Simplicial sets and Nash components: An elementary proof.

Dieter Balkenborg[*] Dries Vermeulen[†]

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Abstract

In this short note we characterize Nash equilibrium components in terms of their topological properties. As is well known, every Nash equilibrium component is a closed semi-algebraic set and can hence be triangulated. More precisely, it is homeomorphic to a finite connected simplicial complex. Conversely, we provide a simple construction showing that every simplicial complex is homoeomorphic to a Nash equilibrium component. Consequently, Nash equilibrium components provide a very rich class of topological spaces including all compact connected topological manifolds.

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1 Introduction

In non-cooperative game theory the claim that Nash equilibrium components can indeed have any conceivable shape they can reasonably be expected have seems to have the status of a "folk theorem". Few researchers in the field doubt this claim, yet no proof seems to be available, and it is unclear what is meant by "every conceivable shape". In this paper we provide the following "folk theorem" on the topological structure of Nash equilibrium components. The "only if" result is, of course, well known.

Theorem 1.1 *A connected topological space is homeomorphic to a Nash equilibrium component if and only if it is homeomorphic to a simplicial complex.*

Thus a Nash equilibrium component has topologically no additional structure beyond what follows directly from the definition and LLojasiewicz' famous triangulation result (1964). The "only if" results thus, of course, well known. For the "if' direction we notice first that every

[*] Department of Economics, University of Exeter, Rennes Drive, Streatham Court, Exeter, EX, UK. E-mail: `D.G.Balkenborg@exeter.ac.uk`

[†] Corresponding author; Department of Quantitative Economics, University Maastricht, P.O. Box 616, 6200 MD Maastricht, The Netherlands. Email: `d.vermeulen@maastrichtuniversity.nl`

simplicial complex is homeomorphic to the union of finitely many faces of the standard simplex in some vectorspace $\mathbb{R}^n$ of sufficiently high dimension. We start with such a simplicial complex and call it a simplicial set. We will project the standard simplex one-to-one onto the union of certain faces of the standard hypercube in the same $\mathbb{R}^n$ such that faces are mapped onto unions of faces. The hypercube can be viewed as the space of mixed strategy combinations of a game with n players where each player has two strategies. We then set the payoff of all players at a pure strategy combination s, i.e. a vertice of the hypercube, equal to 1 if s is the image of a point in the given simplicial complex under the projection. Otherwise we set the payoff of all players zero. Our given simplicial complex is thus seen to be homeomorphic to the set of strategy combinations that maximizes a joint expected utility function and is hence homeomorphic to a closed set of Nash equilibria. So far, so simple. However, we challenge the reader to find a short simple proof showing that this closed set of Nash equilibria is in fact a Nash equilibrium component, i.e. to prove that there are no further Nash equilibria nearby. We have so far not found such a short argument. Our proof relies on a rather lengthy argument using high-order Taylor expansions made earlier by one of the authors ([Balkenborg:1994]) and published as Proposition 4 in [BalkenborgSchlag:2007]. The difficulty might be due to the fact that, although the sets we describe are geometrically simple, they form algebraically the level set of a highly degenerate function containing many often non-isolated and degenerate critical points. On the positive side, because we construct sets which are obviously strict equilibrium sets in the sense of [BalkenborgSchlag:2007], we obtain from their Theorem 6.

Theorem 1.2 *A connected topological space is homeomorphic to a Nash equilibrium component which is asymptotically stable under the replicator dynamics if and only if it is homeomorphic to a simplicial complex.*

We now replicate the arguments in slightly more detail and conclude with a few open questions.

2 Preliminaries

It will hence be useful to have some basic notation and terminology available.

A finite normal form game consists of a finite set of players $N = \{1, \ldots, n\}$, and for each player $i \in N$ a finite pure strategy set S_i and a payoff function $u_i \colon S \to \mathbb{R}$ on the set $S := \prod_{i\in N} S_i$ of pure strategy profiles. We denote the game by (N, u), where $u = (u_i)_{i\in N}$ is the vector of payoff functions. A mixed strategy σ_i of player i is a vector $(\sigma_i(s_i))_{s_i\in S_i}$ that assigns a probability $\sigma_i(s_i) \geq 0$ to each pure strategy $s_i \in S_i$. We denote the set of mixed strategies of player i by

Σ_i. The set of all profiles $\sigma = (\sigma_i)_{i \in N}$ of mixed strategies is denoted by Σ. The support of a mixed strategy σ_i is the set of all pure strategies s_i with $\sigma_i(s_i) > 0$. The multilinear extension of the payoff function u_i of player i to the set Σ of all strategy profiles is

$$u_i(\sigma) = \sum_{s \in S} \prod_{j \in N} \sigma_j(s_j) u_i(s).$$

By $u_i(\sigma \mid s_i)$ we denote the payoff to player i when player i plays pure strategy $s_i \in S_i$ while his opponents adhere to the mixed strategy profile σ. A strategy profile $\sigma \in \Sigma$ is a Nash equilibrium when $u_i(\sigma) \geq u_i(\sigma \mid s_i)$ holds for every player i and every pure strategy s_i of player i.

BINARY GAMES. A binary game is a finite normal form game (N, u) such that $S_i = M = \{A, B\}$ for every player $i \in N$, and

$$u_i(s) \in \{0, 1\}$$

for all strategy profiles $s = (s_1, \ldots, s_n)$. To simplify notation for binary games we write σ_{iA} for the probability that player i plays pure strategy A and σ_{iB} for the probability that player i plays to pure strategy B.

3 Simplicial sets

A collection $\{v_1, \ldots, v_k\}$ of points in $\mathbb{R}^n$ is called independent if for all $a_1, \ldots, a_k$ in $\mathbb{R}$

$$a_1 \cdot v_1 + \cdots + a_k \cdot v_k = 0$$

implies $a_1 = \cdots = a_k = 0$. A collection $\{v_0, \ldots, v_k\}$ of points in $\mathbb{R}^n$ is *geometrically* (or *affine*) it independent if the collection $\{v_1 - v_0, \ldots, v_k - v_0\}$ is independent in $\mathbb{R}^n$. Note that an independent set is automatically geometrically independent. The convex hull

$$\{t_0 \cdot v_0 + \cdots + t_k \cdot v_k \mid t_i \leq 0 \text{ for all } i \text{ and } \sum_i t_i = 1\}$$

of a geometrically independent set $\{v_0, \ldots, v_k\}$ is called a *simplex*. We also say that $\{v_0, \ldots, v_k\}$ *spans* the simplex. For a geometrically independent set $\{v_0, \ldots, v_k\}$ we denote by $[v_0, \ldots, v_k]$ the simplex that is spanned by $\{v_0, \ldots, v_k\}$.

A collection $\mathcal{S} = \{S_1, \ldots, S_p\}$ of simplices in $\mathbb{R}^n$ is called a simplicial complex if for every two simplices $S_i = [v_0, \ldots, v_k]$ and $S_j = [w_0, \ldots, w_m]$ in $\mathcal{S}$ the intersection $S_i \cap S_j$ is the simplex spanned by the geometrically independent set

$$\{v_0, \ldots, v_k\} \bigcap \{w_0, \ldots, w_m\}.$$

(In the usual definition of a simplicial complex the extra requirement is that also $S_i \cap S_j \in \mathcal{S}$ in that case. We however do not necessarily need that in our paper.) The set $S = \cup_i S_i$ is called the carrier of the simplicial complex $\mathcal{S}$. A set $S \subset \mathbb{R}^n$ is called a simplicial set if it is the carrier of a simplicial complex.

Let e_i denote the i^{th} unit vector in $\mathbb{R}^n$. For a set $I \subset N$, a standard simplex Δ^I in $\mathbb{R}^N$ is the simplex that is spanned by a collection

$$\{e_i \mid i \in I\}$$

of unit vectors in $\mathbb{R}^N$. A simplicial set is called standard if it is the carrier of a simplicial complex $\mathcal{S} = \{S_1, \ldots, S_p\}$ of standard simplices S_j in $\mathbb{R}^N$. Note that any standard simplicial set is a union of faces Δ^I of the standard simplex Δ^N. The following is shown in most standard textbooks on algebraic topology.

Proposition 3.1 *Any simplicial complex is homeomorphic to a standard simplicial set.*

Proof. ∎

The following triangulation theorem is a celebrated result in geometry first proved by Llojasiewicz [Llojasiewicz:1964]. A more general proof by Hironaka can be found in Algebraic Geometry [Hironaka:1975].

Theorem 3.2 *Every compact semi-algebraic set is homeomorphic to a standard simplicial set.*

By definition, Nash equilibrium components are compact semi-algebraic sets, and hence this theorem applies. Thus, from a topological point of view, the collection of simplicial sets already captures all of the variety that we could reasonably expect to achieve in the setting of strategic form games.

4 Cubistic sets

For a set $T \subset N$, define the characteristic vector $e_T \in \mathbb{R}^N$ by

$$e_{Ti} = \begin{cases} 1 & \text{if } i \in T \\ 0 & \text{otherwise.} \end{cases}$$

The standard hypercube K^N in $\mathbb{R}^N$ is the convex hull of all characteristic vectors in $\mathbb{R}^N$. A subset $F \subset K^N$ of the standard hypercube is a face if there are sets Z and P in N such that

$$F = \{x \in K^n \mid x_i = 0, x_j = 1 \text{ for all } i \in Z \text{ and } j \in P\}.$$

If $P \neq \varnothing$, we say that the face F is an upward face. Let K_+^N be the union of all upward faces and for a subset $I \subseteq N$ we let $K_+^I = K_+^N \cap \{x \in \mathbb{R}^n \mid x_i = 0 \text{ for } i \notin I\}$. A subset C of K^N is called a *cubistic set* if

[1] C is the union of faces of K^N

[2] C is *n-convex* in the sense that for all points $(y_i, x_{-i}) \in C$, $(z_i, x_{-i}) \in C$ and all scalars $0 \leq \alpha \leq 1$ the convex combination

$$(1-\alpha)(y_i, x_{-i}) + \alpha(z_i, x_{-i})$$

is in C. [1]

An *upward cubistic set* is a cubistic set contained in K_+^N.

In order to connect the standard simplex with the "upper part" K_+^N of the standard hypercube, it is useful to remind ourselves of two norms for vectors $x = (x_1, \ldots, x_n) \in \mathbb{R}^N$, namely of the norm

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$

and of the supremum norm

$$\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|.$$

Consider now a non-zero vector $x \in \mathbb{R}^N$ with non-negative components and the ray

$$Lx = \{\lambda \cdot x \mid \lambda \geq 0\}$$

it generates. This ray intersects the standard simplex Δ^N in the point $\frac{x}{\|x\|_1}$ and the set K_+^N in the point $\frac{x}{\|x\|_\infty}$. This relationship defines a homeomorphism Φ between Δ^N and K_+^N. Moreover, each face Δ^I for $I \subseteq N$ is mapped one-to-one onto K_+^I. We obtain the following result.

Theorem 4.1 *Every standard simplicial set is homeomorphic to an upward cubistic set.*

Proof. Clearly, Φ maps a standard simplicial set C onto a union D of upward faces of the standard cube. To show that C is n-convex, take two different points (y_i, x_{-i}), $(z_i, x_{-i}) \in D$. We assume w.l.o.g. that $y_i > z_i$. Let I be the set of non-zero entries in (y_i, x_{-i}). Since $\frac{(y_i, x_{-i})}{\|(y_i, x_{-i})\|_1} \in C$, the simplex Δ^I is contained in C and contains both points $\frac{(y_i, x_{-i})}{\|(y_i, x_{-i})\|_1}$ and $\frac{(z_i, x_{-i})}{\|(z_i, x_{-i})\|_1}$ and all their convex combinations.

[1] This extends the notion of biconvexity in [AumannHart:1986].

Because $z_i < 1$, (y_i, x_{-i}) and (z_i, x_{-i}) have a joint component $x_j = 1$ for $j \neq i$. Thus

$$\|(y_i, x_{-i})\|_\infty = \|(z_i, x_{-i})\|_\infty = \|((1-\alpha)y_i + \alpha z_i, x_{-i})\|_\infty = 1$$

and therefore $((1-\alpha)y_i + \alpha z_i, x_{-i}) \in K_+^N$ is for all $0 \leq \alpha \leq 1$ the image of a point in C. ■

5 Nash components

We now interpret N as the set of players in a game and interpret K^N as the set of mixed strategies in a binary game where each player has the pure strategies 0 and 1. Fix a union of faces C of K^N and let T be the set of pure strategy combinations it contains. Let $u\colon K^N \to \mathbb{R}$ be the multilinear extension of the function $u\colon \{0,1\}^n \to \mathbb{R}$ defined by

$$u(s) = \begin{cases} 1 & \text{for} \quad s \in T \\ 0 & \text{for} \quad s \notin T \end{cases}$$

It is straightforward to check the following Lemma which describes cubistic sets as maximizers of potentials.

Lemma 5.1 *The equality of sets*

$$C = \left\{x \in K^N \mid u(x) = 1\right\}$$

holds if and only if C is a cubistic set.

Now we suppose that C is a cubistic set and look at the game Γ_C where each player $i \in N$ has the utility function $u_i = u$. Then C is a strict equilibrium set of this game as defined in [BalkenborgSchlag:2007]. This means that if $(y_i, x_{-i}) \in C$ and if z_i is a best reply of player i against x_{-i}, then $(z_i, x_{-i}) \in C$. [BalkenborgSchlag:2007] show that every strict equilibrium set is a finite union of Nash equilibrium components which consists of stable rest points and is an asymptotically stable set of the replicator dynamics. As a corollary of their results we obtain

Theorem 5.2 *Every cubistic set C is a finite union of Nash components of the binary game Γ_C. C is asymptotically stable and consists of stable restpoints for the replicator dynamics.*

6 Summary and conclusions

In our construction the number of strategies per player is as small as possible. However, it needs as many players as the simplicial complex has vertices. [**?**] provides a construction which identifies the set of Nash equilibria for an n-player ($n \geq 3$) game with the set of Nash equilibria in a 3-player game. Thus connected simplicial complexes are homeomorphic to Nash equilibrium components in 3-player games. The question whether this also holds for 2-player games is currently open.

The restriction to components is not essential in our argument. Our argument shows that every cubistic set (and hence, up to homeomorphy, every simplicial complex) is a finite union of Nash equilibrium components of a binary game. However, typically the game constructed will have additional Nash equilibria. This *must* be the case if the Euler characteristic of the set is not 1. If the complex has Euler characteristic 1, can we construct a game such that the complex is homeomorphic to to the set of *all* Nash equilibria in this game?

We considered here only the topological structure. One might more generally ask whether any compact connected semi-algebraic set is differomorphic or even algebraically equivalent to a Nash equilibrium component.

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