, thus fails to satisfy the equality property. On the other hand, Hylland and Zeckhauser (1979) propose a *Point Voting Rule*[4] to be used for the contribution to public goods, with perfectly divisible points and focus on providing an (arbitrary) social choice function that induces the truthful revelation of preferences.

Finally, when we imagine a way in which politicians give more weight to a particular position we immediately think of *logrolling* or *vote trading*. This occurs whenever two voters bilaterally agree on voting against one's position on some non salient issues which are salient for the other voter. The result is that both voters will have gained support on their salient issues at the cost of losing non-salient ones. The relationship and gains of QV with respect to that particular way of expressing the *willingness to influence* will become clear in section 6.1.

Prior to setting out the general model, the next section provides three examples which will introduce our main results and will hopefully shed some light on the applicability of QV in the real world. The first two show how QV allows two players to simultaneously declare how do they rank two different issues and, consequently, allows them to reach the only *ex-ante*, *interim* and *ex-post* efficient allocation. The third example, instead, reflects the main contribution of QV in a setting with more than two players. In particular, in a setting with

[2] The main references are Gibbard (1973) and Satterthwaite (1975). In essence, these works are a formal treatment of the Arrow's Impossibility Theorem from a mechanism design perspective.

[3] "In a *scoring rule*, each voter's ballot is a vector that specifies some number of points that this voter is giving to each of the candidates that are competing in the election. These vote-vectors are summed over, to determine who wins the election.", Myerson (1998)

[4] Brams and Taylor (1996) propose a *Point Voting Rule* (the *Adjusted Winner Procedure*) that is essentially our voting system in a setting of a conflict resolution. Their weakness, though, is that they do not take into account the strategic interactions and restrict players to be truthful on their casted votes.

three players it is shown that whenever non-indifferent players rank issues differently enough it is optimal to let a strong minority decide over a weak majority (heuristically, a voter that strongly opposes an issue should decide over two voters that weakly supports it).
Theorems two and three formally state both results in a setting with uniform priors on the voters' preferences. The dependence of these results on the priors is shown to be critical in theorem four, where it is shown that the allocation reached by MR is the optimal one whenever we allow for any possible information structure on the voters' preferences. Thus, MR cannot be questioned in general grounds but just in some specific situations.
Given that we undertake a mechanism design approach, prior to the optimality analysis argued above we need to define which allocations are implementable. On these lines, theorem one states that any implementable allocation should be only contingent on the voters' relative intensity between the issues. That is, an implementable mechanism cannot treat very enthusiastic voters better than apathetic ones. Otherwise, the ones with weaker preferences will have an incentive to pretend they are also very enthusiastic. From it, we can say that any implementable SCF cannot undertake a direct interpersonal comparison of utility; it can only look to the relative valuations of the individuals. Ultimately, as the mechanism is to aggregate individual preferences it needs to do some comparison across individuals but it is based on the primary intrapersonal one.
The selection of the agenda is showed to be an important issue that arises when analysing QV. It is clear that there are clear incentives to manipulate the agenda in order to induce particular outcomes. The whole of the paper remains silent on this aspect and takes the agenda as given until section seven where the subject is discussed. Finally, section eight concludes and draws the lines for future research.

2 Some Examples

2.1 Example 1: *A Night Out*

Anna and John are to go out on a Friday night and have decided they will first go for dinner and then to the movies. It's their first date so, above all, they want to be together though they do not come to an agreement. Anna wants to see a horror film and fancies having dinner

in a new Italian restaurant and John prefers a comedy film and eating sushi in a Japanese restaurant. If they were to vote on each of the issues, nothing would be decided and they would have to stay at home (we assume that option not to be optimal by either of them).
Additionally, suppose that Anna really cares about which restaurant to go and John, instead, cares more about the film. It seems sensible that, as being good friends, each of them will give up on their least preferred option. That is, they will both go to the Italian restaurant and then to the comedy film.
From a game theoretic perspective, they are both coordinating on the only Pareto optimal allocation that yields a strictly positive utility to both players (in the sense that each one will win on their most preferred issue and lose on their least preferred one).
In this setting, QV will just be a mechanism that will allow John and Anna to non-cooperatively coordinate on the agreed outcome. While this example seems too trivial and no one expects friends to be voting what to do at all times, it reflects the main intuition behind the voting rule we are proposing. It allows players to non-cooperatively declare which their preferred alternative is and thus reach the Pareto optimal allocation.

2.2 Example 2: *Conflict Resolution*

A more realistic version of the previous example may take the shape of a conflict resolution situation. In that case, two parties that have agreed on all concurring issues are to resolve on some dissenting ones. In that context it seems sensible not to expect the nice behaviour we observed in the previous example. Now, parties may see any concession as a loss and (given the sequential nature of the bargaining) may never truthfully declare their preferred alternatives leading to the deferring of any decision.
Imagine a familiar enterprise that, after being badly managed for two generations, is in a very delicate situation and decides to hire a CEO to redirect their business. The new CEO's team does a comprehensive analysis of the situation and concludes that the image of the firm has to be updated and two proposals are made. On the one hand a restyling of the logotype will change the consumer's perception of their brand at a very low cost. On the other hand, a deep improvement of their main product would also be beneficial to improve consumer's perception and, furthermore, it will gain the attention of the press.

The owners are against any change in their product because this is, from their point of view, the essence of their business. Similarly, they cannot contemplate a restyling of their logotype because it was designed by one of their ancestors and they feel emotionally attached to it.
The negotiations between both parts are at a deadlock and, as was highlighted before, any concession is seen as lost. Next again, the parties rank the issues differently. The CEO realises that the first policy is interesting given its low costs but will have no persistent effect on the public and he sees the latter as the essential move to refloat the firm. Instead, the family owners realise that something has to change but wouldn't like to be unfaithful to their ancestor and, above all, want to keep their logotype. We have ended up in a *Prisoner's Dilemma* situation: whatever the opponent does any player will always be better off not ceding and declaring both issues to be equally important (it is *dominant* to do so). And, as it is always the case, the unique equilibria is a Pareto dominated one.
QV allows the players to unlock the negotiation and non-cooperatively choose the Pareto optimal allocation. Let's analyse its functioning: the CEO and the family are endowed with V votes each and will invest all votes in their preferred issue. The reason being that, given the binary nature of the situation, winning one issue will imply losing the remaining one. Hence, the optimal strategy is to ensure the no-loss of the most preferred issue.
So far, we have described two situations in which we have omitted any reference to the information held by the players. Preferences are usually private information and we cannot expect agents to use such information honestly -strategic voting is a common feature in voting games.
A particular feature of the two cases presented so far (conflict resolution situation with two issues) is that they are robust to any kind of prior in the voters' preferences. QV allows to unlock the negotiation
We can device many different situations where such situations occur and where side payments may not be possible (or may be forbidden): a divorce settlement, an international dispute, a bilateral agreement in arms/pollution reduction, a country having the two chambers governed by opposing parties,[5] a clash between the management and the union of a particular firm,

[5]The US Congress and Senate have repeatedly been in a situation where one chamber had a Republican majority and the other a Democrat one. Consequently, many bills have been vetoed by one chamber to the

etc...[6]

It can be argued that the proposed situations are very likely not to fit the two issues case. Instead they correspond to a classical problem in Game Theory, the *Colonel Blotto Game*. One form of this game reads as follows:

> Colonel Blotto and his opponent each have 100 divisions and are going to fight over 10 regions. They therefore divide their forces up into 10 parts and send each part to one region. Ten fights take place, and the one with the larger force wins the region. The winner of the battle is the one with the most won territory.[7]

In our setting a similar explanation can be built taking into account that now the colonel is not indifferent between winning two different regions. Hence the payoff of the game is not only contingent on how many regions he has won or lost but precisely on which regions he has lost or won. Using the result on Milgrom-Weber (1985) we know that it is (essentially) sufficient to have an atomless distribution on the preferences' priors to ensure the existence of pure strategy equilibria in the transformed game. The present paper will only provide a general result on implementability of this kind of game but will remain silent on its optimality questions.

2.3 Example 3: *Committee Meeting*

Imagine a religious association which is composed by three factions which all have the same voting power at the annual committee. In that committee they need to update the association's position in two major biological scientific advances: human cloning and the use of stem cells. Imagine that each of the members of the committee has no clue about their opponents' preferences but privately know their own. The most progressive faction has no

other so that no decision have been easily made. QV could have made the decision process more efficient allowing each party to support those bills which its electorate felt more strongly about.

[6] We have not stressed a crucial distinction among some of these examples. Some of the cases regard the dispute over a *public* decision (non rival and non excludable decision) and the rest regard the dispute over a *private* one (the ownership of some goods). As it will become clear later when we formally introduce the model, any result can be applied to both situations because they are analogous in the sense that the approval or dismissal of any of the issues will correspond to one player increasing or decreasing its payoff more than his opponent (technically, they correspond to different linear transformations of the payoff).

[7] This quote is taken from Jonathan R. Partington's webpage where an e-mail experiment on the game is run. (www.amsta.leeds.ac.uk/~pmt6jrp/personal/blotto.html)

strong position in any of the issues but it is mostly in favour of both. Each of the other two strongly opposes a different issue and recognises that the positive aspects of the remaining one outweighs their moral prejudices and hence favours it. The next diagram captures their position:

	Human cloning	**Use of stem cells**
F1	*agree*	*agree*
F2	*strongly disagree*	*agree*
F3	*agree*	*strongly disagree*

If they vote through MR, both issues will be approved: a weak majority will impose its will over a strong minority. Is that situation *optimal*? Of course the word *optimal* seems not to be the most adequate since the outcome is indeed optimal from faction one's perspective.
In section 5.2 we will show that from an *ex-ante* perspective (i.e. before players know what they are going to vote) the MR outcome may not be optimal. If the difference between the strength of the *strongly disagree* and the *agree* positions is wide enough, it will be optimal to allow the enthusiastic minorities decide over the apathetic majorities.
QV is again a system where agents are able to increase the probability of winning their preferred issue investing all their votes on that issue (there also exists another equilibrium that replicates the allocation achieved by MR having all players evenly splitting their votes between the issues).
Following the example above, the first faction will evenly split its votes, the second will invest all of them in the first issue and the third will do the same in the second one (as depicted in the table below).

	Human cloning	**Use of stem cells**
F1	$\frac{V}{2}$	$\frac{V}{2}$
F2	$-V$	0
F3	0	$-V$

The outcome will be now the opposite to the one before, both issues will be dismissed. As stated above, from an utilitarian point of view this outcome will dominate the previous one as long as non-indifferent voters rank issues differently enough.
Note that information is now playing a very relevant role. In case voters were holding perfect

information about their opponents' preferences QV may have no pure strategy equilibrium.[8] For instance, in the example above, faction one has incentives to deviate and invest all of its votes into one of the issues cancelling out the votes invested by one of the opposed factions.[9] The non-existence of equilibrium is related to section six where the relation between QV and *logrolling* is analysed. It illustrates a big drawback on the proposed voting rule and an argument in favour of MR. As will be generalised later, the only implementable allocation (i.e. there exists a mechanism that implements such allocation) robust to any prior is MR. Hence, a general argument in favour of MR is that it will always be implementable and will always lead, in our setting, to a honest voting behaviour. Nevertheless, this does not undermine the point of this article: there are some situations in which one can strictly Pareto improve the allocation achieved by MR through a simple mechanism we have called QV.

This last example has captured the main contribution of QV as a decision rule. It allows minorities to decide on some issues they regard as being very important whenever the majority does not feel very strongly against it. The optimality of this decision rule as compared to MR will depend on the intensity of voters' preferences being different enough

3 The Model

A *voting game* is defined as a situation where I players have to dismiss or approve N issues and no monetary transfers are allowed. Players privately know their preference profile across the N issues and the prior distributions from which these preferences are drawn are common knowledge (note that this allows for deterministic priors, i.e. commonly known preferences). From a mechanism design perspective this is a multidimensional problem with multilateral asymmetric information and no transfers.

Players and issues are denoted $i \in \{1, 2, ..., I\}$ and $n \in \{1, 2, ...N\}$, respectively. Player i's valuation towards issue n is θ_n^i (throughout, superscripts denote agents and subscripts denote

[8]Similarly to what is observed in the *public goods* literature, voters may now have an incentive to free-ride on the votes casted by some of their opponents. The perception of how pivotal one's votes may be, will induce a highly strategic interaction and the non-existence of equilibria will follow in some of the cases.

[9]The behaviour of the voting rule in case of ties has not yet been addressed. It will be proved that the only optimal tie breaking rule in the case of three voters is the usual majority. The intuition works as follows: if an issue cannot be approved on the grounds of overall intensity, overall support should decide.

issues). The preference vector of player i is $\theta^i = \left(\theta_1^i, ..., \theta_N^i\right)$. At the moment, we will impose no restriction on the range of possible types, i.e. $\theta^i \in \Theta \subseteq \mathsf{R}^N, \forall i = 1 \div I$; neither on their prior distribution.

Preferences should be interpreted as follows: a positive type ($\theta_n^i > 0$) wills the approval of the issue, a negative one ($\theta_n^i < 0$) wills its dismissal and its absolute value ($|\theta_i^n|$) captures the intensity of the preference towards that particular issue.

Player i's payoff on a given voting procedure n is described as follows,

$$\begin{cases} \theta_n^i \text{ if the issue is approved} \\ 0 \text{ if nothing is decided} \\ -\theta_n^i \text{ if the issue is dismissed} \end{cases}$$

and the total payoff is the sum of the individual payoffs across the N voting procedures.[10]

An allocation is a N-tuple of probabilities that corresponds to the probability of approving each of the N issues. The set of allocations is defined as $\mathcal{X} = \{(p_1, ..., p_N) : p_1, ..., p_N \in [0,1]\}$ where p_n is the probability of issue n to be approved. Hence, given any allocation $p \in \mathcal{X}$, a player with preferences θ^i obtains the following utility from it:

$$u\left(p, \theta^i\right) := \sum_{i=1}^{N} p_n \theta_n^i + (1 - p_n)\left(-\theta_n^i\right) = \sum_{i=1}^{N} (2p_n - 1)\,\theta_n^i.$$

Note that we are in a setting of private values where each agent's utility depends only on his own types and utilities are multilinear.

[10]The definition of the payoff is implicitly assuming that issues are independently valued. That is, there are no complementarities or substitutabilities between the issues. Nevertheless, provided that issues are independently valued, results can be extended to different payoffs. For instance, we can consider a situation where what is to be decided is the property of N different goods and the payoff in each issue is:

$$\begin{cases} \theta_n^i \text{ if the good is given to player } i \\ 0 \text{ otherwise.} \end{cases}$$

4 Implementability Result

Given the Revelation Principle we know that the allocation achieved by any indirect mechanism can be replicated by truthful revelation in a *direct mechanism*. That is, given any mechanism that aggregates individuals' preferences into social choices, we can construct a mechanism where agents are supposed to reveal sincerely their type and the mechanism designer will assign the same allocation as with the indirect mechanism. This methodology allows us to characterise all implementable allocations through indirect mechanisms and focus our attention on those allocations that will be achievable.

The analysis is thus restricted to the study of *direct mechanisms*. A *direct mechanism* is a function (p) that maps any revelation of the agents types into an allocation. Such mapping is usually called a Social Choice Function (SCF).

$$p : \Theta^I \to \mathcal{X}$$
$$\text{i.e. } p\left(\theta^1, ..., \theta^I\right) = \left(p_1\left(\theta^1, ..., \theta^I\right), ..., p_N\left(\theta^1, ..., \theta^I\right)\right).$$

As stated above, we need to check which SCFs induce truthful revelation at the stage where each agent privately knows his own type, but only knows the prior distribution of his opponents. This is the stage where he has to reveal his type in the direct mechanism and cast his votes in the indirect mechanism, the *interim stage*. The *interim utility* of a player that declares $\hat{\theta}^i$, being θ^i, is defined as:

$$U^i\left(\hat{\theta}^i, \theta^i\right) := E_{\theta_{-i}}\left\{u\left(p\left(\hat{\theta}^i, \theta^{-i}\right), \theta^i\right) \mid \theta^i\right\}$$

where, $\theta^{-i} := \left(\theta^1, ..., \theta^{i-1}, \theta^{i+1}, ..., \theta^I\right)$. Note that this is nothing but the expectation of his utility taking into account that his opponents will truthfully reveal their type. To simplify the notation let's also define the *interim* probability as:[11]

$$P_n^i\left(\hat{\theta}^i\right) := E_{\theta^{-i}}\left\{2p_n\left(\hat{\theta}^i, \theta^{-i}\right) - 1 \mid \theta^i\right\}.$$

[11] Note that the *interim* probability is the expectation of a linear transformation of the SCF, hence it is not a well defined probability. In particular, its domain lies on $[-1, 1]$.

Hence, the *interim* utility is:[12] $U^i(\cdot,\theta) = \sum_{n=1}^{N} P_n^i(\cdot)\,\theta_n$.

In order to characterise all implementable SCFs we just need to impose the *Incentive Compatibility* constraints (IC). That is, that it is optimal for any player to truthfully reveal his type:

$$\theta \in \arg\max_{\hat{\theta}\in\Theta} U^i\left(\hat{\theta},\theta\right), i = 1,...,I.$$

Such constraints are usually known as the

Assuming that $P_n^i(\cdot)$ is continuous and twice differentiable for almost all $\theta \in \Theta$ we have that the necessary first and second order conditions are:

$$\begin{cases} \frac{\partial}{\partial\hat{\theta}} U^i\left(\hat{\theta},\theta\right)\Big|_{\hat{\theta}=\theta} = 0 \\ \frac{\partial^2}{\partial\hat{\theta}^2} U^i\left(\hat{\theta},\theta\right)\Big|_{\hat{\theta}=\theta} \text{ is negative semidefinite} \\ \text{for } i = 1,...,I. \end{cases}$$

The next proposition is just an extension to the usual technique used in one-dimensional screening problems and is a first step to simplify the first and second order conditions above in order to characterise all the implementable allocations.

Proposition 1 $\theta \in \arg\max_{\hat{\theta}\in\Theta} U^i\left(\hat{\theta},\theta\right) \Longleftrightarrow \begin{cases} \nabla U^i(\theta) = P^i(\theta) \textit{ for almost every } \theta \in \Theta & (1) \\ U^i \textit{ is convex on } \Theta. & (2) \end{cases}$

where $U^i(\theta) := U^i(\theta,\theta)$, $\nabla U^i(\theta) := \left(\frac{\partial U^i(\theta)}{\partial\theta_1},...,\frac{\partial U^i(\theta)}{\partial\theta_N}\right)$ *and* $P^i(\theta) := \left(P_1^i(\theta),...,P_N^i(\theta)\right)$.

An analogous result can be found in Rochet and Chone (1998). The proof (which is provided in the appendix) consists in extending the one-dimensional technique: the envelope theorem gives us the first condition and a combination of the first order and second order conditions gives us the convexity result. An implication of the latter result is the usual implication that the *interim* probability is, *ceteris paribus*, weakly increasing on the player's type on that issue (i.e. $\dfrac{\partial P_n^i(\theta)}{\partial\theta_n} \geq 0, \forall n, i$).

Note that the first condition together with the definition of $U^i(\cdot)$ imply that any imple-

[12] To easy the notation we drop the superscript i on the preference vector θ^i.

mentable SCF should satisfy the following linear first-order partial differential equation:

$$\nabla U^i(\theta) \cdot \theta = U^i(\theta).$$

Euler's theorem tells us that it is sufficient that $U^i(\cdot)$ is homogeneous of degree one (HD1, i.e. $U^i(t \cdot \theta) = t \cdot U^i(\theta), t \in \mathsf{R}, t > 0$) for the previous condition to be satisfied. Something which is not normally stated is that "Euler's theorem on homogeneous functions is invertible, that is, only homogeneous functions of degree one satisfy the equation".[13] The next result follows:

Theorem 1 *A SCF $p : \Theta^I \rightarrow X$ is implementable iff the derived interim utilities for each player are HD1 and convex.*

The homogeneity of degree one on the *interim* utilities implies that the *interim* probabilities are homogeneous of degree zero (HDO).[14] That is, all proportional types are treated equally (bunched) or, in other words, the *interim* probabilities are only sensitive to the relative valuation between the issues. One can interpret this as if there cannot be any direct interpersonal comparison of utilities and any aggregation procedure should be preceded by an intrapersonal one -an apathetic voter and an enthusiastic one are essentially treated in the same manner. This extends the equality argument presented in the introduction: it is not only that wealth effects can play no role in a *voting game* but neither the preference endowment of each individual can do so. Whilst the former argument is an axiomatic one (imposed by ethical or practical reasons) the latter is a necessary condition for the *voting game* to be implementable.

It is worth pointing out that the theorem is a general result allowing for any prior in the preferences. This is, it permits for correlation between issues, individuals, etc. Its weak point is that it is a result on the *interim* SCF and consequently the priors are playing a role.[15]

[13] Elsgolts (1970), pg 264.

[14] The partial derivative of a HD1 function is a HD0 function.

[15] Its weakness is illustrated for instance through Fubbini's theorem. Given that $\nabla U^i(\theta) = P^i(\theta)$ Fubbini's theorem implies that

$$\frac{\partial P_n^i(\theta)}{\partial \theta_m} = \frac{\partial P_m^i(\theta)}{\partial \theta_n}; \forall n, m, i.$$

Note that given a SCF p the priors will play a very relevant role for the former equality to be true.

Note that we had a multidimensional mechanism design problem with multilateral asymmetric information without transfers. Its main difficulty (and main contribution with respect to the existing literature) relied on not allowing for any transfers. Consequently, we introduced an endogeny problem in the sense that we could no longer associate a high transfer to a high type declaration in order to induce truthful revelation of the preferences. Now, as a response to a player declaring an issue being highly preferred, the SCF should not only increase the probability of him winning that issue but the associated cost should be in terms of decreasing the probability of him winning any other issue and this should (intuititively) complicate the analysis terribly. However, as opposed to what is expected, having no transfers simplifies the analysis because the first order partial differential equation that arises from imposing truthful revelation (IC) is now solvable.

To illustrate such property, imagine a setting with only two issues. Then, for a SCF to be implementable it should only care about the direction of the preference vector and should be invariant to its modulus. In a way we have reduced the dimensionality of our problem in one dimension.[16] Furthermore, if the setting is unidimensional, the HDO implies that *interim* probabilities should be invariant to the intensity of the preferences and should only care about its sign (i.e. if the voter wants the approval or the dismissal of the issue).

Hence, the argument usually endorsed by political scientists that "the introduction of an intensity dimension attacks political equality in ways not permissible within the context of democratic theory"[17] is wrong. Intensity can be taken into account as long as we broaden the usual limits and we bundle together the voting of more than one issue. In other words, allowing agents to express the intensity of their preferences whenever they vote goes hand-to-hand with the argument of analysing voting games in multidimensional settings.

4.1 Implementability of Qualitative Voting

We have seen that what matters is the relative intensity of preferences. Hence we should device a mechanism that allows such expression. The most natural way is to give to each voter

[16] Such concept becomes crystal clear if we consider polar coordinates. In that setting, the interim probabilities should only care about the angular coordinate (angular coordinates if the setting has more than two issues) and neglect the radial coordinate.

[17] Spitz (1984), pg 30.

a budget constraint so that they trade off their voting power between the issues according to their relative valuation. The natural answer is an indirect mechanism with the flavour of QV: endow the voters with V votes and allow them to freely distribute them.

The pursue of a simple mechanism suggests the indivisibility[18] of such votes. What is more, restricting the set of possible actions to be finite allows us to apply Milgrom and Weber (1985) to insure the existence of pure strategy equilibria in the general game with I issues, N players and V votes as long as the informational structure of the game satisfy some *agreeable*[19] conditions.

5 Optimality Results

All the optimality analysis will be done under three simplifying assumption. Particular extensions and the robustness of some of our results to these assumptions will be argued were necessary. We will restrict our analysis to a setting with:

- Two issues: $N = 2, n \in \{1, 2\}$
- Two valuations: $\theta_n^i \in \{\pm 1, \pm \theta\}, \theta \in (0, 1)$[20]
- A uniform and pairwise independent prior:

$$\begin{cases} \Pr\left\{\left|\theta_n^i\right| = 1\right\} = \frac{1}{2} \\ \Pr\left\{\theta_n^i > 0\right\} = \frac{1}{2} \\ \text{Pairwise independence across issues and players.} \end{cases}$$

From the viewpoint of the designer of the mechanism it is sensible to ask if a voting rule he would like to implement is on average the best one. That is, if weighting all possible combination of types (given the prior distributions of them) the voting rule reaches the best possible allocation.

As Holmstrom and Myerson (1983) first pointed out, "the proper object for welfare analysis in an economy with incomplete information is the decision rule, rather than the actual decision

[18] In section 4.3 the robustness of our results when we allow votes to be perfectly divisible is argued.

[19] One form of such conditions may read as follows: conditional on some state of nature, players types are independent and the priors are atomless.

[20] Note that without loss of generality and in order to simplify the notation we have assumed the high issue to take a value equal to one. The analysis is totally analogous to the more general setting where $\theta_n^i \in \left\{\pm\bar{\theta}, \pm\underline{\theta}\right\}, \bar{\theta} > \underline{\theta} > 0$.

or allocation ultimately chosen (...) a decision rule is efficient if and only if no other feasible decision rule can be found that may make some individuals better off without ever making any other individuals worse off." In our setting this means that we do not have to compare the set of final allocations but the set of implementable mappings from preference profiles to allocations (i.e. implementable SCFs). It would be useless to provide a welfare analysis regardless of incentive compatibility because strategic manipulation of privately held information will almost surely lead to a different allocation than the expected one. For instance, in our setting the SCF that maximizes the sum of *ex-ante* utilities (i.e. before agents know their own type) is one that approves any particular issue if the sum of preferences towards it is positive (and dismisses it if the sum is negative). Needless to say, this SCF is undertaken a direct interpersonal comparison of utility thus cannot be truthfully implementable. In other words, it is subject to strategic misrepresentation of the preferences and the final allocation may not be optimal any more.

Henceforth we will adopt the criteria that any optimality analysis will be made out of the set of implementable SCFs. That is, following the result in theorem one, the set of SCFs that induce an *interim* utility HD1 and convex. We denote such set $\mathcal{P}$ (i.e. $\mathcal{P} := \left\{p : \Theta^I \to X : p \text{ is implementable}\right\}$).

We can now define the welfare criteria we are interested in: the set of SCFs that reach a Pareto optimal allocation at the *ex-ante* stage. First, a definition for the ex-ante utility for player i given the SCF p:

Definition 1 $U^i(p) := E_{\theta^i}\left\{E_{\theta^{-i}}\left\{u\left(p\left(\theta^i,\theta^{-i}\right),\theta^i\right) \mid \theta^i\right\}\right\}$

Definition 2 *An* ***ex-ante efficient*** *SCF $p : \Theta^I \to X$ is an implementable SCF such that there is no $\hat{p} : \Theta^I \to X$ such that makes some voters better off without worsening off any other, i.e.*

$$p \textit{ is } \textbf{\textit{ex-ante efficient}} \Leftrightarrow \nexists \hat{p} \in \mathcal{P} : U^i(\hat{p}) \geq U^i(p) \text{ for all } i = 1 \div I$$
$$\text{and } U^i(\hat{p}) > U^i(p) \text{ for some } i \in \{1,...,I\}.$$

It is essential to consider SCFs that are *ex-ante* efficient[21] so that they are stable in the sense

[21]Our concept of *ex-ante* efficiency corresponds to the notion of *ex-ante incentive efficient* in Holmstrom

that players will never want to jointly deviate and chose the Pareto optimal rule. Similarly, this argument holds for the *interim* stage: we want mechanisms to be robust once agents privately know their types. It can be proved that *ex-ante* efficiency implies *interim* efficiency, hence our welfare criteria will also imply the stability of the voting rule at the *interim* stage. The first example (*A night out*) illustrates that MR is in some cases not *interim* efficient. In that example, John and Anna had incentives to cede on their least preferred issue and both go to the Italian restaurant and the comedy film. It follows that MR is not *ex-ante* efficient and that both friends will unanimously agree on resolving their dissenting issues through alternative methods.

We will further restrict the analysis so that we focus on mappings that satisfy some appealing properties. On the one hand we will impose that the SCF should satisfy a *unanimity* property. That is, if all voters want the approval or dismissal of a particular issue, that issue should be approved or dismissed accordingly.[22] On the other hand, we will impose standard conditions on the direct mechanisms:[23] *neutrality* and *anonymity*. The first relates to the fact that the SCF should have no tendency towards the approval or dismissal of the issue; and the second relates to the fact that the SCF should treat all individuals in the same manner (all individuals are in an equal or symmetric position). Formally the previous restrictions are defined as follows:

Definition 3 *A SCF $p : \Theta^I \to X$ satisfies the* ***unanimity property*** *iff*

$$p_n\left(\theta^1, ..., \theta^I\right) = \begin{cases} 1 & \text{if } \theta_n^i > 0, \forall i = 1 \div I \\ 0 & \text{if } \theta_n^i < 0, \forall i = 1 \div I \end{cases}, \forall n = 1 \div N.$$

Definition 4 *A SCF $p : \Theta^I \to X$ satisfies the* ***neutrality property*** *iff*

$$p_n\left(\theta^1, ..., \theta^I\right) = 1 - p_n\left(-\theta^1, ..., -\theta^I\right), \forall n = 1 \div N.$$

and Myerson (1983).

[22]Intuitively it seems as if the *unanimity* condition is superfluous once we analyse the *ex-ante* efficient allocations -unanimity seems necessary for the SCF to be *ex-ante efficient* (no one would be worse off if we impose the unanimous criteria whenever it does not hold). Nonetheless, the previous intuition is misleading because it is not taking into account incentive compatibility. It could be the case that imposing unanimity whenever it is not satisfied fails to induce players to truthfully reveal their types.

[23]As introduced by May (1952).

Definition 5 *A SCF $p : \Theta^I \to X$ satisfies the* ***anonymity property*** *iff*

$$p_n\left(\theta^1, ..., \theta^I\right) = p_n\left(\theta^{\sigma(1)}, ..., \theta^{\sigma(I)}\right), \forall n = 1 \div N, \forall \sigma \in S_I.^{24}$$

We will also impose two other restrictions so that the SCF will behave analogously in every issue and will be neutral across issues in the sense that it will be invariant with respect to the sign of the preferences on the remaining issues.

Definition 6 *A SCF $p : \Theta^I \to X$ satisfies the* ***symmetry across issues property*** *iff*

$$p_n\left(\theta^1, ..., \theta^I\right) = p_{\sigma(n)}\left(\left(\theta^1_{\sigma(1)}, ..., \theta^1_{\sigma(N)}\right), ..., \left(\theta^I_{\sigma(1)}, ..., \theta^I_{\sigma(N)}\right)\right), \forall n = 1 \div N, \forall \sigma \in S_N.$$

Definition 7 *A SCF $p : \Theta^I \to X$ satisfies the* ***neutrality across issues property*** *iff*

$$p_n\left(\theta^1, ..., \theta^I\right) = p_n\left(\left(\pm\theta^1_1, ..., \theta^1_n, ... \pm \theta^1_N\right), ..., \left(\pm\theta^I_1, ..., \theta^I_n, ... \pm \theta^I_N\right)\right), \forall n = 1 \div N.$$

The *symmetry across issues* property implies that the probability of approving an issue is independent of the particular ordering of the remaining issues (we just need to consider the set of permutations that leave n fixed: $\sigma(n) = n$). Additionally, the functional form of winning a particular issue is identical (we just need to consider the transposition (n, m)).
It will be useful to define a SCF as being *reasonable* whenever it satisfies the five properties above. That is,

Definition 8 *A SCF $p : \Theta^I \to X$ is* **reasonable** *if and only if it satisfies the unanimity, neutrality, anonymity, symmetry across issues and neutrality across issues properties.*

Note that the set of *reasonable* SCFs that are implementable is not empty; MR or the Unanimity Rule[25] are particular examples of implementable SCF that satisfy the appealing previous properties.
The *symmetry across issues* property together with the *neutrality* and *anonymity* properties allow us, given the uniform and independent priors, to focus the welfare analysis on max-

[25] Under the Unanimity Rule, an issue is approved (dismissed) if and only if all the players will the approval (dismissal) of that issue. Otherwise nothing is decided and the issue is delayed.

imising the expected payoff of the *interim* utility on one issue which is positively valued. In other words, the *ex-ante* efficient SCF will be a maximizer of the following expression:

$$E_{\theta^i}\left\{E_{\theta^{-i}}\left\{\left(2p_n\left(\left(\theta^i,\theta^{-i}\right)\right)-1\right)\cdot\theta_n^i \mid \theta^i\right\}\right\}=E_{\theta^i}\left\{P_n^i\left(\theta^i\right)\cdot\theta_n^i\right\}.$$

Note that given that all players have the same priors we can drop the i superscript in the *interim* probabilities. Similarly, to easy the notation we will also drop the i superscript from the preference vector and the preference type on issue n (i.e. we will write θ and θ_n instead of θ^i and θ_n^i).

5.1 Two Players

Note that this is a more general case than the pure conflict resolution one because we are not imposing preferences to be opposed between both players. Nonetheless, the main intuition in this case is analogous to the one depicted in example two.

5.1.1 The Indirect Mechanism: Qualitative Voting

Players are endowed with V votes that can be freely distributed between the two issues. We assume that V is even[26] so that they can evenly split the votes between the two issues if necessary. The submitted votes can have a positive or negative value capturing the will of the voter towards the approval or dismissal of the issue.

Given the uniform and independent prior on the opponent's preferences it is dominant to truthfully declare the right sense of the preferences. In case any player loses one of the issues he will for sure win the remaining one. This is because of the binary nature of our setting with only two issues: losing an issue means having opposed preferences to the opponent on that issue and having invested less votes hence, the player at hand, will have more votes on the remaining issue. It can be easily proved that it is optimal to insure the non-loss of the most preferred issue and consequently the optimal strategy followed by a non-indifferent player is investing all his votes on his most preferred issue.

[26]Without entering the question of whether 0 is even, we further assume that $V > 0$.

Instead, an indifferent player will be indifferent between playing any of the strategies. It will be assumed that he will evenly split his votes. This adoption of strategy can be seen as the medium point between the strategies followed by the mixed types and allows to reach the Pareto optimal allocation.

It is relevant to state what happens when the votes invested on one of the issues cancel out, i.e. ties occur. In that case we need to define a tie breaking rule that would preserve unanimity: the unanimous preference will be implemented in case it exists, otherwise the decision on the issue will be delayed until the next meeting (or analogously, given the risk neutrality of our players, a coin will be tossed).[27]

Examples one and two highlighted the fact that QV allows any voter to cede on his least preferred issue and whenever that issue is highly preferred by his opponent, the opponent's will is implemented. It is immediate to realise that such voting system Pareto dominates the allocations achieved by MR. Moreover, when we analyse the direct mechanism we will prove that QV is not only superior to MR but it reaches *the* optimal allocation. The remaining of this section formally proves the equilibria of the indirect mechanism.

The game is defined as follows: there are two players ($i \in \{1, 2\}$); the players privately observe their type θ^i ($\theta^i \in \Theta := \{\pm 1, \pm \theta\}^2$); types' priors are pairwise independent and uniform as noted above; the action space is a *voting profile* v

$$v \in \mathcal{V}, \mathcal{V} := \Big\{(v_1, v_2) \in \big\{-V, ..., -1, 0^-, 0^+, 1, ..., V\big\}^2 : |v_1| + |v_2| = V\Big\}$$

and, finally, a strategy is a mapping from the preference vectors to the *voting profiles.*

The action space is defined so that a positive (negative) vote indicates that the voter wills the approval (dismissal) of the issue. Furthermore, there has been a small abuse of notation and the action space has been defined so that investing zero votes is informative about the sign of the voters' preferences in order to implement unanimous wills (i.e. 0^- and 0^+ have

[27]Note that given the uniform priors there is no ex-ante welfare effect if we were to toss a coin whenever there is an unanimous will. Nevertheless it would not be consistent with the *unanimity* property imposed on the analysis of the direct mechanism.

negative and positive sign, repetitively). The behaviour of QV is captured below:[28]

$$\text{QV:} \begin{cases} v_n^1 + v_n^2 > 0 & \text{The issue is approved} \\ v_n^1 + v_n^2 < 0 & \text{The issue is dismissed} \\ \text{Otherwise,} & \text{the issue is delayed} \end{cases}$$

As noted above, the uniform and independent priors imply that to vote in the right sense is dominant (i.e. $sign\left(v_n^i\right) = sign\left(\theta_n^i\right)$). Without loss of generality, we analyse the optimal strategy of player i whenever he has positive preferences. His payoff is:

$$\left(\frac{1}{2} + \frac{1}{2}\tilde{P}_1\left(v_1^i \mid \theta_1^j < 0\right)\right)\theta_1^i + \left(\frac{1}{2} + \frac{1}{2}\tilde{P}_2\left(v_2^i \mid \theta_2^j < 0\right)\right)\theta_2^i$$

where $v_2^i = V - v_1^i$.

The previous expression captures that unanimous wills are implemented and $\tilde{P}_{\cdot}\left(\cdot\right)$ are the equivalent to the *interim* probabilities in the direct mechanism. They are defined as follows (the conditional probabilities are omitted for simplicity):

$$\tilde{P}_1\left(v_1^i\right) := 2\left\{\Pr\left(v_1^i - \left|v_1^j\right| > 0\right) + \tfrac{1}{2}\Pr\left(v_1^i - \left|v_1^j\right| = 0\right)\right\} - 1$$
$$\tilde{P}_2\left(v_2^i\right) := 2\left\{\Pr\left(-\left(V - \left|v_1^j\right|\right) + v_2^i > 0\right) + \tfrac{1}{2}\Pr\left(-\left(V - \left|v_1^j\right|\right) + v_2^i = 0\right)\right\} - 1.$$

Simple calculations allow us to rewrite the payoff of player i as

$$\frac{1}{2}\theta_2^i + \left\{\Pr\left(v_1^j + v_1^i > 0\right) + \frac{1}{2}\Pr\left(v_1^j + v_1^i = 0\right)\right\}\left(\theta_1^i - \theta_2^i\right).$$

It is immediate to conclude from the previous expression that player i will want to maximise the expression inside the curly brackets if $\theta_1^i > \theta_2^i$ (i.e. $v_1^i = V$)[29] and minimise it in case

[28] Note how MR would be defined in this context:

$$\text{MR:} \begin{cases} v_n^1, v_n^2 > 0 & \text{The issue is approved} \\ v_n^1, v_n^2 < 0 & \text{The issue is dismissed} \\ \text{Otherwise,} & \text{the issue is delayed.} \end{cases}$$

[29] This is because player i wants to set v_1^i strictly higher (if possible) than the absolute value of his opponent's invested votes on the first issue. Taking into account that player j will also play accordingly, we have that the only equilibria has non-indifferent players investing all their voting power on their preferred issue.

that $\theta_1^i < \theta_2^i$ (i.e. $v_1^i = 0^+$). Finally, he is indifferent to which strategy to play whenever he is indifferent (i.e. $\theta_1^i = \theta_2^i$).

5.1.2 The Direct Mechanism

Any direct mechanism is defined by 512 parameters. That is, all possible combinations of both players' types times the number of issues we are considering. The five restrictions imposed before simplify the analysis into six parameters because, as stated before, we just need to define the SCF on a particular issue when both players' preferences on that issue clash and regardless of the sign of the remaining issue.

More precisely, the *neutrality* property defines the value of the SCF whenever players have analogous preferences (i.e. whenever both players coincide on how strongly they prefer each issue[30]) and allows us to focus on positively valued issues (the agent we analyse wants the approval of the issue); the *symmetry across issues* allows us to focus on a particular issue (say, issue one) and the *neutrality across issues* property reduces the possible types we have to analyse into four because the SCF is invariant with respect to the sign of the remaining issue. Finally, *unanimity* implies that we just have to consider the cases when the opponent wants the dismissal of issue one. The next table depicts the six parameters that uniquely define any SCF given the properties above:

$(1,\theta)$	$\frac{1}{2}$	A	B	C
$(\theta,1)$		$\frac{1}{2}$	D	E
$(1,1)$			$\frac{1}{2}$	F
(θ,θ)				$\frac{1}{2}$
	$(-1,\theta)$	$(-\theta,1)$	$(-1,1)$	$(-\theta,\theta)$

Note that these parameters are probabilities of approving an issue hence they lie in the interval $[0,1]$.

We define the *interim* probabilities given the four possible declarations as $P(1,\theta), P(\theta,1),$

[30]Note that when both players rank equally both issues and just differ on desiring the approval or dismissal of it, the SCF should be equal to $\frac{1}{2}$. That is, it should play a fair gamble. As stated before, given the risk neutrality of our voters, playing a fair gamble is analogous to delaying the decision until the next meeting -providing them with zero utility.

$P(1,1)$ and $P(\theta,\theta)$. It follows that the utilities of each of the three types of voter given truthful revelation are:

- A non-indifferent type: $P(1,\theta)\cdot 1 + P(\theta,1)\cdot\theta$
- A high type: $P(1,1)\cdot 1 + P(1,1)\cdot 1$
- A low type: $P(\theta,\theta)\cdot\theta + P(\theta,\theta)\cdot\theta$

The next proposition tell us which are the conditions that any *reasonable* SCF should satisfy in order to be implementable (i.e. induce truthful revelation of the players preferences).

Proposition 2 *A reasonable SCF $p:\Theta^I\to X$ is implementable if and only if the next four conditions are satisfied*

$$\begin{array}{llll} \textit{1.} & P(1,1)=P(\theta,\theta) & \textit{3.} & P(1,1)\geq \dfrac{P(\theta,1)+P(1,\theta)}{2} \\ \textit{2.} & P(1,\theta)\geq P(\theta,1) & \textit{4.} & P(1,1)\leq \dfrac{P(\theta,1)\,\theta+P(1,\theta)}{1+\theta}. \end{array}$$

We can relate these four conditions to the general conditions on implementability. The first one is the equivalent to the HD1 condition with continuous types. The remaining three correspond to the convexity condition. Observe that the increasing property we observed before ($\dfrac{\partial P_n(\theta)}{\partial\theta_n}\geq 0,\forall n$) can also be derived out of the four conditions and will now appear as: $P(1,1)>P(\theta,1)$ and $P(1,\theta)>P(\theta,\theta)$.

The proof of the proposition is an immediate consequence of imposing the conditions for truthtelling. For instance, the first condition is a consequence of considering a high type not having incentives to deviate declaring he is a low type together with a low type not deviating declaring he is a high type. The rest of the conditions follow from considering the remaining deviations.

One can easily define the *interim* probabilities given the four possible declarations in terms of the six parameters.[31] From these expressions we can easily transform the four conditions

[31] For instance, $P(1,\theta)=2\left\{E_{\tilde\theta}\left(p\left((1,\theta),\left(\tilde\theta\right)\right)\right)\right\}-1=2\cdot\frac{1}{8}\left\{\frac{1}{2}+A+B+C+4\right\}-1.$

in the proposition in terms of the six parameters:

1. $-B + C - D + E + 2F - 1 = 0$

2. $2A + B + C - D - E - 1 \geq 0$

3. $-6B - 2C - 6D - 2E + 4F + 6 \geq 0$

4. $-A - 2B - C - D + F + 2 + (A - B - 2D - E + F + 1)\,\theta \leq 0$

We are now in position to characterise the *reasonable* and *ex-ante efficient* SCF. To do so we just need to compute the ex-ante expected payoff in terms of the six parameters

$$\begin{aligned} E_{(\theta_1,\theta_2)}\left\{U\left(\theta_1,\theta_2\right)\right\} &= 4\left[\left(P\left(1,\theta\right)+P\left(1,1\right)\right)+\left(P\left(\theta,1\right)+P\left(1,1\right)\right)\theta\right] \\ &= 8\left[3+A+C-D+F+\left(4-A-B+E+F\right)\theta\right] \end{aligned}$$

and maximise it subject to the previous four constraints, i.e.

$$\begin{gathered} \max_{A,B,C,D,E,F\in[0,1]} E_{\left(\theta^1,\theta^2\right)}\left\{U\left(\theta^1,\theta^2\right)\right\} \\ \text{subject to conditions (1), (2), (3) and (4).} \end{gathered}$$

Solving the linear program we get that $A = C = B = 1, D = E = 0$ and $F = 1/2$. A proper analysis of these values lead us to one of the main results in this paper:

Theorem 2 *In a setting with two issues and two players, QV is ex-ante optimal.*

In order to prove the theorem we just need to show that QV replicates the same allocation as the one achieved by the only optimal SCF. In order to interpret the *ex-ante* optimal SCF, let's first plug the parameter values into the table below:

$(1,\theta)$	$\frac{1}{2}$	1	1	1
$(\theta,1)$		$\frac{1}{2}$	0	0
$(1,1)$			$\frac{1}{2}$	$\frac{1}{2}$
(θ,θ)				$\frac{1}{2}$
	$(-1,\theta)$	$(-\theta,1)$	$(-1,1)$	$(-\theta,\theta)$

What is this SCF doing? In the first place, whenever players equally rank both issues or whenever both are indifferent, ties occur. Instead, if players rank issues differently, the individual that is not indifferent between the issues wins its preferred issue and loses the remaining one. Note that this is exactly the functioning of QV.

It is worth pointing out that QV is replicating the only ex-ante optimal SCF but it is not the only indirect mechanism that can do so; QV is just one possible alternative but no other mechanism could do better.

Recall that incentive compatibility implied that indifferent players should be treated analogously at the *interim* stage no matter they held strong or weak preferences. We have now proved that this is not only the case at the *interim* stage but also at the *ex-post* stage. The fact that the last two columns in the table above are identical shows that agents should always be treated in terms of ranking issues differently or in terms of being indifferent. The absolute value does not matter, or following the argument in section four, the optimal implementable SCF does not undertake *ex-post* interpersonal comparisons of utility.

Finally it is clear from the above analysis (and the first two examples) that Majority Rule is not *ex-ante efficient* given that it is not *interim* efficient. Hence, QV *ex-ante* and *interim* Pareto dominates MR. That is, in some situations agents would unanimously like to deviate and decide their discerning issues through QV. Moreover, it can be proved that the allocations achieved by QV are also *ex-post* efficient.

5.2 Three Players

We depart now from the conflict resolution situation and consider a setting with three players. At this point it will not only be relevant to counteract the votes invested by the opponent(s) but the pivotability of one's votes will also play a very relevant role. One could be tempted to say that this pivotability could also be interpreted in terms of a *free-riding* problem but such aspect is out of the scope of the analysis in this section given the independent and uniform prior that we have assumed.

5.2.1 The Tie Breaking Rule

In the case with only two voters we already introduced the relevance of the tie breaking rule to insure that unanimous wills are implemented but we noted that it had no welfare effects. Instead, with three players the tie breaking rule plays a crucial role.

Consider the pivotability of a voter under MR and the uniform priors assumed throughout: a player wanting the approval of an issue will be *very* pivotal given that the issue could only be dismissed if the remaining two voters oppose to it -that event has a probability $\frac{1}{4}$ of happening. Imagine now, that the tie breaking rule under QV is the toss of a coin, i.e. the issue is approved with probability $\frac{1}{2}$. This implies that any player becomes much less pivotal ($\frac{1}{2} < \frac{3}{4}$) than he was under MR and it can be showed that QV is no longer optimal and MR does better.

The optimal tie breaking rule is fairly intuitive: in case of ties issues should be decided through the usual MR. QV becomes then a voting rule that allows issues to be decided on the grounds of overall intensity and, in case this is not decisive, the issue is approved depending on overall support (MR). QV happens to be a natural extension[32] of the usual voting rule where voters declare their position with respect to the approval or dismissal of an issue and then invest extra votes to reflect their *willingness to influence*.

5.2.2 The Indirect Mechanism: Qualitative Voting

Players are endowed again with an even number of votes V. Provided the uniform and independent priors it will still be dominant to declare the right sense on their casted votes. We will now focus on *symmetric* pure strategy equilibria. The *symmetry* should be interpreted as usual in voting theory: the three voters play the same strategy.

We will further assume that a non-indifferent player votes analogously regardless of the labelling of the issue or the sign of it. That is, his strategy is summarised by a parameter $v \in \{0, 1, ..., V\}$ which should be interpreted (together with the corresponding positive or negative sign) as the votes invested in his most preferred issue. The votes invested in his least preferred issue are $(V - v)$ or $(v - V)$ depending on whether he desires the approval or dismissal of it.

Indifferent players are also assumed to play a strategy which is alike for a positive or negative valuation of any of the issues. Hence, their strategy is summarised by a parameter $v_i \in \{0, 1, ..., V\}$ which denotes the absolute value of the votes invested on the first issue . The next lemma tell us that in a symmetric equilibria indifferent voters divide equally their voting

[32]Given this tie breaking rule, MR is just a particular case of QV with $V = 0$.

power.

Lemma 1 *In a setting with two issues and three players, any symmetric pure strategy equilibria has indifferent voters spliting evenly their voting power.*

The proof (which is provided in the appendix) mostly relies on showing that an equilibrium with an *unbalanced* behaviour by indifferent voters cannot be sustained. Imagine, for instance, that there exists an equilibria with $v_i^* > V/2$. Then any player will be better off by deviating and playing, for instance, the complementary strategy: $\tilde{v}_i = V - v_i$. In this way, a player will swift some votes from the first issue to the second and increase in this way their pivotability. Given the defined setting, an equilibrium to our game is uniquely defined by a pair $(v^*, v_i^*) \in \{0, ..., V\}^2$. The previous lemma tell us that, in fact, we are just left with determining v^*. Extrapolating the increasing property we derived from the implementability conditions (i.e. the *interim* probability of any direct mechanism to approve an issue is weakly increasing in the valuation of any individual towards that issue) we have something that looks fairly intuitive: $v^* \geq v_i^*$. That is, the number of votes invested on a high valued issue should be at least as big as the number of votes invested on that issue whenever the voter is indifferent. The next proposition tell us which are *essentially* the three equilibria that one can find.

Proposition 3 *In a setting with two issues and three players, there are essentially three symmetric pure strategy equilibria. These are:*

$$(v^*, v_i^*) = \begin{cases} \left(V, \frac{V}{2}\right) & \textit{-all voting power into preferred issue.} \\ \left(\frac{V}{2}, \frac{V}{2}\right) & \textit{-equivalent to MR} \\ \textit{for } \theta = \frac{1}{2}, \left(\frac{3}{4}V, \frac{V}{2}\right) \end{cases}$$

The first equilibria is essentially the equilibria we observed in the two players case where non-indifferent voters invest all their voting power in their preferred issue so that strong minorities impose their will over weak majorities. The second equilibria replicates the MR allocation. For further reference both will be called Equilibria QV (EqQV) and Equilibria MR (EqMR), respectively.

Finally, the third equilibria is not much relevant given that it only holds for a particular value of theta.[33] This equilibrium can be seen as a mid point between the other two where a weak member of a majority needs just an indifferent voter to overcome a strong minority (instead of a voter with strong positions as would be the case in the EqQV). The non-divisibility of the votes together with the evenness of V may imply that this equilibria (and only this one) may not exist.

The proof of the proposition is quite tedious and is left to the appendix. Its difficulties rely on the *essentiality* aspect of it. By this we mean that there are multiple equilibria but all of them replicate only three sets of allocations. This is because we can device loads of possible combinations of votes where no individual will be better off by deviating but where the last votes will not be pivotal and hence can be placed in any of the issues. Their non-pivotability will imply no changes in the final allocation.

Note that we have been a bit misleading in the intuition behind the proof of lemma 1. The *essentiality* argument also holds there and what is needed is that voters split their *effective* power evenly (i.e. they could put slightly more votes in one issue or another but these latter votes are not pivotal in the sense that if (v^*, v_i^*) is an equilibria, then $\left(v^*, \frac{V}{2}\right)$ is also an equilibria and, more importantly, it replicates the same allocation).

Two relevant issues are left to consideration. On the one hand, the fact that the proposition holds for any number of votes indicates that it may also hold whenever we consider votes to be perfectly divisible.[34] On the other hand, it is worth pointing out that the proposition shows that QV has multiple equilibria and one of them replicates the outcome reached by MR. Henceforth we will focus our attention on the first equilibria. It does not seem worth proposing a slightly more complicated voting system than the traditional MR if it just replicates the same allocation and introduces no gains.[35]

[33]This equilibria disappears whenever we consider the continuous valuation of the issues (see section 5.3). There are two reasons for this to be the case: (1) the relative intensity for which it holds will have a measure zero in the continuous case (given uniform preferences) and (2) the strategy followed by indifferent players is crucial for this equilibria to hold and these voters have also a zero measure in the continuous case.

[34]It is shown in section 5.3 that in the case of a continuous valuation of the issues and perfectly divisible votes, only the EqQV and EqMR can be sustained.

[35]The multiplicity of equilibria when analysing different mechanisms is usually eluded selecting the best equilibria in each possible situation. Note that this approach would benefit our analysis because MR would never be able to do better than QV given that the latter also contemplates the allocation reached by the former. Therefore, focussing on the first equilibria makes our optimality analysis more dificult.

5.2.3 The Direct Mechanism

Any direct mechanism is now defined by 8192 parameters. It is worth pointing that fact to realise how the complexity exponentially escalates with the degree of the problem. Nevertheless, restricting the analysis to *reasonable* SCFs rends the problem tractable and simplifies the analysis into 40 parameters belonging to the interval $[0, 1]$.

Equivalently to the case with only two players, we define the *interim* probabilities given the four possible declarations as $P(1, \theta), P(\theta, 1)$, $P(1, 1)$ and $P(\theta, \theta)$. The utilities of each of the three types of voter given truthful revelation will be identical and proposition 2 will still hold. As a matter of fact, that result holds for any number of players as long as we only consider two issues at hand. The only difference relies on the form of the *interim* utilities.

In the appendix the parameters are defined and the program that an *ex-ante* efficient SCF has to satisfy is stated in terms of these parameters. The following theorem captures the main result.

Theorem 3 *In a setting with two issues and three players and whenever issues are valued differently enough, QV is ex-ante optimal. Moreover, in that case MR is not optimal.*

What do we mean by "*issues are valued differently enough*"? It is proved in the appendix that for $\theta \in \left(0, \frac{1}{3}\right)$ the equilibria we have previously called EqQV is the only optimal allocation. For $\theta \in \left(\frac{1}{2}, 1\right)$ the EqMR is the optimal one. Recall what the EqQV means: the will of a strong minority is implemented as long as there is not a majority that strongly opposes to it. It is sensible for this equilibria to be ex-ante optimal whenever issues are valued sufficiently differently -agents will like to commit to such a rule before knowing their preferences because their possibly strong views will not be silenced by possibly indifferent majorities. Furthermore, MR is not optimal in this case because it fails to satisfy that intuition.

In the interval $\theta \in \left(\frac{1}{3}, \frac{1}{2}\right)$ the third kind of equilibria is the optimal one. That equilibria is a mid point between the other two where the will of a strong minority is only heard if its two opponents feel stronger about the remaining issue. Whenever one the opponents is indifferent between the two, the minority cannot impose its will anymore. In the indirect mechanism that equilibria was only sustained by $\theta = 1/2$. For smaller values of theta, any non-indifferent player had a profitable deviation investing all his votes in the most preferred issue thus increasing the probability of winning his preferred issue. Still this equilibria is sustainable within the direct mechanism analysis given that we can prevent such deviation happening.

Summarising, for θ small enough QV is optimal. Instead, for theta big enough MR is the one that is optimal. In between, there is an interval where firstly QV does better than MR, later MR does better than QV and, overall, both fail to be optimal. It is worth saying that, once again, indifferent players are treated equally *ex-post*.

The core of this section's result rests on the parameter theta: the optimality of QV relies now on a particular range of this parameter as opposed to what was happening in the case with only two voters. As a matter of fact, the main argument for proposing an alternative voting rule was allowing voters to express their *willingness to influence* so that, whenever possible, the final allocation could take into account that aspect. Implicitly we were assuming that gains could only be possible as long as voters differed on which issue was the most relevant. In that case, QV allows voters to achieve the best possible allocation.

The theorem naturally brings up a concern on the selection of the agenda. Namely, how issues are to be bundled together so that one or another voting system should be employed and how this process could be manipulated to alter the final voting outcome. Section seven addresses such matter.

5.3 Robustness Checks

It could be argued that our results rely on the binary nature of the preferences. This is not the case.

Whenever we consider preferences to belong to the interval $[-1, 1]$ and pairwise independent and uniform prior, players will still follow the described strategy: they will invest all of their votes on their most preferred issue.[36] Regarding the direct mechanism it can be proved that QV is not ex-ante optimal if we consider all possible SCFs. For instance in the case with two players and two issues it can be proved that it is not optimal to allow ties to happen whenever voters have opposed preferences but equally rank the importance of the issues: preserving IC

[36]Note that some of the analysis is now simpler because there are no more indifferent players between the issues (more precisely, they have a zero measure).

one can take into account when a voter cares more about an issue than his opponent. From an *ex-ante* perspective this will of course lead to a welfare improvement.

Nevertheless, if we restrict our analysis to *deterministic* SCF (in the sense that issues can only be approved, dismissed or delayed) it can be numerically proved that QV is optimal. Limiting the analysis to *deterministic* SCFs is relevant in our setting given that any *voting game* yields a *deterministic* outcome.

It could also be argued that the equilibria in the voting game is driven by the non-divisibility of points. As was suggested by proposition three, this is not true. In the appendix it is proved that the strategy of investing all of the points on the most preferred issue is robust when we allow for perfectly divisible votes. Besides, together with the equilibria that replicates the MR outcome, this is the only possible equilibria.

The reason for not having players diversifying their vote share according to their intensity is that for players that value both issues *very* differently it will always be optimal to invest all of their voting power on their most preferred issue. This happens for a positive measure of players. Hence whenever a different player thinks of investing a small portion of his votes on the least preferred issue in order to marginally break a possible tie and win that issue, he is incurring a non marginal cost given that he will lose against a non-negligible measure of players that may invest all of their voting power on their most preferred issue. The likelihood of the loss and the one of the gain are of the same order but the gain occurs in the least preferred issue and the loss in the most preferred one. It follows that such deviation cannot be optimal. There is a contagious behaviour by the more *radical* players that explains the non diversification of voting power by the non-indifferent voters.

The former intuition is compatible with the existence of the MR equilibria. We said that it is optimal for a *radical* voter to invest all of his votes on his most preferred issue. More specifically, if his opponents are spliting evenly their voting power, he is indifferent between any of the strategies because he is only to be pivotal whenever votes add up to zero. Given the tie breaking rule, in this case he is equally pivotal with any number of votes. Particularly, it is optimal for him to evenly split his votes.[37]

[37] The proof of the equilibria when continuous preferences and divisible votes with two and three players can be found in the appendix. The case of non-divisible votes and continuous preferences is just a particular

6 An Argument in Favour of Majority Rule

All optimality issues above have rested on the assumption of having a uniform and pairwise independent prior. It seems natural to relax such assumptions and check whether the main optimality results are affected by such change. As was firstly pointed out in the examples, one can see that a more precise knowledge of the opponents preferences may lead to the non-existence of equilibria in the game induced by QV. This can be easily seen if we consider the *Colonel Blotto* game with commonly known preferences and more than two issues. In that setting it is almost sure that no equilibria in pure strategies will exist, the intuition works as follows: for the voting profiles to be an equilibrium no player should invest a single vote in an issue he is going to lose; consequently, a single vote should be sufficient to win any issue; overcoming the single vote invested by an opponent will happen almost surely.

Hence, relaxing the priors may lead to some critical problems in the applicability of QV. The next theorem captures the fact that the optimal, *reasonable* and implementable SCFs robust to any prior is MR. In fact, we have that the set of *reasonable* SCF that are robust to any possible prior distribution on the voters' preferences is very reduced and just standard voting rules like unanimity or MR and Unanimity Rule satisfy the set of conditions. Out of them, MR does the best.

In a setting with two issues and two players, QV is ex-ante optimal.

Theorem 4 *In a setting with two issues and two or three voters, the only ex-ante optimal and reasonable SCF that is implementable regardless of the specification of the setting (priors and valuation of the low valued issue) coincides with the allocation achieved by MR.*

The proof of this result is provided in the appendix. It consists on writing the ICs in terms of the non-uniform priors and check that the only implementable SCF that is robust to any change on the parameters of the model needs not to depend on the intensity of the preferences. It follows that MR is the optimal one (note that in the two players case MR and Unanimity Rule are analogous voting rules).

Briefly, we have seen that the more biassed the prior may be the more strategic voters will become. Consequently, it will be more difficult to achieve a truthful revelation of preferences

case.

and such interaction may outweigh the welfare gains we expect from a voting rule that internalises the voters' *willingness to influence*. In addition, the strategic use of private information leads to the non-existence of a general voting rule that can take such information into account.

6.1 Relation to *Logrolling*

The previous setting is related to *logrolling* in the sense that logrolling occurs in a situation where a certain knowledge of the opponent's preferences exists but there still is scope for understatement of one's preferences and, of course, violation of the agreement once it is made. The latter can be easily overcome through some kind of reputation argument but the former generates major difficulties in modelling theoretically such phenomenon.[38]

Following the analysis above, let's assume we have a setting with three voters, two issues and binary preferences. When is there going to be scope for *vote trading*? This will happen whenever there is a weak majority that supports each of the issues and different voters oppose strongly such outcomes. Example three (*Committee Meeting*) described precisely that situation. Recall which were the preferences in that case:

	Human cloning	**Use of stem cells**
F1	*agree*	*agree*
F2	*strongly disagree*	*agree*
F3	*agree*	*strongly disagree*

Though we may assume there is not a perfect knowledge of the opponent's preferences, something should be known given that voters will barter their votes and this will be directed to whichever faction has complementary views on both issues -in our example, faction two and three are the ones that will try to agree on a different outcome than the one resulting from MR. Given that monetary transfers are banned, there will not be a misstatement of preferences because nothing could be gained out of it. A different matter would be if we were

[38]The lack of a satisfactory theoretical treatment of logrolling supports such assertion. The most relevant work is by Wilson (1969) where votes are assumed to be perfectly divisible and tradeable goods and agents interact in an exchaneg economy framework.

to consider a situation with more than two issues. In that case, a player could claim that in order to vote against his own position on a particular issue the colluding faction should give up more than one issue.

Let's analyse the equilibria of the *Committee Meeting* example under MR and allowing for *logrolling* (i.e. coalitions of two players that jointly agree on the votes towards the issues). Departing from the MR rule outcome (bots issues are approved), faction two and three will collude and vote against their own position on the issue they regard as being less important changing the final allocation into the dismissal of both issues. But this cannot be an equilibrium because faction one will now have an incentive to convince any of the remaining factions (say faction two) to vote against the first issue and in favour of the second one. Both agents will be better off. Finally, faction three will in turn have an incentive to deviate jointly with faction one and replicate the initial MR outcome -approve both issues.

Actually the described situation corresponds to a very well known paradox in voting theory: the *Condorcet Paradox*. This happens whenever the aggregation of the players preferences through MR leads to non-transitive social preferences and pairwise comparison results in a cycle.

In the light of the theorem four, it can be shown that the only equilibria of QV in that setting implements the same allocation as MR.

It can be argued that there is not much scope in addressing the *logrolling* issue on the setting studied in this paper because of the non-existence of equilibria. However, this example is intended to highlight the main difficulties regarding *logrolling* and point out which modelling characteristic may be needed in a future research of such real phenomenon: not all possible coalitions should be allowed. That fact is totally supported by empirical evidence; we can observe that there is a natural tendency of some particular groups to form coalitions and there is a factual impossibility for some groups to collude (for instance one may not expect a unionist and a republican party to form a coalition in a Northern Ireland government).

Lastly, we could say that QV is to *logrolling* what monetary economies are to barter. It eases the ways through which agents can express their *willingness to influence* given that it does not require a double coincidence of wants. Furthermore, it seems reasonable to expect that this increased freedom in the available strategies should prevent agents trading their votes because under QV a vote in an issue can never be seen as something useless since it can be unilaterally moved to a more relevant one.

7 The Selection of the Agenda

In 1989 the OECD started the negotiations on criminalising bribery at an international level. Divergent political and legal situations in the country members led to a resolution not earlier than December 1997. As noted in Metcalfe (2000), there were ten main actors in the negotiations and the setting of the agenda monopolised the negotiations: it wasn't until early 1997 that an agenda with seven issues was defined (differentiating the criminalisation to foreign officials or parliamentarians; the nature of the sanctions; the supervisory measures; etc.).[39] During the negotiations on the final agenda, the United States forced the introduction of an issue that became *divisive*, the criminalisation of *bribery to political parties and party officials*. Metcalfe (2000) emphasizes on the perversive effect that the introduction of a *divisive* issue may have in a negotiation: it creates a conflict between two factions that strongly disagree on the outcome of such issue and prevents any agreement being reach on the remaining ones.

It is fair to say that the scope of this last example is much broader than what we have analysed in this paper though it highlights most of the problems that arise in a negotiation. First and most important, is the selection of the agenda. The introduction of a new bill can drastically change the action taken by a particular individual and so is the case with QV. How, who and when should the issues be selected? There is a clear incentive to manipulate the agenda and to bundle issues that will benefit some particular groups. Nevertheless, the literature lacks tractable models of agenda formation given the somehow dubious knowledge of the opponent's preferences that is needed to correctly manipulate it. In a different setting Dutta *et al.* (2003) define and prove the existence of an *equilibrium for agenda formation* when one alternative has to be selected out of many (the agenda determines the pairwise

[39]In our model each issue could only be approved or dismissed. Instead, in the *Convention Combating Bribery of Foreign Public Officials in International Business Transactions* there where up to four resolution degrees.

comparisons that have to be undertaken to successively delete one alternative).[40]

In our case, we need to rely on those cases in which the agenda is exogenous (e.g. the goods to be split in a divorce settlement) and also in those situations in which after some *magical* negotiations an agenda agreed by all voters is reached.

Finally, the example above also highlights how the agenda can affect the Pareto improvement achieved by QV when compared to MR. Imagine that in the agenda there is an issue which is much more valued by any of the players than any remaining one -a *divisive* issue. This fact will lead players to focalise on that issue and QV will just replicate the allocation achieved by MR.

8 Conclusion

We have proposed an alternative to the usual voting rule which is simple and allows voters to express their *willingness to influence*. A mechanism which seems the most natural extension to MR and that is proved to be not only superior to MR but also a mechanism that achieves the best possible allocation and induces truthful revelation of the voters' preferences in some general settings.

The main findings of this article can be summarised through its four theorems: (1) whenever a public decision has to be made and no transfers are allowed, only the agents' relative intensities between the issues can be considered; (2) QV unlocks conflict resolution situations allowing each of the opponents to trade off their voting power between the various divergent issues; (3) in a situation with more than two voters, QV allows *very* enthusiastic minorities to decide on those issues majorities are mostly indifferent about; and (4) the allocation reached by MR is the optimal one whenever we permit for any possible information structure on the voters' preferences. Thus, MR cannot be questioned in general grounds but just in some specific situations.

The driving force on our results and our main contribution to the existing literature relies on forbidding any kind of transfers between voters. This has been built on *equality* arguments so that no endowments effects can ever play a role in *voting games*. Furthermore, we

[40] Other literature in agenda formation can be found in Dutta *et al.* (2003) related literature.

have extended such concept when analysing the direct mechanisms imposing the *anonymity* property: any aggregating device should not benefit any particular individual. Finally, departing from these axiomatic properties we derived an equilibrium result (theorem one) that extends the *equality* argument in the sense that no direct interpersonal comparison of utility can be undertaken; it is not solely because a voter values more strongly one issue (or because he shouts louder) that he should be given more voting power on it, i.e. the *preference endowments* shouldn't play a role either.

Precisely, the *equality* argument in the three forms expressed above is crucial to insure the stability of any aggregating mechanism as it is stated in the following quote by professor Lionel Robins:

> "I do not believe, and I never have believed, that in fact men are necessarily equal or should always be judged as such. But I do believe that, in most cases, political calculations which do not treat them *as if* they were equal are morally revolting."

Given how the complexity of the problem escalates when we consider more general settings, the next step in our research agenda consists on experimenting QV in a more complex setting with diverse issues and players to realise how do people may react to different information structures. It is premature to say but it seems sensible to expect that the more issues or players the more disperse the information about the opponents' preferences will be. Consequently, similar results to the ones stated in this paper should follow. This is congruent with the notion that voters may not be able to react rationally to some complex situations given their lack of time, knowledge or aptitude to do so. Hence we may observe a less strategic misrepresentation of preferences and voters may use their private information almost truthfully. Be that as it may, this and further considerations are left for further analysis.

Wrapping up, when could this system be proposed as an optimal alternative with respect to MR? Two situations can be clearly differentiated. First, we have seen that, whenever two players disagree on two main issues, QV achieves the only *ex-ante*, *interim* and *ex-post* optimal allocation. Second, when we consider a setting with three players we need non-indifferent players to value one issue in a much stronger way than the remaining one (at least three times more). In that setting it is relevant to realise the importance of the information structure. The more it is commonly known about the voters preferences the more strategic they will react and this may outweigh the gains we anticipate from the use of the agents' *willingness to influence*. Even so, motivations aligned with reputational arguments -or honest bevaviour- may be observed whenever agents interact frequently and more information about their tastes is known. Once again, an experimental analysis is fundamental to support such assertions.

As a final point we would like to bring to mind the article "Political Money" by James Coleman (1970). There, it is argued how political institutions have evolved so little as compared to economic ones in the last centuries. In order to introduce some debate around it he proposed new political forms and among them some that changed "the income of power to representatives" with a "fixed set of decisions" and "fungible votes". In other words, he was proposing what we have now called QV and he was pointing out how "the operating characteristics of such an arrangement are difficult to foresee fully". Hopefully we have built the first step towards its theoretical analysis and its justification as a possible alternative to the traditional MR.

9 Appendix

Proof of Proposition 1.

Sufficiency. The envelope theorem implies that $\left.\frac{\partial U^i(\hat{\theta},\theta)}{\partial \theta}\right|_{\hat{\theta}=\theta} = \nabla U^i(\theta) = P^i(\theta)$ which directly gives the first condition. On the other hand, given that the FOC is satisfied for all $\theta \in \Theta$ it can be differentiated wrt to θ yielding: $\frac{\partial^2 U^i(\theta,\theta)}{\partial \hat{\theta}^2} + \frac{\partial^2 U^i(\theta,\theta)}{\partial \hat{\theta}\partial \theta} = 0$. Note that the SOC tell us that the first matrix is negative semidefinite, hence the second one should be positive semidefinite.

Necessity. One can easily reverse the previous reasoning to get the local conditions. We just need to prove that the conditions are global. A continuously differentiable function $U^i : \Theta \mapsto \mathsf{R}$ is convex iff $U^i(\theta) \geq U^i\left(\hat{\theta}\right) + \nabla U^i(\hat{\theta})\left(\theta - \hat{\theta}\right), \forall \theta, \hat{\theta} \in \Theta$. Using the first condition and the definition of $U^i(\theta)$ we can get the global condition:

$$U^i(\theta) \geq U^i\left(\hat{\theta}\right) + P^i(\hat{\theta})\left(\theta - \hat{\theta}\right), \forall \theta, \hat{\theta} \in \Theta$$

$$P^i(\theta) \cdot \theta \geq P^i\left(\hat{\theta}\right) \cdot \hat{\theta} + P^i(\hat{\theta}) \cdot \left(\theta - \hat{\theta}\right), \forall \theta, \hat{\theta} \in \Theta$$

$$P^i(\theta) \cdot \theta \geq P^i(\hat{\theta}) \cdot \theta, \forall \theta, \hat{\theta} \in \Theta$$

$$U^i(\theta, \theta) \geq U^i\left(\hat{\theta}, \theta\right), \forall \hat{\theta}, \theta \in \Theta$$

■

Proof of Lemma 1.

Assume that there is an equilibrium (v, v_i) such that indifferent players do not evenly split their voting power. That is, such that it reaches a different allocation than $\left(v, \frac{V}{2}\right)$.[41] Without loss of generality we assume that $v_i > \frac{V}{2}$. Given that the only equilibria with $v = v_i$ needs to be $\left(\frac{V}{2}, \frac{V}{2}\right)$ we have that $v > v_i > \frac{V}{2}$. Furthermore, we analyse the candidate to equilibria from the perspective of player one and we assume that he has positive preferences (i.e. he desires the approval of both issues)and is indifferent between the issues.

[41]Note that we are using the *essentiality* argument that we introduce in proposition three. That is, we only need to consider the equilibria that are *essentially* different from $\left(v, \frac{V}{2}\right)$ in the sense that they reach a different final allocation -i.e we can neglect those equilibria where non-pivotal votes are placed in any of the issues.

Player one can face thirty six possible situations depending on the strategy played by both his opponents. In some situations the votes casted by his opponents are higher or equal than zero in which case, regardless of his strategy, the issue is approved. Similarly, if the invested votes are smaller or equal than $-V$ the issue is dismissed. The table below depicts such situations with a positive and negative sign, respectively. The remaining cells capture the total number of votes casted by players two and three:

ISSUE 1

v	$+$	$+$	$+$	$+$	$+$	$+$
v_i	$v_i - v$	$+$	$+$	$+$	$+$	$+$
$(V - v)$	$V - 2v$	$V - v - v_i$	$+$	$+$	$+$	$+$
$-(V - v)$	$-$	$-V + v - v_i$	$-2(V - v)$	$+$	$+$	$+$
$-v_i$	$-$	$-$	$-V + v - v_i$	$V - v - v_i$	$+$	$+$
$-v$	$-$	$-$	$-$	$V - 2v$	$v_i - v$	$+$
	$-v$	$-v_i$	$-(V - v)$	$(V - v)$	v_i	v

ISSUE 2

v	$+$	$+$	$+$	$+$	$+$	$+$
$(V - v_i)$	$V - v - v_i$	$+$	$+$	$+$	$+$	$+$
$(V - v)$	$V - 2v$	$v_i - v$	$+$	$+$	$+$	$+$
$-(V - v)$	$-$	$-2V + v + v_i$	$-2(V - v)$	$+$	$+$	$+$
$-(V - v_i)$	$-$	$-2(V - v_i)$	$-2V + v + v_i$	$v_i - v$	$+$	$+$
$-v$	$-$	$-$	$-$	$V - 2v$	$V - v - v_i$	$+$
	$-v$	$-(V - v_i)$	$-(V - v)$	$(V - v)$	$(V - v_i)$	v

We can now compute the final allocation in each possible situation whenever player one follows the proposed strategy and whenever he unilaterally deviates and invests $(V - v_i)$ votes in the first issue. As noted in the main text, we want to consider a deviation where player one, realising that both his opponents invest more voting power on the first issue, deviates and casts more votes on the second one. Furthermore, the considered deviation does not change his payoff when he faces non-indifferent players. In order to compute the expected interim payoffs we define the following parameters:

$$\begin{array}{lcl} a=1 & \Leftrightarrow & V-2v+v_i \geq 0 \\ a=-1 & \Leftrightarrow & V-2v+v_i < 0 \end{array} \qquad \begin{array}{lcl} c=1 & \Leftrightarrow & 2V-v-2v_i \geq 0 \\ c=-1 & \Leftrightarrow & 2V-v-2v_i < 0 \end{array}$$

$$\begin{array}{lcl} b=1 & \Leftrightarrow & -2V+2v+v_i > 0 \\ b=-1 & \Leftrightarrow & -2V+2v+v_i \leq 0 \end{array} \qquad \begin{array}{lcl} d=1 & \Leftrightarrow & -2V+3v_i \leq 0 \\ d=-1 & \Leftrightarrow & -2V+3v_i > 0 \end{array}$$

Weighting each possible situation times its probability[42] we have that the expected payoffs when non-deviating and deviating are respectively

$$\begin{cases} \Pi := \frac{1}{64}\left[(27-2b+4c-a)+(31+2a+b)\right] \\ \text{and} \\ \Pi_d := \Pi + \frac{1}{64}\left[8-4c+4d\right]. \end{cases}$$

Now we just need to consider all possible combinations of parameters to realise whether it is strictly better to deviate.

Whenever $d=1$ or $c=-1$ it is strictly better to deviate. Instead, when $d=-1$ and $c=1$ both strategies yield the same expected payoff. Nevertheless in that situation some of the hypothesis are violated (for $b=-a=1$ and $b=-a=-1$, (v,v_i) is *essentially* equal to $\left(v,\frac{V}{2}\right)$; instead, when $a=b=1$, (v,v_i) does not constitute an equilibrium because a non-indifferent player that prefers issue one is strictly better off playing v_i votes on the first issue; finally, the case $a=b=-1$ can never happen). ■

Proof of Proposition 3.

Given that indifferent players invest $\frac{V}{2}$ votes in each issue we have that all possible combinations of casted votes in any of the issues by two players that follow the strategy $\left(v,\frac{V}{2}\right)$ is

[42] Given the uniform and independent priors, all columns (alternatively rows) occur with probability $\frac{1}{8}$ except columns two and five which occur with probability $\frac{1}{4}$.

depicted in the matrix below:

$$\begin{array}{c|cccccc}
v & + & + & + & + & + & + \\
\frac{V}{2} & \frac{V}{2}-v & + & + & + & + & + \\
(V-v) & V-2v & \frac{V}{2}-v & + & + & + & + \\
-(V-v) & - & -\frac{3}{2}V+v & -2(V-v) & + & + & + \\
-\frac{V}{2} & - & - & -\frac{3}{2}V+v & \frac{V}{2}-v & + & + \\
-v & - & - & - & V-2v & \frac{V}{2}-v & + \\
\hline
 & -v & -\frac{V}{2} & -(V-v) & (V-v) & \frac{V}{2} & v
\end{array}$$

As we did before, we define the following four parameters:

$$\begin{array}{lll}
a=1 & \Leftrightarrow & \bar{v}\geq 2v-V \\
a=-1 & \Leftrightarrow & \bar{v}<2v-V
\end{array}
\qquad
\begin{array}{lll}
c=1 & \Leftrightarrow & \bar{v}>\frac{3}{2}V-v \\
c=-1 & \Leftrightarrow & \bar{v}\leq \frac{3}{2}V-v
\end{array}$$

$$\begin{array}{lll}
b=1 & \Leftrightarrow & \bar{v}\geq v-\frac{V}{2} \\
b=-1 & \Leftrightarrow & \bar{v}<v-\frac{V}{2}
\end{array}
\qquad
\begin{array}{lll}
d=1 & \Leftrightarrow & \bar{v}>2V-2v \\
d=-1 & \Leftrightarrow & \bar{v}\leq 2V-2v
\end{array}$$

where $\bar{v}$ indicates the number of votes invested in issue one by the remaining player. Without loss of generality we assume that this player has positive preferences and strictly prefers the first issue.

$\left(v,\frac{V}{2}\right)$ is an equilibrium if and only if it is optimal for the remaining player to invest exactly v votes on the first issue (i.e. $\bar{v}=v$ should be optimal).

The way to proceed is to define all possible cases so that the conditions that define the four parameters are well ordered. For instance, whenever $v>\frac{5}{6}V$ we have that $0\leq 2V-2v\leq v-\frac{V}{2}\leq \frac{3}{2}V-v\leq 2v-V\leq V$ and it can be easily shown that $\bar{v}=v$ is an optimal response for player one. Hence, $\left(v,\frac{V}{2}\right)$ is a symmetric equilibria as long as $v\in\left(\frac{5}{6}V,V\right]$. This set of equilibria are *essentially* identical to $\left(V,\frac{V}{2}\right)$.

A further analysis shows that there exists no symmetric equilibria where $v\in\left(\frac{3}{4}V,\frac{5}{6}V\right]$. The case in which $v=\frac{3}{4}V$ implies that $0<v-\frac{V}{2}<2V-2v=2v-V<\frac{3}{2}V-v<V$ and a symmetric equilibria can be sustained if and only if $\theta=\frac{1}{2}$. If $\theta<\frac{1}{2}$, player one prefers investing more voting power on his preferred issue and, inversely, he prefers to split more

equally his votes whenever $\theta > \frac{1}{2}$. Hence we conclude that $\left(\frac{3}{4}V, \frac{V}{2}\right)$ is an equilibria if and only if $\theta = \frac{1}{2}$. Moreover, that equilibria can be sustained by any $v \in \left(\frac{2}{3}V, \frac{3}{4}V\right]$.
Finally, $v \in \left(\frac{V}{2}, \frac{2}{3}V\right)$ can constitute a symmetric equilibria only when $\theta \geq \frac{1}{2}$; when $\theta < \frac{1}{2}$, a non-indifferent player knows that by deviating and investing all of his voting power on his preferred issue he gains that issue when he is confronted with an indifferent player and a low one (instead he loses it if he invests v votes). This equilibria reaches the same allocation as MR. In fact, $\left(\frac{V}{2}, \frac{V}{2}\right)$ is trivially an equilibria for any θ because any player is equally pivotal with any number of votes (in particular with $v = \frac{V}{2}$). ■

Proof of Theorem 3.

Any direct mechanism is now defined by 8192 parameters. Restricting the analysis to *reasonable* SCFs rends the problem *tractable* and *simplifies* the analysis into 44 parameters belonging to the interval $[0, 1]$. The following tables define such parameters depending on the preferences of each individual. Note that given that we have three players the final allocation should be a three dimensional table. Hence, in order to depict it we provide four tables each one corresponding to a different preference profile of player one (as is assumed throughout, player one has positive preferences towards both issues).

	$(1,\theta)$	A	B	C	D	1	1	1	1
	$(\theta,1)$	E	F	G	H	1	1	1	1
	$(1,1)$	I	J	K	L	1	1	1	1
	(θ,θ)	M	N	O	P	1	1	1	1
$\theta^1 = (1,\theta)$,	$(-\theta,\theta)$	1-M	Q	R	S	P	L	H	D
	$(-1,1)$	1-I	T	U	R	O	K	G	C
	$(-\theta,1)$	1-E	V	T	O	N	J	F	B
	$(-1,\theta)$	1-A	1-E	1-I	1-M	M	I	E	A
		$(-1,\theta)$	$(-\theta,1)$	$(-1,1)$	$(-\theta,\theta)$	(θ,θ)	$(1,1)$	$(\theta,1)$	$(1,\theta)$

$\theta^1 = (\theta, 1)\,,$

$(1,\theta)$	E	F	G	H	1	1	1	1
$(\theta,1)$	1-V	a	b	c	1	1	1	1
$(1,1)$	1-T	d	e	f	1	1	1	1
(θ,θ)	1-Q	g	h	i	1	1	1	1
$(-\theta,\theta)$	1-N	1-g	j	k	i	f	c	H
$(-1,1)$	1-J	1-d	l	j	h	e	b	G
$(-\theta,1)$	1-F	1-a	1-d	1-g	g	d	a	F
$(-1,\theta)$	1-B	1-F	1-J	1-N	1-Q	1-T	1-V	E
	$(-1,\theta)$	$(-\theta,1)$	$(-1,1)$	$(-\theta,\theta)$	(θ,θ)	$(1,1)$	$(\theta,1)$	$(1,\theta)$

$\theta^1 = (1, 1)\,,$

$(1,\theta)$	I	J	K	L	1	1	1	1
$(\theta,1)$	1-T	d	e	f	1	1	1	1
$(1,1)$	1-U	1-l	n	o	1	1	1	1
(θ,θ)	1-R	1-j	p	q	1	1	1	1
$(-\theta,\theta)$	1-O	1-h	1-p	r	q	o	f	L
$(-1,1)$	1-K	1-e	1-n	1-p	p	n	e	K
$(-\theta,1)$	1-G	1-b	1-e	1-h	1-j	1-l	d	J
$(-1,\theta)$	1-C	1-G	1-K	1-O	1-R	1-U	1-T	I
	$(-1,\theta)$	$(-\theta,1)$	$(-1,1)$	$(-\theta,\theta)$	(θ,θ)	$(1,1)$	$(\theta,1)$	$(1,\theta)$

$\theta^1 = (\theta, \theta)\,,$

$(1,\theta)$	M	N	O	P	1	1	1	1
$(\theta,1)$	1-Q	g	h	i	1	1	1	1
$(1,1)$	1-R	1-j	p	q	1	1	1	1
(θ,θ)	1-S	1-k	1-r	s	1	1	1	1
$(-\theta,\theta)$	1-P	1-i	1-q	1-s	s	q	i	P
$(-1,1)$	1-L	1-f	1-o	1-q	1-r	p	h	O
$(-\theta,1)$	1-H	1-c	1-f	1-i	1-k	1-j	g	N
$(-1,\theta)$	1-D	1-H	1-L	1-P	1-S	1-R	1-Q	M
	$(-1,\theta)$	$(-\theta,1)$	$(-1,1)$	$(-\theta,\theta)$	(θ,θ)	$(1,1)$	$(\theta,1)$	$(1,\theta)$

Similarly to the proof of theorem two, we just need to compute the interim probabilities in terms of these parameters and maximise the ex-ante utility of any of the players subject to the constraints for truthtelling. The interim probabilities are proportional to:

$$\begin{aligned}
P(1,\theta) = & -9 + A + 2D + 2B + 2C + 2F + 2G + 2H + 2J + 2K+ \\
& +2L + 2N + 2O + 2P + 2Q + 2R + S + 2T + U + V. \\
P(\theta,1) = & \; 2 + 2E - B + 2G + 2H - 2J - 2N - 2Q - 2T - 2V+ \\
& +2i + a + 2b + 2c + 2f + 2h + 2j + l + k + 2e. \\
P(1,1) = & \; 9 + 2I + 2J + 2L - 2T + 2d + 2f - 2U - 2l + n+ \\
& +2o - 2R - 2j + 2q - 2O - 2h - 2G - C - b + r. \\
P(\theta,\theta) = & \; 12 - 2L - 2f - 2R - 2j + 2O - D - 2H - 2Q - 2S+ \\
& +2h + 2N - 2k + 2g - c - o + 2p + 2M + s - 2r.
\end{aligned}$$

The *reasonable* and *ex-ante efficient* SCF is the one that maximizes the *ex-ante* expected utility subject to the truthtelling constraints and the feasibility ones (i.e. the forty parameters need to belong to the interval $[0,1]$).

$$\max E_{\left(\theta^1,\theta^2,\theta^3\right)} \left\{ U\left(\theta^1,\theta^2,\theta^3\right) \right\}$$

$$\text{subject to} \left\{ \begin{array}{c} P(1,1) = P(\theta,\theta) \\ P(1,\theta) \geq P(\theta,1) \\ P(1,1) \geq \dfrac{P(\theta,1) + P(1,\theta)}{2} \\ P(1,1) \leq \dfrac{P(\theta,1)\,\underline{\theta} + P(1,\theta)\,\bar{\theta}}{\bar{\theta} + \underline{\theta}}. \end{array} \right.$$

The end of the proof relies on writing the program in terms of the forty parameters and to solve it one can proceed as follows: step by step assume whether or not any of the constraints are binding. Once this is done we are just left with some tedious (though trivial) linear programs. And it can be proven that for different values of θ the corner solution varies. More specifically, all parameters are equal to one but the ones specified below:

- $\theta \in \left[0, \frac{1}{3}\right]$: $R = S = U = b = c = j = k = l = 0.$
- $\theta \in \left[\frac{1}{3}, \frac{1}{2}\right]$: $Q = R = S = T = U = j = k = l = r = 0.$
- $\theta \in \left[\frac{1}{2}, 1\right]$: $Q = R = S = T = U = V = j = k = l = r = 0.$

A proper analysis of such allocations tells us that they coincide with the allocations achieved by the strategies where a non-indifferent player invests V, $\frac{3}{4}V$ and $\frac{V}{2}$ votes on his preferred issue, respectively. ■

The equilibria when continuous preferences and divisible votes (2 players).

The proof relies on extending the reasoning on section 5.1.1.

We restrict the analysis to pure strategy equilibria. Remember that Theorem 1 tell us that the optimal strategy $_j$ is only contingent on the relative intensities of the preferences and, moreover, it is well behaved (monotonic) with respect to them. In order to simplify the analysis we assume a uniform distribution on the relative intensities rather than on the preferences themselves i.e.

$$\theta_n^i \in \{\pm 1, \pm \theta\} : \begin{cases} \Pr\left\{\left|\theta_n^i\right| = 1\right\} = \frac{1}{2} \\ \Pr\left\{\theta_n^i > 0\right\} = \frac{1}{2} \\ \theta \sim U[0,1] \\ \text{Pairwise independence across issues and players.} \end{cases}$$

We analyse the equilibria from the perspective of a player with positive preferences. The *interim* expected payoff of player i when he invests $v^i \in [0, V] \subset \mathsf{R}$ votes on the first issue is:

$$\tilde{P}_1\left(v^i\right) \cdot \theta_1^i + \tilde{P}_2\left(V - v_1^i\right) \cdot \theta_2^i$$

$$\text{where, } \begin{cases} \tilde{P}_1\left(v^i\right) = \Pr\left(v^i + v^j > 0 \mid \theta_1^j < 0\right) + \frac{1}{2}\Pr\left(v^i + v^j = 0 \mid \theta_1^j < 0\right) \\ \tilde{P}_2\left(1 - v^i\right) = \Pr\left(v^i + v^j < 0 \mid \theta_2^j < 0\right) + \frac{1}{2}\Pr\left(v^i + v^j = 0 \mid \theta_2^j < 0\right) \end{cases}$$

Simple calculations allow us to rewrite the *interim* expected payoff of player i as:[43]

$$\frac{1}{2}\theta_2^i + \left\{ \Pr\left(v^i + v^j > 0\right) + \frac{1}{2}\Pr\left(v^i + v^j = 0\right) \right\} \cdot \left(\theta_1^i - \theta_2^i\right).$$

Hence, an indifferent player will be indifferent between playing any of the strategies (as was done in the binary case, we can assume that he plays the undominated strategy $v_i = \frac{V}{2}$) and a non-indifferent player (say he prefers issue one) wants to maximise the expression inside the curly brackets. In the case that $v^j(\cdot)$ induces an atomless distribution on $[0, V]$ it is dominant for player one to set $v^i = V$ Otherwise, if the induced distribution on the invested votes by player j on issue one is not atomless, v_i will always be strictly higher (if possible) than the absolute value of the lowest possible value of v^j. Closing up, the only possible equilibria has non-indifferent players investing all their voting power on their preferred issue.

Finally note that the proof can also be applied to the case of continuous preferences and non-divisible votes. We just need to restrict the set of strategies of player i. ■

The equilibria when continuous preferences and divisible votes (3 players).

The setting is analogous to the one described in the proof above. We just need to add the restriction that we focus our analysis on *symmetric* equilibria (i.e. the three voters play the same strategy) and (as was done in section 5.2.2) we further assume that a voters behave equivalently regardless of the labelling or the sign of the issue.

This proof is a bit more complicated than the one above because now we need to consider whether each of them is in favour or against the approval of each of the issues in order to assign the appropriate sign to the casted votes. Once we take this into account we have that the *interim* probabilities read as follows:

- $\tilde{P}_1\left(v^i\right) = \frac{1}{2}\Pr\left(v^j + v^k < v^i \mid \theta_1^j, \theta_1^k < 0\right) + \Pr\left(v^j - v^k \leq v^i \mid \theta_1^j, -\theta_1^k < 0\right) - \frac{1}{2}$
- $\tilde{P}_2\left(1 - v^i\right) = \frac{1}{2}\Pr\left(v^j + v^k > V + v^i \mid \theta_1^j, \theta_1^k < 0\right) + \Pr\left(v^j - v^k \leq V - v^i \mid \theta_1^j, -\theta_1^k < 0\right) - \frac{1}{2}.$

where $v^j, v^k \geq 0$.

[43] The conditional probabilities are omitted for simplicity.

Note that the tie breaking rule is now playing a role because player i just needs to equate the sum of his opponents votes whenever only one of them desires the dismissal of the issue. Given the assumption that voters play equivalently regardless of the sign of his preferences we have that v^j and $\left(1-v^j\right)$ have the same induced distribution (the same can be said about player k's strategy). That implies that v^j is symmetrically distributed around $\frac{V}{2}$. In order to simplify the notation we define $X := v^j + v^k$ (which, accordingly, is symmetrically distributed around V i.e. $\Pr\left(X < k\right) = \Pr\left(X > 2V - k\right)$ for $k \in [0, 2V]$). Using such symmetry and the fact that $\left(v^j + \left(1 - v^k\right)\right)$ is distributed as X, we can write the *interim* expected payoff for a player that prefers issue one as follows

$$ct + \frac{1}{2}\Pr\left(X < v^i\right) \cdot \left\{\frac{1}{2} - \theta\right\} + \Pr\left(X \leq V + v^i\right) \cdot \left\{1 - \frac{1}{2}\theta\right\}.$$

First note that whenever both opponents are spliting their voting power evenly (the case of MR), player i is indifferent between playing any of the strategies. In particular $v^i = \frac{V}{2}$ is a best response. Hence, a symmetric equilibria has all players always spliting their voting power equally among both issues.

In the remaining of the proof we show that there exists only one more (and only one) equilibria which corresponds to the one in which non-indifferent players invest all their voting power on their preferred issue.[44]

Any other equilibria will have non-indifferent players investing more than $\frac{V}{2}$ votes on their preferred issue. Consequently, any voter with $\theta \in \left[0, \frac{1}{2}\right)$ clearly invests all his voting power on his preferred issue. Suppose now that there are some voters with $\theta \in \left[\frac{1}{2}, 1\right]$ such that $v^i < V$. Theorem 1 tell us that the optimal strategy should a well behaved function (decreasing with respect to θ) thus we can consider a parameter $\tilde{\theta} \in \left[\frac{1}{2}, 1\right]$ such that any player with $\theta^+ > \tilde{\theta}$ invests strictly less votes on his preferred issue ($v^i\left(\theta^+\right) < V$)and any player with $\theta^- < \tilde{\theta}$ sticks to the strategy $v^i = V$.

[44] The behaviour of indifferent voters does not need to be specified because they have a zero measure. Nevertheless, it can be shown that their best response to any of the equilibria is spliting their voting power evenly.

Given that both are acting optimally we have that the next two inequalities should hold:

$$\left(\Pr\left(X<V\right)-\Pr\left(X<v^{i}\left(\theta^{-}\right)\right)\right)\cdot\left\{\theta-\tfrac{1}{2}\right\}\leq\left(\Pr\left(X\leq 2V\right)-\Pr\left(X\leq V+v^{i}\left(\theta^{-}\right)\right)\right)\cdot\left\{2-\theta^{-}\right\}$$

$$\left(\Pr\left(X<V\right)-\Pr\left(X<v^{i}\left(\theta^{-}\right)\right)\right)\cdot\left\{\theta^{+}-\tfrac{1}{2}\right\}\geq\left(\Pr\left(X\leq 2V\right)-\Pr\left(X\leq V+v^{i}\left(\theta^{-}\right)\right)\right)\cdot\left\{2-\theta^{+}\right\}$$

Given that the optimal function is a well behaved function we have that we should consider two possible cases: (1) the function is smooth at $\tilde{\theta}$ (i.e.$\lim_{\varepsilon\to 0}v^{i}\left(\tilde{\theta}+\varepsilon\right)=V$) and (2) there is a discontinuity (i.e.$\lim_{\varepsilon\to 0}v^{i}\left(\tilde{\theta}+\varepsilon\right)=\bar{v}<V$). Consequently, taking limits as θ^{-} and θ^{+} tend to θ in the previous inequalities lead to two possible equalities depending on the behaviour of the optimal strategy at $\tilde{\theta}$:

1- $\left(\Pr\left(X<V\right)-\Pr\left(X<V\right)\right)\cdot\left\{\tilde{\theta}-\frac{1}{2}\right\}=\left(\Pr\left(X\leq 2V\right)-\Pr\left(X<2V\right)\right)\cdot\left\{2-\tilde{\theta}\right\}.$

2- $\left(\Pr\left(X<V\right)-\Pr\left(X<\bar{v}\right)\right)\cdot\left\{\tilde{\theta}-\frac{1}{2}\right\}=\left(\Pr\left(X\leq 2V\right)-\Pr\left(X\leq V+\bar{v}\right)\right)\cdot\left\{2-\tilde{\theta}\right\}.$

Trivially, the first equality cannot be met because there is a positive measure of types playing the non-diversification strategy thus $\Pr\left(X=2V\right)>0$. The second case also leads to a contradiction given the following inequalities and the fact that one of them will always be strict:

$$2\tilde{\theta}-1\leq 2-\tilde{\theta}$$

$$\Pr\left(X<V\right)-\Pr\left(X<\bar{v}\right)\leq 2\cdot\left(\Pr\left(X\leq 2V\right)-\Pr\left(X\leq V+\bar{v}\right)\right).$$

The second inequality needs some clarification. The term in brackets on the RHS accounts for all those cases in which both opponents are investing strictly more than $(V+\bar{v})$ votes (i.e. $X\in\left(V+\bar{v},2V\right]$). That is, those cases in which both players have a type belonging to the interval $\left[0,\tilde{\theta}\right)$. Hence this occurs with probability ρ^{2} where $\rho:=\Pr\left\{\theta\in\left[0,\tilde{\theta}\right)\right\}$. Instead, the LHS accounts for those cases in which X belongs to $[\bar{v},V)$. A necessary condition for that event is that none of the players should invests V votes i.e. it occurs with a probability lower than $1-\rho$. Given that θ is uniformly distributed, we know that $\rho\geq\frac{1}{2}$.

Finally, we just need to see that the second inequality is strict for $\rho>\frac{1}{2}$ and the first one is strict for $\rho=\frac{1}{2}$. ■

Proof of Theorem 4.

We just need to see that if we allow for any possible prior, the only implementable allocation cannot take into account the intensity of the preferences. It follows that the optimal one is MR.

We consider the simplest setting with two issues, two players. We want the *reasonable* SCF to be implementable for any possible prior and any value of the relative intensity, i.e $\forall \theta \in [0,1]$. We will deviate from our previous setting assuming that the first issue on player i's opponent is distributed as follows:

$$\begin{cases} \Pr\left\{\left|\theta_1^j\right| = 1\right\} = \alpha \in [0,1] \\ \Pr\left\{\theta_n^i > 0\right\} = \beta \in [0,1] \\ \text{Pairwise independent.} \end{cases}$$

Any reasonable SCF is defined by six parameters (see section 5.1) from which we can compute the *interim* probabilities for each issue and the incentive constraints for truthtelling. It results that the six parameters should be equal to $\frac{1}{2}$. In other words, a SCF can only implement unanimous wills in a situation with two players and two issues. ■

References

[1] Brams, S. and Taylor, A. (1996) *Fair Division : From Cake-Cutting to Dispute Resolution*, Cambridge University Press.

[2] Casella, A. (2002), "Storable Votes", *NBER Working Paper* No w9189.

[3] Coleman, J. (1970), "Political Money", *The American Political Science Review*, Vol 64, Issue 4.

[4] Dutta, B., Jackson, M.O. and Le Breton, M. (2003), "Equilibrium Agenda Formation", mimeo.

[5] Elsgolts, L. (1970), *Differential Equations and the Calculus of Variations*, MIR Publishers.

[6] Gibbard, A. (1973), "Manipulation of voting schemes", *Econometrica* 41.

[7] Holmstrom, B. and Myerson, R. (1983), "Efficient and Durable Decision Rules with Incomplete Information", *Econometrica*, Vol 51 Issue 6.

[8] Jackson, M.O. and Sonnenschein, H.F. (2003), "The Linking of Collective Decisions and Efficiency", mimeo.

[9] Metcalfe, D. (2000), "The OECD Agreement to Criminalize Bribery: A Negotiation Analytic Perspective", *International Negotiation: A journal of Theory and Practice*, Vol 5 No 1.

[10] Myerson, R. (1999), "Theoretical Comparisons of Electoral Systems", *European Economic Review*, Vol 43.

[11] Robins, L. (1938), "Interpersonal Comparisons of Utility: A Comment", *The Economic Journal*, Vol 48, Issue 192.

[12] Rousseau, J-J., *The Social Contract*, London: Dent 1993.

[13] Satterthwhaite, M.A. (1975), "Strategy-proofness and Arrow's conditions: Existence and correspondence theorems for voting procedures and social welfare functions", *Journal of Economic Theory* 10.

[14] Spitz, E. (1984), *Majority Rule* Chatham, N.J.

[15] Tideman, T. and Tullock, G. (1976), "A New and Superior Process for Making Social Choices", *The Journal of Political Economy*, Vol 84, Issue 6.

[16] Wilson, R. (1969), "An Axiomatic Model of Logrolling", *The American Economic Review*, Vol 59, Issue 3.