On coalitional semivalues ⋆

M. Josune Albizuri[a,∗], José M. Zarzuelo[a]

[a]*Department of Applied Economics IV; Faculty of Economics and Business Administration; University of the Basque Country; Av. Lehendakari Agirre, 83; 48015 BILBAO, Spain.*

Abstract

In this paper we propose a characterization of the coalitional value for transferable utility games (Owen, 1977), and we define and study coalitional semivalues, which are generalizations of semivalues (Dubey, Neyman and Weber, 1981).

Key words: coalitional value, semivalues

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∗ Corresponding author.
Email addresses: elpalirm@bs.ehu.es (M. Josune Albizuri), elpzazaj@bs.ehu.es (José M. Zarzuelo).
URL: www.ehu.es/zarzuelo (José M. Zarzuelo).

1 Introduction

In 1977 Owen defined and axiomatized the coalitional value for games with transferable utility, providing a generalization of the Shapley value to the coalitional framework, and later Hart and Kurz (1983) gave an alternative axiomatization by considering games with an infinite universe of players. On the other hand Dubey, Neyman and Weber (1981) provided a different generalization of the Shapley value by removing Efficiency from the classical axiomatization of Shapley (1953).

The aim of this paper is twofold. First, we propose an alternative axiomatization of the coalitional value by means of three axioms. Two of them, Carrier and Additivity, were already used by Hart and Kurz (1983). The third one can be seen as a modification of the well known axiom of Anonymity.

Second, as the title of the work already suggests, we marry the two generalizations of the Shapley value proposed by Owen (1977) and Dubey et al. (1981), i.e. we define coalitional semivalues, providing a generalization of semivalues to the coalitional context. We will follow the axiomatic procedure of Dubey et al. (1981), i.e. we will take Efficiency out of the system proposed in the present paper. In addition we will use the translations to the coalitional framework of some axioms used also by Dubey et al. (1981), and we will require an additional axiom which is specific to the coalitional context and is satisfied by the coalitional value.

As we will describe in the preliminaries, Owen (1977) defined the coalitional value of a game by applying the Shapley value twice. First, the Shapley value is employed at the level of the coalitions of the coalitional structure, to define a new game on each one of those coalitions. Subsequently, the Shapley value is applied to these new games. This procedure yields precisely the coalitional value of the original game. So, in certain sense we can say that the coalitional value is obtained by means of a “composition” of the Shapley value with itself. In this work we will show that the coalitional semivalues defined in this paper can also be obtained by means of a “composition” of two arbitrary semivalues. Furthermore, if one additional axiom is considered in the system proposed here, the resulting coalitional semivalues are “compositions” of a semivalue with itself. Finally, we point out that if we remove Efficiency from the system proposed by Hart and Kurz (1983) we do not obtain all the “compositions” of semivalues, but only those in which a semivalue is “composed” with the Shapley value.

The paper is organized as follows. In Preliminaries we present notation, and previous definitions and results which are needed in the course of the paper. In Section 3 we provide the new characterization of the coalitional value. In Section 4 we define coalitional semivalues and obtain an explicit formula for them. In Section 5 we prove that coalitional semivalues are “compositions” of semivalues.

2 Preliminaries

Let U be an infinite set which denotes the *universe of players*. A *coalition* is a non-empty subset of U. A *transferable utility game* (a *game* for short) is a function $v : 2^U \to \mathbb{R}$ such that $v(\emptyset) = 0$, where 2^U denotes the family of all subsets of U. A set $N \subseteq U$ is a *carrier* of v, if $v(S \cap N) = v(S)$ for all $S \subseteq U$. By G we denote the space of all games on U with finite carrier and by G^N

the subspace of G of games with carrier N. It is well known that a basis of G (resp. G^N) is formed by *unanimity games* u_R, where $R \subset U$ (resp. $R \subseteq N$) is finite, defined by $u_R(S) = 1$ if $R \subseteq S$, and $u_R(S) = 0$ otherwise.

A game v is *monotonic* if $v(S) \leq v(T)$ when $S \subseteq T$. A game $v \in G^N$ is *additive* if for every $i \in N$ there exists $a_i \in \mathbb{R}$ such that $v(S) = \sum_{i \in N} a_i$ for every $S \subseteq N$. By AG and AG^N we denote respectively the subspaces of G and G^N formed by additive games.

Each finite partition $\mathfrak{B} = \{B_1, \ldots, B_h\}$ of U is called a *coalitional structure.* If N is a coalition, $\mathfrak{B}_N$ denotes the partition of N induced by $\mathfrak{B}$, i.e., $\mathfrak{B}_N = \{B_p \cap N : B_p \cap N \neq \emptyset, B_p \in \mathfrak{B}\}$. The set of all pairs $(v, \mathfrak{B})$, where $v \in G$, and $\mathfrak{B}$ is a coalitional structure is denoted by X; and X^N denotes the subset of X for which N is a carrier of v. If ψ is a mapping from X into AG, we denote the restriction of ψ to X^N by ψ^N.

Let $\pi : U \to U$ be a mapping. If $v \in G$, denote by πv the game defined by $\pi v(S) = v(\pi^{-1} S)$. If $\mathfrak{B}$ is a coalitional structure, denote $\pi\mathfrak{B} = \{\pi B_p : B_p \in \mathfrak{B}\}$. Notice that $\pi\mathfrak{B}$ is not necessarily a coalitional structure. Denote $\mathfrak{B}^\pi = \pi\mathfrak{B} \cup \{U \backslash \pi U\}$ if $\pi U \neq U$ and $\mathfrak{B}^\pi = \pi\mathfrak{B}$ otherwise. Notice that $\mathfrak{B}^\pi$ is a coalitional structure if and only if $\pi B_q \cap \pi B_r = \emptyset$ whenever $q \neq r$.

In 1977 Owen defined the *coalitional value* for TU games in the following way.

Let $v \in G^N$, and $\mathfrak{B} = \{B_1, \ldots, B_h\}$ be a coalitional structure, and let $B_p \in \mathfrak{B}$ be fixed. For every $S \subseteq B_p \cap N$, let

$$\mathfrak{B}(S) = \{B_1, \ldots, B_{p-1}, S, B_{p+1}, \ldots, B_h\} \tag{1}$$

and $v^{\mathfrak{B}(S)}$ the game on $\mathfrak{B}(S)$ defined by

$$v^{\mathfrak{B}(S)}(T) = v\Big(\bigcup_{t\in T} t\Big), \qquad \text{for each } T \subseteq \mathfrak{B}(S), \tag{2}$$

that is, $v^{\mathfrak{B}(S)}$ is the game v restricted to the field generated by $\mathfrak{B}(S)$ (i.e., considering $\mathfrak{B}(S)$ as set of players.)

Now consider a new game $v_p^{\mathfrak{B}} \in G^{B_p \cap N}$ defined for every $S \subseteq B_p \cap N$ by

$$v_p^{\mathfrak{B}}(S) = Sh_S\left(v^{\mathfrak{B}(S)}\right),$$

where Sh denotes the Shapley value; i.e., $v_p^{\mathfrak{B}}(S)$ is the Shapley value of "player" S in the game $v^{\mathfrak{B}(S)}$.

Owen (1977) defined the *coalitional value* of player $i \in B_p \cap N$, which we denote by $\phi_i(v, \mathfrak{B})$, as the Shapley value of player i in game $v_p^{\mathfrak{B}}$. Formally,

$$\phi_i(v, \mathfrak{B}) = Sh_i\left(v_p^{\mathfrak{B}}\right), \quad \text{for all } i \in B_p \cap N.$$

Owen (1977) characterized this value and gave the following explicit formula.

Proposition 1 *(Owen, 1977). If $(v, \mathfrak{B}) \in X^N$, and $i \in B_p$, then*

$$\phi_i(v, \mathfrak{B}) = \sum_{\substack{T \subseteq \mathfrak{B}_N \setminus \{B_p \cap N\} \\ S \subseteq B_p \cap N \setminus \{i\}}} \frac{|T|! \cdot (|\mathfrak{B}_N| - |T| - 1)!}{|\mathfrak{B}_N|!} \cdot \frac{|S|! \cdot (|B_p \cap N| - |S| - 1)!}{|B_p \cap N|!} \cdot \Big[v\Big(A_T \cup S \cup \{i\}\Big) - v(A_T \cup S)\Big],$$

where $A_T = \bigcup_{B_q \in T} B_q \cap N$.

Remark 2 *The coalitional value $\phi_i(v, \mathfrak{B})$ is independent of the carrier N considered for game v. Actually, Owen (1977) only considered games on a finite set of players when he defined and characterized this value.*

Alternatively, Hart and Kurz (1983) characterized the coalitional value by using the following axioms. Let $\psi : X \to AG$.

Carrier: If N is a carrier of v then

(i) $\sum_{i\in N} \psi_i(v, \mathfrak{B}) = v(N)$.

(ii) $\mathfrak{B}^1_N = \mathfrak{B}^2_N$ implies $\psi(v, \mathfrak{B}^1) = \psi(v, \mathfrak{B}^2)$.

Additivity: $\psi(v + w, \mathfrak{B}) = \psi(v, \mathfrak{B}) + \psi(w, \mathfrak{B})$.

Anonymity: If $\pi : U \to U$ is one-to-one, then for all $i \in U$ it holds that $\psi_i(v, \mathfrak{B}) = \psi_{\pi i}(\pi v, \mathfrak{B}^\pi)$.

Inessential Game: If $v\left(\bigcup_{B_p\in T} B_p\right) = \sum_{B_p\in T} v(B_p)$, for every $T \subseteq \mathfrak{B}$, then $\sum_{i\in B_p} \psi_i(v, \mathfrak{B}) = v(B_p)$, for every $B_p \in \mathfrak{B}$

Theorem 3 *(Hart and Kurz, 1983) There is a unique mapping $\psi : X \to AG$ satisfying Carrier, Additivity, Anonymity, and Inessential Game, and it is the coalitional value ϕ.*

On the other hand, Dubey et al. (1981) defined the *semivalues* for TU games as those mappings $\psi : G \to AG$ that satisfy the following properties.

(P1) ψ is linear;

(P2) If $\pi : U \to U$ is one-to-one, then for each $i \in U$ it holds that $\psi_{\pi i}(\pi v) = \psi_i(v)$;

(P3) If $v \in G$ is monotonic, then $\psi(v)$ is monotonic;

(P4) If $v \in AG$, then $\psi(v) = v$.

These axioms are commonly referred to as Linearity, Symmetry, Monotonicity and Projection axioms, see e.g. Aumann and Shapley (1974), but here we reserve these names for the corresponding axioms in the coalitional context.

Dubey et al. (1981) gave an explicit formula for semivalues. Consider a family of vectors $p = (p^n)_{n\in\mathbb{N}}$, where $p^n = \left(p_0^n, \ldots, p_{n-1}^n\right) \in \mathbb{R}^n$, such that

$$\sum_{s=0}^{n-1} \binom{n-1}{s} \cdot p_s^n = 1, \tag{3}$$

$$p_s^n \geq 0, \qquad 0 \leq s \leq n-1, \tag{4}$$

and

$$p_s^n = p_s^{n+1} + p_{s+1}^{n+1}, \qquad 0 \leq s \leq n-1. \tag{5}$$

Denote by ϕ^p the mapping from G into AG defined for each $v \in G^N$ and every $i \in U$ by

$$\phi_i^p(v) = \sum_{S\subseteq N\setminus\{i\}} p_s^n \cdot \left[v\Big(S \cup \{i\}\Big) - v(S)\right], \tag{6}$$

where $s = |S|$ and $n = |N|$. One can easily check that ϕ^p is well defined, that is, $\phi_i^p(v)$ does not depend on the carrier N chosen for v.

Theorem 4 *(Dubey et al., 1981) A mapping $\psi : G \to AG$ is a semivalue if and only if there exists a collection of vectors $p = (p^n)_{n\in\mathbb{N}}$, where $p^n \in \mathbb{R}^n$, satisfying (3), (4) and (5) for every $n \in \mathbb{N}$, such that $\psi = \phi^p$. Moreover, the correspondence $p \to \phi^p$ is one-to-one.*

3 A characterization of the coalitional value

In this section we are going to characterize the coalitional value by replacing Anonymity and Inessential Game axioms in Theorem 3, by the following one. Let $\psi : X \to AG$.

Rearrangement: Let $\pi : U \to U$ such that $\pi B_q \cap \pi B_r = \emptyset$ whenever $q \neq r$. If $\pi : B_p \to \pi B_p$ is one-to-one, then

$$\psi_{\pi i}\left(\pi v, \mathfrak{B}^{\pi}\right) = \psi_i\left(v, \mathfrak{B}\right) \quad \text{for all } i \in B_p.$$

This axiom is a stronger version of the Anonymity axiom. Notice that if π were restricted to beeing one-to-one, Rearrangement becomes the Anonymity axiom of Hart and Kurz (1983). But in Rearrangement we let π be any mapping. In this way, this new axiom can also be seen as a kind of *consistency* property. Indeed, what π does, apart from renaming players in U, is to maintain the size of B_p and to reduce (or maintain) the size of other coalitions in $\mathfrak{B}$, as if some of the players belonging to other members of the coalitional structure had decided to act really as a single player. Thus Rearrangement requires that the value of a player in B_p should not be affected after renaming players in U and/or reducing (or maintaining) the sizes of the other coalitions in $\mathfrak{B}$.

Theorem 5 *There is a unique mapping $\psi : X \to AG$ that satisfies Carrier, Additivity and Rearrangement, and it is the coalitional value ϕ.*

PROOF. First let us see that ϕ verifies the above axioms. By Theorem 3, it only remains to prove that ϕ satisfies Rearrangement. So let $(v, \mathfrak{B}) \in X^N$, and $\pi : U \to U$ such that $\pi B_q \cap \pi B_r = \emptyset$ if $q \neq r$, and $\pi : B_p \to \pi B_p$ is one-to-one. By Proposition 1 if $i \in B_p$ it holds that

$$\phi_{\pi i}\left(\pi v, \mathfrak{B}^{\pi}\right) = \sum_{\substack{T \subseteq \mathfrak{B}^{\pi}_{\pi N} \setminus \{\pi B_p \cap \pi N\} \\ S \subseteq (\pi B_p \cap \pi N) \setminus \{\pi i\}}} \frac{|T|! \cdot \left(|\mathfrak{B}^{\pi}_{\pi N}| - |T| - 1\right)!}{|\mathfrak{B}^{\pi}_{\pi N}|!} \cdot$$
$$\frac{|S|! \cdot \left(|\pi B_p \cap \pi N| - |S| - 1\right)!}{|\pi B_p \cap \pi N|!} \cdot \left[v\left(A_T \cup S \cup \{i\}\right) - v(A_T \cup S)\right],$$

where $A_T = \bigcup_{\pi B_q \in T} (\pi B_q \cap \pi N)$. Since $\pi : B_p \to \pi B_p$ is one-to-one, the second term in this equality is equal to

$$\sum_{\substack{T \subseteq \mathfrak{B}_N \setminus \{B_p \cap N\} \\ S \subseteq (B_p \cap N) \setminus \{i\}}} \frac{|T|! \cdot \Big(|\mathfrak{B}_N| - |T| - 1\Big)!}{|\mathfrak{B}_N|!} \cdot \frac{|S|! \cdot \Big(|B_p \cap N| - |S| - 1\Big)!}{|B_p \cap N|!} \cdot \Big[v\Big(C_T \cup S \cup \{i\}\Big) - v(C_T \cup S)\Big],$$

where $C_T = \bigcup_{B_q \in T} B_q \cap N$. But this summation is precisely $\phi_i(v, \mathfrak{B})$.

Next we prove that these three axioms fully determine ϕ.

By Additivity it is sufficient to consider the games $c \cdot u_R$, where $c \in \mathbb{R}$ and $R \subset U$ is finite. Let $\mathfrak{B} = \{B_1, \ldots, B_h\}$ be a coalitional structure of U.

Let $i \notin R$. Since $R \cup \{i\}$ and R are carriers of $c \cdot u_R$, by Carrier (i)

$$\sum_{j \in R \cup \{i\}} \psi_j(c \cdot u_R, \mathfrak{B}) = c \cdot u_R\Big(R \cup \{i\}\Big) = c \cdot u_R(R) = \sum_{j \in R} \psi_j(c \cdot u_R, \mathfrak{B}).$$

Therefore, $\psi_i(c \cdot u_R, \mathfrak{B}) = 0$.

Now assume that $\mathfrak{B}_R = \{B'_1, \ldots, B'_\ell\}$, and without loss of generality that $\left|B'_p\right| \leq \left|B'_q\right|$ if $p < q$. To complete the proof it is enough to show that for each $i \in B'_p$, $1 \leq p \leq \ell$, it holds that

$$\psi_i(c \cdot u_R, \mathfrak{B}) = \frac{c}{\ell \cdot \left|B'_p\right|}.$$

Denote $B'_{\ell+1} = U \setminus R$. Thus, the set $\mathfrak{B}' = \Big\{B'_p : p = 1, \ldots, \ell + 1\Big\}$ is a coalitional structure of U.

Since $\mathfrak{B}_R = \mathfrak{B}'_R$, Carrier (ii) implies

$$\psi_i(c \cdot u_R, \mathfrak{B}) = \psi_i(c \cdot u_R, \mathfrak{B}') \quad \text{for all } i \in R. \tag{7}$$

Let us consider a mapping $\pi_1 : U \to U$, such that $\pi_1 B'_q \cap \pi_1 B'_r = \emptyset$ if $q \neq r$, and such that $\pi_1 i = i$ for every $i \in B'_1$, and $\left|\pi_1 B'_p\right| = |B'_1|$, for every $p = 1, \ldots, \ell$. Applying Rearrangement

$$\psi_i\left(c \cdot u_R, \mathfrak{B}'\right) = \psi_i\Big(\pi_1\left(c \cdot u_R\right), \mathfrak{B}'^{\pi_1}\Big) \quad \text{for all } i \in B'_1. \tag{8}$$

Now notice that $\pi_1 R$ is a carrier of $\pi_1\left(c \cdot u_R\right)$, and from Carrier (i) it follows that $\psi_j\Big(\pi_1\left(c \cdot u_R\right), \mathfrak{B}'^{\pi_1}\Big) = 0$ for all $j \notin \pi_1 R$. Notice also that players in $\pi_1 R$ are all identical (since all the $\pi_1 B'_p$ are of the same size, for all $1 \leq p \leq \ell$). Applying Carrier (i) and Rearrangement again (actually, its weaker version of Anonymity) it holds that

$$\psi_i\Big(\pi_1\left(c \cdot u_R\right), \mathfrak{B}'^{\pi_1}\Big) = \frac{c}{\ell \cdot |B'_1|}, \qquad \text{for every } i \in \pi_1 R, \tag{9}$$

From (7), (8) and (9), we conclude that for every $i \in B'_1$ it holds that

$$\psi_i\left(c \cdot u_R, \mathfrak{B}\right) = \frac{c}{\ell \cdot |B'_1|}.$$

Next consider $k \leq \ell$, and, by induction, assume that

$$\psi_i\left(c \cdot u_R, \mathfrak{B}\right) = \frac{c}{\ell \cdot \left|B'_p\right|} \qquad \text{for all } i \in B'_p, \text{ and all } p \in \{1, \ldots, k-1\}. \tag{10}$$

Let $\pi_k : U \to U$ such that $\pi_k B'_q \cap \pi_k B'_r = \emptyset$ if $q \neq r$, and such that $\pi_k i = i$ for every $i \in B'_1 \cup \cdots \cup B'_k$, and $\left|\pi_k B'_q\right| = |B'_k|$ for $q = k, \ldots, \ell$. By Rearrangement

$$\psi_i\left(c \cdot u_R, \mathfrak{B}'\right) = \psi_i\Big(\pi_k\left(c \cdot u_R\right), \mathfrak{B}'^{\pi_k}\Big) \quad \text{for all } i \in B'_1 \cup \ldots \cup B'_k. \tag{11}$$

Since $\pi_k R$ is a carrier of $\pi_k\left(c \cdot u_R\right)$, it follows that $\psi_j\Big(\pi_k\left(c \cdot u_R\right), \mathfrak{B}'^{\pi_k}\Big) = 0$ for all $j \notin \pi_k R$. Since all the $\pi_k B'_p$ are of the same size, for $p = k \ldots, \ell$, it follows that all the players in $\pi_k\left(B'_k \cup \ldots \cup B'_\ell\right)$ are identical. Applying the induction hypothesis, Carrier (i) and Rearrangement (again its weaker version of Anonymity) it holds that

$$\psi_i\left(\pi_k\left(c \cdot u_R\right), \mathfrak{B}'^{\pi_k}\right) = \frac{c}{\ell \cdot |B'_k|}, \qquad \text{for every } i \in \pi_k\left(B'_k \cup \ldots \cup B'_\ell\right). \quad (12)$$

Finally by (7), (11), and (12), for all $i \in B'_k$ it holds that

$$\psi_i\left(c \cdot u_R, \mathfrak{B}\right) = \frac{c}{\ell \cdot |B'_k|}.$$

And the proof is complete. $\square$

Remark 6 *In the characterization of Hart and Kurz (1983) an infinite population is needed. However as the reader can easily check, we do not need an infinite population to state Theorem 5, that is, U can be a finite set.*

4 Coalitional semivalues

As we mentioned in Preliminaries, Dubey et al. (1981) defined semivalues by removing Efficiency from the classical characterization of the Shapley value, or more precisely by removing Efficiency and adding Monotonicity. Our aim in this section is to obtain a generalization of semivalues to the coalitional framework following their procedure. So, we will eliminate Efficiency (actually Carrier (i)) from the new axiom system proposed in the present work, and we will add three other axioms which are satisfied by the coalitional value. Two of them are adaptations of the Monotonicity and Projection axioms of Dubey et al. (1981) (properties (P3) and (P4) in Preliminaries), and the third is specific to the coalitional context.

A mapping $\psi : X \to AG$ will be called a *coalitional semivalue* if it satisfies:

Carrier (ii);

Linearity: $\psi(c_1 \cdot v + c_2 \cdot w, \mathfrak{B}) = c_1 \cdot \psi(v, \mathfrak{B}) + c_2 \cdot \psi(w, \mathfrak{B}),\ c_1, c_2 \in \mathbb{R}$;

Rearrangement;

Monotonicity: If $v \in G$ is monotonic, then $\psi(v, \mathfrak{B})$ is also monotonic;

Projection: If $v \in AG$, then $\psi(v, \mathfrak{B}) = v$

Coalitional Partnership: Let $(v, \mathfrak{B}) \in X^N$ such that B_p is formed by veto players in v. Let $\pi : U \to U$ such that $\pi B_q \cap \pi B_r = \emptyset$, and $|\pi B_p| = 1$. Then

$$\psi_i(v, \mathfrak{B}) = \psi_i\Big(\psi_{\pi B_p}(\pi v, \mathfrak{B}^{\pi}) \cdot u_{B_p}, \mathfrak{B}\Big), \qquad \text{for all } i \in B_p.$$

(A player i is *veto* in game v if $v(S) = 0$ whenever $i \notin S$).

To interpret this axiom we will assume that $B_p \subseteq N$, otherwise v is the zero game. Since coalition B_p is formed by veto players in game v, all its subcoalitions are powerless. In this sense B_p acts as a single player, so we can say that B_p behaves in v as in u_{B_p}, since players in $U \backslash B_p$ are also null players in both games. And this is at the root of the Coalitional Partnership axiom. What π does in this axiom is to focus attention on B_p by formally turning this coalition into one individual. Thus we obtain the semivalue of the "single player" B_p in πv, and then consider the unanimity game $\psi_{\pi B_p}(\pi v, \pi \mathfrak{B}) \cdot u_{B_p}$. This axiom requires the semivalue of any player $i \in B_p$ in this unanimity game to coincide with this semivalue in the former game v.

Notice also that a coalition formed by veto players is a coalition of partners (Kalai and Samet, 1987) and that the Coalitional Partnership axiom has a parallelism with the *Partnership* axiom used by these authors to characterize the weighted Shapley values.

In what follows we provide an explicit formula for coalitional semivalues.

Let $(a^n)_{n\in\mathbb{N}}$ and $(b^n)_{n\in\mathbb{N}}$ be two collections of vectors, with $a^n, b^n \in \mathbb{R}^n$, satisfying (3), (4) and (5) for every $n \in \mathbb{N}$. Define $\phi_{a,b} : X \to AG$ for every finite coalition N, every $(v, \mathfrak{B}) \in X^N$, and every $i \in B_p$ by

$$\left(\phi_{a,b}^N\right)_i (v, \mathfrak{B}) = \sum_{\substack{T \subseteq \mathfrak{B}_N \setminus \{B_p \cap N\} \\ S \subseteq B_p \cap N \setminus \{i\}}} a_t^{|\mathfrak{B}_N|} \cdot b_s^{|B_p \cap N|} \cdot \left[v\Big(A_T \cup S \cup \{i\}\Big) - v\left(A_T \cup S\right) \right], \tag{13}$$

where $A_T = \bigcup_{B_q \in T} B_q \cap N$, and $t = |T|$ and $s = |S|$.

Theorem 7 *A mapping $\psi : X \to AG$ is a coalitional semivalue if and only if there exist two collections of vectors $(a^n)_{n\in\mathbb{N}}$ and $(b^n)_{n\in\mathbb{N}}$, with $a^n, b^n \in \mathbb{R}^n$, satisfying (3), (4) and (5) for every $n \in \mathbb{N}$, such that $\psi = \phi_{a,b}$.*

PROOF. This theorem is a consequence of propositions 8 and 11. □

Proposition 8 *Let $(a^n)_{n\in\mathbb{N}}$ and $(b^n)_{n\in\mathbb{N}}$ be two families of vectors satisfying (3), (4) and (5) for every $n \in \mathbb{N}$. Then mapping $\phi_{a,b}$ is a coalitional semivalue on X.*

PROOF. First let us see that $\phi_{a,b}$ is well defined, that is for every $(v, \mathfrak{B}) \in X^N \cap X^M$ it holds that $\phi_{a,b}^N (v, \mathfrak{B}) = \phi_{a,b}^M (v, \mathfrak{B})$. Clearly, it suffices to prove that: if $k \notin N$, then $\left(\phi_{a,b}^N\right)_i (v, \mathfrak{B}) = \left(\phi_{a,b}^{N\cup\{k\}}\right)_i (v, \mathfrak{B})$ for every $i \in N$. So let $i \in B_p \cap N$ and let us distinguish three cases.

i) If $k \notin B_p$, and $k \in B_r$ with $B_r \cap N \in \mathfrak{B}_N$. By (13)

$$\left(\phi_{a,b}^{N\cup\{k\}}\right)_i (v, \mathfrak{B}) = \sum_{\substack{T\subseteq \mathfrak{B}_N\setminus\{B_p\cap N\} \\ B_r\cap N\notin T \\ S\subseteq B_p\cap N\setminus\{i\}}} a_t^{|\mathfrak{B}_N|} \cdot b_s^{|B_p\cap N|} \cdot \Big[v\Big(A_T \cup S\cup\{i\}\Big) - v\left(A_T\cup S\right)\Big] + \sum_{\substack{T\subseteq \mathfrak{B}_N\setminus\{B_p\cap N\} \\ B_r\cap N\in T \\ S\subseteq B_p\cap N\setminus\{i\}}} a_t^{|\mathfrak{B}_N|} \cdot b_s^{|B_p\cap N|} \cdot \Big[v\Big(A_T \cup\{k\}\cup S\cup\{i\}\Big) - v\left(A_T\cup\{k\}\cup S\right)\Big],$$

where $A_T = \bigcup_{B_q\in T} B_q \cap N$, $t = |T|$ and $s = |S|$.

Since k is a null player in v, it follows that $v\Big(A_T\cup\{k\}\cup S\cup\{i\}\Big) = v\Big(A_T \cup S\cup\{i\}\Big)$, and $v\Big(A_T\cup\{k\}\cup S\Big) = v\left(A_T\cup S\right)$. Hence by (13) the latter sum is equal to $\left(\phi_{a,b}^{N}\right)_i (v, \mathfrak{B})$.

ii) If $k \notin B_p$ and $k \notin \bigcup_{B_q\cap N\in\mathfrak{B}_N} B_q$; applying (13)

$$\left(\phi_{a,b}^{N\cup\{k\}}\right)_i (v, \mathfrak{B}) = \sum_{\substack{T\subseteq \mathfrak{B}_N\setminus\{B_p\cap N\} \\ S\subseteq B_p\cap N\setminus\{i\}}} \Bigg(a_t^{\left|\mathfrak{B}_{N\cup\{k\}}\right|} \cdot b_s^{|B_p\cap N|} \cdot \Big[v\Big(A_T\cup S\cup\{i\}\Big) - v\left(A_T\cup S\right)\Big] + a_{t+1}^{\left|\mathfrak{B}_{N\cup\{k\}}\right|} \cdot b_s^{|B_p\cap N|} \cdot \Big[v\Big(A_T\cup\{k\}\cup S\cup\{i\}\Big) - v\Big(A_T\cup\{k\}\cup S\Big)\Big]\Bigg),$$

where $A_T = \bigcup_{B_q\in T} B_q \cap N$, $t = |T|$ and $s = |S|$.

Since k is a null player in v it follows that

$$\left(\phi_{a,b}^{N\cup\{k\}}\right)_i (v, \mathfrak{B}) = \sum_{\substack{T\subseteq \mathfrak{B}_N\setminus\{B_p\cap N\} \\ S\subseteq B_p\cap N\setminus\{i\}}} \left(a_t^{|\mathfrak{B}_N|+1} \cdot b_s^{|B_p\cap N|} + a_{t+1}^{|\mathfrak{B}_N|+1} \cdot b_s^{|B_p\cap N|}\right) \cdot \Big[v\Big(A_T\cup S\cup\{i\}\Big) - v\left(A_T\cup S\right)\Big].$$

Since a^n satisfies (5), this is equal to $\left(\phi_{a,b}^{N}\right)_i (v, \mathfrak{B})$.

iii) If $k \in B_p$, then applying (13) once more

$$\left(\phi_{a,b}^{N\cup\{k\}}\right)_i (v, \mathfrak{B}) = \sum_{\substack{T\subseteq \mathfrak{B}_N\setminus\{B_p\cap N\}\\ S\subseteq B_p\cap N\setminus\{i\}}} \Bigg(a_t^{|\mathfrak{B}_N|} \cdot b_s^{|B_p\cap N|+1} \cdot \Big[v\Big(A_T \cup S \cup \{i\}\Big) - v\left(A_T \cup S\right) \Big]$$
$$+ a_t^{|\mathfrak{B}_N|} \cdot b_{s+1}^{|B_p\cap N|+1} \cdot \Big[v\Big(A_T \cup \{k\} \cup S \cup \{i\}\Big) - v\left(A_T \cup \{k\} \cup S\right) \Big] \Bigg),$$

where $A_T = \bigcup_{B_q\in T} B_q \cap N$, $t = |T|$ and $s = |S|$. Since k is a null player in v, and b^n satisfies condition (5), it follows that the sum above is equal to $\left(\phi_{a,b}^{N}\right)_i (v, \mathfrak{B})$.

Let us now see that $\phi_{a,b}$ is a coalitional semivalue. It is clear that $\phi_{a,b}$ satisfies Carrier (ii) and Linearity. Monotonicity and the Projection axioms follow respectively since a^n and b^n satisfy (3), (4) for every $n \in \mathbb{N}$. Checking Rearrangement is as in Theorem 5. So it only remains to prove that $\phi_{a,b}$ satisfies Coalitional Partnership.

Let $(v, \mathfrak{B}) \in X^N$ such that B_p is formed by veto players in v. Also let $\pi : U \to U$, such that $\pi\left(B_q\right) \cap \pi\left(B_r\right) = \emptyset$, if $q \neq r$ and $|\pi B_p| = 1$. If B_p is not contained in N, then $v = 0$ and the result follows immediately. So suppose that $B_p \subseteq N$. Applying (13) for each $i \in B_p$ it holds that

$$\left(\phi_{a,b}\right)_i (v, \mathfrak{B}) = \sum_{\substack{T\subseteq \mathfrak{B}_N\setminus\{B_p\cap N\}\\ S\subseteq B_p\cap N\setminus\{i\}}} a_t^{|\mathfrak{B}_N|} \cdot b_s^{|B_p|} \cdot \Big[v\Big(A_T \cup S \cup \{i\}\Big) - v\left(A_T \cup S\right) \Big]$$
$$= \sum_{T\subseteq \mathfrak{B}_N\setminus\{B_p\}} a_t^{|\mathfrak{B}_N|} \cdot b_{|B_p|-1}^{|B_p|} \cdot v\Big(A_T \cup B_p\Big), \quad (14)$$

where $A_T = \bigcup_{B_q \in T} B_q \cap N$, $t = |T|$ and $s = |S|$. Also by (13),

$$\begin{aligned}
&(\phi_{a,b})_{\pi B_p}(\pi v, \mathfrak{B}^\pi) = \\
&\sum_{\substack{T \subseteq \mathfrak{B}^\pi_{\pi N} \setminus \{\pi B_p\} \\ S \subseteq \{\pi B_p\} \cap \pi N \setminus \{\pi B_p\}}} a_t^{\left|\mathfrak{B}^\pi_{\pi N}\right|} \cdot b_s^{|\pi B_p \cap \pi N|} \cdot \Big[\pi v\Big(C_T \cup S \cup \{\pi B_p\}\Big) - \pi v(C_T \cup S)\Big] \\
&= \sum_{T \subseteq \mathfrak{B}^\pi_{\pi N} \setminus \{\pi B_p\}} a_t^{\left|\mathfrak{B}^\pi_{\pi N}\right|} \cdot b_0^1 \cdot \Big[\pi v\Big(C_T \cup \{\pi B_p\}\Big)\Big] = \\
&\sum_{T \subseteq \mathfrak{B}^\pi_{\pi N} \setminus \{\pi B_p\}} a_t^{\left|\mathfrak{B}^\pi_{\pi N}\right|} \cdot \Big[\pi v\Big(C_T \cup \{\pi B_p\}\Big)\Big], \qquad (15)
\end{aligned}$$

where $C_T = \bigcup_{\pi B_q \in T} \pi B_q \cap \pi N$, and the last equality follows since b^n satisfies condition (3), and consequently $b_0^1 = 1$.

Finally, if α is any real number, by (13),

$$\begin{aligned}
(\phi_{a,b})_i \Big(\alpha \cdot u_{B_p}, \mathfrak{B}\Big) &= \sum_{\substack{T \subseteq \mathfrak{B}_N \setminus \{B_p\} \\ S = B_p \setminus \{i\}}} a_t^{|\mathfrak{B}_N|} \cdot b_s^{|B_p|} \cdot \alpha = \\
&b_{|B_p|-1}^{|B_p|} \cdot \alpha \cdot \sum_{t=0}^{|\mathfrak{B}_N|-1} \binom{|\mathfrak{B}_N| - 1}{t} \cdot a_t^{|\mathfrak{B}_N|} = b_{|B_p|-1}^{|B_p|} \cdot \alpha, \qquad (16)
\end{aligned}$$

where the last equality holds since a^n satisfies (3).

Hence, from (14), (15) and (16) it follows that

$$\begin{aligned}
&(\phi_{a,b})_i \Big((\phi_{a,b})_{\pi B_p} (\pi v, \mathfrak{B}^\pi) \cdot u_{B_p}, \mathfrak{B}\Big) = \\
&= b_{|B_p|-1}^{|B_p|} \cdot \sum_{T \subseteq \mathfrak{B}^\pi_{\pi N} \setminus \{\pi B_p\}} a_t^{\left|\mathfrak{B}^\pi_{\pi N}\right|} \cdot \Big[\pi v \left(C_T \cup \pi B_p\right)\Big] = \\
&b_{|B_p|-1}^{|B_p|} \cdot \sum_{T \subseteq \mathfrak{B}_N \setminus \{B_p\}} a_t^{|\mathfrak{B}_N|} \cdot \Big[v \left(A_T \cup B_p\right)\Big] = (\phi_{a,b})_i (v, \mathfrak{B}).
\end{aligned}$$

And the proof is complete. □

The next two lemmas will be used in the proof of the following proposition. Their proofs are located in the appendix. First some definitions.

Let N be a fixed finite coalition. For each pair of positive integers $\widehat{t},\widehat{s}\in\mathbb{N}$ such that $\widehat{t}+\widehat{s}\leq|N|+1$, let $\rho^{N,\widehat{t},\widehat{s}}=\left(\rho_{t,s}^{N,\widehat{t},\widehat{s}}\right)_{\substack{t=0,\ldots,\widehat{t}-1\\ s=0,\ldots,\widehat{s}-1}}$ be a matrix of real numbers. We will require the following conditions for every matrix $\rho^{N,\widehat{t},\widehat{s}}$

$$\sum_{t=0}^{\widehat{t}-1}\sum_{s=0}^{\widehat{s}-1}\binom{\widehat{t}-1}{t}\cdot\binom{\widehat{s}-1}{s}\cdot\rho_{t,s}^{N,\widehat{t},\widehat{s}}=1, \tag{17}$$

and

$$\rho_{t,s}^{N,\widehat{t},\widehat{s}}\geq 0. \tag{18}$$

Let $\rho^N=\left\{\rho^{N,\widehat{t},\widehat{s}}:\widehat{t}+\widehat{s}\leq|N|+1\right\}$ be the collection of such matrices, and let $\phi^{\rho^N}:X^N\rightarrow AG^N$ defined for each $(v,\mathfrak{B})\in X^N$ and every $i\in B_p$ by

$$\phi_i^{\rho^N}(v,\mathfrak{B})=\sum_{\substack{T\subseteq\mathfrak{B}_N\backslash\{B_p\cap N\}\\ S\subseteq B_p\cap N\backslash\{i\}}}\rho_{t,s}^{N,|\mathfrak{B}_N|,|B_p\cap N|}\cdot\left[v\Big(A_T\cup S\cup\{i\}\Big)-v\left(A_T\cup S\right)\right], \tag{19}$$

where $A_T=\bigcup_{B_q\in T}B_q\cap N$, and $t=|T|$ and $s=|S|$.

Lemma 9 *Let $\psi:X\rightarrow AG$ be a mapping that satisfies the Carrier (ii), Linearity, Rearrangement, Monotonicity and Projection axioms. Then for every finite coalition $N\subset U$ there exists a collection of matrices $\rho^N=\left\{\rho^{N,\widehat{t},\widehat{s}}\right\}$, satisfying (17) and (18), such that $\psi^N=\phi^{\rho^N}$.*

Lemma 10 *Let ψ be a coalitional semivalue on X, and $N,M\subset U$ be two finite coalitions. Let ρ^N,ρ^M be the respective collections of matrices according to Lemma 9. If $N\subseteq M$, then $\rho^{N,\widehat{t},\widehat{s}}=\rho^{M,\widehat{t},\widehat{s}}$ for every $\widehat{t},\widehat{s}$ such that $\widehat{t}+\widehat{s}\leq|N|+1$.*

Proposition 11 *If ψ is a coalitional semivalue on X, then there exist two collections $(a^n)_{n\in\mathbb{N}}$, $(b^n)_{n\in\mathbb{N}}$, with $a^n,b^n\in\mathbb{R}^n$, satisfying (3), (4) and (5) for every $n\in\mathbb{N}$, such that $\psi=\phi_{a,b}$.*

PROOF. Let ψ be a coalitional semivalue on X. By Lemma 9, for each finite coalition $N \subset U$ there exists a collection of matrices $\left\{\rho^{N,\widehat{t},\widehat{s}}\right\}$ satisfying (17) and (18), and such that $\psi^N = \phi^{\rho^N}$.

Clearly, Anonymity (Rearrangement) implies $\rho^{N,\widehat{t},\widehat{s}} = \rho^{N',\widehat{t},\widehat{s}}$ for every pair of finite coalitions $N, N' \subset U$ such that $|N| = |N'| = n$. So let us denote $\rho^{n,\widehat{t},\widehat{s}} = \rho^{N,\widehat{t},\widehat{s}}$ and let us prove that there exist two collections $(a^n)_{n\in\mathbb{N}}$ and $(b^n)_{n\in\mathbb{N}}$, satisfying (3), (4) and (5) for every $n \in \mathbb{N}$, and such that $\rho^{n,\widehat{t},\widehat{s}}_{t,s} = a^{\widehat{t}}_t \cdot b^{\widehat{s}}_s$.

For each $n \in \mathbb{N}$, define $a^n_t = \rho^{n,n,1}_{t,0}$, $0 \le t \le n-1$, and $b^n_s = \rho^{n,1,n}_{0,s}$, $0 \le s \le n-1$.

Now let us see that a^n and b^n satisfy conditions (3), (4) and (5).

Since $\rho^{n,n,1}$ and $\rho^{n,1,n}$ satisfy (17) and (18), it immediately follows that a^n and b^n both satisfy conditions (3) and (4).

Now let us see that a^n satisfies (5). Let $N \subset U$ be a finite coalition, and $v \in G^N$. Let $k \notin N$ and consider a coalitional structure $\mathfrak{B}$ such that $k \notin B_p$ and $k \notin \bigcup_{B_q \cap N \in \mathfrak{B}_N} B_q$. Then for every $i \in B_p$ it holds

$$\psi_i^{N\cup\{k\}}(v, \mathfrak{B}) = \sum_{\substack{T \subseteq \mathfrak{B}_N \setminus \{B_p \cap N\} \\ S \subseteq B_p \cap N \setminus \{i\}}} \left(\rho^{n+1,|\mathfrak{B}_N|+1,|B_p\cap N|}_{t,s} + \rho^{n+1,|\mathfrak{B}_N|+1,|B_p\cap N|}_{t+1,s}\right) \cdot \left[v\Big(A_T \cup S \cup \{i\}\Big) - v\left(A_T \cup S\right)\right],$$

where $A_T = \bigcup_{B_q \in T} B_q \cap N$, $t = |T|$ and $s = |S|$. And by Carrier (ii)

$$\psi_i^{N\cup\{k\}}(v, \mathfrak{B}) = \psi_i^N(v, \mathfrak{B}) = \sum_{\substack{T \subseteq \mathfrak{B}_N \setminus \{B_p \cap N\} \\ S \subseteq B_p \cap N \setminus \{i\}}} \rho^{n,|\mathfrak{B}_N|,|B_p\cap N|}_{t,s} \cdot \left[v\Big(A_T \cup S \cup i\Big) - v\left(A_T \cup S\right)\right].$$

Consequently if $0 \le t \le |\mathfrak{B}_N| - 1$ and $0 \le s \le |B_p \cap N| - 1$, then

$$\rho^{n+1,|\mathfrak{B}_N|+1,|B_p\cap N|}_{t,s} + \rho^{n+1,|\mathfrak{B}_N|+1,|B_p\cap N|}_{t+1,s} = \rho^{n,|\mathfrak{B}_N|,|B_p\cap N|}_{t,s}.$$

Taking $\mathfrak{B}$ such that $|\mathfrak{B}_N| = n-1$ and $|B_p \cap N| = 1$, this amounts to $a_t^{n+1} + a_{t+1}^{n+1} = a_t^n$ for every $0 \le t \le n$, and consequently a^n satisfies (5).

Next let us see that b^n satisfies (5). Again let $N \subset U$ be a finite coalition, and $v \in G^N$. Let $k \notin N$ and consider now a coalitional structure $\mathfrak{B}$ such that $k \in B_p$. Then for every $i \in B_p \cap N$ it holds that

$$\psi_i^{N\cup\{k\}}(v,\mathfrak{B}) = \sum_{\substack{T\subseteq \mathfrak{B}_N \setminus \{B_p\cap N\} \\ S\subseteq B_p\cap N\setminus\{i\}}} \Bigg(\rho_{t,s}^{n+1,|\mathfrak{B}_N|,|B_p\cap N|+1} \cdot \Big[v\Big(A_T \cup S \cup \{i\}\Big) - v\left(A_T \cup S\right) \Big] + \rho_{t,s+1}^{n+1,|\mathfrak{B}_N|,|B_p\cap N|+1} \cdot \Big[v\Big(A_T \cup \{k\} \cup S \cup \{i\}\Big) - v\left(A_T \cup \{k\} \cup S\right) \Big] \Bigg),$$

where $A_T = \bigcup_{B_q \in T} B_q \cap N$, $t = |T|$ and $s = |S|$. And by Carrier (ii)

$$\psi_i^{N\cup\{k\}}(v,\mathfrak{B}) = \psi_i^N(v,\mathfrak{B}) = \sum_{\substack{T\subseteq \mathfrak{B}_N \setminus \{B_p\cap N\} \\ S\subseteq B_p\cap N\setminus\{i\}}} \rho_{t,s}^{n,|\mathfrak{B}_N|,|B_p\cap N|} \cdot \Big[v\Big(A_T \cup S \cup \{i\}\Big) - v\left(A_T \cup S\right) \Big].$$

Consequently if $0 \le t \le |\mathfrak{B}_N| - 1$ and every $0 \le s \le |B_p \cap N| - 1$ then

$$\rho_{t,s}^{n+1,|\mathfrak{B}_N|,|B_p\cap N|+1} + \rho_{t,s+1}^{n+1,|\mathfrak{B}_N|,|B_p\cap N|+1} = \rho_{t,s}^{n,|\mathfrak{B}_N|,|B_p\cap N|}.$$

Let $\widehat{t} + \widehat{s} \le n+1$, and take $\mathfrak{B}$ such that $|\mathfrak{B}_N| = \widehat{t}$ and $|B_p \cap N| = \widehat{s}$, then the equality above amounts to

$$\rho_{t,s}^{n+1,\widehat{t},\widehat{s}+1} + \rho_{t,s+1}^{n+1,\widehat{t},\widehat{s}+1} = \rho_{t,s}^{n,\widehat{t},\widehat{s}}, \qquad 0 \le t \le \widehat{t} - 1, \quad 0 \le s \le \widehat{s} - 1. \tag{20}$$

Choosing $\widehat{t} = 1$ and $\widehat{s} = n$, these equalities imply that b^n satisfies condition (5).

To complete the proof it is enough to show that if $\widehat{t} + \widehat{s} \le n+1$ then

$$\rho_{t,s}^{n,\widehat{t},\widehat{s}} = a_t^{\widehat{t}} \cdot b_s^{\widehat{s}}, \qquad 0 \le t \le \widehat{t} - 1, \quad 0 \le s \le \widehat{s} - 1.$$

We will proceed by induction on $\widehat{s}$.

In the case $\widehat{s} = 1$, Lemma 10 and the definition of a^n imply that if $\widehat{t} \leq n$ it holds that

$$\rho_{t,0}^{n,\widehat{t},1} = \rho_{t,0}^{\widehat{t},\widehat{t},1} = a_t^{\widehat{t}} = a_t^{\widehat{t}} \cdot b_0^1, \qquad 0 \leq t \leq \widehat{t} - 1.$$

So let us assume that the statement is true if $\widehat{s'} < \widehat{s}$ and let us prove it for $\widehat{s}$.

To prove that $\rho_{t,s}^{n,\widehat{t},\widehat{s}} = a_t^{\widehat{t}} \cdot b_s^{\widehat{s}}$ for each $s = 1, \ldots, \widehat{s} - 1$ we will proceed by reverse induction on s. We will first show that $\rho_{t,\widehat{s}-1}^{n,\widehat{t},\widehat{s}} = a_t^{\widehat{t}} \cdot b_{\widehat{s}-1}^{\widehat{s}}$ if $0 \leq t \leq \widehat{t} - 1$.

Let us consider a finite coalition $N \subset U$, such that $|N| = n$, and let $\mathfrak{B}$ be any coalitional structure such that $|\mathfrak{B}_N| = \widehat{t}$, and $|B_p \cap N| = \widehat{s}$, and $B_p \subseteq N$. Let $v \in G^N$ be any game for which B_p is formed by veto players. If $i \in B_p$, by Lemma 9 it holds that

$$\psi_i(v, \mathfrak{B}) = \sum_{\substack{T \subseteq \mathfrak{B}_N \setminus (B_p \cap N) \\ S \subseteq B_p \cap N \setminus \{i\}}} \rho_{t,s}^{n,\widehat{t},\widehat{s}} \cdot \Big[v\Big(A_T \cup S \cup \{i\}\Big) - v(A_T \cup S) \Big] =$$

$$\sum_{T \subseteq \mathfrak{B}_N \setminus \{B_p\}} \rho_{t,\widehat{s}-1}^{n,\widehat{t},\widehat{s}} \cdot v(A_T \cup B_p), \quad (21)$$

where $A_T = \bigcup_{B_q \in T} B_q \cap N$, and $t = |T|$ and $s = |S|$.

Let $\pi : U \to U$ be a mapping as in the statement of the Coalitional Partnership axiom; that is, such that $\pi B_q \cap \pi B_r = \emptyset$, if $q \neq r$ and $|\pi B_p| = 1$. Also let $\theta : U \to U$ such that $\Big|\theta\Big(\pi(B_q \cap N)\Big)\Big| = 1$ for all $B_q \in \mathfrak{B}$ and $\theta(\pi B_q) \cap \theta(\pi B_r) = \emptyset$

if $q \neq r$. Applying Rearrangement we have

$$\psi_{\pi B_p}\left(\pi v, \mathfrak{B}^{\pi}\right) = \psi_{\theta(\pi B_p)}\left(\theta\left(\pi v\right), \left(\mathfrak{B}^{\pi}\right)^{\theta}\right) = \sum_{T \subseteq \mathfrak{B}_N \setminus \{B_p\}} \rho_{t,0}^{|\theta(\pi N)|, \left|\left(\mathfrak{B}^{\pi}\right)^{\theta}_{\theta(\pi N)}\right|, 1}$$

$$\cdot \left[\theta\left(\pi v\right)\left(\theta\left(\pi A_T\right) \cup \theta\left(\pi B_p\right)\right) - \theta\left(\pi v\right)\left(\theta\left(\pi A_T\right)\right)\right] =$$

$$\sum_{T \subseteq \mathfrak{B}_N \setminus \{B_p\}} \rho_{t,0}^{\widehat{t}, \widehat{t}, 1} \cdot v\left(A_T \cup B_p\right) = \sum_{T \subseteq \mathfrak{B}_N \setminus \{B_p\}} a_t^{\widehat{t}} \cdot v\left(A_T \cup B_p\right), \quad (22)$$

where $A_T = \bigcup_{B_q \in T} B_q \cap N$.

Furthermore, since B_p is a carrier of u_{B_p}, by (19) for all $\alpha \in \mathbb{R}$ it holds that

$$\psi_i\left(\alpha \cdot u_{B_p}, \mathfrak{B}\right) = \psi_i^{B_p}\left(\alpha \cdot u_{B_p}, \mathfrak{B}\right) =$$

$$\sum_{S \subseteq B_p \setminus \{i\}} \rho_{0,s}^{\widehat{s},1,\widehat{s}} \cdot \left[\alpha \cdot u_{B_p}\left(S \cup \{i\}\right) - \alpha \cdot u_{B_p}\left(S\right)\right] = \rho_{0,\widehat{s}-1}^{\widehat{s},1,\widehat{s}} \cdot \alpha = b_{\widehat{s}-1}^{\widehat{s}} \cdot \alpha. \quad (23)$$

Choosing $\alpha = \sum_{T \subseteq \mathfrak{B}_N \setminus B_p} a_t^{\widehat{t}} \cdot v\left(A_T \cup B_p\right)$, by the Coalitional Partnership axiom, and taking into account (21), (22) and (23), we can conclude that for any $v \in G^N$ for which B_p is formed by veto players, it holds that

$$\sum_{T \subseteq \mathfrak{B}_N \setminus B_p} \rho_{t,\widehat{s}-1}^{n,\widehat{t},\widehat{s}} \cdot v\left(A_T \cup B_p\right) = b_{\widehat{s}-1}^{\widehat{s}} \cdot \sum_{T \subseteq \mathfrak{B}_N \setminus B_p} a_t^{\widehat{t}} \cdot v\left(A_T \cup B_p\right).$$

Therefore, if $\widehat{t} + \widehat{s} \leq n+1$ and $0 \leq t \leq \widehat{t} - 1$, it holds that $\rho_{t,\widehat{s}-1}^{n+1,\widehat{t},\widehat{s}} = a_t^{\widehat{t}} \cdot b_{\widehat{s}-1}^{\widehat{s}}$.

Now assume that if $\widehat{t} + \widehat{s} \leq n+1$ and $0 \leq t \leq \widehat{t} - 1$ it holds that

$$\rho_{t,s'}^{n,\widehat{t},\widehat{s}} = a_t^{\widehat{t}} \cdot b_{s'}^{\widehat{s}}, \qquad \text{for all } s' > s, \quad (24)$$

and let us see that $\rho_{t,s}^{n,\widehat{t},\widehat{s}} = a_t^{\widehat{t}} \cdot b_s^{\widehat{s}}$.

Indeed, if $\widehat{t} + \widehat{s} \leq n+1$, from (20) it follows that

$$\rho_{t,s}^{n,\widehat{t},\widehat{s}} + \rho_{t,s+1}^{n,\widehat{t},\widehat{s}} = \rho_{t,s}^{n-1,\widehat{t},\widehat{s}-1}, \qquad 0 \leq t \leq \widehat{t} - 1,$$

and, applying the induction hypothesis,

$$\rho_{t,s}^{n,\widehat{t},\widehat{s}} + \rho_{t,s+1}^{n,\widehat{t},\widehat{s}} = a_t^{\widehat{t}} \cdot b_s^{\widehat{s}-1} \qquad 0 \le t \le \widehat{t} - 1.$$

This together with (24) implies that

$$\rho_{t,s}^{n,\widehat{t},\widehat{s}} = a_t^{\widehat{t}} \cdot \left(b_s^{\widehat{s}-1} - b_{\widehat{s}+1}^{\widehat{s}} \right) \qquad 0 \le t \le \widehat{t} - 1.$$

Since b^n satisfies (5)

$$\rho_{t,s}^{n+1,\widehat{t},\widehat{s}} = a_t^{\widehat{t}} \cdot b_s^{\widehat{s}} \qquad 0 \le t \le \widehat{t} - 1,$$

and the proof is complete. $\square$

Example 12 *In this example we will show that Coalitional Partnership is independent from the rest of the axioms used in Theorem 7. Indeed, for each* $n \in \mathbb{N}$ *and* $t \in \{0, \dots, n-1\}$ *define*

$$\alpha_t^n = \begin{cases} 1 & \textit{if} \quad n = 1 \\ \frac{1}{2} & \textit{if} \quad n > 1 \quad \textit{and} \quad (t = 0 \ \textit{or} \ t = n-1) \\ 0 & \textit{otherwise}, \end{cases}$$

$$\beta_t^n = \begin{cases} \delta & \textit{if} \quad n > 1 \quad \textit{and} \quad t = n-1 \\ -\delta & \textit{if} \quad n > 1 \quad \textit{and} \quad t = 0 \\ 0 & \textit{otherwise}, \end{cases}$$

where $\frac{-1}{2} < \delta < \frac{1}{2}$*. Let us consider the mapping* $\xi : X \to AG$ *defined for each* $N \subset U$ *finite by* $\xi^N = \phi^{\rho^N}$ *where for each* $\widehat{t}, \widehat{s} \in \mathbb{N}$ *such that* $\widehat{t} + \widehat{s} \le |N| + 1$

$$\rho_{t,s}^{N,\widehat{t},\widehat{s}} = \alpha_t^{\widehat{t}} \cdot \alpha_s^{\widehat{s}} + \beta_t^{\widehat{t}} \cdot \beta_s^{\widehat{s}} \qquad 0 \le t \le \widehat{t} - 1, \quad 0 \le s \le \widehat{s} - 1.$$

One can easily check that ξ *is well defined and satisfies the Carrier (ii), Lin-*

earity, Rearrangement, Monotonicity, and Projection axioms. However it does not satisfy Coalitional Partnership. Otherwise by Theorem 7, there are two families of vectors of real numbers $(a^n),(b^n)$ *such that* $\rho_{t,s}^{N,\widehat{t},\widehat{s}} = a_t^{\widehat{t}} \cdot b_s^{\widehat{s}}$. *As in the proof of Proposition 11 it holds that* $a_t^{\widehat{t}} = \rho_{t,0}^{n,n,1} = \alpha_t^n$ *and* $b_t^{\widehat{t}} = \rho_{0,s}^{n,1,n} = \alpha_t^n$ *when* $\widehat{t} \neq 1$ *and* $\widehat{s} \neq 1$. *But this contradicts the definition of* $\rho_{t,s}^{N,\widehat{t},\widehat{s}}$.

Remark 13 *Dubey et al., (1981) also considered semivalues defined on games with a fixed finite carrier* N *in the following way. A semivalue on* G^N *is a function* $\psi^N : G^N \to AG^N$ *satisfying properties (P1), (P2*^{N}*), (P3) and (P4), where (P2*^{N}*) is (P2) restricted to mappings* π *preserving* N. *We have the following result.*

Theorem 14 *(Dubey et al., 1981) Let* $N \subset U$ *be a finite coalition and* $n = |N|$. *For each vector* p^n *satisfying (3) and (4), the mapping* $\psi^{p^n} : G^N \to AG^N$ *defined by*

$$\left(\psi^{p^n}\right)_i (v) = \sum_{S \subseteq N \setminus \{i\}} p_s^n \cdot \left[v\left(S \cup \{i\}\right) - v\left(S\right) \right],$$

is a semivalue on G^N. *Moreover, every semivalue on* G^N *is of this form, and the mapping* $p^n \to \psi^{p^n}$ *is one-to-one.*

By examining the proof of Theorem 14 above one can realize that it is not necessary to consider that N *is included in the infinite set* U.

For coalitional semivalues we have a similar result.

Let M *be a finite coalition with* $|M| = m$. *A mapping* $\psi : X^M \to AG^M$ *is said to be a coalitional semivalue on* M *if it satisfies Carrier (ii), Linearity, Rearrangement*^{M}*, Monotonicity, Projection, and Coalitional Partnership*^{M} *(where the Rearrangement*^{M} *and Coalitional Partnership*^{M} *axioms stand respectively for Rearrangement and Coalitional Partnership when the games are* *in G^M; the coalitional structures are partitions of M; the mappings π going from M into itself; and $\mathfrak{B}^\pi$ are defined accordingly).*

Let $a = (a^n)_{n=1}^m$ and $b = (b^n)_{n=1}^m$ be two collections of vectors, with $a^n, b^n \in \mathbb{R}^n$, satisfying (3), (4) for every $n = 1, \ldots, m$, and (5) for every $n = 1, \ldots, m-1$ (Notice that by (5) we only need to specify a^m and b^m to completely determine a and b.) Define the mapping $\psi^{a,b} : X^M \to AG^M$ for every $(v, \mathfrak{B}) \in X^M$ and every $i \in B_p$ by

$$\left(\psi^{a,b}\right)_i (v, \mathfrak{B}) = \sum_{\substack{T \subseteq \mathfrak{B} \setminus \{B_p\} \\ S \subseteq B_p \setminus \{i\}}} a_t^{|\mathfrak{B}|} \cdot b_s^{|B_p|} \cdot \left[v\Big(A_T \cup S \cup \{i\}\Big) - v\left(A_T \cup S\right) \right],$$

where $A_T = \bigcup_{B_q \in T} B_q$, and $t = |T|$ and $s = |S|$.

The proof of the following theorem is very similar to that of Theorem 7.

Theorem 15 *For each pair of collections of vectors a and b as before, mapping $\psi^{a,b} : X^M \to AG^M$ is a coalitional semivalue on M. Moreover, every coalitional semivalue on M has this form.*

5 Coalitional semivalues as "compositions" of semivalues

As mentioned in Preliminaries, the coalitional value of a game is obtained by applying the Shapley value twice as follows. First we define a new game by means of the Shapley value, and later we apply the Shapley value to the new game. In this sense we say that the coalitional value is a "composition" of the Shapley value with itself. This section is devoted to proving that coalitional semivalues can be obtained in a similar way. That is, every coalitional semivalue is the "composition" of two semivalues. And on the other hand, the "composition" of two semivalues will yield a coalitional semivalue.

Let $\psi : G \to AG$ be a semivalue, N a finite coalition and $v \in G^N$. Let $\mathfrak{B}$ be a coalitional structure and let us fix $B_p \cap N \in \mathfrak{B}_N$. Denote by $v_p^{\psi,\mathfrak{B}}$ the game in $G^{B_p \cap N}$ defined for each $S \subseteq B_p \cap N$ by

$$v_p^{\psi,\mathfrak{B}}(S) = \psi_S\left(v^{\mathfrak{B}(S)}\right),$$

where $\mathfrak{B}(S)$ and $v^{\mathfrak{B}(S)}$ are defined in (1) and (2) respectively. That is, $v_p^{\psi,\mathfrak{B}}(S)$ is the semivalue of "player" S in game $v^{\mathfrak{B}(S)}$.

Proposition 16 *Let (a^n) and (b^n) be two families of vectors satisfying (3), (4) and (5) for every $n \in \mathbb{N}$. Let ψ^1 and ψ^2 be respectively the semivalues defined by these collections according to Theorem 4. Then for every $(v, \mathfrak{B}) \in X^N$ and every $i \in B_p \cap N$ it holds that*

$$(\phi_{a,b})_i (v, \mathfrak{B}) = \psi_i^2\left(v_p^{\psi^1,\mathfrak{B}}\right).$$

PROOF. Applying (6) twice we obtain

$$\psi_i^2\left(v_p^{\psi^1,\mathfrak{B}}\right) = \sum_{S \subseteq B_p \cap N \setminus \{i\}} b_s^{|B_p \cap N|}\left[\left(v_p^{\psi^1,\mathfrak{B}}\right)\left(S \cup \{i\}\right) - \left(v_p^{\psi^1,\mathfrak{B}}\right)(S)\right] =$$

$$\sum_{S \subseteq B_p \cap N \setminus \{i\}} b_s^{|B_p \cap N|}\left(\sum_{T \subseteq \mathfrak{B}_N \setminus \{B_p \cap N\}} a_t^{|\mathfrak{B}_N|}\left[v\left(A_T \cup S \cup \{i\}\right) - v(A_T)\right] - \right.$$

$$\left.\sum_{T \subseteq \mathfrak{B}_N \setminus \{B_p \cap N\}} a_t^{|\mathfrak{B}_N|}\left[v(A_T \cup S) - v(A_T)\right]\right) =$$

$$= \sum_{\substack{T \subseteq \mathfrak{B}_N \setminus \{B_p \cap N\} \\ S \subseteq B_p \cap N \setminus \{i\}}} a_t^{|\mathfrak{B}_N|} b_s^{|B_p \cap N|}\left[v\left(A_T \cup S \cup \{i\}\right) - v\left(A_T \cup S\right)\right] = (\phi_{a,b})_i (v, \mathfrak{B}),$$

where $A_T = \bigcup_{B_q \in T} (B_q \cap N)$, $t = |T|$ and $s = |S|$. $\square$

The next Corollary states that a coalitional semivalue is the "composition" of two semivalues.

Corollary 17 *A mapping* $\psi : X \to AG$ *is a coalitional semivalue if and only if there exist two semivalues* $\psi^1, \psi^2 : G \to AG$ *such that for every finite coalition* N*,and every* $(v, \mathfrak{B}) \in X^N$ *it holds that*

$$\psi_i (v, \mathfrak{B}) = \psi_i^2 \left(v_p^{\psi^1, \mathfrak{B}} \right) \qquad \text{for all } i \in B_p \cap N.$$

PROOF. This is a direct consequence of Theorem 7 and Proposition 16. □

Remark 18 *As in Remark 13 this corollary can be adapted easily to the case in which the carrier is a fixed finite set.*

Let M *be a finite coalition and* $m = |M|$*. First notice that if* $(v, \mathfrak{B}) \in X^M$*, then games* $v^{\mathfrak{B}}$ *and* $v_p^{\psi^1, \mathfrak{B}}$ *have carriers with cardinality lower than* $|M|$*. Hence we can consider both games as included in* G^M*, and this is how the situation has to be understood in the following theorem, whose proof is omitted.*

Theorem 19 *A mapping* $\psi : X^M \to AG^M$ *is a coalitional semivalue on* M *if and only if there exist two semivalues on* M*,* $\psi^1, \psi^2 : G^M \to AG^M$ *such that for every* $(v, \mathfrak{B}) \in X^N$ *it holds that*

$$\psi_i (v, \mathfrak{B}) = \psi_i^2 \left(v_p^{\psi^1, \mathfrak{B}} \right) \qquad \text{for all } i \in B_p.$$

Now for every finite coalition $N = \{i_1, \ldots, i_n\}$, denote

$$\mathfrak{B}^S(N) = \Big\{ \{i_1\}, \ldots, \{i_n\}, U \backslash N \Big\}, \quad \text{and} \quad \mathfrak{B}^T(N) = \{N, U \backslash N\},$$

that is, in $\mathfrak{B}^S(N)$ we have partitioned N into singletons, and in $\mathfrak{B}^T(N)$ all the members in N are "together" in one coalition.

In the next theorem we identify the semivalues ψ^1, ψ^2 of Corollary 17.

Theorem 20 *If ψ is a coalitional semivalue, then the mappings $\psi^s, \psi^t : G \to AG$ defined for every $v \in G^N$ by*

$$\psi^s(v) = \psi\left(v, \mathfrak{B}^S(N)\right) \quad \text{and} \quad \psi^t(v) = \psi\left(v, \mathfrak{B}^T(N)\right)$$

are semivalues and for every $(v, \mathfrak{B}) \in X^N$ it holds that

$$\psi_i(v, \mathfrak{B}) = \psi_i^t\left(v_p^{\psi^s, \mathfrak{B}}\right) \qquad \text{for all } i \in B_p \cap N.$$

PROOF. Let ψ be a coalitional semivalue and (a^n) and (b^n) be the two families of vectors associated with ψ according to Theorem 7. From Proposition 16 it suffices to show that ψ^s and ψ^t are the semivalues associated respectively with (a^n) and (b^n). Indeed if $v \in G^N$ and $i \in N$

$$\begin{aligned} (\psi^s)_i^N(v) = \psi_i^N\left(v, \mathfrak{B}^S(N)\right) = \\ \sum_{\substack{T \subseteq \mathfrak{B}_N^S \setminus \{\{i\} \cap N\} \\ S = \emptyset}} a_t^{|\mathfrak{B}_N|} \cdot b_0^1 \cdot \left[v\left(A_T \cup S \cup \{i\}\right) - v\left(A_T \cup S\right)\right] \\ = \sum_{T \subseteq \mathfrak{B}_N^S \setminus \{i\}} a_t^n \cdot v\left(A_T \cup \{i\} - v\left(A_T\right)\right). \end{aligned}$$

So ψ^s is the semivalue associated with (a^n). And similarly for ψ^t and (b^n). □

Remark 21 *In the theorem above we have seen that the mapping ψ^s (respectively ψ^t) that assigns its coalitional semivalue to every game $v \in G$ when all the players in the carrier of v are separated (respectively together), is a semivalue; it is in this sense that coalitional semivalues can be considered as generalizations of semivalues.*

According to Corollary 17 coalitional semivalues are "compositions" of two semivalues, but these two semivalues do not necessarily coincide. Next we characterize the subfamily of coalitional semivalues that are the "composition" of a semivalue with itself. We need the following axiom, where $\psi : X \to AG$.

Coalitional Structure Equivalence: $\psi_i\left(v, \mathfrak{B}^S\right) = \psi_i\left(v, \mathfrak{B}^T\right)$.

This axiom requires ψ to yield the same results when all the players in the carrier of v act together as when each of them acts on his own.

Theorem 22 *A mapping $\psi : X \to AG$ satisfies Carrier (ii), Linearity, Rearrangement, Monotonicity, Projection, Coalitional Partnership, and Coalitional Structure Equivalence if and only if there exists a semivalue ξ on G such that for every $(v, \mathfrak{B}) \in X^N$, if $i \in B_p \cap N$ it holds that*

$$\psi_i(v, \mathfrak{B}) = \xi_i\left(v_p^{\xi, \mathfrak{B}}\right).$$

PROOF. This is an immediate consequence of Theorem 7 and Theorem 20. □

Remark 23 *Notice that the mapping defined in Example 12 satisfies Coalitional Structure Equivalence. Therefore, Coalitional Partnership is independent from the other axioms in the previous theorem.*

To finish this section we will consider the characterization of Hart and Kurz (1983) given in Theorem 3. It turns out that if we remove Efficiency from the system proposed by these authors, we do not obtain all the "compositions" of semivalues, but only those in which a semivalue is "composed" with the Shapley value.

Theorem 24 *A mapping $\psi : X \to AG$ satisfies the Carrier (ii), Linearity, Anonymity, Inessential Game, Monotonicity, and Projection axioms if and* *only if there exists a semivalue ξ on G such that for every $(v, \mathfrak{B}) \in X^N$,*

$$\psi_i(v, \mathfrak{B}) = Sh_i\left(v_p^{\xi, \mathfrak{B}}\right) \quad \textit{for all } i \in B_p \cap N. \tag{25}$$

PROOF. First of all it is clear that if ξ is a semivalue, the mapping defined by (25) satisfies these axioms.

To prove the converse consider a mapping $\psi : X \to AG$, that satisfies these axioms. Let $\psi^s : G \to AG$ be the mapping defined by

$$\psi^s(v) = \psi\left(v, \mathfrak{B}^S(N)\right), \quad \text{for every } v \in G^N.$$

Clearly, by Carrier (ii) and (27) the mapping ψ^s is well defined. And since ψ satisfies the Linearity, Anonymity, Monotonicity and Projection axioms, it immediately follows that ψ^s satisfies (P1), (P2), (P3) and (P4). By Theorem 4 the mapping ψ^s is a semivalue on G.

Now let $\mathfrak{B} = \{B_1, \ldots, B_h\}$ be an arbitrary coalitional structure and consider the set $H = \{i_1, \ldots, i_h\}$, where $i_p \in B_p$ for each $p = 1, \ldots, h$. Consider also the family of games $\left\{v^{\mathfrak{B}} : v \in G\right\}$, where $v^{\mathfrak{B}}$ is the game v restricted to the field generated by $\mathfrak{B}$ (i.e., considering $\mathfrak{B}$ as set of players). Obviously we can identify set $\left\{v^{\mathfrak{B}} : v \in G\right\}$ with G^H. So ψ induces a mapping $\psi^{\mathfrak{B}} : G^H \to AG^H$ defined for each $v^{\mathfrak{B}} \in G^H$ and each $i_p \in H$ by

$$\psi_{i_p}^{\mathfrak{B}}\left(v^{\mathfrak{B}}\right) = \psi(v, \mathfrak{B})(B_p),$$

where $\psi(v, \mathfrak{B})(B_p) = \sum_{i \in B_p} \psi_i(v, \mathfrak{B})$.

Mapping $\psi^{\mathfrak{B}}$ is well defined, since $v^{\mathfrak{B}} = w^{\mathfrak{B}}$ implies $(v - w)^{\mathfrak{B}} = 0$ and, applying Linearity and Inessential Game, for every $B_p \in \mathfrak{B}$ it holds that

$$\psi(v, \mathfrak{B})(B_p) = \psi(w, \mathfrak{B})(B_p).$$

On the other hand notice that for every $v^{\mathfrak{B}} \in G^H$ it holds $\left(v^{\mathfrak{B}}\right)^{\mathfrak{B}} = v^{\mathfrak{B}}$. Hence, by Carrier (ii) for every $v^{\mathfrak{B}} \in G^H$ it holds that

$$\psi_{i_p}^{\mathfrak{B}}\left(v^{\mathfrak{B}}\right) = \psi_{i_p}^{\mathfrak{B}}\left(\left(v^{\mathfrak{B}}\right)^{\mathfrak{B}}\right) = \psi\left(v^{\mathfrak{B}}, \mathfrak{B}\right)(B_p) = \psi\left(v^{\mathfrak{B}}, \mathfrak{B}^S(H)\right)(B_p) = \psi_{i_p}\left(v^{\mathfrak{B}}, \mathfrak{B}^S(H)\right) = \psi_{i_p}^{s}\left(v^{\mathfrak{B}}\right). \quad (26)$$

Hence $\psi^{\mathfrak{B}} = \psi^s$ on G^H.

Now let us show equality (25) for $\xi = \psi^s$, and the proof will be completed. Taking into account that for every $v, w \in G$ it holds that $v_p^{\psi^s,\mathfrak{B}} + w_p^{\psi^s,\mathfrak{B}} = (v+w)_p^{\psi^s,\mathfrak{B}}$, and from Linearity, we only need to consider unanimity games. So let u_R be a unanimity game, with $R \subset U$ finite. First we show that

$$\psi_i^N(u_R, \mathfrak{B}) = 0 \quad \text{for all } i \notin N. \quad (27)$$

So let $i \notin N$. By Monotonicity $\psi_i(u_R, \mathfrak{B}) \geq 0$. Now consider the game $w = -u_R + \sum_{j\in R} u_{\{j\}}$. Since w is monotonic it follows that $\psi_i(w, \mathfrak{B}) \geq 0$. By the Projection and Linearity axioms $\psi_i(w, \mathfrak{B}) = \psi_i\left(-u_R + \sum_{j\in R} u_{\{j\}}, \mathfrak{B}\right) = -\psi_i(u_R, \mathfrak{B})$, and therefore, $\psi_i(w, \mathfrak{B}) \leq 0$. Consequently it holds that $\psi_i(w, \mathfrak{B}) = -\psi_i(u_R, \mathfrak{B}) = 0$.

Then we have

$$\sum_{i\in B_p\cap R} \psi_i(u_R, \mathfrak{B}) = \sum_{i\in B_p} \psi_i(u_R, \mathfrak{B}) = \psi_{i_p}^{\mathfrak{B}}\left(u_R^{\mathfrak{B}}\right) = \psi_{i_p}^{s}\left(u_R^{\mathfrak{B}}\right) = \psi_{B_p}^{s}\left(u_R^{\mathfrak{B}(B_p)}\right) = (u_R)_p^{\psi^s,\mathfrak{B}}(B_p) = \sum_{i\in B_p\cap R} Sh_i\left((u_R)_p^{\psi^s,\mathfrak{B}}(B_p)\right),$$

where the first equality follows from (27) and since $u_R \in G^R$; the second by definition of $\psi_{i_p}^{\mathfrak{B}}$; the third by (26); the fourth equality by the definition of $u_R^{\mathfrak{B}(B_p)}$; the fifth equality by the definition of $(u_R)_p^{\psi,\mathfrak{B}}$; and the last one from the fact that the Shapley value is efficient.

Since the Shapley value is symmetric and ψ satisfies Anonymity we obtain the desired result.

6 Appendix

Proof of Lemma 9:

PROOF. Let $\mathfrak{B}$ be a coalitional structure. First notice that if ψ satisfies the Linearity, Monotonicity and Projection axioms, then for every unanimity game u_R, where $R \subseteq N$, it holds that

$$\psi_i^N(u_R, \mathfrak{B}) = 0 \quad \text{for all } i \notin R. \tag{28}$$

The proof of this statement is identical to the proof of (27), so we omit it.

Now let $B_p \in \mathfrak{B}$ such that $B_p \cap N \neq \emptyset$. Consider the vector space formed by the linear mappings from G^N into $AG^{B_p \cap N}$. Notice that for every mapping ψ on X that satisfies Linearity, the composition of the mapping $\psi^N(\cdot, \mathfrak{B})$ and the projection $Pr : AG^N \to AG^{B_p \cap N}$ defined by $Pr(w) = w_{|B_p \cap N}$, belongs to the latter vector space. This composition will be denoted by $\psi^N(\cdot, \mathfrak{B})_{|B_p \cap N}$. Furthermore, the mappings $\psi^N(\cdot, \mathfrak{B})_{|B_p \cap N}$, where ψ satisfies the Carrier (ii), Linearity, Rearrangement, Monotonicity and Projection axioms, generate a subspace, which will be denoted by $F_{B_p}^{\mathfrak{B}}$. Next we prove that $\dim F_{B_p}^{\mathfrak{B}} = |B_p \cap N| \cdot |\mathfrak{B}_N|$.

Since unanimity games $\{u_R : R \subseteq N\}$ form a basis of G^N, every element $f \in F_{B_p}^{\mathfrak{B}}$ is fully determined by its values on these games. In fact, due to (28) and by Anonymity (Rearrangement actually) it is enough to specify $f_i(u_R)$ for a single player $i \in B_p \cap R$ and for every unanimity game u_R, with $R \subseteq N$.

Let us see now that if u_{R_1}, u_{R_2} are two unanimity games with $R_1, R_2 \subseteq N$ such that $|B_p \cap R_1| = |B_p \cap R_2|$ and $|\mathfrak{B}_{R_1}| = |\mathfrak{B}_{R_2}|$, then

$$f_i(u_{R_1}) = f_j(u_{R_2}), \qquad \text{for all } i \in B_p \cap R_1, \text{ and all } j \in B_p \cap R_2. \tag{29}$$

Assume that $f = \sum_{k=1}^{m} \lambda_k \cdot \left(\psi^k\right)^N_{|B_p \cap N}$, where ψ^k satisfies the Carrier (ii), Linearity, Rearrangement, Monotonicity and Projection axioms. Let $H = \{i_q : B_q \cap R_1 \neq \emptyset, q \neq p\}$, where $i_q \in B_q \cap R_1$ for each $q \neq p$, and consider the following coalitional structures:

$$\mathfrak{B}^\ell = \mathfrak{B}_{R_\ell} \cup \{U \backslash R_\ell\}, \quad \ell = 1, 2, \quad \text{and}$$

$$\mathfrak{B}' = \{B_p \cap R_1\} \cup \Big\{ \{i_q\} : i_q \in H \Big\} \cup \Big\{ U \backslash (B_p \cap R_1) \backslash H \Big\}.$$

And for each $\ell i = 1, 2$ let $\pi_\ell : U \to U$ such that $\pi_\ell B_q \cap \pi_\ell B_r = \emptyset$ if $q \neq r$, and

(1) $\pi_\ell (B_p \cap R_\ell) = B_p \cap R_1$, and

(2) $\pi_\ell (B_q \cap R_\ell) = \{i_q\}$ if $q \neq p$.

Then we have for every $i \in B_p \cap R_1$ and every $j \in B_p \cap R_2$

$$\begin{aligned} f_i(u_{R_1}) &= \sum_{k=1}^{m} \lambda_k \left(\psi^k\right)^N_i (u_{R_1}, \mathfrak{B}) = \sum_{k=1}^{m} \lambda_k \left(\psi^k\right)^N_i \left(u_{R_1}, \mathfrak{B}^1\right) = \\ &\sum_{k=1}^{m} \lambda_k \left(\psi^k\right)^N_i (\pi_1 u_{R_1}, \mathfrak{B}') = \sum_{k=1}^{m} \lambda_k \left(\psi^k\right)^N_j (\pi_2 u_{R_2}, \mathfrak{B}') = \\ &\quad \sum_{k=1}^{m} \lambda_k \left(\psi^k\right)^N_j \left(\pi_2 u_{R_2}, \mathfrak{B}^2\right) = \sum_{k=1}^{m} \lambda_k \left(\psi^k\right)^N_j (u_{R_2}, \mathfrak{B}) = f_j(u_{R_2}), \end{aligned}$$

where the 2nd and 6th equalities follow from Carrier (ii); the 3rd and 5th from Rearrangement and the 4th from the fact that $\pi_1 u_{R_1} = \pi_2 u_{R_2}$. Consequently we have proved (29).

So to specify $f_i(u_R)$ for a player $i \in B_p \cap R$, it is enough to know how many players are there in $B_p \cap R$ and how many coalitions in $\mathfrak{B}$ intersect coalition R.

That is, $f_i(u_R)$ only depends on two numbers: $|B_p \cap R|$ and $|\mathfrak{B}_R|$. Since R can be any nonempty coalition in N intersecting B_p, the number $|B_p \cap R|$ ranges from 1 to $|B_p \cap N|$, and the number $|\mathfrak{B}_R|$ ranges from 1 to $|\mathfrak{B}_N|$. Consequently the dimension of $F^{\mathfrak{B}}_{B_p}$ is at most $|B_p \cap N| \cdot |\mathfrak{B}_N|$.

Now for each $\alpha \in \{0, \ldots, |\mathfrak{B}_N| - 1\}$ and $\beta \in \{0, \ldots, |B_p \cap N| - 1\}$, consider the family of matrices $\rho^N(\mathfrak{B}, p, \alpha, \beta) = \left\{\rho^{N,\widehat{t},\widehat{s}}(\mathfrak{B}, p, \alpha, \beta) : \widehat{t} + \widehat{s} \leq |N| + 1\right\}$, where $\rho^{N,\widehat{t},\widehat{s}}(\mathfrak{B}, p, \alpha, \beta) = 0$ if $\widehat{t} \neq |\mathfrak{B}_N|$ or $\widehat{s} \neq |B_p \cap N|$, and

$$\rho^{N,|\mathfrak{B}_N|,|B_p\cap N|}_{t,s}(\mathfrak{B}, p, \alpha, \beta) = \begin{cases} \binom{|\mathfrak{B}_N|-1}{t}^{-1} \cdot \binom{|B_p\cap N|-1}{s}^{-1} & \text{if } (t,s) = (\alpha,\beta) \\ 0 & \text{otherwise.} \end{cases}$$

Let $\phi^{\rho^N(\mathfrak{B},p,\alpha,\beta)}$ be the associated mapping defined by (19). For the sake of simplicity denote this mapping by $\phi^{\mathfrak{B},p}_{\alpha,\beta}$. Clearly the family $\left\{\phi^{\mathfrak{B},p}_{\alpha,\beta}\right\}_{\alpha,\beta}$ is linearly independent in $F^{\mathfrak{B}}_{B_p}$. So dim $F^{\mathfrak{B}}_{B_p} = |\mathfrak{B}_N| \cdot |B_p \cap N|$ and this family is a basis for this subspace.

Let us consider now any mapping ψ that satisfies the Carrier (ii), Linearity, Rearrangement, Monotonicity and Projection axioms. Then there exist real numbers $c^{\mathfrak{B},p}_{\alpha,\beta}$ such that for every $v \in G^N$

$$\psi^N(v, \mathfrak{B})_{|B_p\cap N} = \sum_{\alpha=0}^{|\mathfrak{B}_N|-1} \sum_{\beta=0}^{|B_p\cap N|-1} c^{\mathfrak{B},p}_{\alpha,\beta} \cdot \phi^{\mathfrak{B},p}_{\alpha,\beta}(v, \mathfrak{B})_{|B_p\cap N}. \tag{30}$$

If $i \in B_p \backslash N$, then $i \notin R$ for all u_R such that $R \subseteq N$, and therefore (28) implies that $\psi_i(v, \mathfrak{B}) = 0$ for every $v \in G^N$. As $\left(\phi^{\mathfrak{B},p}_{\alpha,\beta}\right)_i(v, \mathfrak{B}) = 0$, and taking into account (30), for all $v \in G^N$ and all $i \in B_p$ it holds that

$$\psi^N_i(v, \mathfrak{B}) = \sum_{\alpha=0}^{|\mathfrak{B}_N|-1} \sum_{\beta=0}^{|B_p\cap N|-1} c^{\mathfrak{B},p}_{\alpha,\beta} \cdot \left(\phi^{\mathfrak{B},p}_{\alpha,\beta}\right)_i(v, \mathfrak{B}).$$

Now let us show that if $\mathfrak{C}$ is a coalitional structure such that $|\mathfrak{C}_N| = |\mathfrak{B}_N|$ and

$|C_q \cap N| = |B_p \cap N|$, then $c_{\alpha,\beta}^{\mathfrak{C},q} = c_{\alpha,\beta}^{\mathfrak{B},p}$.

Indeed, w. l. o. g. we can assume that $\mathfrak{C}_N = \{C_1 \cap N, \ldots, C_\ell \cap N\}$, and $\mathfrak{B}_N = \{B_1 \cap N, \ldots, B_\ell \cap N\}$. Denote $\mathfrak{C}' = \mathfrak{C}_N \cup \{U \backslash N\}$, and $\mathfrak{B}' = \mathfrak{B}_N \cup \{U \backslash N\}$. Assume that $\min\left\{ |C_r \cap N| \,:\, r \neq q\right\} \geq \min\left\{ |B_r \cap N :| \, r \neq p\right\} = K$, and consider a mapping $\theta_1 : U \to U$ such that $\theta_1 (C_r \cap N) \cap \theta_1 (C_{r'} \cap N) = \emptyset$, whenever $r \neq r'$, and

(1) $\theta_1 N \subseteq N$,

(2) $\theta_1 h = h$, for all $h \in C_q \cap N$,

(3) $\left|\theta_1 (C_r \cap N)\right| = K$, for all $r \in \{1, \ldots, \ell\} \backslash \{q\}$.

Consider also the game $v_{\alpha,\beta}^{\mathfrak{C},q} \in G^N$ defined by

$$v_{\alpha,\beta}^{\mathfrak{C},q}(R) = \begin{cases} 1 & \text{if } |\{C_r \in \mathfrak{C} : r \neq q,\, \emptyset \neq C_r \cap N \subseteq R\}| > \alpha \\ 1 & \text{if } |\{C_r \in \mathfrak{C} : r \neq q,\, \emptyset \neq C_r \cap N \subseteq R\}| = \alpha \\ & \qquad \text{and } |R \cap C_q \cap N| > \beta \\ 0 & \text{otherwise.} \end{cases} \tag{31}$$

If $j \in C_q \cap N$ it holds that

$$c_{\alpha,\beta}^{\mathfrak{C},q} = \psi_j^N \left(v_{\alpha,\beta}^{\mathfrak{C},q}, \mathfrak{C} \right) = \psi_j^N \left(v_{\alpha,\beta}^{\mathfrak{C},q}, \mathfrak{C}' \right) = \psi_j^N \left(\theta_1 v_{\alpha,\beta}^{\mathfrak{C},q}, \mathfrak{C}'^{\theta_1} \right) = \psi_j^N \left(v_{\alpha,\beta}^{\mathfrak{C}'^{\theta_1},q}, \mathfrak{C}'^{\theta_1} \right), \tag{32}$$

where the second equality follows from Carrier (ii), the third from Rearrangement and the last from the fact that $\theta_1 v_{\alpha,\beta}^{\mathfrak{C},q} = v_{\alpha,\beta}^{\mathfrak{C}'^{\theta_1},q}$.

Now let $i \in B_p \cap N$ and let us consider a mapping $\theta_2 : U \to U$ such that

(1) $\theta_2(B_r \cap N) = \theta_1(C_r \cap N)$, $1 \le r \le \ell,\ r \ne p$,

(2) θ_2 is one-to-one from $B_p \cap N$ to $C_q \cap N$, and

(3) $\theta_2(U \backslash N) = U \backslash \bigcup_{r=1}^{\ell} \theta_2(B_r \cap N)$.

Note that $\theta_2 v_{\alpha,\beta}^{\mathfrak{B},p} = \theta_1 v_{\alpha,\beta}^{\mathfrak{C},q}$ and $\mathfrak{C}'^{\theta_2} = \mathfrak{C}'^{\theta_1}$. Therefore, Carrier (ii) and Rearrangement imply

$$c_{\alpha,\beta}^{\mathfrak{B},p} = \psi_i^N\left(v_{\alpha,\beta}^{\mathfrak{B},p}, \mathfrak{B}\right) = \psi_i^N\left(v_{\alpha,\beta}^{\mathfrak{B},p}, \mathfrak{B}'\right) = \psi_{\theta_2 i}^N\left(\theta_2 v_{\alpha,\beta}^{\mathfrak{B},p}, \mathfrak{C}'^{\theta_2}\right) =$$
$$\psi_{\theta_2 i}^N\left(\theta_1 v_{\alpha,\beta}^{\mathfrak{C},q}, \mathfrak{C}'^{\theta_1}\right) = \psi_{\theta_2 i}^N\left(v_{\alpha,\beta}^{\mathfrak{C}'^{\theta_1},q}, \mathfrak{C}'^{\theta_1}\right) = c_{\alpha,\beta}^{\mathfrak{C},q},$$

where the last equality follows from (32).

If we show that $c_{\alpha,\beta}^{\mathfrak{B},p} \ge 0$ for all α, β, and $\sum_{\alpha=0}^{|\mathfrak{B}_N|-1} \sum_{\beta=0}^{|B_p \cap N|-1} c_{\alpha,\beta}^{\mathfrak{B},p} = 1$, the proof will be completed just by taking

$$\rho_{t,s}^{N,\widehat{t},\widehat{s}} = \frac{c_{t,s}^{\mathfrak{B},p}}{\binom{|\mathfrak{B}_N|-1}{t} \cdot \binom{|B_p \cap N|-1}{s}},$$

for a coalitional structure $\mathfrak{B}$ such that $\widehat{t} = |\mathfrak{B}_N|$, and $\widehat{s} = |B_p \cap N|$.

Since $v_{\alpha,\beta}^{\mathfrak{B},p} \in G^N$ is monotonic, it follows that $0 \le \psi_i^N\left(v_{\alpha,\beta}^{\mathfrak{B},p}, \mathfrak{B}\right) = c_{\alpha,\beta}^{\mathfrak{B},p}$.

Now let us fix some $i \in B_p \cap N$. On one hand, by the Projection axiom, $\psi_i^N\left(u_{\{i\}}, \mathfrak{B}\right) = 1$. On the other hand, $\psi_i^N\left(u_{\{i\}}, \mathfrak{B}\right) = \sum_{t=0}^{|\mathfrak{B}_N|-1} \sum_{s=0}^{|B_p \cap N|-1} c_{t,s}^{\mathfrak{B},p}$. Consequently $\sum_{t=0}^{|\mathfrak{B}_N|-1} \sum_{s=0}^{|B_p \cap N|-1} c_{t,s}^{\mathfrak{B},p} = 1$. □

Proof of Lemma 10:

PROOF. Clearly, it suffices to prove that if $k \in U \backslash N$, then $\rho^{N,\widehat{t},\widehat{s}} = \rho^{N \cup \{k\},\widehat{t},\widehat{s}}$, for $\widehat{t}, \widehat{s}$ such that $\widehat{t} + \widehat{s} \le |N| + 1$.

First notice that from condition (17), it follows that $\rho_{0,0}^{N,1,1}$=1 for every finite coalition $N \subset U$. So we can assume $|N| \geq 2$.

Consider a coalitional structure $\mathfrak{B}$ such that $|\mathfrak{B}_N| = \widehat{t}$, and $|B_p \cap N| = \widehat{s}$, and $k \in B_r$ for some $r \neq p$ such that $B_r \cap N \neq \emptyset$. By Lemma 9, applying (19), if $v \in G^N$ and $i \in B_p$ it holds that

$$\psi_i^N(v, \mathfrak{B}) = \sum_{\substack{T \subseteq \mathfrak{B}_N \backslash \{B_p \cap N\} \\ S \subseteq B_p \cap N \backslash \{i\}}} \rho_{t,s}^{N,\widehat{t},\widehat{s}} \Big[v\Big(A_T \cup S \cup \{i\}\Big) - v\left(A_T \cup S\right) \Big], \tag{33}$$

where $A_T = \bigcup_{B_q \in T} B_q \cap N$, and $t = |T|$, and $s = |S|$.

On the other hand, since k is a null player in v it holds that

$$\psi_i^{N \cup \{k\}}(v, \mathfrak{B}) = \sum_{\substack{T \subseteq \mathfrak{B}_N \backslash \{B_p \cap N\} \\ S \subseteq B_p \cap N \backslash \{i\}}} \rho_{t,s}^{N \cup \{k\},\widehat{t},\widehat{s}} \cdot \Big[v\Big(A_T \cup S \cup \{i\}\Big) - v\left(A_T \cup S\right) \Big], \tag{34}$$

where $A_T = \bigcup_{B_q \in T} B_q \cap N$.

Since $\psi_i^N(v, \mathfrak{B}) = \psi_i^{N \cup \{k\}}(v, \mathfrak{B})$ for every $v \in G^N$, from equalities (33) and (34), it follows that $\rho_{t,s}^{N,\widehat{t},\widehat{s}} = \rho_{t,s}^{N \cup \{k\},\widehat{t},\widehat{s}}$, and the proof is complete. □

7 References

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