Dynamic Relational Contracts under Limited Liability

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Abstract

This paper considers a long-term relationship between two agents who undertake costly actions or investments which produce a joint benefit. Agents have an opportunity to expropriate some of the joint benefit for their own use. The question asked is how to structure the investments and division of the surplus over time so as to avoid expropriation. It is shown that investments may be either above or below the efficient level and that actions and the division of the surplus converges to a stationary solution at which either both investment levels are efficient or both are below the efficient level. [100 words]

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1. Introduction

It is often difficult to enforce contracts. This may be because the terms of the contracts are difficult to specify precisely or because they are difficult to specify in a way verifiable to a court. It may be that there is no legal authority to enforce a contact. When relationships are repeated it is possible to include an element of self-enforcement in the contract by designing terms so that any short-term incentive to renege is offset by a long-term advantage to adhering to the contract.

We consider such a self-enforcing or relational contract in the case where two risk-neutral agents make repeated relation-specific investments or actions a_i that produce a stochastic joint output $y(a_1, a_2, s)$ to be shared at each date. We shall further assume that both agents have limited liability so that the consumption of the agents is bounded below. Contracts cannot be enforced and in the event of disagreement agent i receives a gross breakdown payoff of $\phi_i(a_1, a_2, s)$.[1] We shall assume that theses investments or actions are complementary. In this case if agent 1's investment is increased more must be offered to agent 2 to prevent him reneging. Thus although the joint surplus may be increased, agent 1 may have no incentive to increase her investment or action as there may be no division of the surplus which simultaneously prevents agent 2 from reneging

[1] The breakdown payoffs are assumed to be feasible, $\phi_1(a_1, a_2, s) + \phi_2(a_1, a_2, s) \leq y(a_1, a_2, s)$. The exact details of the model will be specified in Section 2.

and compensates agent 1 for the increased investment. Thus agents will face a *hold-up* problem and investment may be inefficiently low.

The question we therefore address is how the investments or actions of the agents and the division of the joint output can be structured over time to avoid agents' incentives to deviate and ameliorate the efficiency loss caused by the hold-up problem. That is we shall be interested in finding and characterizing the set of Pareto-efficient self-enforcing contracts.

We establish three main results. Firstly we derive a backloading result (Theorem 3). In general it is well known that an important property of optimum self-enforcing contracts is the *backloading principle* (for a general argument see Ray 2002). If we consider a case where agents are risk-neutral and one agent is able to commit to the contract, then the backloading principle says that transfers to the agent who cannot commit should be backloaded into the future. The intuition is the following. Suppose that of the two agents, agent 1 can commit to the contract but agent 2 cannot. Further suppose that agent 2 is getting a relatively low discounted utility from the contract. This may impose an efficiency cost on the contract as the investment of agent 1 needs to be kept low to limit the gains to agent 2 from expropriation. Since both agents are risk-neutral they are concerned only with the discounted value of utility (transfers net of action costs) and not the actual timing of utility received. Thus the best way to discourage agent 2 from reneging is to backload transfers to agent 2 whilst keeping the discounted sum of transfers unchanged. This provides a carrot in the future which would be forgone if agent 2 reneged. Such a change doesn't worsen current incentives but improves future incentives by increasing agent 2's continuation utilities and hence allowing future increases in agent 1's investment. In our context where neither agent can commit, the operation of this principle is less clear. We show however, that this principle remains partially valid and that we have backloading of consumption for the agents whose self-enforcing constraints are most difficult to meet.

Furthermore Theorem 3 shows that as the backloading principle applies to utilities and not simply consumption it might be optimal to increase investment beyond the efficient level. This allows more output to allocated to the other agent and thus more backloading. Of course there is an efficiency loss in overinvesting so it will always be desirable to backload transfers as much as possible before backloading utility by altering actions. The result however, has the implication that the optimum self-enforcing contract will typically involve overinvestment in the initial periods by one of the agents despite the hold-up suggesting that there will be underinvestment. Nevertheless we shall show that it will never be the case that both agents overinvest in any equilibrium. Equally we are able to show that in the case where only one agent takes an action (as in much of the existing literature) there is never overinvestment.

Secondly, we establish that the contract converges to a stationary phase in finite time with probability one (Theorem 5). We show that this stationary phase corresponds to the self-enforcing contract which maximizes current net surplus. Although it is to be expected that should we ever reach a stationary state the contract will maximize the current joint surplus amongst all feasible self-enforcing contracts, convergence itself is more surprising. In particular we show that convergence holds even when the default payoffs and production technologies fluctuate through time or when the action choices are made sequentially at alternative periods rather than simultaneously. Furthermore we show that in the stationary phase for a given state either both agents are investing efficiently or both are underinvesting. Likewise, unless the first best is attainable, in this stationary state we show that both agents are simultaneously constrained in the sense that they are both at the point of reneging on the contract and taking their default payoffs.

Thirdly we show that if the optimum contract is non-trivial with positive investment at each date then it will exhibit a two-phase property (Theorem 6). In the first phase there is backloading with zero consumption and overinvestment by one of the agents. This first phase may not exit although we shall present an example where it does. In the second phase (which occurs with probability one) there will be no overinvestment. In this second phase, there can be a first one-period transition in which one of the agents is investing efficiently and thereafter either both actions are efficient or both actions are inefficient and both agents are indifferent to reneging on the contract. The subsequent part of this phase is stationary and joint utility maximizing.

The model we present here covers or is related to many models of repeated bilateral relationships in the literature. The models of Thomas and Worrall (1994) on foreign direct investment, Sigouin (2003) on international financial flows and Albuquerque and Hopenhayn (2004) on credit constrained firm growth might all be considered as special cases where only one party to the contract undertakes an action. Although slightly more general in allowing for multidimensional actions, the model of Ray (2002) also has investment by only one agent.[2] The model we present is significantly more general as our structure allows for actions to be taken alternately by each agent and allows for limited commitment by both sides.

Although this paper significantly extends existing results it does so by adopting a different approach. The literature just cited uses a dynamic programming approach to characterize optimum self-enforcing contracts. In our context the dynamic programming approach has the disadvantage that the resulting problem may be non-convex and, even when it is convex, it is known (see e.g. Thomas and Worrall 1994) that the value function may not be differentiable. Thus the use of first-order conditions is typically problemat-

[2]In Ray (2002) the agent not taking the action, the principal, is able to commit to the contract.

ical.[3] We avoid these issues by deriving results from variational methods and our main results will not require that set of constrained efficient contracts be convex.[4] This allows us to derive our results in more generality than in some of the existing literature.

Our model is also related to two other relevant literatures. First there is a literature on risk-sharing and two-sided limited commitment and no actions (see e.g. Kocherlakota 1996, Ligon et al. 2002, Thomas and Worrall 1988). This literature shows that the optimum risk-sharing contract exhibits two important properties. First transfers depend both on the current income shocks and the past history of income shocks. Secondly the contract evolves toward a stationary but typically non-degenerate distribution of future expected utilities. However, the distribution of utilities, or the distribution of the implied Pareto-weights, does not converge.[5] This is in contrast to the results of the current paper. Moreover, in the current case if efficiency cannot be sustained in the stationary phase then both agents are constrained and this feature again makes the model qualitatively distinct from the risk-sharing models with no actions where in any non-degenerate contract only (at most) one agent's constraint will bind at a time.

A second related literature which takes a slightly different approach is that of Levin (2003) and others which builds upon the work of Macleod and Malcomson (1989). In that work output accrues to individual agents with subsequent non-contractual transfers being made. This is captured in our model by interpreting $\phi_i(a_1, a_2, s)$ as the individual outputs and assuming that breakdown payoffs exhaust output. The model of Levin (2003) has recently been generalized by Rayo (2007) who considers the multiple agent case and by Doornik (2006) who allows for two-sided moral hazard. There are two key differences between these works and our paper. Firstly, these recent papers assume that effort is unobserved so that there is an asymmetry of information whereas we assume observability (but non-verifiability) of actions. Secondly, they do not assume that agents have limited liability. Hence stationary contracts are optimum (at least after an initial period). This is in contrast to the current paper where we show how actions and transfers are structured along the path to a stationary state.

Perhaps closest in terms of the model of our paper is the work of Garvey (1995) and Halonen (2002). However, they consider the minimum discount factor that will allow the

[3]If both agents are investing, the value function will be differentiable but need not be concave. If only one agent is investing the value function will not be differentiable in general even if it is concave. These points of non-differentiability can also be an important part of the solution so that even with concavity a sub-differential analysis must be used. This is in contrast to the dynamic moral hazard problem analyzed by Pavoni (2004) who is able use a first-order approach despite points of non-differentiability by showing that such points are almost never reached at the optimum.

[4]Although it would be possible to convexify the problem by allowing for random contracts, we prefer to concentrate on pure strategy equilibria, partly because our results show that even in this case strong convergence results can be established.

[5]The two-sided lack of commitment is crucial to this result. If there is only one-sided lack of commitment the distribution of utilities will also converge to a degenerate distribution.

efficient investments to be sustained under different assumptions about the breakdown payoffs where the breakdown payoffs themselves may depend on whether there is joint or single ownership of production. Thus they do not consider the inefficiency in investments or the temporal structure of investment which of central importance here. Moreover these models do not explicitly allow for uncertainty.

The paper proceeds as follows. The next section describes the model and optimum self-enforcing contracts. Section 3 provides the main results of the paper. Section 4 considers the important special case where only one agent contributes to production. Section 5 concludes.

2. Model

We consider a dynamic model of joint production where agents repeatedly undertake some action or investment that generates a joint output. Once produced agents have the opportunity to unilaterally expropriate some of the joint output for their own benefit. Agents have limited liability in the sense that their consumption cannot be reduced below some lower bound. In this section we shall describe the economic environment, the joint production and action sets, the part of joint output that can be expropriated and the set of self-enforcing contracts. In addition we shall define a game played by the two agents and identify self-enforcing contracts with the subgame perfect equilibria of that game. Our interest will be in optimal self-enforcing contracts or equivalently the Pareto-efficient subgame perfect equilibria.

2.1. Economic environment

Time is discrete and indexed by $t = 0, 1, 2, \ldots, \infty$. The environment is uncertain and at the start of each date a state of nature s is realized from a finite state space $\mathcal{S} = \{s_1, s_2, \ldots, s_n\}$. The state evolves according to a time homogeneous Markov process with transition matrix $[\pi_{sr}]$ where $\sum_{r \in \mathcal{S}} \pi_{sr} = 1$, and we assume some initial state s_0 has probability one. We shall assume that the Markov chain is irreducible so that every state communicates with every other. It is important to emphasize that this is a very general structure and encompasses the case with no uncertainty where $n = 1$ and the possibility that some $\pi_{rs} = 0$.[6] We shall denote the state at date t by s_t and the history of states will be denoted $s^t = \{s_0, s_1, s_2, \ldots, s_t\}$.[7]

There are two agents and at each date both agents choose an action or investment a_i from $\Re_+$, $i = 1, 2$ (we shall use the notation $\Re_+$ to denote the non-negative orthants

[6]A number of these assumptions are inessential and made for convenience and simplicity. The important property is that the stochastic process is Markovian. Finiteness of the state space is also not essential and most of the results would go through if the state space were continuous. Equally we could we specify a distribution over the set of initial states rather than assuming there is some initial state s_0. For many results it is possible to assume that there is some finite time horizon T although we shall be interested in convergence properties of optimum contract and these results will require an infinite time horizon.

[7]Where we write s^t we shall assume this is a positive probability event unless otherwise stated.

of the real numbers and $\Re_{++}$ to denote its interior). States are perfectly observed and actions are taken simultaneously and after the state is realized. Actions lead to an output $y(a_1, a_2, s_t)$ that may depend upon the current state. Output and actions are observed by both agents. We make the following assumption about the production function.

ASSUMPTION 1: The function $y(\cdot, \cdot, s_t) \colon \Re^2_+ \to \Re_+$ satisfies the following conditions:(i) *Output at zero*: $y(0, 0; \cdot) = 0$; (ii) *Continuity*: it is continuous in a on $\Re^2_+$; (iii) *Differentiability*: it is twice continuously differentiable in a on $\Re^2_{++}$; (iv) *Monotonicity*: it is *either* constant in a_i on $\Re_+$ *or* strictly increasing in a_i on $\Re_+$ for each $i = 1, 2$ and is strictly increasing for at least one $i = 1, 2$ and some $s_t \in \mathcal{S}$; (v) *Concavity*: it is strictly concave in a on $\Re^2_{++}$ with $\partial^2 y / \partial a_i^2 < 0$ whenever $\partial y / \partial a_i > 0$; (vi) *Boundedness*: $\lim_{\alpha \to \infty} y(\alpha a_1, \alpha a_2, s_t)/\alpha < a_1 + a_2$ for all $a \in \Re^2_+$; (vii) *Complementarity*: $\partial^2 y / \partial a_1 \partial a_2 \geq 0$ for all $a \in \Re^2_+$.

With the exception of complementarity these are standard assumptions on the production function. The assumption of strategic complements is made as we want to analyze situations where there is mutual benefit from cooperation so that increasing the action of one agent increases the marginal benefit of the other's action. It is important to realize that our assumptions will enable us to consider the case where only one action matters for production. In this case although we allow both agents to choose an action this will be the same in an efficient equilibrium as imposing the restriction that the action is zero. We impose Assumption 1(vi) so that in conjunction with the other conditions the set of action choices that yield non-negative surplus, $\tilde{A}(s_t) = \{(a_1, a_2) \in R^2_+ \mid y(a_1, a_2, s_t) \geq a_1 + a_2\}$ is compact. Since all assumptions apply state-wise we shall often be able to drop the notational dependence on the state where this is convenient.

REMARK: The production technology and stochastic structure is extremely general. Thus we may have some states where only agent 1 takes an action and in other states only agent 2 takes an action. Although in the model actions are chosen simultaneously by agents each period, this allows us to cover the case say, where one agent takes an action in even periods and the other agent takes an action in odd periods. To examine that situation we could use a two state transition matrix

$$\mathbf{\Pi} = \begin{pmatrix} 0, 1 \\ 1, 0 \end{pmatrix}$$

where there are two states but they alternate between even and odd periods. Similarly we may have a situation where in some states neither agent takes an action and output is zero. All these cases can be handled by the above specification.

Since actions are chosen simultaneously and independently we define the *conditionally efficient actions* $a_i^*(a_j, s_t)$ such that

$$a_i^*(a_j, s_t) \in \operatorname*{arg\,max}_{a_i \in \Re_+} y(a_1, a_2, s_t) - a_i.$$

We then have the following standard result.[8]

Lemma 1: *Given Assumption 1 the conditionally efficient actions are continuous (single-valued) non-decreasing functions of the other agent's action. Moreover, these functions cross at most once on* $\Re_{++}$.

We define an *efficient action pair* $a^*(s_t) = (a_1^*(s_t), a_2^*(s_t))$ to be

$$a^*(s_t) \in \operatorname*{arg\,max}_{a \in \Re_+^2} y(a_1, a_2, s_t) - a_1 - a_2.$$

The efficient action pair will correspond to a point where $a_i^*(s_t) = a_i^*(a_j^*(s_t), s_t)$ for $i \neq j = 1, 2$. As we have not imposed a profitability condition that there exists a vector a such that $y(a_1, a_2, s_t) > a_1 + a_2$ it may be that $a^*(s_t) = (0, 0)$ is the efficient action pair in some state or indeed in all states.[9] The concavity conditions are however sufficient to rule out the possibility that there are multiple non-zero efficient action pairs.

The joint output will be divided up between the two agents in a way which is described in the next sub-section. For the moment we shall simply think of a consumption and action plan for each agent that depends on the history s^t. Denote the consumption of agent i in history s^t by $c_i(s^t)$ and the action by $a_i(s^t)$. Critically, we shall assume that agents have limited liability so that consumption must be non-negative. There is no storage, and therefore the feasible set of consumptions at time t in state s is

$$\begin{aligned} C(a, s_t) = \{(c_1(s^t), c_2(s^t))\colon\, & c_i(s^t) \geq 0 \text{ for } i = 1, 2 \\ & \text{and } \sum_{i=1}^{2} c_i(s^t) \leq y(a_1(s^t), a_2(s^t), s_t)\}. \end{aligned}$$

Agents have preferences over consumption and action streams. We assume that agents have time separable utility functions, are risk neutral and that action costs are linear:[10]

[8] All proofs are given in the Appendix.

[9] Our results will apply (trivially) in this case.

[10] As is fairly standard this linearity assumption is made for convenience and the analysis will carry through if actions costs are convex. Thus suppose $w_i = c_i - g_i(a_i)$ where g_i is strictly increasing and convex and $g(0) = 0$. Letting h_i denote the inverse of g_i we have $a_i = h_i(g_i)$ where h_i is strictly increasing and concave. Hence agents can be viewed as choosing g_i and the reduced-form production function is $f(g_1, g_2, s_t) = y(h_1(g_1), h_2(g_2), s_t)$ which will satisfy Assumption 1 with g_i replacing a_i and f replacing y. In this case the net surplus is $f(g_1, g_2, s_t) - g_1 - g_2$.

agent i's utility at t is given by

$$w_i(s^t) = c_i(s^t) - a_i(s^t).$$

We assume that both agents discount the future by a common factor $\delta \in (0,1)$ and that agents are interested in maximizing expected discounted utility

$$\mathrm{E}\left[\sum_{t=0}^{\infty} \delta^t w_i(s^t) \mid s_0\right].$$

For a given pair of actions the *current net surplus* generated at history s^t is

$$y(a_1(s^t), a_2(s^t), s_t) - (a_1(s^t) + a_2(s^t))$$

and we define the *feasible* set of per-period utility payoffs as

$$\begin{aligned} W(a, s_t) = \{ & (w_1(s^t), w_2(s^t)) \colon w_i(s^t) \geq -a_i(s^t) \ \text{ for } \ i = 1, 2 \\ & \text{and } \ \sum_{i=1}^{2} w_i(s^t) \leq y(a_1(s^t), a_2(s^t), s_t) - (a_1(s^t) + a_2(s^t))\}. \end{aligned} \tag{1}$$

2.2. *The breakdown game*

In this section we specify how agents agree on the division of the surplus and what happens in the event of disagreement. In each period agents must decide how to act and how to divide up the subsequent surplus. We shall suppose that each period is split into two stages with actions being determined at the first stage and the division of the surplus taking place at the second stage after output is known (recall that uncertainty is resolved before the action decision).[11]

The game played by the agents will involve them choosing actions each period contingent upon the past play of the game. We have somewhat more latitude in specifying the game at the division stage and we suppose that at the division stage the agents play a Nash demand game.[12] In this Nash demand game both agents simultaneously announce utility claims (w_1, w_2). If these claims are feasible, *viz*, $(w_1, w_2) \in W(a, s_t)$, then this determines the split of the surplus. If they are not feasible, then agents receive a *breakdown*

[11]While it is useful to think of the actions being taken, and observed before the agents decide on their demands, it is equivalent in terms of the subgame-perfect equilibria to a model in which actions and demands are determined simultaneously.

[12]What we want to capture is that there is an agreement on how output should be split, and failure to abide by it will lead to breakdown. The Nash demand game is a simple way of operationalising this idea.

payoff[13] given by

$$\phi_i(a_1, a_2, s_t) - a_i$$

for agent i in state s_t as a function of the actions taken.[14] These breakdown payoffs show how the payoff to agent i in breakdown depends on his own and the other agent's contribution. They may reflect the property rights of the two agents over output, for example, specifying a fixed percentage split. Analogous to Assumption 1 we shall assume that the breakdown payoffs satisfy:

ASSUMPTION 2: The function $\phi_i(\cdot, \cdot, s_t) \colon \Re^2_+ \to \Re_+$ is non-decreasing, twice continuously differentiable and strictly concave on $\Re^2_{++}$. Where both actions play a role we assume that $\partial^2 \phi_i(a_1, a_2)/\partial a_1 \partial a_2 \geq 0$. In addition, the ϕ_i are feasible, i.e. $(\phi_1(a, s_t), \phi_2(a, s_t)) \in C(a, s_t)$, $\partial y(a_1, a_2, s_t)/\partial a_i > 0$ implies $\partial \phi_j(a_1, a_2, s_t)/\partial a_i > 0$, $j \neq i$ and

$$\frac{\partial \phi_1(a_1, a_2, s_t)}{\partial a_i} + \frac{\partial \phi_2(a_1, a_2, s_t)}{\partial a_i} \leq \frac{\partial y(a_1, a_2, s_t)}{\partial a_i} \qquad \forall \; s_t \text{ and } i = 1, 2. \tag{2}$$

It should be noted that equation (2) shows that the increase in the total breakdown payoff cannot exceed the marginal product and given the assumption of $y(0, 0, \cdot) = 0$ implies that breakdown payoffs are in fact feasible, $(\phi_1(a, s_t), \phi_2(a, s_t)) \in C(a, s_t)$ and $\phi(0, 0, \cdot) = 0$ for $i = 1, 2$.

REMARK: We refer to the assumption that $\partial y(a_1, a_2, s_t)/\partial a_i > 0$ implies $\partial \phi_j(a_1, a_2, s_t)/\partial a_i > 0$ as our *hold-up* assumption. It is made to avoid the case where i's contribution to output does not increase j's claim on output. In such a case hold-up and underinvestment by j cannot occur in any efficient equilibrium, and ruling it out allows us to simplify the arguments below.

Assumption 2 places relatively few restrictions on the breakdown payoffs. They are restricted to be feasible as shown by equation (2) but we do not require that these payoffs exhaust available output. For example, disagreement may incur a cost, such as lawyers' fees, which produces some loss for one or both agents. There are many special cases which satisfy Assumption 2. Here we mention just two. First, we may have that each agent can extract a simple percentage (perhaps depending upon the state) of output in the breakdown. In this case $\phi_i(a_1, a_2, s_t) = \theta_i(s^t) y(a_1, a_2, s_t)$ and Assumption 2 is satisfied provided $\theta_i(s_t) > 0$ and $\sum_{i=1}^{2} \theta_i(s_t) \leq 1$. Secondly, suppose that out-

[13] The idea of going immediately to breakdown if the surplus is not split appropriately (rather than, say, renegotiation) is in the spirit of repeated game analysis in which deviations from agreed courses of action are punished with severe continuations. In general, of course, renegotiation proofness will not be satisfied here.

[14] An alternative formulation would be to assume that a deviation at the action stage can be punished independently of going to the breakdown position after output is realized. The idea would be that a deviation is observed by the other agent who may be able to take measures that affect output or breakdown payoffs. Such a formulation, by making a deviation at the action stage more easily punishable (leaving aside issues of renegotiation proofness) may shift the emphasis towards the distribution stage of the game.

put is an additive function of inputs such that $y(a_1, a_2, s_t) = f_1(a_1, s_t) + f_2(a_2, s_t)$ and $\phi_i(a_1, a_2, s_t) = \theta_{i1}(s_t) f_1(a_1, s_t) + \theta_{i2}(s_t) f_2(a_2, s_t)$. In this case agents make separate contributions to joint output and can capture some of their own and some of the other agent's contribution in the breakdown. Then Assumption 2 is satisfied provided $\theta_{ij}(s_t) \geq 0$ with this being strict for $i \neq j$, and $\sum_{i=1}^{2} \theta_{ij}(s_t) \leq 1$, $j = 1, 2$. This latter additive structure includes the case which has been predominantly studied in the literature where only the action of one agent is productive and the other can extract the entire output in the breakdown; for example, $f_2 = 0$ and $\theta_{21} = 1$ and this case is considered in Section 4.

One might imagine some situations where the breakdown payoffs do not satisfy equation (2). For example both may be grab the entire output provided they are first to do so. This is not the situation we consider here.

REMARK: In this paper we treat the breakdown payoffs as exogenously given. However, a number of papers in the relational contracting tradition assume that the breakdown payoffs are a consequence of the legal framework or ownership decision and study the effect of different default structures. For example Halonen (2002) considers a model where the breakdown payoffs allow either one agent to expropriate the entire output if there is single ownership or both agents to expropriate half of the output if there is joint ownership. Solving for efficient contracts, as we do here, is a necessary prior step. Extending our analysis to endogenize the breakdown payoffs is an avenue for potential future research.

An important part of the analysis will be related to the best response in the breakdown game. Denote the best-reply functions (functions because of the strict concavity assumptions)

$$a_i^N(a_j, s_t) = \underset{\tilde{a} \in \Re_+}{\arg\max} \{\phi_i(\tilde{a}, a_j, s_t) - \tilde{a}\}.$$

LEMMA 2: *Given Assumption 2, $a_i^N(a_j, s_t)$ is weakly increasing in a_j. Moreover we have $a_i^N(a_j, s_t) \leq a_i^*(a_j, s_t)$ for each a_j and every state s_t with strict inequality whenever $a_i^*(a_j, s_t) > 0$.*

The optimized breakdown payoffs (which result when the defaulting agent chooses the best-response action in the breakdown game) can be written as follows:

$$\phi_1(a_1^N(a_2, s_t), a_2, s_t) - a_1^N(a_2, s_t) \quad \text{and} \quad \phi_2(a_1, a_2^N(a_1, s_t), s_t) - a_2^N(a_1, s_t).$$

A Nash equilibrium of the breakdown game occurs where the best-response functions intersect. We denote a Nash equilibrium as a pair $(a_1^{NE}(s_t), a_2^{NE}(s_t))$. Although the Nash equilibrium need not be unique, all Nash equilibria can be Pareto-ranked (by the non-decreasing reaction functions) and we let $(a_1^{NE}(s_t), a_2^{NE}(s_t))$ denote the dominant Nash equilibrium in this case.

REMARK: We can use the univalence approach of Gale-Nikaido to establish conditions for the uniqueness of the Nash equilibrium of the best-response functions. Letting

$$\mathbf{J} = \begin{pmatrix} \frac{\partial^2 \phi_1}{\partial a_1 \partial a_1}, \frac{\partial^2 \phi_1}{\partial a_1 \partial a_2} \\ \frac{\partial^2 \phi_2}{\partial a_2 \partial a_1}, \frac{\partial^2 \phi_2}{\partial a_2 \partial a_2} \end{pmatrix}$$

be the matrix of partial derivatives, then the Nash equilibrium is unique provided J is negative quasi-definite, that is $J + J^T$ is negative definite. This condition is satisfied in the case where breakdown payoffs are proportionate to outputs $\phi_i(a_1, a_2, s_t) = \theta_i(s_t) y(a_1, a_2, s_t)$ given $\theta_i(s_t) > 0$.

2.3. *Equilibria*

To specify what happens in the dynamic game played by the agents we shall assume that there is reversion to the Nash equilibrium of the breakdown game after any deviation and compute equilibria relative to these punishments. Suppose that a is the current recommended action vector. If agent i is to deviate then it is clear that the best the agent can do is to choose the best response action $a_i^N(a_j, s_t)$. Then write $D_i(a_j, s_t)$ to denote the best non-cooperative discounted payoff that i can get starting from agent j's action a_j in the history s^t, given that she will choose the current best-response and will be punished thereafter by Nash reversion. During the Nash reversion phase both agents choose their best responses and hence both will play the Nash equilibrium of the breakdown game.[15] We refer to $D_i(a_j, s_t)$ as the *deviation payoff* which can be defined recursively as

$$D_i(a_j, s_t) = \phi_i(a_i^N(a_j, s_t), a_j, s_t) - a_i^N(a_j, s_t) + \delta \sum_{s_{t+1} \in \mathcal{S}} \pi_{s_t s_{t+1}} D_i\left(a_j^{NE}(s_{t+1}), s_{t+1}\right).$$

Given our hold-up assumption it follows that the deviation payoff is strictly increasing in the action of the other agent when the other agent's action increases output. This and other properties of the deviation payoff are stated in the following lemma.

LEMMA 3: *The deviation payoff $D_i(a_j, s_t)$ is a continuous, differentiable, non-decreasing and concave function of a_j. $D_i(a_j, s_t) \geq 0$. If $\partial y(a_1, a_2)/\partial a_j > 0$ then $D_i(a_j, s_t)$ is strictly increasing in a_j.*

REMARK: For many of our results the deviation payoff could be taken as a primitive of the model. Alternatively the deviation payoffs could be derived from different assumptions

[15]If there are multiple Nash equilibria, this could be any of them, and we are arbitrarily assuming that this is to the Pareto-dominant one, although it is only sufficient to assume that the continuation equilibrium selected is fixed in each state. In fact all results go through if one models the post default situation as involving termination of the relationship and some state dependent outside options being taken which offer a utility no more than repetition of the breakdown Nash. Likewise reversion to the worst subgame perfect continuation, which may be more severe than what we are assuming, does not affect the results, although in the existing literature the two coincide. Repeated Nash reversion is a subgame perfect equilibrium (each agent can just *demand* the whole output each period) and one cannot be held below this in any equilibrium in which breakdowns do not occur.

about the nature of the breakdown game. For example, if there are outside options which can only be taken after *the end* of the current period (so agents are locked in for a period after observing the current state), but it is never efficient to take them (i.e., it is not part of an efficient equilibrium to terminate the relationship), then the characterization we give will still apply. See Bond (2003) for a model of this type in a related context.

We consider pure-strategy subgame-perfect equilibria of the above game. Let the putative outcome path of an equilibrium be represented by $\{a(s^t), w(s^t)\}_{t=0}^{\infty}$, where $a(s^t)$ and $w(s^t)$ are the respective actions and actual payoff divisions (not demands) at time t along the equilibrium path. This includes the possibility that breakdown has occurred in any period, in which case $w(s^t) = \phi(a(s^t), s_t) - a(s^t)$. The outcome path $\{a(s^t), w(s^t)\}_{t=0}^{\infty}$ is feasible if $w(s^t) \in W(a(s^t), s_t)$ for every history s^t where $W(a(s^t), s_t)$ is defined in equation (1).

As has been stated we assume that there is Nash reversion after any deviation and compute equilibria relative to these deviation payoffs. It is clear that we need only consider deviations at the choice of action stage since if an agent were to contemplate deviation at the surplus division stage the breakdown payoff would be the same except that her action would not be optimized to maximize the breakdown payoff. Thus the agent can always do no worse than deviate at the action choice stage, choosing the action from the best-reply action. Then necessary and sufficient conditions for this path to be equilibrium relative to Nash-reversion is that it is feasible, and for $i = 1, 2$, for every s^t,

$$V_i(s^t) \equiv w_i(s^t) + \mathrm{E}\left[\sum_{\tau=t+1}^{\infty} \delta^{\tau-t} w_i(s^\tau) \mid s^t\right] \geq D_i(a_j(s^t), s_t). \tag{3}$$

The payoff $V_i(s^t)$ is the discounted payoff to t that i anticipates from the equilibrium, while the right hand side of (3) is the deviation payoff she would get from deviating from the recommended action $a(s^t)$ after the history s^t. We shall refer to the $V_i(s^t)$ as the *continuation utilities*. Whenever (3) holds with equality we say that agent i is *constrained* — any reduction in her on-equilibrium path payoff would lead her to deviate at s^t; otherwise we say that agent i is unconstrained. We refer to such paths that satisfy the inequalities in (3) as *self-enforcing* and the inequalities themselves as the *self-enforcing* or *incentive constraints*. Then along such a path, even though there is no legal enforcement, the payoffs and actions are supported by the deviation payoffs so neither agent has an incentive to deviate.

A self-enforcing agreement or contract Γ, specifies history contingent actions and utilities $a(s^t)$ and $w(s^t)$ at each date in each state, $\Gamma = \{a(s^t), w(s^t)\}_{t=0}^{\infty}$ that is both feasible and self-enforcing, i.e. satisfies both (1) and (3). A self-enforcing agreement then corresponds to a pure strategy sub-game perfect equilibrium of the game. We shall denote the restriction of the self-enforcing contract after the history s^t by $\Gamma(s^t)$ where this corresponds to an action-utility profile $\{a(s^\tau \mid s^t), w(s^\tau \mid s^t)\}_{\tau=t}^{\infty}$ that is feasible and self-enforcing for every date and history $\tau > t$ contingent on s^t. We define the set of self-enforcing agreements as $\mathcal{G}$. Because of our Markov assumption and because all the self-enforcing constraints are forward looking and the time-horizon is infinite the set of self-enforcing agreements depends only on the current state s at a particular date t and is independent of the history s^t. We shall denote this set of self-enforcing agreements given current state s by $\mathcal{G}_s$. Associated with each $\Gamma(s^t) \in \mathcal{G}_{s_t}$ are the discounted payoffs to the two agents $(V_1(s^t), V_2(s^t))$ given in equation (3). We shall let $\mathcal{V}$ denote the set of payoffs (V_1, V_2) which correspond to self-enforcing agreements Γ, and $\mathcal{V}_{s_t}$ denote the set of equilibrium payoffs $(V_1(s^t), V_2(s^t))$. Again where no confusion arises and the state s occurs at date t we shall write these continuation utilities as $(V_{1,s}, V_{2,s})$.

The sets $\mathcal{G}$ and $\mathcal{V}$ are not necessarily convex because of the presence of $a(s^t)$ on the right hand side of equation (3). This potential non-convexity does not affect our main characterization results and therefore we do not impose further restriction on the model to guarantee convexity.[16] We define the Pareto-frontier of the payoff set by the set

$$\Lambda(\mathcal{V}) = \{(V_1, V_2) \in \mathcal{V} \mid \nexists(\tilde{V}_1, \tilde{V}_2) \in \mathcal{V} \text{ with } (\tilde{V}_1, \tilde{V}_2) \geq (V_1, V_2) \text{ and } \tilde{V}_i > V_i \text{ for } i = 1 \text{ or } 2\}$$

with $\Lambda(\mathcal{V}_s)$ denoting the Pareto-frontier in state s. As our objective is to characterize the set Pareto-efficient self-enforcing agreements (when looked at from the outset of the game) we shall be interested in the set $\Lambda(\mathcal{V})$. We shall say that agreements that correspond to this Pareto-frontier are *optimum* or optimum contracts and refer to the corresponding actions as optimum actions.

3. Results

This section provides the main results of the paper. The existence of a optimum contracts is established in Section 3.1. Section 3.2 demonstrates when actions will be inefficient and Section 3.3 proves the backloading principle. The long-run properties are examined in Section 3.4 and Section 3.5 which show convergence to a stationary phase which maximizes surplus amongst all self-enforcing contracts. Finally Section 3.6 will consider an example with no uncertainty to illustrate our results. The case where only one agent contributes to output is considered in Section 4.

3.1. Existence

We first establish that efficient self-enforcing contracts do exist. This follows from a straightforward argument showing that the payoff set is compact.

Lemma 4: *The set of pure-strategy subgame perfect equilibrium payoffs $\mathcal{V}$ is non-empty and compact. Hence optimum contracts exist.*

[16] There are special cases where the sets are convex. The additive production technology case is one such example.

We shall say that a self-enforcing contract is trivial if $a(s^t) = 0$ for all t. From Assumption 1(i) on the production function, it follows that this corresponds to a point $(0,0) \in \mathcal{V}$. Lemma 4 does not imply the existence of an optimum non-trivial contract so that it is possible that $\Lambda(\mathcal{V}) = (0,0)$ and all our results will apply (trivially) in this case. Nevertheless we shall establish in the next section that in any optimum self-enforcing contract the actions are never below the Nash reaction functions and never below the Nash equilibrium actions so that if the Nash equilibrium actions are positive a non-trivial contract will exist.

3.2. *Actions at a particular date*

We shall consider the dynamic path of actions in sections 3.3 and 3.5 but in this section we consider actions at a given date and how they relate to the Nash best-response and conditionally efficient actions. Our method is to argue by contradiction, changing an assumed optimal contract at a particular date after a particular history. If this change satisfies the self-enforcing and feasibility constraints for both agents at that date, and a Pareto-improvement has been generated, then all prior self-enforcing constraints also hold as by construction the future utility entering these constraints has not been decreased. Equally all future constraints must continue to hold. Hence this leads to a Pareto-superior contract—contrary to the assumed optimality of the original contract.

As we are considering only a particular date we shall, in what follows, we can suppress the history s^t or state s_t without creating any ambiguity. We shall also use s for the current state and r to index the state next period where necessary. We first show that actions cannot be below the Nash best-response functions $a_i^N(a_j)$. The intuition is that if any agent's action is below the Nash best-response action, the action can be increased and surplus divided in such a way that neither agent has an incentive to move to the breakdown and this increase in action will increase output and utility. To see this suppose that a_2 is below the reaction function (but a_1 is not). As agent 2's action is increased, suppose we give agent 1 the increase in her deviation payoff to stop her reneging. Since this will be approximately the share she can appropriate of the extra output, giving the remainder to agent 2 gives him what he would get from increasing his action in the breakdown game. Since he is below his optimal breakdown action this will increase his utility too. hence both agents can be made better off.

THEOREM 1: *In any optimum self-enforcing contract, after any positive probability history s^t, $a_i \geq a_i^N(a_j)$, and $(a_1, a_2) \geq a^{NE}$ where a^{NE} is the dominant Nash equilibrium of the breakdown game.*

We would also like to say how the optimum actions relate to the conditionally efficient actions. This is less clear cut as we shall show that optimum actions can be above or below the conditionally efficient actions. We can however, show that an agent's action is only under-efficient if the other agent's self-enforcing constraint binds and is only over-efficient if they are at their subsistence consumption (of zero).

THEOREM 2: *In an optimum contract after some positive probability history (i) If agent i is unconstrained, i.e.* $V_i > D_i(a_j)$, *then* $a_j \geq a_j^*(a_i)$; *(ii) If agent i has positive consumption* $c_i > 0$, *then* $a_i \leq a_i^*(a_j)$.

REMARK: Theorem 2 relates the optimum actions to the conditionally efficient level. It is therefore, unlike Theorem 1, completely independent of the default structure we have specified.

The intuition behind the proof is simple. If the self-enforcing constraint is not binding then there is no cost to increasing the other agent's action or investment up to the conditionally efficient level. Equally if an action, say agent 1's action, is above the conditionally efficient level it will be profitable to reduce it. However this will reduce the output and some consumption must be reassigned to the agent 2 if his utility is not to fall. The only circumstances where this transfer cannot be made is if the agent 1 already has zero consumption and his limited liability constraint is binding.

There are two straightforward implications of Theorem 2(ii). Firstly it is impossible in an optimum non-trivial contract that both agents overinvest (they cannot both have zero consumption). Secondly an agent cannot be permanently overinvesting (i.e., with probability one) as this would imply that her consumption is always zero, which cannot be self-enforcing.

3.3. Backloading

As discussed in the introduction there is a well known backloading principle that applies when commitment by one agent is limited. This principle says that *ceteris paribus* transfers to that agent should be backloaded into the future if the commitment constraint is binding, to provide a carrot in the future that would be forgone if the agent reneged. The operation of this principle in our environment where both agents undertake an action or investment and neither can commit is more subtle because of the actions the other agent may choose in the breakdown game. Nevertheless we shall show that backloading applies to this case and has the additional implication that one agent may overinvest in the early periods of a optimum contract.

We start by showing that allocating all of the current output to an agent guarantees, under certain conditions, that this agent's self-enforcing constraint is not violated. This is proved in the next lemma and is true of any self-enforcing contract (whether optimum or not). The intuition is straightforward: an agent can get no higher current period payoff by defaulting no matter how big a share she can claim in breakdown, as she is already getting 100%, and so has no short-run gain, although since in default the agent may be able to choose a more advantageous action some care is needed to make this argument.

Suppose agent 2 gets allocated all the current output. Consider starting from agent 2's best-response (in the breakdown game) to agent 1's action, and hold the latter fixed; if 2's action is increased and he is being allocated all of output, then his utility will be rising until his efficient response is reached, where his payoff is maximized. Provided a_2 is above or equal to his best-response, but not higher than the conditionally efficient action, he is weakly better off at a_2 getting all consumption than best responding—even if he can claim all of the output—in the breakdown. Hence the self-enforcing constraint is satisfied, even if deviating leads to no future losses. This property is important for our backloading result as it will enable us to check that the self-enforcing constraint for an agent is satisfied by checking that that agent is receiving all the available output. (Note it does not refer to optimum contracts.)

LEMMA 5: *If $c_2 = y(a_1, a_2)$, $a_2^N(a_1) \leq a_2 \leq a_2^*(a_1)$ and $V_{2,r} \geq D_2(a_{1,r}^{NE}, r)$, all $r \in \mathcal{S}$, then*

$$c_2 - a_2 + \delta \sum_{r \in \mathcal{S}} \pi_{sr} V_{2,r} \geq D_2(a_1); \tag{4}$$

moreover the inequality is strict if $a_2 > 0$ and $y(a_1, a_2) > 0$. Likewise with the agent indices swapped.

We now present our main backloading result. For notational convenience we will now treat actions and consumptions at a particular date as random variables and write a_i^t and c_i^t for $a_i(s^t)$ and $c_i(s^t)$ etc.

THEOREM 3: *(i) If at $\tilde{t}$ in an optimum contract (after positive probability history $s^{\tilde{t}}$), agent 1, say, is unconstrained* and *$a_1^{\tilde{t}} < a_1^*(a_2^{\tilde{t}})$, then at all previous dates $t < \tilde{t}$ on the same path, $c_2^t = 0$; (ii) If at $\tilde{t}$ in an optimum contract (after positive probability history $s^{\tilde{t}}$), say agent 2 has $a_2^{\tilde{t}} > a_2^*(a_1^{\tilde{t}})$, then at all previous dates $t < \tilde{t}$ on the same path, $c_2^t = 0$.*

Theorem 3(i) shows that if in any optimum contract agent 1 is under-investing but unconstrained then agent 2 will have been held to his subsistence consumption level in all previous periods along the history to that point. The idea is that if today agent 1 is unconstrained and her action is inefficiently low, while agent 2 has positive consumption earlier, agent 1's action can be increased and at the same time consumption can be transferred at the current date to agent 2 to stop him reneging; agent 1 can be compensated for her increased effort by agent 2 transferring consumption at the earlier date. This backloading of agent 2's consumption allows his later constraint to be relaxed. Intuitively, keeping agent 1's action inefficiently low will help prevent agent 2 from reneging. However, this is inefficient if the action is already below its conditionally efficient level. Since agents are risk neutral they do not care about the timing of consumption flows (keeping the action plans fixed) if the expected discounted value is the same. Thus it will be desirable to backload consumption to the future to provide a carrot for sticking to the contract. It is important though that agent 1 is unconstrained for this result to hold and we will show later that it may not apply if agent 1 is also constrained at the later date.

Theorem 3(ii) shows that the backloading principle extends to actions as well as consumption. Backloading of actions implies that actions may be above the conditionally efficient levels in the early periods of an efficient contract. This however involves an efficiency loss not incurred by backloading consumption. So reducing consumption is more efficient than increasing the action. Nevertheless it may be optimal on the margin to increase the action as for a small increase the loss will be of second-order and it will enable the action of the other agent to be increased without violating the self-enforcing constraint. Theorem 3 therefore demonstrates that if ever agent 2 is overinvesting then consumption has already been backloaded in all previous periods.

3.4. *Surplus Maximization*

In the next section we shall show that the optimum contract converges to the point that maximizes joint utility. In this section we shall show that the maximization of joint utility involves choosing the actions that maximizes current joint surplus subject to the self-enforcing and feasibility constraints.

We begin by defining the actions which maximize current joint surplus and the joint utility maximizing contract.

Definition 1: An action pair a in state s_t at date t is *current joint surplus maximizing* if $a \in \arg\max_{a \in \Re^2_+} \{y(a_1, a_2, s_t) - a_1 - a_2$: $\exists$ a self-enforcing contract $\Gamma(s^t) \in \mathcal{G}_{s_t}$ starting at date t with $a^t = a\}$.

Definition 2: A self-enforcing contract $\Gamma(s^t) \in \mathcal{G}_s$ at date t in state s_t is *joint utility maximizing* if the sum of the corresponding continuation utilities is maximized across all self-enforcing contracts: $(V_1, V_2) \in \arg\max_{(V_1,V_2) \in \mathcal{V}_{s_t}} (V_1 + V_2)$.

Note that a current joint surplus maximizing a is found by looking across all self-enforcing contracts starting from s and picking one that maximizes the surplus in the first period of the contract, irrespective of what happens later.

To show the connection between a joint utility maximizing and current joint surplus maximizing self-enforcing contract, it will be useful to consider an intermediate case where the action profile maximizes current joint surplus for a given set of continuation utilities.

Definition 3: We say that the action vector a is *myopic efficient* at date t in state s relative to continuation utilities $(V_{1,r}, V_{2,r})_{r \in \mathcal{S}}$ if there is an associated consumption vector c such that $(a, c) \in \arg\max_{(a,c) \in R^4_+} (y(a_1, a_2, s) - a_1 - a_2)$ s.t. $c_i - a_i + \delta \sum_{r \in \mathcal{S}} \pi_{sr} V_{i,r} \geq D_i(a_j, s)$, and $c_1 + c_2 \leq y(a_1, a_2)$ for $i = 1, 2$, $j \neq i$.

Myopic efficient actions are not necessarily optimum since they take the continuation utilities as given and do not take into account the trade-off between actions today and actions in the future. Neither are optimum actions necessarily myopic efficient since one agent may be worse off if the myopic efficient actions were chosen. The current joint surplus maximizing actions are, however, myopic efficient for the corresponding future continuation utilities. In the next two lemmas we show that at the myopic efficient actions it is always possible to find some division of the current surplus that satisfies the current self-enforcing constraints for any continuation utilities.

LEMMA 6: *If the action vector $\tilde{a}$ is myopic efficient for some $(V_{1,r}, V_{2,r})_{r\in\mathcal{S}}$ with $V_{i,r} \geq D_i(a^{NE}_{j,r}, r)$ all $r \in \mathcal{S}$, $i, j = 1, 2$, $j \neq i$, then*

$$y(\tilde{a}_1, \tilde{a}_2, s) - \tilde{a}_i + \delta \sum_{r\in\mathcal{S}} \pi_{sr} D_i(a^{NE}_{j,r}, r) \geq D_i(\tilde{a}_j, s), \tag{5}$$

for $i, j = 1, 2$, $j \neq i$; i.e., giving all output to agent i implies that the current self-enforcing constraint continues to hold for i even if the continuation utilities are replaced with the deviation payoffs.

LEMMA 7: *Take any myopic efficient action a for some $(V_{1,r}, V_{2,r})_{r\in\mathcal{S}}$; then given any alternative continuation utilities $(\hat{V}_{1,r}, \hat{V}_{2,r})_{r\in\mathcal{S}}$ satisfying, for all $r \in \mathcal{S}$, $\hat{V}_{1,r} + \hat{V}_{2,r} \geq V_{1,r} + V_{2,r}$, $\hat{V}_{i,r} \geq D_i(a^{NE}_{j,r}, r)$, there is a division of $y(a_1, a_2)$ such that the self-enforcing constraints are satisfied with the same action a.*

The argument of Lemma 7 is true for any $(\hat{V}_{1,r}, \hat{V}_{2,r})$ satisfying $\hat{V}_{1,r} + \hat{V}_{2,r} \geq V_{1,r} + V_{2,r}$, $\hat{V}_{i,r} \geq D_i(a^{NE}_{j,r}, r)$. Thus in particular it holds for joint utility maximizing points since $a \geq a^{NE}$ and therefore $D_i(a_j, r) \geq D_i(a^{NE}_{j,r}, r)$. So we can establish that any joint utility maximizing equilibrium involves current joint surplus maximization.

THEOREM 4: *Any joint utility maximizing self-enforcing contract starting at date t from s^t, $\Gamma(s^t)$, has with probability one current joint surplus maximizing actions at each date $\tau \geq t$.*

The theorem is intuitive since changing actions away from the current joint surplus maximizing ones will lower utility at that particular date and hence overall utility. What the theorem shows is that there is no additional benefit of changing actions from relaxing one of the self-enforcing constraints.

3.5. Convergence

In this section we show that any optimum contract converges to the joint utility maximizing self-enforcing contract. This is a surprising result given the generality of the model and stochastic structure. We shall show first however, that if at any date t and state s the self-enforcing constraints bind for both agents and there is no over-efficiency of actions, then the contract always involves joint utility maximization from the *next period* onward.

Hence from Theorem 4 this involves the surplus maximizing actions at every subsequent date.

LEMMA 8: *If in an optimum contract s^t has positive probability, both self-enforcing constraints bind at t, and $a_i^t \leq a_i^*(a_j^t)$, $i, j = 1, 2$, $i \neq j$, then the contract must specify joint utility maximization from $t+1$ (i.e., in every positive probability successor state).*

The idea behind Lemma 8 is that if it were not true that every successor state maximized joint utility then it would be possible to replace $(V_{1,r'}, V_{2,r'})$ by $(\hat{V}_{1,r'}, \hat{V}_{2,r'}) \in \mathcal{V}_{r'}$ such that $\hat{V}_{1,r'} + \hat{V}_{2,r'} > V_{1,r'} + V_{2,r'}$ and demonstrate a Pareto-improvement. To show this it is necessary that both agents were previously constrained. If either agent were unconstrained and the inequality were strict then replacing $V_{i,r}$ with $\hat{V}_{i,r}$ might lower utility for one agent.

We now present the convergence result that an optimum contract has actions which converge almost surely to current joint surplus maximizing actions (and, *a fortiori*, joint utility maximization). To show this, we first show that there exists a stopping time such that both $c_1 > 0$ and $c_2 > 0$ at some point before this time, and it is finite almost surely. The argument is intuitive: provided at least one agent has a strictly positive payoff, then one agent, say agent 1, must take a positive action at some point. Thus agent 1 must receive positive consumption at some point not too far after the action was taken, or else her overall payoff would be negative, something which is inconsistent with the self-enforcement and the deviation payoff (an agent can always guarantee herself at least zero by taking a null action each period). Likewise, by the fact that agent 1 took a positive action, the agent 2 can get a positive share of that output by the hold-up assumption on the breakdown payoffs, and hence must have positive continuation utility at this point. Thus agent 2 must also anticipate positive consumption. This situation must happen repeatedly in an optimum contract, and thus positive consumption for both agents occurs with probability one (the proof is only complicated by the need to ensure that the number of periods before positive consumption is received is bounded). Next, once both agents have had positive consumption, our backloading results imply that there cannot be overinvestment, and if either agent is unconstrained then actions are at the efficient level. Alternatively, if both agents are constrained, we know that joint utility maximization occurs thereafter by the previous lemma.

THEOREM 5: *For any optimum contract, there exists a random time T which is finite with probability one such that for $t \geq T$, a^t is current joint surplus maximizing.*

We know from Theorem 3(ii) that if both agents have had positive consumption at some date prior to $\hat{t}$, then for $t > \hat{t}$, $a_i^t \leq a_i^*(a_j^t)$ for $i = 1, 2$. However, if agent i is unconstrained at t, then it follows from Theorem 3(i) that $a_i^t \geq a_i^*(a_j^t)$, otherwise the consumption of agent j could not have been positive prior to $\hat{t}$. Equally from Theorem 2(i),

$a_j^t \geq a_j^*(a_i^t)$ as there is no need to hold agent j's investment below the efficient level if agent i is unconstrained. Thus we can conclude that the only case of inefficiency occurs when both agents are constrained. We thus have the following corollary.

COROLLARY 1: *There exists a random time T, finite with probability one, such that for $t \geq T$, and for any state $s_t \in \mathcal{S}$ in which efficiency a^* is not achievable for any division of the surplus, then both self-enforcing constraints bind and there is underinvestment by both agents.*

The next theorem considers the canonical two-sided action case in which both actions are always positive. This allows us to present the sharpest results in terms of optimum action levels relative to unconstrained efficient levels. It shows that for the case where we are guaranteed to have positive actions each period, there will be two phases, one (which may not exist) is a backloading phase with zero consumption and overinvestment by one of the agents (the same agent throughout the phase), and the other phase (which exists with probability one) will have no overinvestment, but consists of an initial transition period which is then followed by either efficient actions, if they can be sustained in equilibrium in that state, or both constraints binding and positive consumption.

THEOREM 6: *Whenever the Nash actions $a_{i,s}^{NE}$ are positive, $i = 1, 2$, then with probability one an optimum path has two phases, where $i = 1$ or 2:*
***Phase 1**: $c_i^t = 0$, $a_i > a_i^*$ and $a_j \leq a_j^*$, for $0 \leq t < \tilde{t}$, $j \neq i$, where $\infty > \tilde{t} \geq 0$;*
***Phase 2**: $a_1^t \leq a_1^*$ and $a_2^t \leq a_2^*$ for $t \geq \tilde{t}$ and after the first period of phase 2, if a^* is feasible in s_t then $a^t = a^*$; otherwise $a < a^*$, both constraints bind, and $c > 0$.*

The requirement of positive Nash actions is a simple way to ensure that optimum actions at each date are positive by virtue of Theorem 1. We need to assume this to prove Theorem 6 for two reasons. Firstly, even if both actions are productive, it may be that overinvestment does not occur in the backloading phase. This might be the case if $a_i = 0$ and $a_i^* = 0$ and the marginal product at zero is well below one. In this case the optimum action may be at the corner solution where the marginal product below one and the optimum action is zero. Secondly, underinvestment may not occur in the second phase as it possible that the efficient action levels are zero.

If current surplus is not maximized after the first period of Phase 2, then $a \neq a^*$ but by Theorem 6 this implies both constraints bind, in which case current surplus is maximized thereafter. Hence we have the following corollary.

COROLLARY 2: *With probability one, current surplus is* not *maximized in at most two periods of Phase 2.*[17]

[17] In the case of additive production technology this can be strengthened to current surplus being maximized in all but the first period of Phase 2.

3.6. *Example with No Uncertainty*

We consider a simple example to illustrate our results. This will be the simplest possible example with an additive production technology and no uncertainty. The example is very similar to the model of joint production presented by Garvey (1995).[18] For simplicity in our example both agents can grab all the other's output but if they do so they lose their own output and are unable to produce anything on their own.

We assume the production function function is

$$y(a_1, a_2) = f_1(a_1) + f_2(a_2) = 2b\sqrt{a_1} + 2\sqrt{a_2}$$

for a parameter $b \in (0, 1)$. The breakdown payoffs are of the form $\phi_i(a_1, a_2) = \theta_{i1} f_1(a_1) + \theta_{i2} f_2(a_2)$ where the parameters are $\theta_{11} = \theta_{22} = 0$ and $\theta_{12} = \theta_{21} = 1$.

The efficient actions are $a_1^* = b^2$ and $a_2^* = 1$ with a maximal efficient surplus of $f_1(a_1^*) + f_2(a_2^*) - a_1^* - a_2^* = 1 + b^2$. With $\theta_{12} = \theta_{21} = 1$, the breakdown payoffs are

$$\phi_1(a_1, a_2) = f_2(a_2) = 2\sqrt{a_2} \quad \text{and} \quad \phi_2(a_1, a_2) = f_1(a_1) = 2b\sqrt{a_1}.$$

Given the additive technology the Nash best-response functions are dominant strategies which is simply not to invest, $a_1^N = a_2^N = 0$.[19] Hence the deviation payoffs are[20]

$$D_1(a_2) = (1 - \delta)2\sqrt{a_2} \quad \text{and} \quad D_2(a_1) = (1 - \delta)2b\sqrt{a_1}.$$

We consider the special case of $\delta = 1/3$ and $b = \sqrt{3}/3$.[21] The solution can be found by first finding the efficiency stationary solution and then working backwards in time given the above results. The stationary solution can be found by solving the two equations

$$a_1 = b^2 a_2 \quad \text{and} \quad 2(1 - \delta)\sqrt{a_2} + 2b(1 - \delta)\sqrt{a_1} = 2b\sqrt{a_1} - a_1 + 2\sqrt{a_2} - a_2.$$

[18]Garvey (1995) has linearly additive outputs and quadratic cost functions but this is equivalent to our formulation with square-root production functions and linear investment costs. His concern is with finding a legal structure, joint ventures or integration, that is best suited (in terms of a minimum discount factor) to sustaining the efficient investment levels. He does not therefore examine the temporal structure of investments.

[19]Strictly Theorem 6 does not apply since the Nash equilibrium actions are zero. However, the importance of the assumption of positive Nash equilibrium actions was to rule out trivial contracts and non-trivial contracts are not optimal for the parameter values chosen and hence the substance of the theorems does apply.

[20]For the purposes of calculating the example all per-period payoffs have been multiplied by $(1 - \delta)$.

[21]In this example a non-trivial contract is sustainable for any $\delta > \delta_* = 0$ and an efficient stationary solution is sustainable if $\delta \geq \delta^* = 1/2$. The value $\delta = 1/3$ is chosen below this critical value so that the efficient outcome is not sustainable in the the stationary solution but large enough to generate simple but interesting dynamics for the optimum contract. The value of b is simply chosen for convenience.

For the given parameter values of b and δ the solution to this equation is $a_1 = 4b^2\delta^2 = 4/27$ and $a_2 = 4\delta^2 = 4/9$.The net surplus generated is $2b\sqrt{a_1} - a_1 + 2\sqrt{a_2} - a_2 = 32/27$ and this surplus is divided so that $w_1 = 4\delta(1-\delta) = 8/9$ and $w_2 = 4b\delta(1-\delta) = 8/27$.[22]

Having calculated the stationary solution, and given that we know the contract will converge to the stationary solution, it is possible to work backwards and calculate all other points on the frontier. For example for values of V_1 close to the stationary value the contract will move to the stationary solution next period, but still both self-enforcing constraints will bind. Since the self-enforcing constraint for agent 1 binds

$$V_1 = 2(1-\delta)\sqrt{a_2}$$

and a_2 is determined once V_1 is known. Equally w_1 is found from the recursive equation

$$(1-\delta)w_1 + \delta V_1^+ = V_1$$

where $V_1^+ = 4\delta(1-\delta)$ is the continuation value for agent 1 and $V_2^+ = 4b^2\delta(1-\delta)$ is the continuation utility for agent 2. As w_1, a_2 are functions of V_1, a_1 as a function of V_1 can be found as the solution to

$$(1-\delta)(2b\sqrt{a_1} - a_1 + 2\sqrt{a_2} - a_2 - w_1) + \delta V_2^+ = 2b(1-\delta)\sqrt{a_1}.$$

Using this value of a_1 the Pareto-frontier can then be computed from the constraint $V_2 = 2b(1-\delta)\sqrt{a_1}$. Since we have assumed that both self-enforcing constraints bind, the endpoints of this part of the Pareto-frontier function are determined either by a non-negativity condition on the action or at the point where the efficient level of investment can just be sustained. At the left-hand endpoint for example we either have $a_1 = b^2$ or $a_2 = 0$. At the right-hand end of this interval $a_1 = 0$ and the slope of the frontier is $-\infty$ at this point. Thus the right-hand end point is the full extent of the domain of the frontier. However, the frontier is extended to the left and we can now check how the frontier extends. Proceeding as before but taking $a_2 = b^2$ we can again compute the Pareto-frontier. The left-hand endpoint of this section of the frontier is determined where the zero consumption starts to bind which is at the point given by $(1-\delta)(4\delta^2 - b^2) = 2/27$. The slope of the Pareto-frontier at this point is downward sloping (differentiating the Pareto-frontier) and so the frontier extends further to the left, where the consumption of agent 1 is zero. To calculate this part of the frontier we need to use the information just calculated to obtain the continuation values. Analytically this is more complex and involves solving a cubic equation. Nevertheless, it is possible to obtain an analytic solution for that part of the Pareto-frontier.

[22]The corresponding values for consumption are $c_1 = 4\delta((1-\delta) + \delta b^2)) = 28/27$ and $c_2 = 4\delta(\delta + b^2(1-\delta)) = 20/27$.

The solution is illustrated in Figure 1. The upper left part of the diagram draws the Pareto-frontier. This function is differentiable and the left hand endpoint of the frontier is determined at the point where this function has a zero slope.[23] The upper-right part of the diagram plots the net surplus showing that net surplus is maximized at the point on the frontier which maximizes $V_1 + V_2$ (Theorem 4) where the slope of the frontier is -1. The lower-right part of the diagram plots agent 2's action level against V_1 and shows that she is always underinvesting (the efficient level is $a_2^* = 1$). The lower left part of the diagram shows agent 1's action. There is overinvesment for low values of V_1. At this point in the contract consumption $c_1 = 0$ and the continuation value for V_1 will be in the range $(2/27, 4\sqrt{7}/9\sqrt{3})$. Thus next period the action a_1 is chosen efficiently, $a_1 = 1/3$. The next continuation values are the utility maximizing ones at the stationary point. Thus in this example and for the parameters we've used there is a most one period of Phase 1 where there is overinvestment and after two periods the stationary solution where both agents are underinvesting is reached.[24]

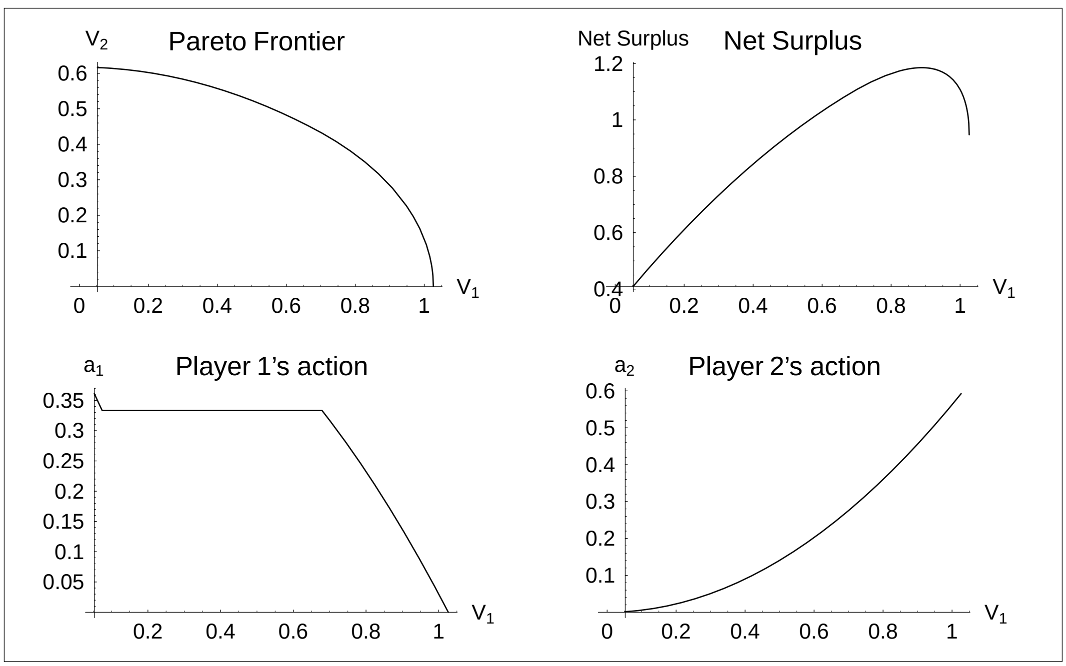

Figure 1: Pareto-Frontier, Net Surplus and Actions

[23]Numerical calculation gives the left hand value for $\underline{V}_1 = 0.0524532$.

[24]For different parameter values there may be more than one period of Phase 1 in which there is overinvestment.

4. One-Sided Investment

In this section we discuss the case where only one of the two agents makes an investment which has been the subject of most of the previous literature (see for example, Albuquerque and Hopenhayn (2004) and Thomas and Worrall (1994)). We shall show that in this case there is no overinvestment by the contributing agent.[25] We'll suppose that this is agent 1 and assume that agent 2 never contributes towards output. We therefore write output as $y(a_1, a_2, s_t) = f(a_1, s_t)$ and the breakdown payoff for agent 1 as $\phi_1(a_1, s_t) \leq f(a_1, s_t)$. In this case agent 1's self-enforcing constraint reduces to a more conventional nonnegative surplus constraint,

$$V_1(s^t) \geq D_1(s_t) = \phi_1(a_1^{NE}(s_t), s_t) - a_1^{NE}(s_t) + \delta \sum_{s_{t+1} \in \mathcal{S}} \pi_{s_t s_{t+1}} D_1(s_{t+1}).$$

We show that agent 1 never overinvests in these circumstances. This is perhaps unsurprising in view of the idea that backloading of utility will only apply to agent 2, the agent whose self-enforcing constraint can prevent efficient actions by agent 1. Overinvestment (and hence $c_1 = 0$) implies a negative current utility for agent 1, and as the future goes against agent 1, this would lead to a negative overall utility, something which would violate agent 1's constraint.

Theorem 7: *In the case of one-sided investment where, say, agent 1 is the only contributor to output, then at any date t and state s, $a_1^t \leq a_1^*$; overinvestment never occurs in an efficient self-enforcing contract.*

Example

We now consider a simple version, with no uncertainty, of the model of foreign direct investment found in Thomas and Worrall (1994).[26] In that model agent 1 is a transnational corporation and agent 2 is a host country. Only the transnational corporation has an action, namely how much to invest in the host country. The host country can though expropriate any output produced by the transnational corporation within its territory. The transnational corporation invests a_1 the output is given by the production function $f(a_1)$. The net surplus function is $f(a_1) - a_1$. The per-period breakdown payoffs are given by $\phi_1(a_1) - a_1 = -a_1$ and $\phi_2(a_1) = f(a_1)$ as in the event of breakdown the transnational corporation loses its investment and the host country expropriates all of output. Clearly at the action stage, if the transnational corporation is going to default it will choose $a_1 = 0$

[25] A differences arises in the one-sided and two-sided cases since when only one agent takes any action it is not always possible to adjust actions to smoothly raise or lower the continuation utilities for both agents as is the case with the two-sided case.

[26] In Thomas and Worrall (1994), investment is taken before the state is known. The current paper thus has more similarities with Albuquerque and Hopenhayn (2004) where investment decisions are taken after the state is revealed. This makes only minor differences in determining the optimum contract and in the absence of uncertainty as in the present example makes no difference at all.

and thus the deviation payoffs are $D_2(a_1) = f(a_1)$ as the host can expropriate all output but only gets the output for the current period and $D_1 = 0$ as the transnational corporation simply has the option to withdraw. The non-negativity constraints for consumption reduce to $w_1 + a_1 = c_1 \geq 0$ and $w_2 = c_2 \geq 0$, where the the former constraint can be written equivalently as $w_2 = c_2 \leq f(a_1)$ since $w_1 + w_2 = f(a_1) - a_1$.

At a stationary solution w_1 and w_2 are constant as is a_1. So feasibility requires that $w_2 \geq (1-\delta)f(a_1)$ and $w_1 \geq 0$. Since $w_1 + w_2 = f(a_1) - a_1$, this reduces to $\delta f(a_1) \geq a_1$ or

$$\frac{f(a_1)}{a_1} - 1 \geq \frac{1}{\delta} - 1 = r \tag{6}$$

where r is discount rate. Thus the constraint reduces to the condition that the return on the investment is no less than the discount rate. As the average product $f(a_1)/a_1$ is greater than the marginal product $f'(a_1)$, the left hand side of (6) is decreasing in a_1, so either the efficient level where $f'(a_1) = 1$ is sustainable or a_1 is increased until the the rate of return is equalized to the discount rate.

Following our previous analysis it is easy to show that if the stationary solution is not reached the period after next then the host country receives no payment from the transnational corporation. Thus in the early periods there is effectively a tax holiday with the transnational corporation taking all profits from the investment. However, investment is low in the initial periods, $f'(a_1) > 1$, to prevent the host country from reneging. Letting V_2^+ be the continuation value for the transnational company, it is also easy to check that it satisfies $V_2^+ = f(a_1)/\delta$ and since $V_2^+ > V_2$ investment is less than efficient but increasing over time. Measuring the slope of the Pareto-frontier as $-dV_1/dV_2$, it is also possible to show that the change in the slope of the frontier is given by $(f'(a_1) - 1)/f'(a_1)$. A full analysis can be found in Thomas and Worrall (1994).

5. Conclusion and Further Work

In this paper we have analyzed the dynamic properties of a relational or self-enforcing contract between two risk-neutral agents both of whom undertake a costly investment or action which yields joint benefits. We have shown that there is convergence to a stationary state at which the net surplus is maximized. Provided the optimum contract is non-trivial it exhibits a two-phase property. In the first phase (which may or may not occur) there is backloading of the utility of one of the agents. In this phase that agent has zero consumption and will overinvest while the other agent will underinvest. In the next phase (which will occur with probability one) there is no overinvestment and after the first period of the this phase there will be either efficient investment, if that is sustainable in that state, or underivestment by both agents and with both agents constrained.

The analysis presented in the paper is applicable to a wide variety of situations. It will apply to situations of joint ventures where two partners expend individual effort or investment to improve profits. It will apply to a labor market situation where both employer and employee invest in improving the productivity of the job match and it could apply to situations of international trade where trading partners undertake investments to reduce the cost or improve the efficiency of trade. It can also apply with some re-working to a public good model where agents have to decide upon their individual contributions to a public good that benefits both agents.

The model can be extended in a number of directions. An obvious extension is to allow for risk aversion. The limited liability assumption introduces some risk aversion but allowing smoothly concave preferences will be an important extension as it will bring together the strand of the literature on self-enforcing contracts which concentrates on risk-sharing with the strand which emphasizes the actions undertaken by agents. It will also broaden the range of applications to include, for example, household behavior and investment decisions in village economies. Another extension is to treat the actions as real investments with capital accumulation such as in a model of sovereign debt.

Appendix

Lemma 1: *Given Assumption 1 the conditionally efficient actions are continuous (single-valued) non-decreasing functions of the other agent's action. Moreover, these functions cross at most once on $\Re_{++}$.*

Proof of Lemma 1: The first part follows from Assumption 1(v). Given complementarity the conditionally efficient action functions are non-decreasing and the strict concavity of the production function means that these functions cross just once on $\Re_{++}$. □

Lemma 2: *Given Assumption 2, $a_i^N(a_j, s_t)$ is weakly increasing in a_j. Moreover we have $a_i^N(a_j, s_t) \leq a_i^*(a_j, s_t)$ for each a_j and every state s_t with strict inequality whenever $a_i^*(a_j, s_t) > 0$.*

Proof of Lemma 2: We drop the notational dependence on s_t as it is inessential. Uniqueness of the best responses follows from Assumption 2 as $\partial^2\phi_i/\partial a_i^2 < 0$. The best-response $a_i^N(a_j, s_t)$ is weakly increasing in a_j as we assume in addition that $\partial^2\phi_i/\partial a_i \partial a_j \geq 0$. To show that $a_i^N(a_j, s_t) \leq a_i^*(a_j, s_t)$ we observe that from the definition of efficiency $\partial y(a_i^*(a_j), a_j)/\partial a_i \leq 1$ with equality if $a_i^*(a_j) > 0$. Hence if $a_i^*(a_j) > 0$, then either $a_i^N(a_j) = 0$ and we're done or

$$1 = \frac{\partial \phi_i(a_i^N(a_j), a_j)}{\partial a_i} = \frac{\partial y(a_i^*(a_j), a_j)}{\partial a_i} > \frac{\partial \phi_i(a_i^*(a_j), a_j)}{\partial a_i}$$

where the final inequality follows from the marginal product condition in Assumption 2. It follows from $\partial^2\phi_i/\partial a_i^2 < 0$ that $a_i^N(a_j) < a_i^*(a_j)$. If on the other hand $a_i^*(a_j) = 0$ and $a_i^N(a_j) > 0$. Then

$$1 = \frac{\partial \phi_i(a_i^N(a_j), a_j)}{\partial a_i} < \frac{\partial y(a_i^N(a_j), a_j)}{\partial a_i}$$

but

$$1 \geq \frac{\partial y(0, a_j)}{\partial a_i} > \frac{\partial y(a_i^N(a_j), a_j)}{\partial a_i}$$

where the last inequality follows from Assumption 1(v) that the function $y(a_i, \cdot, zcdot)$ is strictly concave when increasing. This later condition contradicts the supposed positivity of $a_i^N(a_j)$. Hence $a_i^N(a_j) = 0$ if $a_i^*(a_j) = 0$. □

LEMMA 3: *The deviation payoff $D_i(a_j, s_t)$ is a continuous, differentiable, non-decreasing and concave function of a_j. $D_i(a_j, s_t) \geq 0$. If $\partial y(a_1, a_2)/\partial a_j > 0$ then $D_i(a_j, s_t)$ is strictly increasing in a_j.*

PROOF OF LEMMA 3: That the deviation payoff $D_i(a_j, s_t)$ is a continuous, differentiable and non-decreasing function of a_j follows from Assumption 2 and Lemma 2. Non-negativity follows since the agents can guarantee themselves at least zero by choosing the null action. Using the envelope theorem

$$\begin{aligned} \frac{\partial D_1(a_2, s_t)}{\partial a_2} &= \frac{\partial \phi_1\left(a_1^N(a_2, s_t), a_2), s_t\right)}{\partial a_2} \\ \frac{\partial D_2(a_1, s_t)}{\partial a_1} &= \frac{\partial \phi_2\left(a_1, a_2^N(a_1, s_t), s_t\right)}{\partial a_1}. \end{aligned} \tag{A.1}$$

Therefore if $\partial y(a_1, a_2)/\partial a_i > 0$, it follows from Assumption 2, that $\partial \phi_j(a_1, a_2)/\partial a_i > 0$ for $j \neq i$ and hence $\partial D_j(a_i, s_t)/\partial a_i > 0$ from the above equation. □

LEMMA 4: *The set of pure-strategy subgame perfect equilibrium payoffs $\mathcal{V}$ is non-empty and compact. Hence optimum contracts exist.*

PROOF OF LEMMA 4: Consider the strategy for each agent of always playing the breakdown Nash equilibrium actions $a_i^{NE}(s_t)$, and demanding the entire output. By definition these are short-run mutual best responses if the game ends up in breakdown; this occurs unless output is zero. But in the latter case closedness follows from standard arguments: Briefly, the action-consumption profiles after any history s^t must be bounded in equilibrium. To see this note that assumptions on the action sets and the production function mean that actions can be restricted to some closed and bounded set $\tilde{A}(s_t) \subseteq \Re_+^2$ and hence the per-period utility payoffs also belong to a closed and bounded subset $\tilde{W}(s_t) \equiv \{W(a, s_t) \colon a \in \tilde{A}(s_t)\}$. Thus we can restrict the action-consumption pairs to a compact subset, say $\digamma(s_t) \subset R^4$. Hence the product space $\prod_{s_t} \digamma(s_t)$ is sequentially compact in the product topology as it is a countable product of compact spaces. Thus any limiting sequence of equilibrium payoffs has a convergent sub-sequence of contracts that converges pointwise to the limiting contract. Now consider the payoffs associated with this sequence of contracts. By the *dominated convergence theorem* the limit must satisfy the self-enforcing constraints (3) since payoffs are continuous functions of contracts in this topology with $\delta < 1$, and the constraints are weak inequalities. Thus the limit is an equilibrium, and thus the limiting sequence of equilibrium payoffs has a limit point which corresponds to an equilibrium. It follows the payoff set $\mathcal{V}$ is closed and bounded and hence compact subset of R^2. Since the Pareto-frontier $\Lambda(\mathcal{V})$ is a part of the boundary of this set, it follows that optimum contracts exist. □

THEOREM 1: *In any optimum self-enforcing contract, after any positive probability history* s^t, $a_i \geq a_i^N(a_j)$, *and* $(a_1, a_2) \geq a^{NE}$ *where* a^{NE} *is the dominant Nash equilibrium of the breakdown game.*

PROOF OF THEOREM 1: The proof proceeds in two parts. The first is to show that one cannot simultaneously have $a_2 < a_2^N(a_1)$ *and* $a_1 > a_1^N(a_2)$ or visa-versa. Thus the actions must be either above both reaction functions or below both reaction functions. The next part shows that $a < a^{NE}$ is impossible, ruling out that both are below the reaction functions since the reaction functions are non-decreasing from Lemma 2.
A. Suppose then that at some date t, $a_2 < a_2^N(a_1)$ and $a_1 > a_1^N(a_2)$. Consider a small increase in a_2 of $\Delta a_2 > 0$. The consequent increase in output is approximately $(\partial y(a_1, a_2)/\partial a_2)\, \Delta a_2$ (which is positive by the fact that $a_2 < a_2^N(a_1)$; from Assumptions 1 and 2 $(\partial y(a_1, a_2)/\partial a_2) = 0$ would imply $\phi_2(a_1, a_2') = 0$ for all $a_2' \geq a_2$ (and so $a_2 \geq a_2^N(a_1)$). Change the contract by giving agent 1 the increase in her deviation payoff, which is to a first order approximation $D_1'(a_2)\Delta a_2 = \left(\partial \phi_1(a_1^N(a_2), a_2)/\partial a_2\right) \Delta a_2$ (by the envelope theorem). The remainder of the extra output, approximately

$$\left(\frac{\partial y(a_1, a_2)}{\partial a_2} - \frac{\partial \phi_1(a_1^N(a_2), a_2)}{\partial a_2}\right) \Delta a_2,$$

is given to agent 2. Keep the future unchanged. We now show that these changes meet the constraints and lead to a Pareto-improvement, contrary to the assumed optimality of the contract. First, agent 1 is no worse off (in fact better off) and by construction her constraint is satisfied. For agent 2, the change in current utility to a first-order approximation is

$$\Delta w_2 \simeq \left(\frac{\partial y(a_1, a_2)}{\partial a_2} - \frac{\partial \phi_1(a_1^N(a_2), a_2)}{\partial a_2} - 1\right) \Delta a_2. \tag{A.2}$$

Since $a_2 < a_2^N(a_1)$ and $\partial^2 \phi_2/\partial a_2^2 < 0$ on $\left(a_2, a_2^N(a_1)\right)$ (by Assumption 2, given that $\phi_2(a_1, a_2^N(a_1)) > 0$),

$$\frac{\partial \phi_2(a_1, a_2)}{\partial a_2} > \frac{\partial \phi_2(a_1, a_2^N(a_1))}{\partial a_2} = 1 \tag{A.3}$$

(where the last equality follows by virtue of $a_2^N(a_1) > 0$ so there is an interior solution). Since $a_1 > a_1^N(a_2)$, and $\partial^2 \phi_1/\partial a_1 \partial a_2 \geq 0$, we have

$$\frac{\partial \phi_1(a_1^N(a_2), a_2)}{\partial a_2} \leq \frac{\partial \phi_1(a_1, a_2)}{\partial a_2}. \tag{A.4}$$

Together (A.3), (A.4) and (2) imply the term in brackets in the right hand side of (A.2) is positive, and thus for Δa_2 small enough, $\Delta w_2 > 0$. A symmetric argument applies if $a_1 < a_1^N(a_2)$ and $a_2 > a_2^N(a_1)$.
B. Suppose that $(a_1, a_2) \leq (a_1^{NE}, a_2^{NE})$ with strict inequality for at least one agent, say 2, and consider replacing the actions with the Nash equilibrium actions a_i^{NE} so that output rises from $y(a_1, a_2)$ to $y(a_1^{NE}, a_2^{NE})$. We give $\phi_1(a_1^{NE}, a_2^{NE}) - \phi_1(a_1, a_2)$ of this increase to

agent 1 and the rest, to agent 2. Hence

$$\Delta w_1 = \phi_1(a_1^{NE}, a_2^{NE}) - \phi_1(a_1, a_2) - (a_1^{NE} - a_1) \tag{A.5}$$

and

$$\begin{aligned}\Delta w_2 &= y(a^{NE}) - y(a) - \Big(\phi_1(a^{NE}) - \phi_1(a)\Big) - (a_2^{NE} - a_2) \\ &\geq \phi_2(a^{NE}) - \phi_2(a) - (a_2^{NE} - a_2),\end{aligned}$$

where the inequality follows from integrating equation (2). By definition of (a_1^{NE}, a_2^{NE}),

$$D_i(a_j^{NE}) = \phi_i(a_1^{NE}, a_2^{NE}) - a_i^{NE} + \delta \sum_{r \in \mathcal{S}} \pi_{sr} D_i(a_{j,r}^{NE}, r).$$

Hence for agent 1

$$D_1(a_2^{NE}) - D_1(a_2) = \phi_1(a_1^{NE}, a_2^{NE}) - \phi_1(a_1^N(a_2), a_2)) - (a_1^{NE} - a_1^N(a_2))$$

with a similar expression for agent 2. Agent 1's constraint is not violated as

$$\Delta w_1 - \Big(D_1(a_2^{NE}) - D_1(a_2)\Big) = \phi_1(a_1^N(a_2), a_2)) - \phi_1(a_1, a_2) - (a_1^N(a_2) - a_1) \geq 0,$$

where the inequality follows from the definition of $a_1^N(a_2)$:

$$\phi_1(a_1^N(a_2), a_2) - a_1^N(a_2) \geq \phi_1(a_1, a_2) - a_1, \tag{A.6}$$

all $a_1 \geq 0$. Likewise for agent 2. It remains to show that Δw_1, $\Delta w_2 \geq 0$, with at least one strict inequality. We have that

$$\begin{aligned}\phi_1(a_1^{NE}, a_2^{NE}) - a_1^{NE} &\geq \phi_1(a_1^N(a_2), a_2^{NE}) - a_1^N(a_2) && \text{(A.7)} \\ &\geq \phi_1(a_1^N(a_2), a_2) - a_1^N(a_2) && \text{(A.8)}\end{aligned}$$

where the first inequality follows since by definition a_1^{NE} maximizes $\phi_1(a_1, a_2^{NE}) - a_1$, and the second from the fact that ϕ_1 is nondecreasing in a_2. If $a_1^{NE} > a_1^N(a_2)$, then (A.7) is strict (by the uniqueness of the best response). On the other hand consider $a_1^{NE} = a_1^N(a_2)$. We have $y(a^{NE}) > y\left(a_1^{NE}, a_2\right)$, as if $y(a^{NE}) = y\left(a_1^{NE}, a_2\right)$ then from equation (2), $\phi_2(a^{NE}) = \phi_2\left(a_1^{NE}, a_2\right)$ so $a_2^{NE} > a_2$ could not be a best response to a_1^{NE} (a_2 is cheaper and generates the same breakdown payoff). Then from Assumption 2, $\phi_1(a^{NE}) > \phi_1\left(a_1^{NE}, a_2\right)$, and so (A.8) is strict. Hence

$$\phi_1(a_1^{NE}, a_2^{NE}) - a_1^{NE} > \phi_1(a_1^N(a_2), a_2) - a_1^N(a_2). \tag{A.9}$$

Hence substituting (A.9) and (A.6) into (A.5), the increase in utility for agent 1 is $\Delta w_1 > 0$. By symmetry similar conditions to (A.7) and (A.8) hold for agent 2 and hence $\Delta w_1 \geq 0$. □

THEOREM 2: *In an optimum contract after some positive probability history (i) If agent i is unconstrained, i.e. $V_i > D_i(a_j)$, then $a_j \geq a_j^*(a_i)$; (ii) If agent i has positive consumption $c_i > 0$, then $a_i \leq a_i^*(a_j)$.*

Proof of Theorem 2: (i) If $a_j < a_j^*(a_i)$ then raising a_j by Δa_j sufficiently small will not violate the self-enforcing constraint as $D_i(\cdot)$ is continuous, and will produce more output. Giving this extra output to agent j, the change in his utility is $\Delta w_j \simeq \left(\left(\partial y(a_1,a_2)/\partial a_j\right) - 1\right)\Delta a_j$. Since $a_j < a_j^*(a_i)$, $\left(\partial y(a_1,a_2)/\partial a_j\right) > 1$ and hence utility is improved without violating any constraint. This contradicts the assumed optimality of the initial contract.
(ii) Suppose say $a_1 > a_1^*(a_2)$ and $c_1 > 0$. Consider cutting a_1 and c_1 by the same amount, that is $\Delta c_1 = \Delta a_1 < 0$. The $\Delta w_1 = \Delta c_1 - \Delta a_1 = 0$. Transfer this cut in consumption to agent 2 and reduce his consumption by the reduction in output. That is $\Delta c_2 = -\Delta c_1 + \left(\partial y(a_1,a_2)/\partial a_1\right)\Delta a_1 = -\left(1 - \left(\partial y(a_1,a_2)/\partial a_1\right)\right)\Delta a_1$. As $\partial y(a_1,a_2)/\partial a_1 < 1$, the change in agent 2's utility is $\Delta w_2 = \Delta c_2 > 0$ showing that a Pareto-improvement can be found. □

Lemma 5: *If $c_2 = y(a_1,a_2)$, $a_2^N(a_1) \le a_2 \le a_2^*(a_1)$ and $V_{2,r} \ge D_2(a_{1,r}^{NE}, r)$, all $r \in \mathcal{S}$, then*

$$c_2 - a_2 + \delta \sum_{r\in\mathcal{S}} \pi_{sr} V_{2,r} \ge D_2(a_1); \tag{A.10}$$

moreover the inequality is strict if $a_2 > 0$ and $y(a_1,a_2) > 0$. Likewise with the agent indices swapped.

Proof of Lemma 5: As $a_2^*(a_1) = \arg\max_{a_2'} \{y(a_1,a_2') - a_2'\}$ and $\partial^2 y(a_1,a_2)/(\partial a_2)^2 \le 0$, we have that $y(a_1,a_2') - a_2'$ is weakly increasing in a_2' for $a_2^N(a_1) \le a_2' \le a_2^*(a_1)$. In particular, since $a_2^N(a_1) \le a_2 \le a_2^*(a_1)$,

$$y(a_1,a_2) - a_2 \ge y(a_1,a_2^N(a_1)) - a_2^N(a_1). \tag{A.11}$$

Since in breakdown agent 2 may not receive all output,

$$\begin{aligned} D_2(a_1) &\equiv \phi_2(a_1,a_2^N(a_1)) - a_2^N(a_1) + \delta \sum_r \pi_{sr} D_2(a_{1,r}^{NE}, r) \\ &\le y(a_1,a_2^N(a_1)) - a_2^N(a_1) + \delta \sum_r \pi_{sr} D_2(a_{1,r}^{NE}, r) \\ &\le y(a_1,a_2) - a_2 + \delta \sum_r \pi_{sr} D_2(a_{1,r}^{NE}, r), \end{aligned} \tag{A.12}$$

where the second inequality follows from (A.11), and since $V_{2,r} \ge D_2(a_{1,r}^{NE}, r)$, all $r \in \mathcal{S}$, this implies (A.10) is satisfied. Next suppose $a_2 > 0$ and $y(a_1,a_2) > 0$. If $a_2^N(a_1) = 0$, then it follows from $a_2^*(a_1) \ge a_2 > 0$ that $a_2^*(a_1) > a_2^N(a_1)$ and hence from Assumption 1 $\partial y(a_1,a_2)/\partial a_2 > 0$ and thus $\partial^2 y(a_1,a_2)/(\partial a_2)^2 < 0$ on $(0, a_2^*(a_1))$. Thus (A.11) holds strictly. If $a_2^N(a_1) > 0$ then $y(a_1,a_2^N(a_1)) > 0$ and (A.12) is strict by Assumption 2 which implies that $\phi_2(a_1,a_2^N(a_1)) < y(a_1,a_2^N(a_1))$; so in either case (A.10) holds strictly. □

Theorem 3: *(i) If at $\tilde{t}$ in an optimum contract (after positive probability history $s^{\tilde{t}}$), agent 1, say, is unconstrained* and $a_1^{\tilde{t}} < a_1^*(a_2^{\tilde{t}})$, *then at all previous dates $t < \tilde{t}$ on the same path, $c_2^t = 0$; (ii) If at $\tilde{t}$ in an optimum contract (after positive probability history $s^{\tilde{t}}$), say agent 2 has $a_2^{\tilde{t}} > a_2^*(a_1^{\tilde{t}})$, then at all previous dates $t < \tilde{t}$ on the same path, $c_2^t = 0$.*

Proof of Theorem 3: (i) Assume by contradiction that agent 1 is unconstrained and $a_1^{\tilde{t}} < a_1^*(a_2^{\tilde{t}})$, but that $c_2^t > 0$ after s^t for some $t < \tilde{t}$ (where s^t is composed of the first t

components of $s^{\tilde{t}}$). Assume w.l.o.g. that $c_2^{t'} = 0$ for $t < t' < \tilde{t}$ (i.e., choose t so that this is satisfied on $s^{\tilde{t}}$). We shall change the contract at dates t and $\tilde{t}$ (only), and demonstrate an improvement. Consider a small increase in $a_1^{\tilde{t}}$, $\Delta a_1^{\tilde{t}} > 0$, and let $\tilde{\Delta}$ ($\simeq D_2'(a_1^{\tilde{t}})\Delta a_1^{\tilde{t}}$) be the resulting increase in agent 2's deviation payoff. There are two cases to consider depending on whether $c_1^{\tilde{t}} > 0$ or $c_1^{\tilde{t}} = 0$.

Case (a): $c_1^{\tilde{t}} > 0$. To preserve agent 2's self-enforcing constraint, transfer $\tilde{\Delta}$ from agent 1 at date $\tilde{t}$, so that $\Delta c_2^{\tilde{t}} = \tilde{\Delta}$, but allocate the extra output to agent 1, $\Delta c_1^{\tilde{t}} \simeq (\partial y(a_1^{\tilde{t}}, a_2^{\tilde{t}})/\partial a_1)\Delta a_1^{\tilde{t}} - \tilde{\Delta}$. Agent 1's self-enforcing constraint holds as it was slack initially. Thus both self-enforcing constraints hold at $\tilde{t}$. Since $c_2^t > 0$, the increase in 1's effort can be compensated at t, and the increase in surplus must imply a Pareto-improvement at t. Specifically: cut agent 2's consumption at date t so that $\Delta c_2^t = -\delta^{\tilde{t}-t}\tilde{\pi}\tilde{\Delta} < 0$ where $\tilde{\pi} > 0$ is the probability of reaching $s^{\tilde{t}}$ from s^t. This consumption is given to agent 1 so $\Delta c_1^t = -\Delta c_2^t > 0$. Thus the change in discounted utility for the two agents at date t is

$$\begin{aligned}\Delta V_1(s^t) &= \delta^{\tilde{t}-t}\tilde{\pi}\tilde{\Delta} + \delta^{\tilde{t}-t}\tilde{\pi}(\Delta c_1^{\tilde{t}} - \Delta a_1^{\tilde{t}}) \\ &\simeq \delta^{\tilde{t}-t}\tilde{\pi}\tilde{\Delta} + \delta^{\tilde{t}-t}\tilde{\pi}\left(\left(\frac{\partial y(a_1^{\tilde{t}}, a_2^{\tilde{t}})}{\partial a_1} - 1\right)\Delta a_1^{\tilde{t}} - \tilde{\Delta}\right)\end{aligned}$$

which is positive since $\partial y(a_1^{\tilde{t}}, a_2^{\tilde{t}})/\partial a_1 > 1$ ($a_1^{\tilde{t}}$ is under-efficient by assumption); $\Delta V_2(s^t) = -\delta^{\tilde{t}-t}\tilde{\pi}\tilde{\Delta} + \delta^{\tilde{t}-t}\tilde{\pi}\tilde{\Delta} = 0$. Thus the period t constraints hold as actions are unchanged, and there is a Pareto-improvement. It remains to check the constraints at periods t' for $t < t' < \tilde{t}$: agent 2's future utility (at $\tilde{t}$) has increased so his constraints still hold. However agent 1's utility at $\tilde{t}$ may have fallen, decreasing her payoff at t'. Nevertheless, agent 1 gets all consumption so $c_1^{t'} > 0$ and hence by Theorem 2(ii) $a_1^{t'} \leq a_1^*(a_2^{t'})$, unless $y(a^{t'}) = 0$, in which case $a_1^{t'} = 0$, so again $a_1^{t'} \leq a_1^*(a_2^{t'})$. Also by Theorem 1 $a_1^{t'} \geq a_1^N(a_2^{t'})$. Likewise by Theorem 1 $a_2^{\tilde{t}} \geq a_2^{NE}$, and we have shown that agent 1's constraint holds at $\tilde{t}$, so $\tilde{V}_1(s^{\tilde{t}}) \geq D_1(a_2^{\tilde{t}}) \geq D_1(a_2^{NE})$ as $D_1(\cdot)$ nondecreasing, where $\tilde{V}_1(s^{\tilde{t}}) \equiv V_1(s^{\tilde{t}}) - \Delta c_2^{\tilde{t}}$ is agent 1's new utility. In the other (unreached) states at $\tilde{t}$, the corresponding inequality holds by equilibrium, so continuation utilities after $t' = \tilde{t} - 1$ satisfy $V_{1,r} \geq D_1(a_{2,r}^{NE}, r)$, for all $r \in \mathcal{S}$. Lemma 5 can thus be applied to ensure her constraints hold at $\tilde{t} - 1$. Working backwards, the same holds for all $t' > t$. As all the constraints are met a Pareto-improvement has been found.

Case (b): $c_1^{\tilde{t}} = 0$. Continue to allocate all output to agent 2 as a_1 is increased. We can apply Lemma 5 at $\tilde{t}$: From $c_1^{\tilde{t}} = 0$ we have $c_2^{\tilde{t}} > 0$ and hence $a_2^{\tilde{t}} \leq a_2^*(a_1^{\tilde{t}})$, unless $y(a^{\tilde{t}}) = 0$, in which case $a_2^{\tilde{t}} = 0$, so then again $a_2^{\tilde{t}} \leq a_2^*(a_1^{\tilde{t}})$. Since $a_2^*(\cdot)$ is weakly increasing, and a_2 is held constant at $a_2^{\tilde{t}}$, $a_2^{\tilde{t}} \leq a_2^*(a_1^{\tilde{t}} + \Delta a_1^{\tilde{t}})$. However, it is possible that $a_2^{\tilde{t}} < a_2^N(a_1^{\tilde{t}} + \Delta a_1^{\tilde{t}})$, but in this case also increase $a_2^{\tilde{t}}$ to $a_2^N(a_1^{\tilde{t}} + \Delta a_1^{\tilde{t}})$ and allocate all additional output to agent 2. Thus $a_2^N(a_1^{\tilde{t}} + \Delta a_1^{\tilde{t}}) \leq a_2 \leq a_2^N(a_1^{\tilde{t}} + \Delta a_1^{\tilde{t}})$. Likewise at $\tilde{t} + 1$, $V_{1,r} \geq D_1(a_{2,r}^{NE}, r)$, for all $r \in \mathcal{S}$, by the original equilibrium being optimum (as argued in (a)). Thus from Lemma 5 agent 2's constraint holds at $\tilde{t}$. Agent 1's self-enforcing constraint holds as it was slack initially. Since net surplus has risen by the increase in action(s), agent 1 is more than compensated at t, while keeping agent 2 no worse off (and the constraints at t' for $t < t' < \tilde{t}$ continue to hold), following the logic of case (a).

(ii) We now prove the second part of the theorem. Assume by contradiction that $a_2^{\tilde{t}} >$

$a_2^*(a_1^{\tilde{t}})$ but that $c_2^t > 0$ after s^t for some $t < \tilde{t}$. Assume w.l.o.g. that $c_2^{t'} = 0$ for $t < t' < \tilde{t}$ (we can choose t so that this is satisfied on $s^{\tilde{t}}$). We shall change the contract at dates t and $\tilde{t}$ (only), and demonstrate a Pareto-improvement. By Theorem 2(ii) as $a_2^{\tilde{t}} > a_2^*(a_1^{\tilde{t}})$ we must have $c_2^{\tilde{t}} = 0$ and so $c_1^{\tilde{t}} = y(a_1^{\tilde{t}}, a_2^{\tilde{t}})$. Now consider a small change in $a_2^{\tilde{t}}$ of $\Delta a_2^{\tilde{t}} < 0$, but continue allocating all output to agent 1. If $a_1^{\tilde{t}} > 0$ then since $c_1^{\tilde{t}} = y(a_1^{\tilde{t}}, a_2^{\tilde{t}}) > 0$ and thus $a_1^{\tilde{t}} \leq a_1^*(a_2^{\tilde{t}})$ by Theorem 2(ii), agent 1 is unconstrained by Lemma 5 and a small cut in the consumption of agent 1 will not violate his constraint. If on the other hand $a_1^{\tilde{t}} = 0$, then after the change we have $a_1^{\tilde{t}} \leq a_1^*(a_2^{\tilde{t}} + \Delta a_2^{\tilde{t}})$, while $a_1^N(a_2^{\tilde{t}} + \Delta a_2^{\tilde{t}}) \leq a_1^N(a_2^{\tilde{t}}) \leq a_1^{\tilde{t}}(= 0)$ as $a_1^N(\cdot)$ non-increasing. So Lemma 5 applies and again agent 1's constraint must hold. For agent 2 since $a_1^{\tilde{t}}$ is unchanged and $w_2^{\tilde{t}}$ is increased (the cut in effort implies $\Delta w_2^{\tilde{t}} = -\Delta a_2^{\tilde{t}} > 0$), his self-enforcing constraint is satisfied at $\tilde{t}$. Thus both self-enforcing constraints hold at $\tilde{t}$. Agent 1 is getting all consumption and so satisfies the self-enforcing constraint at all intervening dates t', $t < t' < \tilde{t}$, repeating the argument of the first part of the proof, while agent 2 is better off due to the improvement at $\tilde{t}$, so his constraints are not violated. The increase in surplus at $\tilde{t}$ allows for a Pareto-improvement at t: To compensate agent 1 at date t for any decreased consumption at date $\tilde{t}$, $\Delta c_1^{\tilde{t}} < 0$, let $\Delta c_1^t = -\delta^{(\tilde{t}-t)}\tilde{\pi}\Delta c_2^{\tilde{t}} > 0$, where we denote by $\tilde{\pi} > 0$ the probability of reaching $s^{\tilde{t}}$ at date $\tilde{t}$ starting from t on the same path. We take this increase from agent 2, so $\Delta c_2^t = \delta^{(\tilde{t}-t)}\tilde{\pi}\Delta c_1^{\tilde{t}}$ and since $\Delta c_1^{\tilde{t}} \simeq \left(\partial y(a_1^{\tilde{t}}, a_2^{\tilde{t}})/\partial a_2\right)\Delta a_2^{\tilde{t}}$ we have that the change in discounted utility for agent 2 is

$$\Delta V_2(s^t) \simeq \delta^{(\tilde{t}-t)}\tilde{\pi}\left(\left(\partial y(a_1^{\tilde{t}}, a_2^{\tilde{t}})/\partial a_2\right) - 1\right)\Delta a_2^{\tilde{t}} > 0$$

since $\partial y(a_1^{\tilde{t}}, a_2^{\tilde{t}})/\partial a_2 < 1$ (as $a_2^{\tilde{t}} > a_2^*(a_1^{\tilde{t}})$) and $\Delta a_2^{\tilde{t}} < 0$. This Pareto-improvement at date t implies that both self-enforcing constraints hold at t, and moreover the original contract was not optimum. □

LEMMA 6: *If the action vector $\tilde{a}$ is myopic efficient for some $(V_{1,r}, V_{2,r})_{r\in\mathcal{S}}$ with $V_{i,r} \geq D_i(a_{j,r}^{NE}, r)$ all $r \in \mathcal{S}$, $i, j = 1, 2$, $j \neq i$, then*

$$y(\tilde{a}_1, \tilde{a}_2, s) - \tilde{a}_i + \delta \sum_{r\in\mathcal{S}} \pi_{sr} D_i(a_{j,r}^{NE}, r) \geq D_i(\tilde{a}_j, s), \tag{A.13}$$

for $i, j = 1, 2$, $j \neq i$; i.e., giving all output to agent i implies that the current self-enforcing constraint continues to hold for i even if the continuation utilities are replaced with the deviation payoffs.

PROOF OF LEMMA 6: By adapting the proof of Theorem 1 it is easy to check that $\tilde{a}_i \geq a_i^N(\tilde{a}_j)$ for $i, j = 1, 2$, $j \neq i$. Next, suppose $\tilde{a}_i > a_i^*(\tilde{a}_j)$. We shall establish a contradiction. If $c_i > 0$ then lowering both a_i and c_i by an equal small amount is feasible and raises the net surplus (give this extra to agent $j \neq i$), contrary to assumption. Thus $\tilde{a}_i > a_i^*(\tilde{a}_j)$ and $c_i > 0$ is impossible. If $c_i = 0$ and hence $c_j = y(\tilde{a}_1, \tilde{a}_2)$ then lowering a_i will lower the consumption of agent j and hence might violate j's self-enforcing constraint. But $\tilde{a}_i > a_i^*(\tilde{a}_j)$ implies $y(\tilde{a}_1, \tilde{a}_2) > 0$ so $c_j > 0$ and thus $\tilde{a}_j \leq a_j^*(\tilde{a}_i)$ by the above argument. Suppose that a_i is reduced to $a_i^*(\tilde{a}_j)$ with agent j still receiving all the output. Clearly surplus has increased, and i's utility has risen while $D_i(\tilde{a}_j)$ is unchanged, so i's self-enforcing constraint is still satisfied.

(i) If $\tilde{a}_j \leq a_j^*(a_i^*(\tilde{a}_j))$ then leave a_j unchanged at $\tilde{a}_j$. Thus, as a_j^N can only fall with the cut in a_i, $a_j^N(a_i^*(\tilde{a}_j)) \leq \tilde{a}_j \leq a_j^*(a_i^*(\tilde{a}_j))$. All the conditions of Lemma 5 are satisfied at $(a_i^*(\tilde{a}_j), \tilde{a}_j)$, so the current self-enforcing constraint for j holds:

$$y(a_i^*(\tilde{a}_j), \tilde{a}_j) - \tilde{a}_j + \delta \sum_{r \in \mathcal{S}} \pi_{sr} V_{j,r} \geq D_j \left(a_i^*(\tilde{a}_j) \right).$$

(ii) As a_i is reduced $a_j^*(a_i)$ may have fallen below $\tilde{a}_j$ however. If this is the case, cut a_j to $a_j^*(a_i^*(\tilde{a}_j))$. Repeating the argument just given, j's constraint will be satisfied at $(a_i^*(\tilde{a}_j), a_j^*(a_i^*(\tilde{a}_j)))$, while the cut in a_j cannot lead to a violation in i's constraint. So again the changed contract satisfies the self-enforcing constraints.
In both cases (i) and (ii) the reduction in overinvestment leads to an increase in current surplus, contrary to the assumption that $\tilde{a}$ was myopic efficient. We conclude that $\tilde{a}_i > a_i^*(\tilde{a}_j)$ is impossible. Thus $a_i^*(\tilde{a}_j) \geq \tilde{a}_i \geq a_i^N(\tilde{a}_j)$. Thus Lemma 5 can again be appealed to, at $\tilde{a}$ with continuation utilities set *equal* to $D_i(a_{j,r}^{NE}, r)$, establishing (A.13). □

LEMMA 7: *Take any myopic efficient action a for some $(V_{1,r}, V_{2,r})_{r \in \mathcal{S}}$; then given any alternative continuation utilities $\left(\hat{V}_{1,r}, \hat{V}_{2,r}\right)_{r \in \mathcal{S}}$ satisfying, for all $r \in \mathcal{S}$, $\hat{V}_{1,r} + \hat{V}_{2,r} \geq V_{1,r} + V_{2,r}$, $\hat{V}_{i,r} \geq D_i(a_{j,r}^{NE}, r)$, there is a division of $y(a_1, a_2)$ such that the self-enforcing constraints are satisfied with the same action a.*

PROOF OF LEMMA 7: We need to show that both self-enforcing constraints can still hold with some output division $(\hat{c}_1, \hat{c}_2)$, where $\hat{c}_1 + \hat{c}_2 = y(a_1, a_2)$, i.e.,

$$\hat{c}_i - a_i + \delta \sum_r \pi_{sr} \hat{V}_{i,r} \geq D_i(a_j) \tag{A.14}$$

for $i, j = 1, 2,\ j \neq i$. By assumption they hold in the equilibrium supporting a:

$$c_i - a_i + \delta \sum_r \pi_{sr} V_{i,r} \geq D_i(a_j) \tag{A.15}$$

for $i, j = 1, 2; j \neq i$. Let $i = 1$. If (A.14) holds at $\hat{c}_1 = 0$, then setting $\hat{c}_2 = y(a_1, a_2)$ guarantees that (A.14) holds also for agent 2 by Lemma 6 (because $\hat{V}_{i,r} \geq D_i(a_{j,r}^{NE}, r)$). Otherwise choose $\hat{c}_1$ such that (A.14) holds with equality for $i = 1$; by continuity this is possible as (A.14) holds at $\hat{c}_1 = y(a_1, a_2)$ using Lemma 6 again. Suppose that (A.14) is violated for $i = 2$. Summing the left hand side of (A.14) over i thus implies

$$y(a_1, a_2) - \sum_i a_i + \delta \sum_r \pi_{sr} \sum_i \hat{V}_{i,r} < \sum_i D_i(a_j). \tag{A.16}$$

But summing (A.15) over i implies that

$$y(a_1, a_2) - \sum_i a_i + \delta \sum_r \pi_{sr} \sum_i V_{i,r} \geq \sum_i D_i(a_j),$$

and since the left hand side is smaller than the left hand side of (A.16) by $\hat{V}_{1,r} + \hat{V}_{2,r} \geq V_{1,r} + V_{2,r}$, there is a contradiction. Hence we conclude that there is a division of $y(a_1, a_2)$ such that (A.14) holds for both agents. □

THEOREM 4: *Any joint utility maximizing self-enforcing contract starting at date t from* s^t, $\Gamma(s^t)$, *has with probability one current joint surplus maximizing actions at each date* $\tau \geq t$.

PROOF OF THEOREM 4: Consider the following putative equilibrium. At t, in state s_t, set a to a current joint surplus maximizing action compatible with equilibrium in this state. As the set of possible actions can be restricted to a compact set $\tilde{A}(s_t)$ the maximizing actions exist. Let $(V_{1,r}, V_{2,r})_{r\in\mathcal{S}}$ be the corresponding continuation utilities from $t+1$. At time $t+1$, in any state r and regardless of previous history, follow the (or an) equilibrium that maximizes the joint utility from that point onwards, yielding utilities we denote by $\left(\hat{V}_{1,r}, \hat{V}_{2,r}\right)_{r\in\mathcal{S}}$. Clearly a is myopic efficient relative to $(V_{1,r}, V_{2,r})_{r\in\mathcal{S}}$, so by Lemma 7 there is a split of $y(a_1, a_2)$ which sustains this as an equilibrium from t on when $(V_{1,r}, V_{2,r})_{r\in\mathcal{S}}$ is replaced by $\left(\hat{V}_{1,r}, \hat{V}_{2,r}\right)_{r\in\mathcal{S}}$ since the latter have a maximal sum in each state. Note that this must provide maximal joint utility since the current joint surplus is maximized at t, and joint utilities are maximal from $t+1$. Consequently starting from any state s_t, a joint utility maximizing equilibrium must involve a current joint surplus maximizing action compatible with equilibrium in state s_t, for if it did not, replacing it by the equilibrium just constructed would lead to a higher utility sum. At $t+1$, since the utility sum is maximal in each state r, repeating the above argument again confirms that current joint surplus is maximal for state r. So a joint utility maximizing equilibrium must involve a current joint surplus maximizing action compatible with equilibrium in every state and date. □

LEMMA 8: *If in an optimum contract* s^t *has positive probability, both self-enforcing constraints bind at t, and* $a_i^t \leq a_i^*(a_j^t)$, $i, j = 1, 2$, $i \neq j$, *then the contract must specify joint utility maximization from* $t+1$ *(i.e., in every positive probability successor state).*

PROOF OF LEMMA 8: By assumption that both constraints bind we have

$$c_i - a_i + \delta \sum_r \pi_{sr} V_{i,r} = D_i(a_j)$$

for $i, j = 1, 2$, $i \neq 2$. Suppose, to the contrary of the claim, that the pair $\left(V_{1,r'}, V_{2,r'}\right)$ does not maximize joint utility in at least one successor state r'. We can change the contract as follows. Replace $(V_{1,r'}, V_{2,r'})$ by $\left(\hat{V}_{1,r'}, \hat{V}_{2,r'}\right) \in \mathcal{V}_{r'}$ such that $\hat{V}_{1,r'} + \hat{V}_{2,r'} > V_{1,r'} + V_{2,r'}$ (and recall that we must have $\hat{V}_{i,r'} \geq D_i(a_{j,r'}^{NE}, r')$, $i, j = 1, 2, j \neq i$), and choose a division $\hat{c}$ of the current output $y(a_1, a_2)$ (i.e., holding a constant) such that

$$\hat{c}_i - a_i + \delta \sum_{r\neq r'} \pi_{sr} V_{i,r} + \delta \pi_{sr'} \hat{V}_{i,r'} \geq D_i(a_j) \tag{A.17}$$

for $i, j = 1, 2,\ j \neq i$, with a strict inequality for at least one i. This is possible by the fact that if $\hat{c}_i = y(a_1, a_2)$ then, as $a_i \geq a_i^N\left(\tilde{a}_j\right)$ by Theorem 1, $a_i \leq a_i^*(a_j)$ by hypothesis, and $V_{i,r} \geq D_i(a_{j,r}^{NE}, r)$, $r \neq r'$, $\hat{V}_{i,r'} \geq D_i(a_{j,r'}^{NE}, r')$, the self-enforcing constraint for agent i must be satisfied (Lemma 5). The argument then follows the proof of Lemma 7; however the increase in aggregate utility implies the constraint (A.17) is strict for one agent. This is a Pareto-improvement, so the original contract could not have been optimal. □

THEOREM 5: *For any optimum contract, there exists a random time T which is finite with probability one such that for* $t \geq T$, a^t *is current joint surplus maximizing.*

Proof of Theorem 5: First suppose that $\Lambda(\mathcal{V}) \neq (0,0)$; otherwise the proposition is trivial.
(i) Suppose first there exists $(V_1, V_2) \in \mathcal{V}$ with $V_1, V_2 > 0$. Thus in any optimum, with payoffs $(\tilde{V}_1, \tilde{V}_2)$, either $\tilde{V}_1 \geq V_1$, or $\tilde{V}_2 \geq V_2$. We deal first with the former case. In this equilibrium, choosing $t' > 0$ so that $\delta^{t'} \bar{w}/(1-\delta) < V_1/2$, where $\bar{w}$ is an upper bound on equilibrium per-period payoffs (this follows from Assumption 1(vi) on the boundedness of $y(\cdot)$) must be that $c_1^t \geq \mu_1 \equiv V_1/2t'$ for some history s^t which occurs with positive probability with $t < t'$, otherwise V_1 cannot be accumulated. However, $c_1^t \geq \mu_1$ implies that $y(a_1^t, a_2^t, s_t) \geq \mu_1$. For convenience, let $s_t = r$. If $\lim_{a_1 \to \infty} y(a_1, 0, r) < \mu_1$ then define $\underline{a}_2 > 0$ to be such that $\lim_{a_1 \to \infty} y(a_1, \underline{a}_2, r) = \mu_1$ (this exists by y nondecreasing and by continuity); clearly $a_2^t \geq \underline{a}_2$, and agent 2's discounted consumption at s^t must at least equal $\underline{a}_2$ in order for continuation utility to be nonnegative. If $\lim_{a_1 \to \infty} y(a_1, 0, r) < \mu_1$ define $\tilde{a}_1$ so that $y(\tilde{a}_1, 0, r) = \mu_1$. By Assumption 2, $\phi_2(\tilde{a}_1, 0, r) > 0$. Consequently $D_2(\tilde{a}_1, r) \geq \phi_2(\tilde{a}_1, 0, r) > 0$. By continuity there exists $(\underline{a}_1, \underline{a}_2)$ such that $y(\underline{a}_1, \underline{a}_2, r) = \mu_1$, with $D_2(\underline{a}_1, r) \geq D_2(\tilde{a}_1, r)/2$ *and* $\underline{a}_2 > 0$ (note that $\underline{a}_1 = \tilde{a}_1$ and $\underline{a}_2$ is arbitrary if a_2 does not contribute to output). For the case where $\lim_{a_1 \to \infty} y(a_1, 0, r) = \mu_1$ a slight modification of this argument yields the same conclusion. Thus we conclude that $y(a_1^t, a_2^t, s^t) \geq \mu_1$ implies either $a_1^t \geq \underline{a}_1 > 0$ or $a_2^t \geq \underline{a}_2 > 0$. In the former case ($a_1^t \geq \underline{a}_1$), since $D_2(a_1, r) \geq D_2(\underline{a}_1, r) > 0$, positive consumption now or in the future must generate a positive current utility to maintain agent 2's self-enforcing constraint. In the latter case ($a_2^t \geq \underline{a}_2$), since $D_2(a_1, r) \geq 0$, the negative utility from $a_2 > 0$ must be compensated by positive consumption now or in the future. Thus taking these two cases together, agent 2's discounted expected consumption at s^t must at least equal $\min\{D_2(\underline{a}_1, r)/2, \underline{a}_2\} > 0$, which depends only on V_1 (fixed in the proof). So, in the same way we showed that $c_1^t \geq \mu_1$ for some $t < t'$, we can show there exists $t'' \geq t'$ such that $c_2^t \geq \mu_2$ for some $t < t''$ with positive probability. Next, if $\tilde{V}_2 \geq V_2$, we can repeat the argument in a symmetric fashion. Although μ_1 and μ_2 depend on state r, we can take their minima over all $r \in \mathcal{S}$, and over the two cases $\tilde{V}_1 \geq V_1$ and $\tilde{V}_2 \geq V_2$, and we denote these minima by $\underline{\mu}_1$ and $\underline{\mu}_2$ henceforth. Putting this together, for both agents, $c_i^t \geq \min\left\{\underline{\mu}_1, \underline{\mu}_2\right\} > 0$, for some $t < t''$ (not necessarily at the same date t for each agent) with positive probability at least equal to the minimum probability (this is positive by $\mathcal{S}$ finite) $\underline{\pi}$, say, of any t''-period positive probability history emanating from s_0 (where $\min\left\{\underline{\mu}_1, \underline{\mu}_2\right\}$, t'', and $\underline{\pi}$ are all independent of the particular equilibrium).
(ii) If no $(V_1, V_2) \in \Lambda(\mathcal{V}_s)$ exists with $V_1, V_2 > 0$, then there exists either a unique optimum $(V_1 > 0, V_2 = 0)$, or a unique optimum $(V_1 = 0, V_2 > 0)$, or both points exist as optima. In either case the argument above can be repeated *mutatis mutandis*.
Let $\hat{t}$ (random) denote the earliest date such that both consumptions have been positive, i.e., the first period for which $c_1^{\tilde{t}} > 0$ and $c_2^{t'} > 0$ for $\tilde{t}$, $t' \leq \hat{t}$. We first show that $\hat{t}$ is finite almost surely. Note that by optimality, whenever s_0 occurs on a positive probability history, utilities must belong to Λ. From the above argument and given that all states communicate then after any positive probability s^t, as s_0 can be reached with positive probability in at most $n-1$ periods, there is a probability of at least $\hat{\pi}$, $\underline{\pi} \geq \hat{\pi} > 0$, such that $c_1 > 0$ and $c_2 > 0$ within the next $t'' + n - 1$ periods. Consequently $\text{Prob}[c_1^t = 0,\ \forall t$ or $c_1^t = 0\ \forall t] = 0$. We conclude that such $\hat{t}$ exists for almost all sample paths.
After $\hat{t}$, both c_1 and c_2 have been positive at some point in the past. From Theorem 3(ii) we know that for $t > \hat{t}$, $a_i^t \leq a_i^*(a_j^t)$. And if i is unconstrained at t, from Theorem 3(i),

$a_i^t \geq a_i^*(a_j^t)$, $j \neq i$, while from Theorem 2(i), $a_j^t \geq a_j^*(a_i^t)$, hence $a^t = a^*$. So inefficiency can only occur if both agents are constrained. But from Lemma 8 net output is maximized thereafter. We conclude then on any path after $\hat{t}$ either actions are always efficient, or there is at most one date at which actions are not efficient, but this is then followed by the myopic efficient net output maximizing actions thereafter. □

THEOREM 6: *Whenever the Nash actions $a_{i,s}^{NE}$ are positive, $i = 1, 2$, then with probability one an optimum path has two phases, where $i = 1$ or 2:*
***Phase 1**: $c_i^t = 0$, $a_i > a_i^*$ and $a_j \leq a_j^*$, for $0 \leq t < \tilde{t}$, $j \neq i$, where $\infty > \tilde{t} \geq 0$;*
***Phase 2**: $a_1^t \leq a_1^*$ and $a_2^t \leq a_2^*$ for $t \geq \tilde{t}$ and after the first period of phase 2, if a^* is feasible in s_t then $a^t = a^*$; otherwise $a < a^*$, both constraints bind, and $c > 0$.*

PROOF OF THEOREM 6: A. We show first that if at $t \geq 0$, $c_1^t = 0$, we have $a_1^t \geq a_1^*$ and $a_2^t \leq a_2^*$. Moreover, if $t \geq 1$ and either inequality is strict, then if $c_1^{t-1} = 0$, $a_1^{t-1} > a_1^*$. (And likewise if the indices are swapped.) To see the first part of the claim (suppressing the t superscripts), note that by $a_i \geq a_{i,s}^{NE} > 0$, $i = 1, 2$, $y > 0$ and so $c_2 > 0$; thus $a_2 \leq a_2^*$ from Theorem 2(ii). Moreover, by Lemma 5, agent 2 is unconstrained (as $a_2 > 0$); hence $a_1 \geq a_1^*$. For the second part of the claim, we shall consider increasing agent 1's utility at t a small amount by decreasing a_1 and at the same time increasing a_2 so that agent 1's constraint is unaffected (so if it initially binds, it remains satisfied but binding), while holding $c_1 = 0$, and the future contract *fixed*. Let V_i^t denote current (to t) discounted equilibrium utility and $V_{i,r}^{t+1}$ the same at $t+1$ in state r. Then consider the equations

$$\begin{aligned} V_1 + a_1 &= \delta \sum_{r \in \mathcal{S}} \pi_{s_t r} V_{1,r}^{t+1}, \\ V_2 - y(a_1, a_2) + a_2 &= \delta \sum_{r \in \mathcal{S}} \pi_{s_t r} V_{2,r}^{t+1}, \\ V_1 - D_1(a_2) &= V_1^t - D_1\left(a_2^t\right). \end{aligned} \tag{A.18}$$

These are satisfied at $\left(V_1^t, V_2^t, a_1^t, a_2^t\right)$ (noting that the equality $c_2 = y(a_1, a_2)$ has been used to substitute out for c_2 in the second line). As the functions $y(a_1, a_2)$, $D_1(a_2)$ and $D_2(a_1)$ are continuous and differentiable, and $\partial D_1(a_2^t)/\partial a_2 \neq 0$ (as $\partial D_1(a_2^t)/\partial a_2 > 0$ from Lemma 3), observing that $0 < a_2^t \leq a_2^*$ implies $\partial y(a_1^t a_2^t)/\partial a_2 > 0$), the *implicit function theorem* asserts the existence of continuously differentiable functions $a_1(V_1)$, $a_2(V_1)$ and $\tilde{V}_2(V_1)$ in an open interval around V_1^t such that $a_1(V_1^t) = a_1^t$ etc. which satisfy (A.18), and such that

$$\tilde{V}_2'(V_1^t) = -\frac{\partial y(a_1^t, a_2^t)}{\partial a_1} - \frac{\left(1 - \frac{\partial y(a_1^t, a_2^t)}{\partial a_2}\right)}{\frac{\partial D_1(a_2^t)}{\partial a_2}}. \tag{A.19}$$

Agent 2 is unconstrained so remains unconstrained for small changes in V_1 away from V_1^o, the value at the optimum, while the third line of (A.18) ensures that agent 1's constraint holds, so the contracts defined as V_1 is varied are self-enforcing. Now suppose that either inequality is strict, i.e., $a_1^t > a_1^*$ or $a_2^t < a_2^*$. Then $\tilde{V}_2'(V_1^t) > -1$. At $t-1$, suppose that $a_1^{t-1} \leq a_1^*$. A small increase in a_1^{t-1} of $\Delta > 0$ holding $c_1^{t-1} = 0$, leads to an increase in agent 2's payoff of at least, to a first-order approximation, Δ (since $\partial y/\partial a_1 \geq 1$) while agent 1 can be compensated at t by a value of V_1 satisfying $(V_1^o - V_1)/\delta \pi_{s_{t-1} s_t} = \Delta$, so that agent 2's utility changes by approximately $\Delta + \delta\pi_{s_{t-1}s_t}\left(\tilde{V}_2'(V_1)(V_1^o - V_1)/\delta\pi_{s_{t-1}s_t}\right) > 0$. Agent 1's constraint at $t-1$ holds as her deviation payoff is unchanged (a_2^{t-1} is unchanged), and agent 1's constraint must hold by Lemma 5 (recall $c_1^{t-1} = 0$ by hypothesis). By construction of $\tilde{V}_2(V_1)$ the constraints hold from t. Thus a Pareto-improvement has been found for Δ small enough, contrary to the assumed optimality. Hence $a_1^{t-1} > a_1^*$.

B. Suppose on a path that $a_1^{t'} > a_1^*$ for some $t' \geq 0$. Then we have $c_1^t = 0$ for all $t \leq t'$ (by Theorem 2(ii) and Theorem 3(ii)). However, $a_1^{t'} > a_1^*$ then implies by repeated application of part A that $a_1^t > a_1^*$ for all $t < t'$. Moreover as $c_2^t > 0$ for all $t \leq t'$, $a_2^t \leq a_2^*$ for all $t \leq t'$. Hence Phase 1 conditions are satisfied for all $t \leq t'$ and for $i = 1$. By the same logic, if at any point t' Phase 2 conditions hold (i.e., $a^{t'} \leq a^*$), they must hold at all subsequent dates, since a violation (i.e., $a_i^t > a_i^*$ for $t > t'$ and some i) would imply that $a_i^{t'} > a_i^*$ also. Thus any positive probability $s^{t'}$ must satisfy the two-phase property up to t'. The fact that $\tilde{t}$ in the statement of the theorem is a.s. finite follows from the argument in the proof of convergence that the date at which both consumptions have been positive is itself a.s. finite (as both consumptions positive implies $a^t \leq a^*$ by Theorem 3(ii)).

C. Suppose that at $\tilde{t}+1$ in some state r with $\pi_{s_{\tilde{t}}r} > 0$ at least one constraint does not bind. Suppose w.l.o.g. that agent 2 is unconstrained and we can repeat the argument of Part A, with c_1 again being held constant, but now at a possibly positive level. Again we have a locally differentiable relationship between utilities arising from self-enforcing contracts at $\tilde{t}+1$, with slope $\tilde{V}_2'(V_1^{\tilde{t}+1})$ given by (A.19). As agent 2 is unconstrained $a_1^{\tilde{t}+1} \geq a_1^*$, and by Phase 2, $a_1^{\tilde{t}+1} \leq a_1^*$, so $a_1^{\tilde{t}+1} = a_1^*$. Consequently if $a_2^{\tilde{t}+1} < a_2^*$, then $\tilde{V}_2'(V_1^{\tilde{t}+1}) > -1$, and repeating the argument at the end of Part A, $a_1^{\tilde{t}} > a_1^*$, which contradicts the definition of $\tilde{t}$. Thus it must be that $a^{\tilde{t}+1} = a^*$. Thus if $a^{\tilde{t}+1} \neq a^*$, both agents are constrained, and it cannot be that $c_i^{\tilde{t}+1} = 0$ for either $i = 1$ or 2 as that would imply agent $j \neq i$ is unconstrained by Lemma 5 (as $a_j^{\tilde{t}+1} > 0$). Hence the only alternative is that both are constrained, and $c^{\tilde{t}+1} > 0$.

D. Moreover if this latter is the case, it cannot be that $a_i^{\tilde{t}+1} = a_i^*$ for either $i = 1$ or 2. Suppose to the contrary and that w.l.o.g. $a_1^{\tilde{t}+1} = a_1^*$. Then consider the equations

$$\begin{aligned}
V_1 - c_1 + a_1 &= \delta \sum_{r\in\mathcal{S}} \pi_{sr} V_{1,r}^{\tilde{t}+2}, \\
V_2 - y(a_1, a_2) + c_1 + a_2 &= \delta \sum_{r\in\mathcal{S}} \pi_{sr} V_{2,r}^{\tilde{t}+2}, \\
V_1 - D_1(a_2) &= 0, \\
V_2 - D_2(a_1) &= 0,
\end{aligned} \tag{A.20}$$

where $V_{i,r}^{\tilde{t}+2}$ is continuation utility for i on the equilibrium path from $\tilde{t}+2$ in state r. These are satisfied at $\left(V_1^{\tilde{t}+1}, V_2^{\tilde{t}+1}, c_1^{\tilde{t}+1}, a_1^{\tilde{t}+1}, a_2^{\tilde{t}+1}\right)$. As the functions $y(a_1, a_2)$, $D_1(a_2)$ and $D_2(a_1)$ are continuous and differentiable, the *implicit function theorem* asserts, provided that

$$\frac{\partial D_1(a_2^{\tilde{t}+1})}{\partial a_2}\left(1 + \frac{\partial D_2(a_1^{\tilde{t}+1})}{\partial a_1} - \frac{\partial y(a_1^{\tilde{t}+1}, a_2^{\tilde{t}+1})}{\partial a_1}\right) \neq 0, \tag{A.21}$$

the existence of continuously differentiable functions $c_1(V_1)$, $a_1(V_1)$, $a_2(V_1)$ and $\tilde{V}_2(V_1)$ in an open interval around $V_1^{\tilde{t}+1}$ such that $c_1(V_1^{\tilde{t}+1}) = c_1^{\tilde{t}+1}$ etc. which satisfy equation (A.20). As $a_1^{\tilde{t}+1} = a_1^*$, $\partial y(a_1^{\tilde{t}+1}, a_2^{\tilde{t}+1})/\partial a_1 = 1$, and also $\partial D_1(a_2^{\tilde{t}+1})/\partial a_2, \partial D_2(a_1^{\tilde{t}+1})/\partial a_1 > 0$ (from Lemma 3), since $\partial y/\partial a_i > 0$ as $a_i^{\tilde{t}+1} \leq a_i^*$ in Phase 2), so (A.21) holds. Since $c_1^{\tilde{t}+1} > 0$ and $c_2^{\tilde{t}+1} > 0$, nonnegativity constraints on consumption will also hold in an open interval and the self-enforcing constraints hold. Hence holding the future contract fixed, but varying V_1 varies the current contract according to $c_1(V_1)$, $a_1(V_1)$, $a_2(V_1)$, and traces out a series of self-enforcing contracts, such that 2's discounted utility is $\tilde{V}_2(V_1)$, with

$$\tilde{V}_2'(V_1^{\tilde{t}+1}) = -\frac{\frac{\partial D_2(a_1^{\tilde{t}+1})}{\partial a_1}\left(1 + \frac{\partial D_1(a_2^{\tilde{t}+1})}{\partial a_2} - \frac{\partial y(a_1^{\tilde{t}+1}, a_2^{\tilde{t}+1})}{\partial a_2}\right)}{\frac{\partial D_1(a_2^{\tilde{t}+1})}{\partial a_2}\left(1 + \frac{\partial D_2(a_1^{\tilde{t}+1})}{\partial a_1} - \frac{\partial y(a_1^{\tilde{t}+1}, a_2^{\tilde{t}+1})}{\partial a_1}\right)}.$$

As $a_2^{\tilde{t}+1} < a_2^*$, $\partial y(a_1^{\tilde{t}+1}, a_2^{\tilde{t}+1})/\partial a_2 > 1$, so given $\partial y(a_1, a_2)/\partial a_1 = 1$ it follows that $\tilde{V}_2'(V_1^{\tilde{t}+1}) > -1$, which we have shown is impossible as again it would imply $a_1^{\tilde{t}} > a_1^*$. Hence a contradiction, and therefore it is concluded that $a^{\tilde{t}+1} < a^*$. □

THEOREM 7: *In the case of one-sided investment where, say, agent 1 is the only contributor to output, then at any date t and state s, $a_1^t \leq a_1^*$; overinvestment never occurs in an efficient self-enforcing contract.*

PROOF OF THEOREM 7: Suppose to the contrary that $a_1^t > a_1^*$ for some s^t. From Theorem 2(ii), $c_1^t = 0$. Agent 1's optimal current payoff from defaulting is just the Nash breakdown payoff $= \phi_1(a_1^{NE}, s_t) - a_1^{NE}$. We thus have equilibrium current utility, w_1^t, is less than this breakdown payoff, as $a_1^{NE} \leq a_1^* < a_1^t$ and $c_1^t = 0 \leq \phi_1(a_1^{NE}, s_t)$. Denote this negative surplus by $\chi^t \equiv w_1^t - (\phi_1(a_1^{NE}, s_t) - a_1^{NE}) < 0$. Agent 1's discounted utility is $V_1(s^t) = w_1^t + \delta \sum_{s_{t+1} \in \mathcal{S}} \pi_{s_t s_{t+1}} V_1(s^{t+1})$, so defining the discounted surplus as $VS_1(s^t) \equiv V_1(s^t) - D_1(s_t)$ we have

$$VS_1(s^t) = \chi(s^t) + \delta \sum_{s_{t+1} \in \mathcal{S}} \pi_{s_t s_{t+1}} VS_1(s^{t+1}) \geq 0. \tag{A.22}$$

From equation (A.22) it follows that for at least one state at date $t+1$ with $\pi_{s_t s_{t+1}} > 0$ such that $VS_1(s^{t+1}) \geq -\chi(s^t)/\delta > 0$. Suppose that either $a_1^{t+1} = 0$ or $a_1^{t+1} > a_1^*$. In the former case, $f(a_1^{t+1}, s_{t+1}) = 0$, so $w_1^{t+1} = 0$. In the latter case, from Theorem 2(ii), $c_1^{t+1} = 0$, so $w_1^{t+1} < 0$. Consequently, there must, by repeating the earlier logic, be another successor state at date $t+2$ with $\pi_{s_{t+1} s_{t+2}} > 0$ such that continuation surplus $VS_1(s^{t+2}) \geq -\chi(s^t)/\delta^2$. We can repeat this argument if again either $a_1^{t+2} = 0$ or $a_1^{t=2} > a_1^*$. Since continuation surplus must be bounded, this can only happen a fixed number of times. Thus we must have (along such a path) in finite time $t'(> t)$, $0 < a_1^{t'} \leq a_1^*(s_{t'})$ and $VS_1(s^{t'}) > 0$ for the first time. Suppose first this happens at $t' = t+1$. Thus in this state at $t+1$, agent 1 is unconstrained. Consider frontloading 1's utility by increasing her action at $t+1$ in state s_{t+1} by $\Delta > 0$ and reducing it at t by $\delta \pi_{s_t s_{t+1}} \Delta$ to compensate (holding consumption constant). Agent 2's utility changes (to a first-order approximation) by $-\left(\partial f(a_1^t, s_t)/\partial a_1\right) \delta \pi_{s_t s_{t+1}} \Delta + \delta \pi_{s_t s_{t+1}} \left(\partial f(a_1^{t+1}, s_{t+1})/\partial a_1\right) \Delta$ which is positive by virtue of $a_1^t > a_1^*$ (so $\partial f(a_1^t, s_t)/\partial a_1 < 1$) and $0 < a_1^{t+1} \leq a_1^*(s_{t+1})$ (so $\partial f(a_1^{t+1}, s_{t+1})/\partial a_1 \geq 1$). No constraints are violated by this: agent 1 is unconstrained at $t+1$ ($VS_1(s^{t+1}) > 0$) so for Δ small her constraint is maintained; agent 2 receives the extra output at $t+1$ and by Assumption 2 his breakdown payoff increases by at most this amount, so his constraint holds. At t there is a Pareto-improvement and agent 2's breakdown payoff has not increased (and 1's is constant) so again the constraints hold. Thus we have a contradiction. The remaining possibility is that $t' > t+1$. A similar construction will lead to a Pareto-improvement at t, but now we have additionally to worry about constraints for periods $\hat{t}$ between t and $t+1$. By construction $VS_1(s^{\hat{t}}) > 0$ along the entire path, so for Δ small enough 1's continuation surplus remains positive. Agent 2's utility is backloaded, so his constraints are relaxed. Again we have a contradiction. □

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