Potential games and path independence: an alternative algorithm

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Abstract

Constructing a directed graph for any finite game, this paper provides a simple characterization of potential games in terms of the path independence property of this graph. Using this characterization, we propose an algorithm to determine if a game is potential or not. The number of equations required in this algorithm is lower than the number obtained in the algorithms proposed in the recent papers of Hino (2010) and Sandholm (2010).

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1 Introduction

A potential of a game is a function of its strategy profiles such that if a profile is obtained from another through a unilateral deviation by one player, then the difference in the potential between these profiles equals the gain in payoff of the deviating player. If a game admits a potential, it is called a potential game. Introduced by Monderer and Shapley (1996), the notion of potential games has been useful in enhancing the understanding of issues as diverse as network congestion, evolutionary dynamics and strategic complementarity (see, e.g., Ui, 2000; Dubey et al., 2006; Hino, 2010; Sandholm, 2010; and the references therein).

This paper provides a simple characterization of potential games. Consider any game with finitely many players and strategies. For every player, we denote one of its strategy as the "zero strategy" and call the rest "positive strategies". A profile $\widetilde{s}$ is called a predecessor of a profile s if s is obtained from $\widetilde{s}$ via a unilateral deviation of a player from its zero strategy to some positive strategy. We construct a weighted directed graph from the game as follows. The vertices of the graph are the strategy profiles, with the profile with all zero strategies z acting as the origin. For any predecessor-successor pair $(\widetilde{s}, s)$, a directed edge is drawn from $\widetilde{s}$ to s and the weight assigned to this edge is the gain in payoff of the corresponding deviating player. The length of any directed path in this graph is the sum of weights of all edges that appear in the path. For any profile s, the *path independence* property holds if all paths from the origin z to s have the same length. We show that a game admits a potential if and only if the path independence property holds for all vertices of its associated graph.[1]

Based on the characterization above, we construct an algorithm to determine whether a game is potential or not. For all two-person games, the number of equations required in our algorithm is the same as the number required in the ones[2] proposed by Hino (2010) and Sandholm (2010) and for games with more than two players, our algorithm requires a strictly lower number of equations.

We present the model and our results in the next section.

2 The model

Let $\Gamma = \langle N, (S_i)_{i\in N}, (u_i)_{i\in N}\rangle$ be a game in strategic form with a finite number of players, with each player having a finite number of strategies. Let $N = \{1, \ldots, n\}$ be the set of players, S_i the set of strategies of player i and $u_i : S \to R$ the payoff function of player i, where $S = \times_{i\in N} S_i$ is the set of strategy profiles and R the set of real numbers. Also denote $S_{-i} = \times_{j\neq i} S_j$.

Player i has $k_i \geq 1$ strategies, i.e., $|S_i| = k_i$. It will be useful for our analysis to denote $S_i = \{0, \ldots, k_i - 1\}$. Thus each player i has a distinct *zero strategy* ($s_i = 0$) and possibly other *positive strategies* ($s_i > 0$).

Definition 1 (Monderer and Shapley, 1996) The game Γ is a *potential game* if there is a function $P : S \to R$ (called a potential function of Γ) such that for every $i \in N$, $s_i, s_i' \in S_i$

[1] The path independence property is closely related with potential functions. See, e.g., Calvo and Santos (1997).

[2] Both of these algorithms require checking the same number of equations, but Hino's algorithm involves less storage.

and $s_{-i} \in S_{-i}$

$$u_i(s_i, s_{-i}) - u_i(s'_i, s_{-i}) = P(s_i, s_{-i}) - P(s'_i, s_{-i}) \tag{1}$$

We begin with the following simple observation.

Lemma 1 *The game Γ is a potential game if and only if there is a function $P : S \to R$ such that for every $i \in N$, $s_i \in S_i$ and $s_{-i} \in S_{-i}$*

$$u_i(s_i, s_{-i}) - u_i(0, s_{-i}) = P(s_i, s_{-i}) - P(0, s_{-i}) \tag{2}$$

Proof If Γ is a potential game, then (1) holds for all (s_i, s'_i, s_{-i}), so in particular, it holds for all $(s_i, 0, s_{-i})$. To prove the only if part, suppose there is $P : S \to R$ such that (2) holds. Then

$$u_i(s_i, s_{-i}) - u_i(s'_i, s_{-i}) = [u_i(s_i, s_{-i}) - u_i(0, s_{-i})] - [u_i(s'_i, s_{-i}) - u_i(0, s_{-i})]$$

$$= [P(s_i, s_{-i}) - P(0, s_{-i})] - [P(s'_i, s_{-i}) - P(0, s_{-i})] = P(s_i, s_{-i}) - P(s'_i, s_{-i})$$

This completes the proof. ■

In light of Lemma 1, the zero strategy of a player will play a central role in our analysis to determine whether a game is potential or not. Denote the strategy profile where every player plays its zero strategy by z, i.e.,

$$z \equiv (0, \dots, 0) \tag{3}$$

It will be useful to classify the strategy profiles according to the number of zero strategies in a profile. Specifically, for $t = 0, \dots n$, define

$$V_t := \{s \in S | s \text{ has } t \text{ positive strategies}\} \tag{4}$$

Thus, V_t is the set of all strategy profiles where $n - t$ players play their zero strategies and the remaining t players play positive strategies. Note that $V_t \cap V_{t'} = \emptyset$ for $t \neq t'$ and any strategy profile s is an element of V_t for some $t = 0, \dots, n$, so we can partition $S = \cup_{t=0}^{n} V_t$. Also observe that V_0 is the singleton set $\{z\}$.

Definition 2 Let $s \in V_t$. A strategy profile $\widetilde{s} \in V_{t-1}$ is called a *predecessor* of s (and s a *successor* of $\widetilde{s}$) if there is a player i such that (a) $s_i > 0$ and $\widetilde{s}_i = 0$ and (b) $s_j = \widetilde{s}_j$ for all $j \neq i$. The player i for which (a) and (b) hold is called the *critical player* of the pair $(\widetilde{s}, s)$.

Since there are t players who play positive strategies in $s \in V_t$, any such s has exactly t predecessors, each having a different critical player. In particular, z has no predecessor and any $s \in V_1$ has only one predecessor z.

By Definition 2, if $\widetilde{s}$ is a predecessor of s, then $s = (s_i, s_{-i})$ and $\widetilde{s} = (0, s_{-i})$ with $s_i > 0$, where i is the critical player of $(\widetilde{s}, s)$. The profile s can be reached from profile $\widetilde{s}$ if player i makes a unilateral deviation from its zero strategy to the strategy s_i. For any predecessor-successor pair $(\widetilde{s}, s)$ with critical player i, denote

$$\Delta(\widetilde{s}, s) := u_i(s) - u_i(\widetilde{s}) = u_i(s_i, s_{-i}) - u_i(0, s_{-i}) \tag{5}$$

Thus, $\Delta(\widetilde{s}, s)$ presents the gain in payoff of the critical player from its deviation when we move from $\widetilde{s}$ to s.

Consider condition (2) of Lemma 1. Note that if $s_i = 0$, this condition holds for any function P, so let $s_i > 0$. Using Definition 2 and (5) in Lemma 1, it follows that a function $P : S \to R$ is a potential function if and only if *for every* predecessor-successor pair $(\widetilde{s}, s)$,

$$P(s) - P(\widetilde{s}) = \Delta(\widetilde{s}, s) \tag{6}$$

2.1 A directed graph representation of Γ

Motivated by (6), we construct the weighted directed graph $G(\Gamma)$ from the game Γ as follows.

(i) Present each $s \in S = \cup_{t=0}^{n} V_t$ as a vertex.

(ii) For any predecessor-successor pair $(\widetilde{s}, s)$, draw a directed edge which originates at $\widetilde{s}$ and terminates at s. Denote this edge by $(\widetilde{s}, s)$.

(iii) Put the weight $\Delta(\widetilde{s}, s)$ (given by (5)) on the edge $(\widetilde{s}, s)$.

Remark 1 Denote by e_t the number of edges that terminate at some $s \in V_t$. Since any $s \in V_t$ has t predecessors, $e_t = t|V_t|$. The total number of edges[3] in G is $\eta(G) = \sum_{t=0}^{n} e_t = \sum_{t=0}^{n} t|V_t|$.

Definition 3 A sequence $\gamma = (y_0, y_1, \ldots, y_m)$ is a *directed path* of $G(\Gamma)$ if for all ℓ, y_ℓ is a predecessor of $y_{\ell+1}$. The vertex y_0 is called the *origin* and y_m the *terminus* of the path γ. The number of edges in the path γ is m, given by $(y_0, y_1), \ldots, (y_{m-1}, y_m)$. The *length* of the path γ is the sum of weights over these m edges, given by

$$L(\gamma) := \Delta(y_0, y_1) + \Delta(y_1, y_2) + \ldots + \Delta(y_{m-1}, y_m) = \sum_{t=0}^{m-1} \Delta(y_\ell, y_{\ell+1}) \tag{7}$$

If $m = 0$, the path $\gamma = (y_0)$ has only one vertex and no edge, so its length is zero.

One immediate property of directed paths can be noted. By (7), it follows that for two directed paths $\gamma = (y_0, \ldots, y_m)$ and $\gamma' = (y_0, \ldots, y_{m-1})$,

$$L(\gamma) = L(\gamma') + \Delta(y_{m-1}, y_m) \tag{8}$$

2.2 Path independence: a characterization of potential games

Let $s \in V_t$ for some $t \geq 0$. It will be useful for our analysis to consider all directed paths with origin at z and terminus at s. Since the number of players with positive strategies at $s \in V_t$ is t, to reach s from z, we need t unilateral deviations. As these deviations can occur in $t!$ different ways, this is the number of directed paths with origin z and terminus s. Each of these paths has exactly t edges.

[3]If all players have the same number of strategies k, then $|V_t| = \binom{n}{t}(k-1)^t$ and $\eta(G) = n(k-1)k^{n-1}$. Sandholm (2010, p.9) also proposes a construction of a graph from a game as follows: "...one first assigns the function an arbitrary value at an arbitrary initial strategy profile $s_0 \in S$. One then constructs a tree that reaches every other strategy profile in S, and whose edges all correspond to unilateral deviations to adjacent strategies." In this construction, there is an edge between any two adjacent strategies, so the number of edges is $n\binom{k}{2}k^{n-1} \geq \eta(G)$ for any $k \geq 2$. The graph G constructed in this paper has the minimal number of edges and thus involves lower storage. See also Section 2.4.

Definition 4 The *path independence* property (PI) holds for $s \in S$ if all directed paths with origin z and terminus s have the same length.

Remark 2 Note that PI holds vacuously for z (which is the only element of V_0). If $s \in V_1$, its only predecessor is z and there is only path $\gamma = (z, s)$ with origin z and terminus s. By (7) and (5), this path has length

$$\Delta(z, s) = u_i(s) - u_i(z) \tag{9}$$

Therefore, PI trivially holds for any $s \in V_1$.

Theorem 1 *The game Γ is a potential game if and only if the path independence property holds for all $s \in S$.*

Proof The "if part": Suppose PI holds for all $s \in S$. Then all directed paths with origin z and terminus s have the same length. Denote this length by $\lambda(s)$. Define the function $P : S \to R$ as

$$P(s) := \lambda(s) \tag{10}$$

Observe in particular that $P(z) = \lambda(z) = 0$. We prove that P defined in (10) is a potential function of Γ by showing that P satisfies (6) for any predecessor-successor pair $(\widetilde{s}, s)$.

First suppose $s \in V_1$. Then we know from Remark 2 that the only predecessor of s is z and there is only one path with origin z and terminus s. This path has length $\Delta(z, s)$, so for this case, $P(s) = \lambda(s) = \Delta(z, s)$. Since $P(z) = 0$, we have $P(s) - P(z) = \Delta(z, s) = u_i(s) - u_i(z)$, so (6) holds.

Next suppose $s \in V_t$ for some $t \geq 2$. Consider any predecessor $\widetilde{s} \in V_{t-1}$ of s. Let $\gamma = (y_0, \dots, y_{t-1}, y_t)$ be a directed path with origin z and terminus s that passes through $\widetilde{s}$, i.e., $y_0 = z$, $y_{t-1} = \widetilde{s}$ and $y_t = s$. Consider the path $\gamma' = (y_0, \dots, y_{t-1})$ with origin z and terminus $\widetilde{s}$. Then it follows by (8) that

$$L(\gamma) = L(\gamma') + \Delta(y_{t-1}, y_t) = L(\gamma') + \Delta(\widetilde{s}, s) \tag{11}$$

Since PI holds for both s and $\widetilde{s}$, we have $L(\gamma) = \lambda(s) = P(s)$ and $L(\gamma') = \lambda(\widetilde{s}) = P(\widetilde{s})$. Using this in (11), we have $P(s) - P(\widetilde{s}) = \Delta(\widetilde{s}, s)$, so (6) holds for this case as well, which completes the proof of the "if part".

The "only if part": Suppose Γ is a potential game. Then there is a function P such that (6) holds for any predecessor-successor pair $(\widetilde{s}, s)$. By (7), the length of any directed path $\gamma = (y_0, \dots, y_m)$ is $L(\gamma) = \sum_{\ell=0}^{m-1} \Delta(y_\ell, y_{\ell+1})$. Since $(y_\ell, y_{\ell+1})$ is a predecessor-successor pair, by (6), we have $\Delta(y_\ell, y_{\ell+1}) = P(y_{\ell+1}) - P(y_\ell)$ for all ℓ, so that $L(\gamma) = \sum_{\ell=0}^{m-1} [P(y_{\ell+1}) - P(y_\ell)] = P(y_m) - P(y_0)$. Taking $y_0 = z$ and $y_m = s$, all paths with origin z and terminus s has the same length $P(s) - P(z)$, proving that PI holds for s. ∎

2.3 An algorithm to determine a potential game

Theorem 2 *Denote*

$$A_n(k_1, \dots, k_n) := \sum_{t=2}^{n} (t-1)|V_t| \tag{12}$$

To determine whether Γ is a potential game or not, it is sufficient to check A_n equations.

Proof Using Theorem 1, we determine whether Γ is a potential game or not by checking if PI holds for all $s \in S = \cup_{t=0}^{n} V_t$. Recall from Remark 2 that PI always holds for all $s \in V_0 \cup V_1$. We check the property recursively as follows.

Let $t \geq 2$. Suppose PI holds for all $s \in V_\ell$ for $\ell = 0, \ldots, t-1$. Consider any $s \in V_t$. Any directed path with origin z and terminus s must pass through some predecessor $\widetilde{s} \in V_{t-1}$ of s. Consider two such paths $\gamma = (y_0, \ldots, y_{t-1}, y_t)$ and $\gamma' = (y_0', \ldots, y_{t-1}', y_t')$ that pass through the same predecessor $\widetilde{s}$, i.e., $y_0 = y_0' = z$, $y_t = y_t' = s$ and $y_{t-1} = y_{t-1}' = \widetilde{s}$. Denoting $\gamma_1 = (y_0, \ldots, y_{t-1})$ and $\gamma_2 = (y_0', \ldots, y_{t-1}')$, observe that

$$L(\gamma) = L(\gamma_1) + \Delta(\widetilde{s}, s) \text{ and } L(\gamma') = L(\gamma_1') + \Delta(\widetilde{s}, s) \tag{13}$$

Note that both γ_1 and γ_1' have origin z and terminus $\widetilde{s}$. Since $\widetilde{s} \in V_{t-1}$, PI holds for $\widetilde{s}$, so $L(\gamma_1) = L(\gamma_1')$. Denote this common length by $\lambda(\widetilde{s})$. Then it follows from (13) that all directed paths with origin z and terminus s that pass through the same predecessor $\widetilde{s}$ have the same length $\lambda(\widetilde{s}) + \Delta(\widetilde{s}, s)$. Since $s \in V_t$ has exactly t predecessors $\widetilde{s}^1, \ldots, \widetilde{s}^t$, it follows that PI holds for s if and only if

$$\lambda(\widetilde{s}^1) + \Delta(\widetilde{s}^1, s) = \ldots = \lambda(\widetilde{s}^t) + \Delta(\widetilde{s}^t, s)$$

This implies that we need to check $t-1$ equations for every $s \in V_t$. Therefore the number of equations we have to check to see if PI holds for all $s \in V_t$ is $(t-1)|V_t|$.

Applying the recursive argument above for $t = 2, \ldots, n$, the total number of equations we need to check to see if PI holds for all $s \in S$ is $\sum_{t=2}^{n} (t-1)|V_t|$. ■

2.4 Comparison with Hino (2010) and Sandholm (2010)

Consider a game Γ with set of players $N = \{1, \ldots, n\}$ where player i has k_i strategies. In the algorithm proposed in the recent paper of Hino (2010), the number of equations required to verify whether Γ is a potential game is given as follows, where $i, j, \ell \in N$.

$$B_n(k_1, \ldots, k_n) = \sum_{i<j} \theta_{ij} \text{ where } \theta_{ij} := (k_i - 1)(k_j - 1) \prod_{\ell \neq i,j} k_\ell \tag{14}$$

Comparing A_n from (12) with B_n, we have the following result.

Corollary 1

(i) *$B_n(k_1, \ldots, k_n) = A_n(k_1, \ldots, k_n)$ for $n = 2$ and $B_n(k_1, \ldots, k_n) - A_n(k_1, \ldots, k_n) = \sum_{t=3}^{n} \binom{t-1}{2} |V_t| > 0$ for $n \geq 3$.*

(ii) *If $k_i = k$ for all $i \in N$, then for any $n \geq 3$*

$$B_n(k) - A_n(k) = (k-1)k^{n-2}[(k-1)n^2 - (3k-1)n]/2 + k^n - 1$$

$$= (n-1)(n-2)k^n/2 - n(n-2)k^{n-1} + n(n-1)k^{n-2}/2 - 1$$

For $2 \leq k < \infty$, $lim_{n\to\infty}[B_n(k) - A_n(k)] = \infty$ and for $3 \leq n < \infty$, $lim_{k\to\infty}[B_n(k) - A_n(k)] = \infty$.

Proof (i) It is convenient to translate the expression in (14) in terms of the terminologies of our model. Observe that for two players $i < j$, player i has $(k_i - 1)$ positive strategies, j has $(k_j - 1)$ positive strategies and there are $\prod_{\ell \neq i,j} k_\ell$ combinations of strategies from all the remaining players. Thus, θ_{ij} presents the number of strategy profiles where both i and j play positive strategies. The expression in (14) adds up such θ_{ij} profiles over all $i < j$.

Note that any strategy profile that is counted in (14) has at least two players playing positive strategies, i.e., only profiles $s \in V_t$ for $t \geq 2$ are counted. Consider any such $s \in V_t$. In this s, exactly t players play positive strategies. Since out of these t players, two players $i < j$ can be chosen in $\binom{t}{2}$ ways, it follows that the number of times a profile $s \in V_t$ is counted in (14) is $\binom{t}{2}$. Since the number B_n in (14) is the sum of counts of all $s \in \cup_{t=2}^{n} V_t$, we conclude that

$$B_n(k_1, \ldots, k_n) = \sum_{t=2}^{n} \binom{t}{2} |V_t| \tag{15}$$

As $\binom{t}{2} - (t-1)$ equals 0 if $t = 2$ and $\binom{t-1}{2}$ if $t \geq 3$, the result follows from (12) and (15).

(ii) If $k_i = k$ for all $i \in N$ (i.e., all players have the same number of strategies k), $|V_t| = \binom{n}{t}(k-1)^t$ and we have

$$A_n(k) = \sum_{t=2}^{n} (t-1) \binom{n}{t} (k-1)^t = (n-1)k^n - nk^{n-1} + 1,$$

$$B_n(k) = \sum_{t=2}^{n} \binom{t}{2} \binom{n}{t} (k-1)^t = \frac{n(n-1)}{2} k^{n-2}(k-1)^2$$

Using the expressions above, the conclusion of (ii) follows by standard computations. ■

The number of equations required to be checked for the algorithm based on Theorem 3 of Hino (2010) and Theorem 3.5 of Sandholm (2010) is B_n. The number of equations that we require to check in our algorithm is $A_n < B_n$ for $n \geq 3$. We conclude the paper by providing an example.

Example 1 Let $k_i = k$ for all $i \in N$ (i.e., all players have the same number of strategies).

	$A_n(k)$	$B_n(k)$
$n = 3, k = 3$	28	36
$n = 3, k = 4$	81	108
$n = 4, k = 3$	136	216
$n = 3, k = 10$	1701	2430
$n = 10, k = 3$	$3,34,612$	$11,80,980$

The table above indicates that as the number of players or the number of strategies increases, $B_n - A_n$ increases substantially and the algorithm presented in this paper leads to significant reduction in the number of equations to be checked.

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