Bubbles, Crashes and Efficiency with Double Auction Mechanisms*

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April 2011

"[S]ince equilibrium is a relationship between actions, and since the actions of one person must necessarily take place successively in time, it is obvious that the passage of time is essential to give the concept of equilibrium any meaning. This deserves mention since many economists appear to have been unable to find a place for time in equilibrium analysis and consequently have suggested that equilibrium must be conceived as timeless." F.A. Von Hayek (1937)

Abstract

We provide a quantitative boundary on the stepsizes of bid and ask of a double auction (DA) mechanism to answer two questions, when the DA mechanism is efficient and when it creates bubbles and crashes. The main result is that the ratio of the two stepsizes and their spread are the key factors for the DA mechanism to be efficient. Sentiment that leads to a swing in the spread and the ratio of the two stepsizes can result in prices to deviate from the intrinsic value equilibrium. These results are derived from a theoretical analysis of the DA mechanism built on the incremental subgradient method in Nedić and Bertsekas (2001).

Keywords: double auction mechanism, incremental subgradient method, efficient markets hypothesis, investors' sentiment, job matching market, multiple objects

*We thank Mark Satterthwaite, an expert on DA mechanisms, for helpful communications.

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1 Introduction

A double auction (DA) mechanism is a decentralized clearinghouse system under which a buyer or seller submits orders based on his or her private information, without information available on the total market demand and supply. A specialist or a market maker may be needed to set bids and asks to accommodate the two order flows. But orders are either fully or partially filled one at a time. If no complete information is available on the total market demand and supply, how is it possible to tell if a DA mechanism has priced an asset right at equilibrium? Such a question is central to the efficient markets hypothesis (EMH) (Fama (1970)), which has been lately under an attack from behavioral finance for the existence of various financial anomalies such as momentum, bubbles and crashes, which are often observed in the equity, foreign exchange, commodity, futures as well as experimental markets (see, e.g., Baker and Wurgler (2007), Barberis et al. (1998), Shleifer (1999), Smith et al. (1988)).

The benchmark model of this paper has the general form given below (Bertsekas (2010)):

$$\mathcal{P} \qquad \text{minimize } F(y) = \sum_{j=1}^{m} (f_j + g_j)(y)$$

subject to $y \in Y$, and f_j and g_j are real-valued convex functions. A large class of economies and markets can be represented by this form, which will be introduced more precisely in Section 2. For example, let N denote the set of all indivisible shares or contracts[1]. There are m investors or agents who initially own N. The value of owning a portfolio $S \subset N$ to an agent j is his expected private intrinsic value $Eu_j(S, R)$ of cash flows generated by S minus the market value of buying S plus the market value of his initial endowed portfolio, where R is some random variable. Such a quasilinear form has been widely used in the auction and the DA literature. The market prices at equilibrium are determined by agents' expected private intrinsic values and the market clearing condition. It turns out that such an equilibrium is also closely tied to an optimal solution to problem $\mathcal{P}$.

The above model is not necessarily a perfect match with real exchange markets like the NYSE and Nasdaq. But the insight in our study of the DA mechanism should still be useful for these exchange markets and the EMH. Our study is built on the incremental subgradient method studied by Nedić and Bertsekas (2001). In their approach the m agents form a cycle in an arbitrary order, or are chosen randomly with equal probability, and prices are iterated along the cycle one agent at a time, purely based on his revealed private demand or supply. They show that with certain conditions on the stepsize in the iteration process the system can discover the market prices at

[1] Shares or contracts are traded in integer units but orders may be placed in fraction or integer.

equilibrium without knowing the total demand and supply. Moreover, such a system is shown to be robust even with presence of stochastic noises (Ram et al. (2009)). In our DA mechanism there are two different sides: buyers and sellers. Let f_j be the seller side and g_j be the buyer side. Note that an agent j equipped with $f_j + g_j$ in the incremental subgradient method has been separated as j seller and j buyer in the DA mechanism, under which sellers and buyers form two parallel lines in an arbitrary order, or a pair of a buyer and a seller is chosen randomly with equal probability, and prices are iterated according to private information revealed through buy and sell orders, very much like the style of an actual exchange market. Some weight must be assigned how the bid and the ask enter the prices (Chatterjee and Samuelson (1983) and Wilson (1985)).[2] The DA mechanism with bid and ask has an advantage over the incremental subgradient method in one important aspect: It enables one to study how the relative strength of the stepsizes in bid and ask affects the price process. Thus one can see how a bubble is created once buyers dominate sellers in bidding and how a crash is created once sellers dominate buyers in asking. Moreover, we find that the DA mechanism creates a bubble or crash in trends because the sequence of prices generated by the DA mechanism converges to an equilibrium in a unique way, which is determined by the ratio in the limit of the two stepsizes of bid and ask. We also show that these conclusions are robust with the stochastic noises of Ram et al. (2009). When the stepsizes of bid and ask are at equal in the limit and have equal weight of entering the prices, the DA mechanism will converge to an equilibrium determined by the total demand and supply. That is, prices will follow the fundamental to an intrinsic value equilibrium. We see this result as a strong quantitative support for the EMH. With our results, one may also understand it better why there is a bubble or crash both in the experimental market in Smith et al. (1988) and in so many real exchange markets.

The rest of the paper is organized as follows. Section 2 introduces the model. Section 3 discusses the equilibrium and defines bubbles, crashes and efficiency. Section 4 presents the main results with DA mechanisms that do not have stochastic noises in buy and sell orders. Section 5 shows that those results in Section 4 are robust even under the stochastic noises. Section 6 discusses the

[2]A DA mechanism has been modeled since then as a strategic form game with incomplete information, see, e.g., Williasms (1991), Rustichini et al. (1994), Jackson and Swinkels (2005) and Fudenberg et al. (2007). One major concern is the existence of equilibrium (with trade) and how strategic misreporting of private valuations may jeopardize the competitive efficiency of a DA mechanism. Gjerstad and Dickhaut (1998) is a departure from this tradition and they use a belief system formed by buyers and sellers after observing information that has been revealed. They show that their system can reach competitive equilibrium under a DA mechanism after a few periods. Shneyerov and Wong (2010) is another departure and they model a DA mechanism as a sequential bargaining game. A major obstacle with the game theoretical approach of a DA mechanism is how to handle heterogeneous multiple objects or assets and bring time into an equilibrium analysis as requested by Hayek (1937) in his article "Economics and Knowledge". Strategic games of price-quantity in Dubey (1982) and Dubey and Shubik (1978) with bidding are exceptional in one important aspect, in which there can be more than one product for sale and no unit demand and supply are assumed.

literature and makes some remarks.

2 Model

We consider a class of problems that can be described by the following general form (Bertsekas (2010)):

$$\mathcal{P} \qquad \text{minimize } F(y) \equiv f(y) + g(y)$$
$$\text{subject to } y \in Y,$$

where

$$f = \sum_{i=1}^{n} f_i \text{ and } g = \sum_{j=1}^{m} g_j.$$

For all $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, m$, $f_i : Y \to R$ and $g_j : Y \to R$ are real-valued convex functions and Y is a nonempty closed convex set of finite dimensions. As Bertsekas (2010) has demonstrated, such a form covers a large class of problems in the literature: a). Least Squares and Related Inference Problems; b). Dual Optimization in Separable Problem; c). Problems with Many Constraints; d). Minimization of an Expected Value - Stochastic Programming; e). Weber Problem in Location Theory; f). Distributed Incremental Optimization-Sensor Networks.

We use the notation

$$F^* = \inf_{y \in Y} F(y), \qquad Y^* = \{y \in Y | F(y) = F^*\}, \qquad dist(y, Y^*) = \inf_{y^* \in Y^*} \|y - y^*\|$$

where $\|\cdot\|$ denotes the Euclidean norm. For any two vectors x and y in R^m, $< x, y >= \sum_{j=1}^{m} x_j y_j$. For two sets A and B in R^d, we say $A \leq B$ if for any $a \in A$, there exists $b \in B$ such that $a \leq b$.

2.1 Job Matching Market and Economies with Indivisible Goods

We now show why this form of problem $\mathcal{P}$ naturally arises for a large class of economies or markets typified by the job matching model of Kelso and Crawford (1982) and the related economy with indivisible objects of Bikhchandani and Mamer (1997).

For the job matching market of Kelso an Crawford (1982), f is seen as the side of workers and g is seen as the side of employers. For the economy with multiple objects of Bikhchandani and Mamer (1997), f is seen as the side of objects and g is seen as the side of agents. Next we will use the economy with multiple objects in Bikhchandani and Mamer (1997) to illustrate how to define f and g. The job matching model of Kelso and Crawford (1982) and the assignment problem (Shapley and Shubik (1971)) can be done similarly.

Let $N = \{1, 2, \cdots, n\}$ denote the set of objects or assets and $M = \{1, 2, \cdots, m\}$ denote the set of agents. An agent j's utility or intrinsic value function $u_j : 2^N \to R_+$ is defined by a set function $u_j(S)$ over all object bundles $S \subset N$ that satisfies $u_j(\emptyset) = 0$. We say that u_j is weakly monotone if $u_j(S) \leq u_j(T)$ for all $S, T \subset N$ such that $S \subseteq T$. A **feasible allocation** Z is a partition $(Z_0, Z_1, \cdots, Z_m)$ of the set of objects N, in which agent j, $j \in M$, is allocated the bundle Z_j. Z_0 is the unallocated bundle. Let $\bar{Z}$ denote the set of all feasible allocations. Define $e : 2^N \to R^n$ by $e_i(S) = 1$ for $i \in S$ and $e_i(S) = 0$ otherwise. Given a price vector $p \in R^n_+$, agent j's profit function $g_j : 2^N \times R^n_+ \to R$ is defined by

$$g_j(S, p) = u_j(S) - \sum_{a \in S} p_a$$

and his demand correspondence $D_j : R^n \to R^n$ is defined by

$$D_j(p) = \{e(S) : g_j(S, p) \geq g_j(T, p), \forall T \subset N\}.$$

A pair (Z, p) of a feasible allocation Z and a price vector p is a **Walrasian or competitive equilibrium** if $p_a = 0$ for all $a \in Z_0$ and $e(Z_j) \in D_j(p)$ for all agents j. Let Z^* be an optimal allocation. That is,

$$Z^* \in argmax_{Z \in \bar{Z}} \sum_{j \in M} u_j(Z_j).$$

Let $V = \sum_{j \in M} u_j(Z^*_j)$ for some Z^*.

Define $g_j(p) = g_j(S, p)$ for $e(S) \in D_j(p)$ for all $j = 1, 2, \cdots, m$ and $f_i : R^n_+ \to R$ by $f_i(p) = p_i$ for all $i = 1, 2, \cdots, n$. Note that g_j is convex. We obtain the general form $F(p) = f(p) + g(p)$ subject to $p \in R^n_+$.

To make a connection between problem $\mathcal{P}$ and the economic model to study competitive equilibrium, we need two results:

Lemma 2.1 (Ma (1998a)). *a). For all price vectors $p \in R^n_+$, $F(p) = f(p) + g(p) \geq V$; b) A price vector p is a Walrasian equilibrium if and only if $F(p) = f(p) + g(p) = V$.*

Lemma 2.2 (Upper Envelope Theorem)(Ma and Nie (2003)). *For all $j \in M$, $\partial g_j(p) = -\bar{co}D_j(p)$ for all $p \in R^n_+$, where $\bar{co}D_j(p)$ is the closed convex hull of the demand set $D_j(p)$.*

For a proof of Lemmas 2.1 and 2.2, see Ma and Nie (2003). Lemma 2.1 is related to the linear programming approach in Bikhchandani and Mamer (1987) and the dual. $F(p)$ is called the economic rent function by Smith (1965). Lemma 2.2 is basically an application of the Fenchel duality. Thus, a competitive equilibrium is related to those optimal solutions y in Y^* by Lemma 2.2. A vector y is an optimal solution in Y^* if and only if

$$0 \in \sum_{i=1}^{n} \bar{co}S_i(y) - \sum_{j=1}^{m} \bar{co}D_j(y).$$

Moreover a price vector p is competitive only if $0 \in \partial(f+g)(p)$. The GS condition in Kelso and Crawford (1982) is sufficient for the existence of competitive equilibrium. It is shown to be almost necessary by Gul and Stacchetti (1999).

Gross Substitutes (GS) Condition (Kelso and Crawford (1982)). *A utility function $u : 2^N \to R$ satisfies the gross substitutes condition if its demand correspndence $D : R^n \to R^n$, defined by*

$$D(p) = \{e(S) : u(S) - \sum_{a \in S} p_a \geq u(T) - \sum_{b \in T} p_b \ \forall S, T \subset N\},$$

satisfies the following: For any p and q in R^n_+ such that $p \leq q$, for any S such that $e(S) \in D(p)$, there exists $e(T) \in D(q)$ such that $\{i \in S : p_i = q_i\} \subset T$.

Theorem 2.3 (Kelso and Crawford (1982), Gul and Stacchetti (1999)). *Let the GS condition hold for all utility functions u_j, $j = 1, 2, \cdots, m$. Then a competitive equilibrium exists.*

Moreover, under the GS condition, the set Y^* of the optimal solutions of the problem $\mathcal{P}$ coincides with the set of all competitive equilibrium prices (Lemma 2.1). It is a lattice whenever Y is a lattice under $\vee$ and $\wedge$ by the following result and Theorem 4.1 in Topkis (1978) (Gul and Stacchetti (1999)).

Theorem 2.4 (Ausubel and Milgrom (2002)). *Let the GS condition hold for all utility functions u_j, $j = 1, 2, \cdots, m$. Then the function $F(p) = f(p) + g(p)$ is submodular.*

In Section 5, we also consider situations where u_j is affected by some random variable R_j of finite dimensions, which may persist with the DA mechanism all the way.

3 The DA Mechanism

The existence of competitive equilibrium is not a major concern of this paper. It exists if N contains all shares of one single stock or shares of stocks that are substitutes in nature. Henceforth we call an optimal solution in Y^* an (intrinsic value) equilibrium, with understanding that f is the seller side and g is the buyer side.

3.1 The DA Mechanism without Randomization

Let $f = \sum_{i=1}^{m} f_i$ and $g = \sum_{i=1}^{m} g_i$.[3]

The DA mechanism is defined as follows.

Let $\Phi_{0,k} = X_k$. For $i = 1, 2, \cdots, m$, let

$$(DA) \qquad \begin{array}{rcll} \psi_{i,k} & = & \Phi_{i-1,k} - a_k h_{i,k}, \ h_{i,k} \in \partial f_i(\Phi_{i-1,k}) & \\ \varphi_{i,k} & = & \Phi_{i-1,k} - b_k \ell_{i,k}, \ \ell_{i,k} \in \partial g_i(\Phi_{i-1,k}) & \\ \Phi_{i,k} & = & \alpha\psi_{i,k} + (1-\alpha)\varphi_{i,k}, & \alpha \in [0,1]. \end{array}$$

Let $X_{k+1} = \Phi_{m,k}$.

The two sequences $\{a_k\}$ and $\{b_k\}$ are the stepsizes of ask and bid, respectively. In the NYSE specialists have the power of controlling the stepsizes of bid and ask and in the Nasdaq market makers typically control the stepsizes of bid and ask. $h_{i,k}$ and $\ell_{i,k}$ may be considered as the sell and buy orders, respectively, after observing prices $\Phi_{i-1,k}$. Prices are adjusted based on the buy and sell orders and the stepsizes of bid and ask. The ask prices $\psi_{i,k}$ and the bid prices $\varphi_{i,k}$ are weighted according to α (the α-DA in Chatterjee and Samuelson (1983) and Wilson (1985)). There is a question how to keep a record of the orders that are executed. As long as the quasi-linear structure remains, the ownership or change in ownership of the objects will not affect Y^* and the problem $\mathcal{P}$. This is somehow important for our DA mechanism. If objects are seen as shares of some stocks or contracts, the details who own what shares or contracts and by how many will become irrelevant for the fundamental value at equilibrium and the problem $\mathcal{P}$.[4] Because there are changes in ownerships from k to $k+1$, f_i may not be the same for k and $k+1$. Such an issue will not cause problems for our results since what matters is just f in the end, which remains the same across all $k = 0, 1, 2 \cdots$. For simplicity in notation we treat f_i as if it remains the same across all k. Also note that after each round k, buyers and sellers can form parallel lines in an arbitrary order.

[3]The number of objects may not be the same as the number of agents in the economy considered by Bikhchandani and Mamer (1987). But one can alsways add a dummy on either side. If objects in N are initially owned by agents in M (Ma (1998b)), f_i should be defined by $f_i(p) = \sum_{a \in X_i} p_a$ where X_i is agent i's initial endowment of objects.

[4]This implies that a company's intrinsic value does not increase after learning that Warren Buffett owns shares of that company. But share price may go up after the news at least temporarily because such news may create noises in orders. We will study noises in detail in Section 5.

The key is that each buyer or seller must be involved in the price iteration process (see Nedić and Bertsekas (2001)).

A buy or sell order contains the genuine demand or supply information, no matter whether it has been fully or partially executed. Therefore prices are adjusted according to the orders that are placed not what has been executed.

In our DA mechanism above, if we set $\alpha = 0$ and assume $f = 0$, the DA mechanism becomes the incremental subgradient method in Nedić and Bertsekas (2001). If $f \neq 0$ and $\alpha \in (0, 1)$, one can set $\alpha a_k = (1 - \alpha) b_k$ for all $k = 0, 1, 2, \cdots$, and then the DA mechanism also becomes the incremental subgradient method. So the DA mechanism is a natural generalization of the incremental subgradient method from algorithm design aspects. More important, almost all major exchanges of stocks and derivatives use a version of DA mechanisms. The main advantage of the DA mechanism with bid and ask is that we can see how the two sequences of stepsizes of bids $\{b_k\}$ and asks $\{a_k\}$ affect the convergence of the generated price sequence $\{X_k\}$, so that we can study issues like bubbles, crashes, and trends often observed in real exchange markets. As in the incremental subgradient method, the DA mechanism is a decentralized system under which a buyer and a seller reveal their private information at each moment, without knowing what the market demand and supply are.[5] It is somehow amazing to find out such a system can in fact reach a market equilibrium in the end under certain regularities on, almost only on, the stepsizes of bid and ask.

3.2 The DA Mechanism with Randomization

Next we introduce the DA mechanism with randomization, similar to Nedić and Bertsekas (2001). Here we can have more freedom on the sizes of the two sides. Let $f = \sum_{i=1}^{m} f_i$ and $g = \sum_{j=1}^{n} g_j$.

Let w_k be a random variable taking equiprobable values from the set $\{1, 2, \cdots, m\}$ and w'_k be a random variable taking equiprobable values from the set $\{1, 2, \cdots, n\}$. Let $h_{w_k}(X_k) \in \partial f_{w_k}(X_k)$ and $\ell_{w'_k}(X_k) \in \partial g_{w'_k}(X_k)$, where if w_k takes a value j, then the vector $\partial f_{w_k}(X_k)$ is $\partial f_j(X_k)$, similar for g.

Our sequence $\{X_k\}$ is generated by the DA mechanism with randomization as below.

Given X_k, let

$$\psi_{k+1} = X_k - a_k h_{w_k}(X_k), \qquad h_{w_k}(X_k) \in \partial f_{w_k}(X_k)$$

[5]It is more likely that they do not care. It is our academia who cares very much.

(RDA)

$$\varphi_{k+1} = X_k - b_k \ell_{w'_k}(X_k), \qquad \ell_{w'_k}(X_k) \in \partial g_{w'_k}(X_k).$$

And set $X_{k+1} = \alpha\psi_{k+1} + (1-\alpha)\varphi_{k+1}, \qquad \alpha \in [0,1].$

We assume that the stepsizes satisfy the following conditions, which are commonly used for the gradient, subgradient, and incremental subgradient methods in the literature (see, e.g., Correa and Lemaréchal (1993), Nedić and Bertsekas (2001), Ram et al. (2009)).

Assumption 3.1 (Diminishing Stepsizes). *Assume that the two sequences $\{a_k\}$ and $\{b_k\}$ of stepsizes are such that (i). $a_k > 0$ and $b_k > 0$; (ii). $\sum_{k=0}^{\infty} a_k = +\infty$ and $\sum_{k=0}^{\infty} b_k = +\infty$; (iii). $\sum_{k=0}^{\infty} a_k^2 < +\infty$ and $\sum_{k=0}^{\infty} b_k^2 < +\infty$.*

3.3 Equilibrium

Beyond Y^*, there are two related problems $\mathcal{P}(\alpha)$ and $\mathcal{P}(\alpha,\lambda)$, where $\alpha \in [0,1]$ and λ is some positive constant:

$$\mathcal{P}(\alpha) \qquad \text{minimize } F(y,\alpha) \equiv (\alpha f + (1-\alpha)g)(y)$$
$$\text{subject to } y \in Y$$

and

$$\mathcal{P}(\alpha,\lambda) \qquad \text{minimize } F(y,\alpha,\lambda) \equiv (\alpha f + \lambda(1-\alpha)g)(y)$$
$$\text{subject to } y \in Y.$$

We need additional notation:

$$F^*(\alpha) = \inf_{y\in Y} F(y,\alpha), \qquad Y^*(\alpha) = \{y \in Y | F(y,\alpha) = F^*(\alpha)\}$$

and

$$F^*(\alpha,\lambda) = \inf_{y\in Y} F(y,\alpha,\lambda), \qquad Y^*(\alpha,\lambda) = \{y \in Y | F(y,\alpha,\lambda) = F^*(\alpha,\lambda)\}.$$

With the help of the GS condition, it follows from Theorem 4.1 in Topkis (1978) and Theorem 2.4 that both $Y^*(\alpha)$ and $Y^*(\alpha,\lambda)$ are lattice whenever Y is. The following two results tell us how the three sets of equilibria are related for a single asset. Since f is the supply side and g is the demand side, the supply and demand curves are given by $\partial f(y)$ and $-\partial g(y)$, respectively. Note that $Y^* = Y^*(\alpha)$ for $\alpha = \frac{1}{2}$.

Proposition 3.2 *Let $Y = R_+$ and assume $\partial g(y) < 0$ for $y \in Y^*$. Then $Y^* \geq Y^*(\alpha)$ and $relint(Y^*) \cap relint(Y^*(\alpha)) = \emptyset$ for any α such that $1 \geq \alpha > \frac{1}{2}$. Similarly, $Y^* \leq Y^*(\alpha)$ and $relint(Y^*) \cap relint(Y^*(\alpha)) = \emptyset$ for any α such that $\frac{1}{2} > \alpha \geq 0$.*

Proof. Let $Y^*(\alpha) = [\underline{v}, \overline{v}]$. Then we have that $(\alpha\partial f + (1-\alpha)\partial g)(y) > 0$ for any $y > \overline{v}$ and $(\alpha\partial f + (1-\alpha)\partial g)(y) < 0$ for any $y < \underline{v}$. Since

$$\alpha\partial f + (1-\alpha)\partial g = \alpha(\partial f + \partial g) + (1-2\alpha)\partial g,$$

we have that $(1-2\alpha)\partial g(y) \subset (\alpha\partial f + (1-\alpha)\partial g)(y)$ for any $y \in Y^*$. If $\alpha > \frac{1}{2}$, then $(1-2\alpha)\partial g(y) > 0$ for any $y \in Y^*$. So $y \not< \underline{v}$ for any $y \in Y^*$. If there exists $y \in Y^*$ such that $y < \overline{v}$, then $0 = (\alpha\partial f + (1-\alpha)\partial g)(y)$, which implies that $(1-2\alpha)\partial g(y) = 0$, a contradiction. Thus, $Y^* \geq Y^*(\alpha)$ and $relint(Y^*) \cap relint(Y^*(\alpha)) = \emptyset$ for any α such that $1 \geq \alpha > \frac{1}{2}$. Similarly, $Y^* \leq Y^*(\alpha)$ and $relint(Y^*) \cap relint(Y^*(\alpha)) = \emptyset$ for any α such that $\frac{1}{2} > \alpha \geq 0$. This completes the proof.

Similarly, we have the following result. Note that $Y^* = Y^*(\alpha, \lambda)$ whenever $\lambda = \frac{\alpha}{1-\alpha}$ for $\alpha \in (0, 1)$.

Proposition 3.3 *Let $Y = R_+$ and $\partial g(y) < 0$ for $y \in Y^*$. (i). If $\lambda < \frac{\alpha}{1-\alpha}$, then $Y^* \geq Y^*(\alpha, \lambda)$; (ii). If $\lambda > \frac{\alpha}{1-\alpha}$, then $Y^* \leq Y^*(\alpha, \lambda)$.*

One can also follow the proof in Proposition 3.2 to study how an entry of a seller or a buyer may affect the equilibrium prices. Next we define bubbles, crashes and efficiency.

Definition 3.4. *Let $\{X_k\}$ be the sequence of prices generated by a DA mechanism for an asset or good, i.e., all items in N are identical. (i). It is a* **bubble** *if there exists a λ such that $\lambda > \frac{\alpha}{1-\alpha}$ and the sequence $\{X_k\}$ converges to $y \in Y^*(\alpha, \lambda) \setminus Y^*$; (ii). It is a* **crash** *if there exists a λ such that $\lambda < \frac{\alpha}{1-\alpha}$ and the sequence $\{X_k\}$ converges to $y \in Y^*(\alpha, \lambda) \setminus Y^*$; (iii). It is* **efficient** *if it converges to Y^*.*

A bubble is created for an asset or good if its price reaches a high level that is not supported by its fundamental. A crash is created for an asset or good if its price reaches a low level that does not reflect its fundamental. These definitions are consistent with those defined in the literature of finance (see, e.g., Barberis et al. (1998), Shleifer (1999), Lo (2005)) and in experimental markets studied by Smith et al. (1988). A DA mechanism is efficient if the sequence of prices generated converges to an equilibrium supported by its fundamental, i.e., agents' intrinsic values. This definition is consistent with the one defined by Fama (1965):

> "The challenge of the theory of random walks to the proponent of fundamental analysis, however, is more involved. If the random-walk theory is valid and if security exchanges are 'efficient markets', then stock prices at any point in time will represent good estimates of intrinsic or fundamental values."

We will answer this challenge question for problem $\mathcal{P}$, in a style demanded by F.A. Von Hayek (1937). A word of caution is in order: The security exchanges discussed by Fama (1965) may be more complicated than what has been modeled by $\mathcal{P}$. Our DA mechanism may be seen at best an ideal proxy of real DA mechanisms used by various security exchanges. But this does not mean that our study is not useful for these markets. For example, the EMH has been extensively studied in the empirical literature. It should be worthwhile if it can be proved with a rigorous study in theory. Indeed, one has to answer the question if there is a possibility for a DA mechanism to be efficient in an abstract and general framework. Such a question is critical for the EMH since stock prices are, afterall, determined through bids and asks by DA mechanisms. One underlying assumption of the EMH is that a DA mechanism can always find a market equilibrium supported by intrinsic values. On the other hand, the behavioral finance that heavily depends on investors' sentiment may be criticized to be speculative without first answering the question in theory if a DA mechanism can indeed generate bubbles and crashes. The aim below is to find conditions when the DA mechanism defined above is efficient and when it creates bubbles or crashes.

4 Main Results

We need to assume that all subgradients are bounded, as in Nedić and Bertsekas (2001):

Assumption 4.1 (Subgradient Boundness). *There exist scalars $C_1, C_2, \cdots, C_m$ and $D_1, D_2, \cdots, D_m$ such that*

$$\|h\| \leq C_i, \qquad \forall h \in \partial f_i(X_k) \cup \partial f_i(\Phi_{i-1,k}), i = 1, 2, \cdots, m, k = 0, 1, 2, \cdots$$

and

$$\|\ell\| \leq D_i, \qquad \forall \ell \in \partial g_i(X_k) \cup \partial g_i(\Phi_{i-1,k}), i = 1, 2, \cdots, m, k = 0, 1, 2, \cdots.$$

For the economy or market introduced in Subsection 2.1 there are a finite number of objects. Moreover, f and g are polyhedral functions. So this assumption is automatically satisfied. Next we provide a result that is an analogy to Lemma 2.1 in Nedić and Bertsekas (2001).

Lemma 4.2. *Let Assumption 4.1 hold. Let $\{X_k\}$ be the sequence generated by (DA) mechanism above. Then for all $y \in Y$ and $k \geq 0$, we have*

$$\begin{aligned} \|X_{k+1} - y\|^2 &\leq \|X_k - y\|^2 - 2a_k\alpha(f(X_k) - f(y)) - 2b_k(1-\alpha)(g(X_k) - g(y)) \\ &+ (\alpha a_k C + (1-\alpha) b_k D)^2, \end{aligned}$$

where $C = \sum_{i=1}^m C_i$ and $D = \sum_{i=1}^m D_i$.

Proof.

$$\begin{aligned} \|\Phi_{i,k} - y\|^2 &= \|\alpha\psi_{i,k} + (1-\alpha)\varphi_{i,k} - y\|^2 \\ &= \|\alpha(\psi_{i,k} - y) + (1-\alpha)(\varphi_{i,k} - y)\|^2 \\ &= \alpha^2\|\psi_{i,k} - y\|^2 + (1-\alpha)^2\|\varphi_{i,k} - y\|^2 + 2\alpha(1-\alpha)\langle(\psi_{i,k} - y), (\varphi_{i,k} - y)\rangle \\ &= \alpha^2\|\Phi_{i-1,k} - y - a_k h_{i,k}\|^2 + (1-\alpha)^2\|\Phi_{i-1,k} - y - b_k\ell_{i,k}\|^2 \\ &\quad +2\alpha(1-\alpha)\langle(\Phi_{i-1,k} - y - a_k h_{i,k}), (\Phi_{i-1,k} - y - b_k\ell_{i,k})\rangle \\ &= \alpha^2\|\Phi_{i-1,k} - y\|^2 - 2a_k\alpha^2\langle h_{i,k}, (\Phi_{i-1,k} - y)\rangle + \alpha^2 a_k^2\|h_{i,k}\|^2 \\ &\quad +(1-\alpha)^2\|\Phi_{i-1,k} - y\|^2 - 2b_k(1-\alpha)^2\langle \ell_{i,k}, (\Phi_{i-1,k} - y)\rangle + (1-\alpha)^2 b_k^2\|\ell_{i,k}\|^2 \\ &\quad +2\alpha(1-\alpha)\|\Phi_{i-1,k} - y\|^2 - 2\alpha(1-\alpha)a_k\langle h_{i,k}, (\Phi_{i-1,k} - y)\rangle \\ &\quad -2\alpha(1-\alpha)b_k\langle \ell_{i,k}, (\Phi_{i-1,k} - y)\rangle + 2\alpha(1-\alpha)a_k b_k\langle h_{i,k}, \ell_{i,k}\rangle \\ &= \|\Phi_{i-1,k} - y\|^2 - 2\alpha a_k\langle h_{i,k}, (\Phi_{i-1,k} - y)\rangle - 2(1-\alpha)b_k\langle \ell_{i,k}, (\Phi_{i-1,k} - y)\rangle \\ &\quad +\|\alpha a_k h_{i,k} + (1-\alpha)b_k\ell_{i,k}\|^2 \end{aligned}$$

Since we have $\|h_{i,k}\| \leq C_i$, $\|\ell_{i,k}\| \leq D_i$ for all $k = 0, 1, 2, \cdots$, we obtain

$$\|\Phi_{i,k} - y\|^2 \leq \|\Phi_{i-1,k} - y\|^2 - 2\langle(\alpha a_k h_{i,k} + (1-\alpha)b_k\ell_{i,k}), (\Phi_{i-1,k} - y)\rangle + (\alpha a_k C_i + (1-\alpha)b_k D_i)^2.$$

Sum over $i = 1, 2, \cdots, m$, we get

$$\begin{aligned}\sum_{i=1}^{m} \|\Phi_{i,k} - y\|^2 &\leq \sum_{i=1}^{m} \|\Phi_{i-1,k} - y\|^2 - 2\sum_{i=1}^{m} \langle (\alpha a_k h_{i,k} + (1-\alpha) b_k \ell_{i,k}), (\Phi_{i-1,k} - y)\rangle \\ &\quad + \sum_{i=1}^{m} (\alpha a_k C_i + (1-\alpha) b_k D_i)^2.\end{aligned}$$

So we have

$$\begin{aligned}\|X_{k+1} - y\|^2 &\leq \|X_k - y\|^2 - 2\sum_{i=1}^{m} \langle (\alpha a_k h_{i,k} + (1-\alpha) b_k \ell_{i,k}), (\Phi_{i-1,k} - y)\rangle \\ &\quad + \sum_{i=1}^{m} (\alpha a_k C_i + (1-\alpha) b_k D_i)^2.\end{aligned}$$

By the definition of $h_{i,k}$ and $\ell_{i,k}$,

$$\langle h_{i,k}, (y - \Phi_{i-1,k})\rangle \leq f(y) - f(\Phi_{i-1,k})$$

and

$$\langle \ell_{i,k}, (y - \Phi_{i-1,k})\rangle \leq g(y) - g(\Phi_{i-1,k}).$$

Then

$$\begin{aligned}\|X_{k+1} - y\|^2 &\leq \|X_k - y\|^2 - 2\alpha a_k \sum_{i=1}^{m} (f_i(\Phi_{i-1,k}) - f_i(y)) - 2(1-\alpha) b_k \sum_{i=1}^{m} (g_i(\Phi_{i-1,k}) - g_i(y)) \\ &\quad + \sum_{i=1}^{m} (\alpha a_k C_i + (1-\alpha) b_k D_i)^2.\end{aligned}$$

So

$$\begin{aligned}\|X_{k+1} - y\|^2 &\leq \|X_k - y\|^2 - 2\alpha a_k (f(X_k) - f(y)) - 2(1-\alpha) b_k (g(X_k) - g(y)) \\ &\quad - 2\alpha a_k \sum_{i=1}^{m} (f_i(\Phi_{i-1,k}) - f_i(X_k)) - 2(1-\alpha) b_k \sum_{i=1}^{m} (g_i(\Phi_{i-1,k}) - g_i(X_k)) \\ &\quad + \sum_{i=1}^{m} (\alpha a_k C_i + (1-\alpha) b_k D_i)^2.\end{aligned} \tag{1.1}$$

Next we need to estimate $f_i(\Phi_{i-1,k}) - f_i(X_k)$ and $g_i(\Phi_{i-1,k}) - g_i(X_k)$.

Lemma 4.2.1. $\|\Phi_{i-1,k} - X_k\| \leq \sum_{j=1}^{i-1} (\alpha a_k C_j + (1-\alpha) b_k D_j)$.

Proof of Lemma 4.2.1. We show Lemma 4.2.1 by induction. Note that $\Phi_{0,k} - X_k = 0$. Assume that it holds for $i - 1$. Then

$$\|\Phi_{i,k} - X_k\| = \|(\alpha \psi_{i,k} + (1-\alpha) \varphi_{i,k}) - X_k\|$$

$$
\begin{aligned}
&\leq \alpha\|\psi_{i,k} - X_k\| + (1-\alpha)\|\varphi_{i,k} - X_k\| \\
&\leq \alpha\|\Phi_{i-1,k} - a_k h_{i,k} - X_k\| + (1-\alpha)\|\Phi_{i-1,k} - b_k \ell_{i,k} - X_k\| \\
&\leq \|\Phi_{i-1,k} - X_k\| + \alpha a_k C_i + (1-\alpha) b_k D_i \qquad \text{by induction hypothesis} \\
&\leq \sum_{j=1}^{i-1} (\alpha a_k C_j + (1-\alpha) b_k D_j) + \alpha a_k C_i + (1-\alpha) b_k D_i.
\end{aligned}
$$

This completes the proof of Lemma 4.2.1.

So

$$\|f_i(\Phi_{i-1,k}) - f_i(X_k)\| \leq \sum_{j=1}^{i-1} C_i(\alpha a_k C_j + (1-\alpha) b_k D_j)$$

and

$$\|g_i(\Phi_{i-1,k}) - g_i(X_k)\| \leq \sum_{j=1}^{i-1} D_i(\alpha a_k C_j + (1-\alpha) b_k D_j).$$

Plug into (1,1), we have

$$
\begin{aligned}
\|X_{k+1} - y\|^2 \leq\; & \|X_k - y\|^2 - 2\alpha a_k (f(X_k) - f(y)) - 2(1-\alpha) b_k (g(X_k) - g(y)) \\
& + 2\sum_{i=1}^{m} (\alpha a_k C_i + (1-\alpha) b_k D_i) \sum_{j=1}^{i-1} (\alpha a_k C_j + (1-\alpha) b_k D_j) \\
& + \sum_{i=1}^{m} (\alpha a_k C_i + (1-\alpha) b_k D_i)^2 \\
=\; & \|X_k - y\|^2 - 2\alpha a_k (f(X_k) - f(y)) - 2(1-\alpha) b_k (g(X_k) - g(y)) \\
& + (\sum_{i=1}^{m} (\alpha a_k C_i + (1-\alpha) b_k D_i))^2 \\
=\; & \|X_k - y\|^2 - 2\alpha a_k (f(X_k) - f(y)) - 2(1-\alpha) b_k (g(X_k) - g(y)) \\
& + (\alpha a_k C + (1-\alpha) b_k D)^2.
\end{aligned}
$$

This completes the proof of Lemma 4.2.

We are now ready to prove one of our main results. Note that the convergence theorem for the diminishing stepsize in Nedić and Bertsekas (2001) does not apply to the DA mechanism because the relative strength of the two stepsizes of bid and ask for (DA) mechanism matters very much, which is capitured by λ and the additional condition on a_k and b_k. Moreover, (DA) mechanism may not converge to an optimal solution in Y^*, leaving the door open for both bubbles and crashes.

Proposition 4.3. *Let Assumptions 3.1 and 4.1 hold and Y be compact. If there exists a positive constant λ such that*

$$\sum_{k=0}^{\infty} |b_k - \lambda a_k| < +\infty$$

and $\{X_k\}$ be the sequence generated by (DA) mechanism. Then

$$\liminf_{k\to\infty} dist(X_k, Y^*(\alpha,\lambda)) = 0.$$

Proof. From Lemma 4.2, we obtain for all $y^* \in Y^*(\alpha,\lambda)$ and $k \geq 0$,

$$\begin{aligned} \|X_{k+1} - y^*\|^2 &\leq \|X_k - y^*\|^2 - 2a_k[(\alpha f + (1-\alpha)\lambda g)(X_k) - (\alpha f + (1-\alpha)\lambda g)(y^*)] \\ &\quad -2(b_k - \lambda a_k)(1-\alpha)(g(X_k) - g(y^*)) + (\alpha a_k C + (1-\alpha) b_k D)^2. \end{aligned}$$

Since Y is compact, g is continuous, image $(g(Y))$ is bounded. That means there exists $M > 0$ such that $|g(y)| \leq M$ for all $y \in Y$. Hence,

$$\begin{aligned} 0 \leq \|X_{\aleph+1} - y^*\|^2 &\leq \|X_0 - y^*\|^2 - 2\sum_{k=0}^{\aleph} a_k[(\alpha f + (1-\alpha)\lambda g)(X_k) - (\alpha f + (1-\alpha)\lambda g)(y^*)] \\ &\quad +2(\sum_{k=0}^{\aleph} |b_k - \lambda a_k|) \cdot (1-\alpha) \cdot 2M + \sum_{k=0}^{\aleph} (\alpha a_k C + (1-\alpha) b_k D)^2 \\ &= I - II + III + IV. \end{aligned}$$

I is a constant. When $\aleph$ goes to infinity, $III < +\infty$ since $\sum_{k=0}^{\infty} |b_k - \lambda a_k| < +\infty$ and

$$IV \leq 2(\alpha^2 C^2 \sum_{k=0}^{\infty} a_k^2 + (1-\alpha)^2 D^2 \sum_{k=0}^{\infty} b_k^2) < +\infty.$$

Thus, $II < +\infty$. We obtain

$$\liminf_{k\to\infty} [(\alpha f + (1-\alpha)\lambda g)(X_k) - (\alpha f + (1-\alpha)\lambda g)(y^*)] = 0.$$

Otherwise, $\exists \delta > 0$ such that $\forall k \in \aleph$,

$$(\alpha f + (1-\alpha)\lambda g)(X_k) - (\alpha f + (1-\alpha)\lambda g)(y^*) > \delta.$$

And then $II > \delta \sum_{k=0}^{\infty} a_k = +\infty$, a contradiction.

Now take a subsequence $\{X_{n_k}\}$ of $\{X_k\}$ such that

$$0 \leq (\alpha f + (1-\alpha)\lambda g)(X_{n_k}) - (\alpha f + (1-\alpha)\lambda g)(y^*) < \frac{1}{k}.$$

By the fact that Y is compact, $\{X_{n_k}\}$ has at least one accumulation point y_0, say, and since $\alpha f + (1-\alpha)\lambda g$ is continuous, we have that

$$\lim_{k\to\infty} (\alpha f + (1-\alpha)\lambda g)(X_{n_k}) = (\alpha f + (1-\alpha)\lambda g)(y_0).$$

By the definition of $\{X_{n_k}\}$, we know that

$$\lim_{k\to\infty} (\alpha f + (1-\alpha)\lambda g)(X_{n_k}) = (\alpha f + (1-\alpha)\lambda g)(y^*),\ y^* \in Y^*(\alpha, \lambda).$$

Hence, $y_0 \in Y^*(\alpha, \lambda)$. So

$$\liminf_{k\to\infty} dist(X_k, Y^*(\alpha, \lambda)) = 0.$$

This completes the proof.

One can strengthen Proposition 4.3 by applying Proposition 1.3 in Correa and Lemaréchal (1993) as follows.

Proposition 4.4. *Let Assumptions 3.1 and 4.1 hold. Then the sequence $\{X_k\}$ in Proposition 4.3 converges to some optimal solution $y_0 \in Y^*(\alpha, \lambda)$.*

Proof. Let

$$\begin{aligned}\delta_k &= 2a_k[(\alpha f + (1-\alpha)\lambda g)(X_k) - (\alpha f + (1-\alpha)\lambda g)(y_0)] + 2(|b_k - \lambda a_k|)\cdot(1-\alpha)\cdot 2M \\ &\quad +(\alpha a_k C + (1-\alpha) b_k D)^2 > 0.\end{aligned}$$

Then $\sum_{k=0}^{\infty} \delta_k = II + III + IV < +\infty$ (see the proof of Propositin 4.3). We also have that

$$\begin{aligned}\|X_{k+1} - y_0\|^2 &\le \|X_k - y_0\|^2 - 2a_k[(\alpha f + (1-\alpha)\lambda g)(X_k) - (\alpha f + (1-\alpha)\lambda g)(y_0)] \\ &\quad +2(|b_k - \lambda a_k|)\cdot(1-\alpha)\cdot 2M + (\alpha a_k C + (1-\alpha) b_k D)^2 \\ &\le \|X_k - y_0\|^2 + \delta_k.\end{aligned}$$

Then applying Proposition 1.3 in Correa and Lemaréchal (1993) with the result in Proposition 4.3, we have that

$$\lim_{k\to\infty} X_k = y_0.$$

Theorem 4.5 (Efficiency). *Let Assumption 4.1 hold and assume that* $\alpha = \frac{1}{2}$. *Let* $\{a_k\}$ *and* $\{b_k\}$ *be two sequences of stepsizes satisfying Assumption 3.1 and*

$$\sum_{k=0}^{\infty} |b_k - a_k| < +\infty.$$

Then the sequence $\{X_k\}$ *generated by (DA) mechanism converges to an optimal solution in* Y^*.

Note that $Y^* = Y^*(\frac{1}{2})$. Theorem 4.5 follows from Proposition 4.4 by setting $\lambda = 1$. For a single asset, with $\alpha = 0.5$, (DA) mechanism may generate a bubble (crash) if the two sequences $\{a_k\}$ and $\{b_k\}$ of stepsizes satisfy Assumption 3.1 and

$$\sum_{k=0}^{\infty} |b_k - \lambda a_k| < +\infty$$

for some λ that is greater (smaller) that 1.

4.1 What is λ?

We have seen λ above many times. In Proposition 4.3 we need λ to satisfy the condition such that $\sum_{k=0}^{\infty} |b_k - \lambda a_k| < +\infty$. Does it always exist such a λ for any two sequences $\{a_k\}$ and $\{b_k\}$ satisfying Assumption 3.1? Unfortunately the answer is negative. This shows that the relative strength between ask and bid is quite subtle for the DA mechanism. Such an issue does not exist for the incremental subgradient method in Nedić and Bertsekas (2001) because there is only one single sequence of stepsize needed there.

Example 4.6. Let

$$a_k = \begin{cases} \frac{1}{k}, & k \text{ is odd} \\ \frac{1}{k^2}, & k \text{ is even} \end{cases}$$

$$b_k = \begin{cases} \frac{1}{k^2}, & k \text{ is odd} \\ \frac{1}{k}, & k \text{ is even} \end{cases}$$

Then

$$\begin{aligned} \sum_{k=0}^{\infty} |a_k - \lambda b_k| &\geq \sum_{\substack{k \geq [\lambda]+1 \\ k \text{ is odd}}} |a_k - \lambda b_k| \\ &= \sum_{\substack{k \geq [\lambda]+1 \\ k \text{ is odd}}} a_k - \lambda \sum_{\substack{k \geq [\lambda]+1 \\ k \text{ is odd}}} b_k \\ &= \sum_{\substack{k \geq [\lambda]+1 \\ k \text{ is odd}}} \frac{1}{k} - \lambda \sum_{\substack{k \geq [\lambda]+1 \\ k \text{ is odd}}} \frac{1}{k^2} = +\infty \end{aligned}$$

for any λ.

Even if the limit $\lim_{k\to\infty} \frac{b_k}{a_k}$ exits and equals 1, there may not exist λ such that

$$\sum_{k=0}^{\infty} |b_k - \lambda a_k| < \infty.$$

Example 4.7. Let $a_k = \frac{1}{k^{\frac{3}{4}}}$ and $b_k = a_k(1 + a_k^{\frac{1}{3}})$. Then

$$\lim_{k\to\infty} \frac{b_k}{a_k} = \lim_{k\to\infty} (1 + \frac{1}{k^{\frac{1}{4}}}) = 1.$$

But

$$\sum_{k=0}^{\infty} |b_k - a_k| = \sum_{k=0}^{\infty} a_k^{\frac{4}{3}} = \sum_{k=0}^{\infty} \frac{1}{k} = +\infty.$$

Note that, by Proposition 4.8 below, if λ exists, it can only be 1. Hence, λ does not exist here.

The following answers what λ must be.

Proposition 4.8. *Let Assumption 3.1 hold. If $\sum_{k=0}^{\infty} |b_k - \lambda a_k| < +\infty$ for some λ, then the following must hold*

$$\liminf_{k\to\infty} \frac{b_k}{a_k} \le \lambda \le \limsup_{k\to\infty} \frac{b_k}{a_k}.$$

Proof. If there exist $\delta > 0$ and k_0 such that $\frac{b_k}{a_k} - \lambda > \delta$ for all $k \ge k_0$, then

$$\sum_{k=k_0}^{\infty} |b_k - \lambda a_k| \ge \delta \sum_{k=k_0}^{\infty} a_k = +\infty, \qquad \text{a contradiction.}$$

Hence, $\liminf_{k\to\infty} \frac{b_k}{a_k} \le \lambda$ is one necessary condition for $\sum_{k=0}^{\infty} |b_k - \lambda a_k| < +\infty$. Similarly, $\limsup_{k\to\infty} \frac{b_k}{a_k} \ge \lambda$. This completes the proof.

Proposition 4.9. *Let Assumption 3.1 hold. If there exists a λ such that $\sum_{k=0}^{\infty} |b_k - \lambda a_k| < +\infty$, then it must be unique.*

Proof. Suppose, on the contrary, that there are two λ and λ' such that

$$\sum_{k=0}^{\infty} |b_k - \lambda a_k| < +\infty \text{ and } \sum_{k=0}^{\infty} |b_k - \lambda' a_k| < +\infty.$$

Then

$$|\lambda - \lambda'| \sum_{k=0}^{\infty} a_k \leq \sum_{k=0}^{\infty} |b_k - \lambda a_k| + \sum_{k=0}^{\infty} |b_k - \lambda' a_k| < +\infty.$$

But $\sum_{k=0}^{\infty} a_k = +\infty$, a contradiction. This complets the proof.

When an equity is lack of liquidity, the accumulated spreads $\sum_k |b_k - a_k|$ can be large and the competitive efficiency of the DA mechanism may be in jeopardy. Note that these conditions on λ are based on the two sequences $\{a_k\}$ and $\{b_k\}$ only even though the price sequence of the DA mechanism depends on the buy and sell orders. Thus, the efficiency of an exchange such as the NYSE and Nasdaq may be tested through the two stepsizes of bid and ask. Consider a quote from Investopedia.com:

> What's the main difference between a specialist and a market maker? Not much. Both the New York Stock Exchange (NYSE) specialist and the Nasdaq market maker try to increase the liquidity on their respective exchanges and provide more fluid and efficient trading. -http://www.investopedia.com

Investors believe that the NYSE and Nasdaq are the most efficient markets. What does it mean precisely by efficiency here? Is it good enough for the bid−ask spread to be small for being efficiency of an exchange? Our results provide an answer more precisely through the conditions on the two stepsizes of bid and ask. Note the differences. Our conditions, though related, are not on the bid−ask spread. They are on the two "speeds" of bid and ask. In our setup a key condition for a $\frac{1}{2}$-DA mechanism to be efficient is such that $\sum_{k=0}^{\infty} |b_k - a_k| < \infty$ for the two sequences of stepsizes satisfying Assumption 3.1. Moreover, the following must hold

$$\liminf_{k \to \infty} \frac{b_k}{a_k} \leq 1 \leq \limsup_{k \to \infty} \frac{b_k}{a_k}.$$

These conditions apply to the DA mechanism (see Theorem 4.5) and the DA mechanism with randomization below (see Proposition 4.11 with $m = n$), without or with stochastic noises (see Propositions 5.5 and 5.7 with noises and $m = n$). In fact they are quite intuitive. But their proofs are somehow involved technically.

4.2 Main Result for the DA Mechanism with Randomization

The DA mechanism with randomization is defiend in Subsection 3.2. We need an assumption for our result.

Assumption 4.10. a). The sequence $\{w_k\}(\{w'_k\})$ is a sequence of independent random variables, each unifromly distributed over the set $\{1, 2, \cdots, m\}(\{1, 2, \cdots, n\})$. Furthermore, the two sequences $\{w_k\}$ and $\{w'_k\}$ are independent of the sequence $\{X_k\}$.
b). The two sets of subgradients $\{h_{w_k}(X_k), k = 0, 1, 2, \cdots\}$ and $\{\ell_{w'_k}(X_k), k = 0, 1, 2, \cdots\}$ are bounded. That is, there exist some positive constants C_0 and D_0 such that, with probability 1, $\|h_{w_k}(X_k)\| \leq C_0$ and $\|\ell_{w'_k}(X_k)\| \leq D_0, \forall k \geq 0$.

Proposition 4.11 *Let Assumptions 3.1 and 4.10 hold. If there exists a positive constant λ such that*

$$\sum_{k=0}^{\infty} |\frac{b_k}{n} - \lambda \frac{a_k}{m}| < +\infty,$$

then the sequence $\{X_k\}$ generated by (RDA) mechanism converges to some optimal solution in $Y^*(\alpha, \lambda)$.

Proof We obtain for all k and $y \in Y^*(\alpha, \lambda)$, as in the proof of Proposition 4.3,

$$E\{\|X_{k+1}-y\|^2|\mathcal{F}_k\} \leq \|X_k-y\|^2-2\alpha\frac{a_k}{m}(f(X_k)-f(y))-2(1-\alpha)\frac{b_k}{n}(g(X_k)-g(y))+(\alpha a_k C+(1-\alpha)b_k D)^2,$$

where $\mathcal{F}_k = \{X_0, X_1, \cdots, X_k\}$.

Two definitions are in order. A sample path is a sequence of $\{X_k\}$. For each $y^* \in Y^*(\alpha, \lambda)$, let Ω_{y^*} denote the set containing all sample paths $\{X_k\}$ such that

$$2\sum_{k=0}^{\infty}[(\alpha\frac{a_k}{m}f + (1-\alpha)\frac{b_k}{n}g)(X_k) - (\alpha\frac{a_k}{m}f + (1-\alpha)\frac{b_k}{n}g)(y^*)] < +\infty$$

and that $\{\|X_k - y^*\|\}$ converges. By the supermartingale convergence theorem (see Theorem 3.1 in Nedić and Bertsekas (2001) or Appendix 7.1), for each $y^* \in Y^*(\alpha, \lambda)$, we have that Ω_{y^*} is a set of probability 1. Let $\{\nu_i\}$ be a countable subset of the relative interior $relint_i(Y^*(\alpha, \lambda))$ that is dense in $Y^*(\alpha, \lambda)$. Define $\Omega = \bigcap_{i=1}^{\infty} \Omega_{\nu_i}$. Then Ω has probability 1 since

$$Prob(\bigcup_{i}^{\infty} \bar{\Omega}_{\nu_i}) \leq \sum_{i=1}^{\infty} Prob(\bar{\Omega}_{\nu_i}) = 0.$$

Since Y is compact, g is continuous, there exists M such that $|g(y)| \leq M$ for all $y \in Y$. For each sample path in Ω, the sequence $\|X_k - \nu_i\|$ converges so that $\{X_k\}$ is bounded. By

$$2\sum_{k=0}^{\infty}[(\alpha\frac{a_k}{m}f + (1-\alpha)\frac{b_k}{n}g)(X_k) - (\alpha\frac{a_k}{m}f + (1-\alpha)\frac{b_k}{n}g)(y)] \leq K < +\infty,$$

we have

$$\begin{aligned} 2\sum_{k=0}^{\infty}\frac{a_k}{m}[(\alpha f + \lambda(1-\alpha)g)(X_k) \quad &- \quad (\alpha f + \lambda(1-\alpha)g)(y)] \\ &\leq \quad 2\sum_{k=0}^{\infty}|\frac{b_k}{n} - \lambda\frac{a_k}{m}| \cdot |g(X_k) - g(y)| + K \\ &\leq \quad 4M\sum_{k=0}^{\infty}|\frac{b_k}{n} - \lambda\frac{a_k}{m}| + K < +\infty \end{aligned}$$

Thus, we have that

$$\lim_{k\to\infty}[(\alpha f + \lambda(1-\alpha)g)(X_k) - (\alpha f + \lambda(1-\alpha)g)(y^*)] = 0.$$

Otherwise, if there exist $\delta > 0$ such that for all k,

$$(\alpha f + \lambda(1-\alpha)g)(X_k) - (\alpha f + \lambda(1-\alpha)g)(y^*) > \delta,$$

then we have

$$2\sum_{k=0}^{\infty}\frac{a_k}{m}[(\alpha f + \lambda(1-\alpha)g)(X_k) - (\alpha f + \lambda(1-\alpha)g)(y)] > \frac{\delta}{m}\sum_{k=0}^{\infty}a_k = +\infty,$$

which is impossible.

Continuity of $\alpha f + \lambda(1-\alpha)g$ implies that all the limit points of $\{X_k\}$ are belong to $Y^*(\alpha, \lambda)$. Since $\{\nu_i\}$ is a dense subset of Y^* and $\|X_k - v_i\|$ converges, it follows that $\{X_k\}$ cannot have more than one limit point, so it must converge to some vector $y \in Y^*(\alpha, \lambda)$. This completes the proof.

5 The DA Mechanism with Stochastic Noises

Stochastic noises in buy and sell orders are important. Some of the noises may come from the fact that investors may not know precisely the intrinsic value functions u so that the submitted orders may deviate from the "true" orders. Note that these noises may go all the way along with the DA mechanism. The very existence of noises may jeopardize the efficiency of an exchange. But the DA mechanism can perform very well with stochastic noises as we will show below.

Let $f = \sum_{i=1}^m f_i$ and $g = \sum_{i=1}^m g_i$. Assume X_0 is a random initial vector. Let $\epsilon_{i,k}$ and $\delta_{i,k}$ denote two independent random noise vectors, $a_k > 0$ and $b_k > 0$ denote step-size.

The DA mechanism with stochastic noises is defined as follows.

Let $\Phi_{0,k} = X_k$. For $i = 1, 2, \cdots, m$, let

$$\begin{aligned}
\psi_{i,k} &= \Phi_{i-1,k} - a_k(h_{i,k} + \epsilon_{i,k}), \;\; h_{i,k} \in \partial f_i(\Phi_{i-1,k}) \\
\varphi_{i,k} &= \Phi_{i-1,k} - b_k(\ell_{i,k} + \delta_{i,k}), \;\; \ell_{i,k} \in \partial g_i(\Phi_{i-1,k}) \\
\Phi_{i,k} &= \alpha\psi_{i,k} + (1-\alpha)\varphi_{i,k}, \qquad\qquad \alpha \in [0,1].
\end{aligned}$$

Let $X_{k+1} = \Phi_{m,k}$. We define $\mathcal{F}_k^i$ to be the σ-algebra generated by the sequence $\Phi_{0,0}, \Phi_{1,0}, \cdots, \Phi_{m,0}, \cdots, \Phi_{i,k}$. Note that $\mathcal{F}_k^0$ is also denoted as $\mathcal{F}_k$.

Remark: These noise terms are similar to Ram et al. (2009). In fact, if $\alpha = 0$ and $f = 0$, the DA mechanism with stochastic noises above becomes the incremental subgradient method with stochastic errors studied by Ram et al. (2009). However, the results in Ram et al. (2009) do not apply here because there are two sequences of stepsizes that interact together to determine the price iteration process of the DA mechanism.

Assumption 5.1. There exist deterministic scalar sequences $\{\mu_k\}$, $\{\nu_k\}$, $\{\tau_k\}$ and $\{\sigma_k\}$ satisfying the following. For all i and k,

$$\begin{aligned}
\|E[\epsilon_{i,k}|\mathcal{F}_k^{i-1}]\| \le \mu_k, &\qquad \|E[\delta_{i,k}|\mathcal{F}_k^{i-1}]\| \le \tau_k; \\
E[\|\epsilon_{i,k}\|^2|\mathcal{F}_k^{i-1}] \le \nu_k^2, &\qquad E[\|\delta_{i,k}\|^2|\mathcal{F}_k^{i-1}] \le \sigma_k^2.
\end{aligned}$$

Note that $\mu_k \le \nu_k$ and $\tau_k \le \sigma_k$ for all $k = 0, 1, \cdots$.

Assumption 5.2. *There exist scalars $C_1, C_2, \cdots, C_m$ and $D_1, D_2, \cdots, D_m$ such that*

$$\|h\| \le C_i, \qquad \forall h \in \partial f_i(X_k) \cup \partial f_i(\Phi_{i-1,k}), i = 1, 2, \cdots, m, k = 0, 1, 2, \cdots$$

and

$$\|\ell\| \le D_i, \qquad \forall \ell \in \partial g_i(X_k) \cup \partial g_i(\Phi_{i-1,k}), i = 1, 2, \cdots, m, k = 0, 1, 2, \cdots.$$

Let $C = \sum_{i=1}^m C_i$ and $D = \sum_{i=1}^m D_i$.

Lemma 5.3. *Let Assumptions 5.1 and 5.2 hold. Then the sequene $\{X_k\}$ generate by the DA mechanism with stochastic noises is such that for any step size rule and any $y \in Y$,*

$$\begin{aligned} E[\|X_{k+1} - y\|^2 | \mathcal{F}_{k-1}^m] &\leq \|X_k - y\|^2 - 2\alpha a_k (f(X_k) - f(y)) - 2(1-\alpha) b_k (g(X_k) - g(y)) \\ &+2(\alpha a_k \mu_k + (1-\alpha) b_k \tau_k) \sum_{i=1}^{m} E[\|\Phi_{i-1,k} - y\| | \mathcal{F}_{k-1}^m] \\ &+(\alpha a_k C + (1-\alpha) b_k D + \alpha m a_k \nu_k + (1-\alpha) m b_k \sigma_k)^2, \end{aligned}$$

Note that $\mathcal{F}_{k-1}^m = \mathcal{F}_k^0$.

Proof of Lemma 5.3.

$$\begin{aligned} \|\Phi_{i,k} - y\|^2 &= \|\alpha \psi_{i,k} + (1-\alpha)\varphi_{i,k} - y\|^2 \\ &= \|\alpha(\psi_{i,k} - y) + (1-\alpha)(\varphi_{i,k} - y)\|^2 \\ &= \alpha^2 \|\psi_{i,k} - y\|^2 + (1-\alpha)^2 \|\varphi_{i,k} - y\|^2 + 2\alpha(1-\alpha) < \psi_{i,k} - y, \varphi_{i,k} - y > \\ &= \alpha^2 \|\Phi_{i-1,k} - y - a_k h_{i,k} - a_k \epsilon_{i,k}\|^2 + (1-\alpha)^2 \|\Phi_{i-1,k} - y - b_k \ell_{i,k} - b_k \delta_{i,k}\|^2 \\ &\quad +2\alpha(1-\alpha) < \Phi_{i-1,k} - y - a_k h_{i,k} - a_k \epsilon_{i,k}, \Phi_{i-1,k} - y - b_k \ell_{i,k} - b_k \delta_{i,k} > \\ &= \alpha^2 \|\Phi_{i-1,k} - y - a_k h_{i,k}\|^2 + \alpha^2 a_k^2 \|\epsilon_{i,k}\|^2 - 2\alpha^2 < \Phi_{i-1,k} - y - a_k h_{i,k}, a_k \epsilon_{i,k} > \\ &\quad +(1-\alpha)^2 \|\Phi_{i-1,k} - y - b_k \ell_{i,k}\|^2 + (1-\alpha)^2 b_k^2 \|\delta_{i,k}\|^2 \\ &\quad -2(1-\alpha)^2 < \Phi_{i-1,k} - y - b_k \ell_{i,k}, b_k \delta_{i,k} > \\ &\quad +2\alpha(1-\alpha) < \Phi_{i-1,k} - y - a_k h_{i,k}, \Phi_{i-1,k} - y - b_k \ell_{i,k} > \\ &\quad -2\alpha(1-\alpha) < \Phi_{i-1,k} - y - a_k h_{i,k}, b_k \delta_{i,k} > -2\alpha(1-\alpha) < \Phi_{i-1,k} - y - b_k \ell_{i,k}, a_k \epsilon_{i,k} > \\ &\quad +2\alpha(1-\alpha) a_k b_k < \epsilon_{i,k}, \delta_{i,k} > \\ &= \|\alpha(\Phi_{i-1,k} - y - a_k h_{i,k}) + (1-\alpha)(\Phi_{i-1,k} - y - b_k \ell_{i,k})\|^2 + \|\alpha a_k \epsilon_{i,k} + (1-\alpha) b_k \delta_{i,k}\|^2 \\ &\quad -2\alpha < \Phi_{i-1,k} - y - a_k h_{i,k}, \alpha a_k \epsilon_{i,k} + (1-\alpha) b_k \delta_{i,k} > \\ &\quad -2(1-\alpha) < \Phi_{i-1,k} - y - b_k \ell_{i,k}, \alpha a_k \epsilon_{i,k} + (1-\alpha) b_k \delta_{i,k} > \\ &= \|\Phi_{i-1,k} - y\|^2 - 2\alpha a_k < h_{i,k}, (\Phi_{i-1,k} - y) > -2(1-\alpha) b_k < \ell_{i,k}, (\Phi_{i-1,k} - y) > \\ &\quad +\|\alpha a_k h_{i,k} + (1-\alpha) b_k \ell_{i,k}\|^2 + \|\alpha a_k \epsilon_{i,k} + (1-\alpha) b_k \delta_{i,k}\|^2 \\ &\quad -2\alpha < \Phi_{i-1,k} - y - a_k h_{i,k}, \alpha a_k \epsilon_{i,k} + (1-\alpha) b_k \delta_{i,k} > \\ &\quad -2(1-\alpha) < \Phi_{i-1,k} - y - b_k \ell_{i,k}, \alpha a_k \epsilon_{i,k} + (1-\alpha) b_k \delta_{i,k} > \end{aligned}$$

Taking conditional expectations with respect to the σ-field $\mathcal{F}_k^{i-1}$, we further obtain that

$$E[\|\Phi_{i,k} - y\|^2 | \mathcal{F}_k^{i-1}] = \{\|\Phi_{i-1,k} - y\|^2 - 2\alpha a_k < h_{i,k}, (\Phi_{i-1,k} - y) > -2(1-\alpha) b_k < \ell_{i,k}, (\Phi_{i-1,k} - y) >$$

$$
\begin{aligned}
&+\|\alpha a_k h_{i,k} + (1-\alpha) b_k \ell_{i,k}\|^2\} \\
&+\{E[\|\alpha a_k \epsilon_{i,k} + (1-\alpha) b_k \delta_{i,k}\|^2 | \mathcal{F}_k^{i-1}] \\
&-2\alpha < \Phi_{i-1,k} - y - a_k h_{i,k}, E[\alpha a_k \epsilon_{i,k} + (1-\alpha) b_k \delta_{i,k} | \mathcal{F}_k^{i-1}] > \\
&-2(1-\alpha) < \Phi_{i-1,k} - y - b_k \ell_{i,k}, E[\alpha a_k \epsilon_{i,k} + (1-\alpha) b_k \delta_{i,k} | \mathcal{F}_k^{i-1}] >\} \\
= \;& I + II.
\end{aligned}
$$

Consider II first. We have that

$$
\begin{aligned}
II \;\le\; & (\alpha a_k \nu_k + (1-\alpha) b_k \sigma_k)^2 + 2\alpha(\|\Phi_{i-1,k} - y\| + a_k \|h_{i,k}\|)(\alpha a_k \mu_k + (1-\alpha) b_k \tau_k) \\
& +2(1-\alpha)(\|\Phi_{i-1,k} - y\| + b_k \|\ell_{i,k}\|)(\alpha a_k \mu_k + (1-\alpha) b_k \tau_k) \\
= \;& (\alpha a_k \nu_k + (1-\alpha) b_k \sigma_k)^2 + 2\|\Phi_{i-1,k} - y\|(\alpha a_k \mu_k + (1-\alpha) b_k \tau_k) \\
& +2\alpha a_k C_i(\alpha a_k \mu_k + (1-\alpha) b_k \tau_k) + 2(1-\alpha) b_k D_i(\alpha a_k \mu_k + (1-\alpha) b_k \tau_k).
\end{aligned}
$$

Now consider I. We have that

$$
\begin{aligned}
I \;\le\; & \|\Phi_{i-1,k} - y\|^2 - 2\alpha a_k (f_i(\Phi_{i-1,k}) - f_i(y)) - 2(1-\alpha) b_k (g_i(\Phi_{i-1,k}) - g_i(y)) \\
& +\|\alpha a_k C_i + (1-\alpha) b_k D_i\|^2
\end{aligned}
$$

since $h_{i,k} \in \partial f_i(\Phi_{i-1,k})$ and $\ell_{i,k} \in \partial g_i(\Phi_{i-1,k})$.

Taking the expectations contional on $\mathcal{F}_{k-1}^m = \mathcal{F}_k^0$, we obtain from $I + II$ that

$$
\begin{aligned}
E[\|\Phi_{i,k} - y\|^2 | \mathcal{F}_{k-1}^m] \;\le\; & E[\|\Phi_{i-1,k} - y\|^2 | \mathcal{F}_{k-1}^m] - 2\alpha a_k (f_i(X_k) - f_i(y)) - 2(1-\alpha) b_k (g_i(X_k) - g_i(y)) \\
& +2E[\|\Phi_{i-1,k} - y\| | \mathcal{F}_{k-1}^m](\alpha a_k \mu_k + (1-\alpha) b_k \tau_k) + M_{i,k},
\end{aligned}
$$

where

$$
\begin{aligned}
M_{i,k} \;=\; & (\alpha a_k C_i + (1-\alpha) b_k D_i)^2 + (\alpha a_k \nu_k + (1-\alpha) b_k \sigma_k)^2 \\
& +2\alpha a_k C_i(\alpha a_k \mu_k + (1-\alpha) b_k \tau_k) + 2(1-\alpha) b_k D_i(\alpha a_k \mu_k + (1-\alpha) b_k \tau_k) \\
& +2\alpha a_k E[\|f_i(\Phi_{i-1,k}) - f_i(X_k)\| | \mathcal{F}_{k-1}^m] + 2(1-\alpha) b_k E[\|g_i(\Phi_{i-1,k}) - g_i(X_k)\| | \mathcal{F}_{k-1}^m].
\end{aligned}
$$

Note that $\Phi_{0,k} = X_k$ and $\Phi_{m,k} = X_{k+1}$. Taking sum over $i = 1, 2, \cdots, m$, we have that

$$
\begin{aligned}
E[\|X_{k+1} - y\|^2 | \mathcal{F}_{k-1}^m] \;\le\; & \|X_k - y\|^2 - 2\alpha a_k (f(X_k) - f(y)) - 2(1-\alpha) b_k (g(X_k) - g(y)) \\
& +2(\alpha a_k \mu_k + (1-\alpha) b_k \tau_k) \sum_{i=1}^m E[\|\Phi_{i-1,k} - y\| | \mathcal{F}_{k-1}^m] + \sum_{i=1}^m M_{i,k}.
\end{aligned}
$$

Next we consider $\sum_{i=1}^m M_{i,k}$.

Lemma 5.3.1. *We claim that*

$$\|\Phi_{i-1,k} - X_k\| \leq \sum_{j=1}^{i-1}[\alpha a_k C_j + (1-\alpha)b_k D_j + \alpha a_k\|\epsilon_{j,k}\| + (1-\alpha)b_k\|\delta_{j,k}\|].$$

Proof of Lemma 5.3.1. We prove by induction.

$$\begin{aligned}
\|\Phi_{i,k} - X_k\| &= \|(\alpha\psi_{i,k} + (1-\alpha)\varphi_{i,k}) - X_k\| \\
&\leq \alpha\|\psi_{i,k} - X_k\| + (1-\alpha)\|\varphi_{i,k} - X_k\| \\
&= \alpha\|\Phi_{i-1,k} - a_k h_{i,k} - a_k\epsilon_{i,k} - X_k\| + (1-\alpha)\|\Phi_{i-1,k} - b_k\ell_{i,k} - b_k\delta_{i,k} - X_k\| \\
&\leq \|\Phi_{i-1,k} - X_k\| + \alpha a_k\|h_{i,k}\| + (1-\alpha)b_k\|l_{i,k}\| \\
&\quad + \alpha a_k\|\epsilon_{i,k}\| + (1-\alpha)b_k\|\delta_{i,k}\|.
\end{aligned}$$

By induction, we get that

$$\|\Phi_{i,k} - X_k\| \leq \sum_{j=1}^{i}[\alpha a_k C_j + (1-\alpha)b_k D_j + \alpha a_k\|\epsilon_{j,k}\| + (1-\alpha)b_k\|\delta_{j,k}\|].$$

This completes the proof of Lemma 5.3.1.

We now continue the proof of Lemma 5.3. Hence,

$$\begin{aligned}
E[\|f_i(\Phi_{i-1,k}) - f_i(X_k)\||\mathcal{F}_{k-1}^m] &\leq E[C_i\sum_{j=1}^{i-1}(\alpha a_k C_j + (1-\alpha)b_k D_j + \alpha a_k\|\epsilon_{j,k}\| + (1-\alpha)b_k\|\delta_{j,k}\|)|\mathcal{F}_{k-1}^m] \\
&\leq C_i\sum_{j=1}^{i-1}(\alpha a_k C_j + (1-\alpha)b_k D_j + \alpha a_k\nu_k + (1-\alpha)b_k\sigma_k);
\end{aligned}$$

and

$$\begin{aligned}
E[\|g_i(\Phi_{i-1,k}) - g_i(X_k)\||\mathcal{F}_{k-1}^m] &\leq E[D_i\sum_{j=1}^{i-1}(\alpha a_k C_j + (1-\alpha)b_k D_j + \alpha a_k\|\epsilon_{j,k}\| + (1-\alpha)b_k\|\delta_{j,k}\|)|\mathcal{F}_{k-1}^m] \\
&\leq D_i\sum_{j=1}^{i-1}(\alpha a_k C_j + (1-\alpha)b_k D_j + \alpha a_k\nu_k + (1-\alpha)b_k\sigma_k).
\end{aligned}$$

Then

$$\begin{aligned}
\sum_{i=1}^{m} M_{i,k} &\leq \sum_{i=1}^{m}(\alpha a_k C_i + (1-\alpha)b_k D_i)^2 + m(\alpha a_k\nu_k + (1-\alpha)b_k\sigma_k)^2 \\
&\quad + 2\sum_{i=1}^{m}(\alpha a_k C_i + (1-\alpha)b_k D_i)(\alpha a_k\mu_k + (1-\alpha)b_k\tau_k) \\
&\quad + 2\sum_{i=1}^{m}(\alpha a_k C_i + (1-\alpha)b_k D_i)\sum_{j=1}^{i-1}(\alpha a_k C_j + (1-\alpha)b_k D_j + \alpha a_k\nu_k + (1-\alpha)b_k\sigma_k)
\end{aligned}$$

$$
\begin{aligned}
& \text{(since } \mu_k \leq \nu_k \text{ and } \tau_k \leq \sigma_k) \\
\leq\ & \sum_{i=1}^{m}(\alpha a_k C_i + (1-\alpha) b_k D_i + \alpha a_k \nu_k + (1-\alpha) b_k \sigma_k)^2 \\
& +2\sum_{i=1}^{m}(\alpha a_k C_i + (1-\alpha) b_k D_i + \alpha a_k \nu_k + (1-\alpha) b_k \sigma_k) \bullet \\
& \sum_{j=1}^{i-1}(\alpha a_k C_j + (1-\alpha) b_k D_j + \alpha a_k \nu_k + (1-\alpha) b_k \sigma_k) \\
=\ & (\sum_{i=1}^{m}(\alpha a_k C_i + (1-\alpha) b_k D_i + \alpha a_k \nu_k + (1-\alpha) b_k \sigma_k))^2 \\
=\ & (\alpha a_k C + (1-\alpha) b_k D + \alpha m a_k \nu_k + (1-\alpha) m b_k \sigma_k)^2.
\end{aligned}
$$

This completes the proof of Lemma 5.3.

5.1 The DA Mechanism with Stochastic Noises and without Randomization

Assumption 5.4. The following holds:

$$\sum_{k=0}^{\infty} a_k \mu_k < \infty, \sum_{k=0}^{\infty} b_k \tau_k < \infty, \sum_{k=0}^{\infty} a_k^2 \nu_k^2 < \infty, \sum_{k=0}^{\infty} b_k^2 \sigma_k^2 < \infty.$$

Proposition 5.5. *Let Assumptions 3.1, 5.1, 5.2, and 5.4 hold. Assume that Y is compact and $Y(\alpha, \lambda)$ is nonempty for some positive constant $\lambda > 0$ such that*

$$\sum_{k=0}^{\infty} |b_k - \lambda a_k| < +\infty.$$

Then the sequence $\{X_k\}$ generated by the DA mechanism with stochastic noises converges to an optimal solution $y^* \in Y^*(\alpha, \lambda)$, with probability 1.

Proof. By Lemma 5.3, for any $y^* \in Y^*(\alpha, \lambda)$, we have that

$$
\begin{aligned}
E[\|X_{k+1} - y^*\|^2 | \mathcal{F}_{k-1}^m] \leq\ & \|X_k - y^*\|^2 - 2\alpha a_k (f(X_k) - f(y^*)) - 2(1-\alpha) b_k (g(X_k) - g(y^*)) \\
& +2(\alpha a_k \mu_k + (1-\alpha) b_k \tau_k) \sum_{i=1}^{m} E[\|\Phi_{i-1,k} - y^*\| | \mathcal{F}_{k-1}^m] + M_k,
\end{aligned}
$$

where

$$M_k = (\alpha a_k C + (1-\alpha) b_k D + m \alpha a_k \nu_k + m(1-\alpha) b_k \sigma_k)^2.$$

Since

$$E[\|\Phi_{i-1,k} - y^*\| | \mathcal{F}_{k-1}^m] \leq E[\|\Phi_{i-1,k} - X_k\| | \mathcal{F}_{k-1}^m] + \|X_k - y^*\| \qquad \text{by Lemma 5.3.1}$$

$$\leq \sum_{j=1}^{i-1}(\alpha a_k C_j + (1-\alpha)b_k D_j + \alpha a_k \nu_k + (1-\alpha)b_k\sigma_k) + \|X_k - y^*\|.$$

Hence

$$\begin{aligned}
2(\alpha a_k \mu_k &+ (1-\alpha)b_k\tau_k)\sum_{i=1}^{m} E[\|\Phi_{i-1,k} - y^*\||\mathcal{F}_{k-1}^m] \\
&\leq 2(\alpha a_k\mu_k + (1-\alpha)b_k\tau_k)\sum_{i=1}^{m}(\sum_{j=1}^{i-1}(\alpha a_k C_j + (1-\alpha)b_k D_j + \alpha a_k\nu_k + (1-\alpha)b_k\sigma_k) + \|X_k - y^*\|) \\
&\leq 2(\alpha a_k\mu_k + (1-\alpha)b_k\tau_k)\sum_{i=1}^{m}\sum_{j=1}^{i-1}(\alpha a_k C_j + (1-\alpha)b_k D_j + \alpha a_k\nu_k + (1-\alpha)b_k\sigma_k) \\
&\quad + m(\alpha a_k\mu_k + (1-\alpha)b_k\tau_k)(\|X_k - y^*\|^2 + 1).
\end{aligned}$$

And then

$$\begin{aligned}
E[\|X_{k+1} - y^*\|^2|\mathcal{F}_{k-1}^m] &\leq (1 + m(\alpha a_k\mu_k + (1-\alpha)b_k\tau_k))\|X_k - y^*\|^2 \\
&\quad - 2\alpha a_k(f(X_k) - f(y^*)) - 2(1-\alpha)b_k(g(X_k) - g(y^*)) \\
&\quad + (M_k + N_k),
\end{aligned}$$

where

$$\begin{aligned}
N_k &= 2(\alpha a_k\mu_k + (1-\alpha)b_k\tau_k)\sum_{i=1}^{m}\sum_{j=1}^{i-1}(\alpha a_k C_j + (1-\alpha)b_k D_j + \alpha a_k\nu_k + (1-\alpha)b_k\sigma_k) \\
&\quad + m(\alpha a_k\mu_k + (1-\alpha)b_k\tau_k).
\end{aligned}$$

Apply Lemma 3.2 in Ram et al. (2009) (also see Appendix 7.1).

Let $q_k = m(\alpha a_k\mu_k + (1-\alpha)b_k\tau_k)$ and $W_k = M_k + N_k$.

Then

$$\begin{aligned}
\sum_{k=0}^{\infty} q_k &= m\alpha\sum_{k=0}^{\infty} a_k\mu_k + m(1-\alpha)\sum_{k=0}^{\infty} b_k\tau_k < +\infty \\
\sum_{k=0}^{\infty} W_k &= \sum_{k=0}^{\infty}(M_k + N_k).
\end{aligned}$$

Since

$$\begin{aligned}
\sum_{k=0}^{\infty} M_k &= \sum_{k=0}^{\infty}(\alpha a_k C + (1-\alpha)b_k D + m\alpha a_k\nu_k + m(1-\alpha)b_k\sigma_k)^2 \\
&\leq 4\sum_{k=0}^{\infty}[(\alpha a_k C)^2 + ((1-\alpha)b_k D)^2 + (m\alpha a_k\nu_k)^2 + (m(1-\alpha)b_k\sigma_k)^2] \\
&< \infty
\end{aligned}$$

and

$$\begin{aligned}\sum_{k=0}^{\infty} N_k &= \sum_{k=0}^{\infty} 2(\alpha a_k \mu_k + (1-\alpha) b_k \tau_k) \sum_{i=1}^{m} \sum_{j=1}^{i-1} (\alpha a_k C_j + (1-\alpha) b_k D_j + \alpha a_k \nu_k + (1-\alpha) b_k \sigma_k) + \sum_{k=0}^{\infty} q_k \\ &\leq \sum_{k=0}^{\infty} [\sum_{i=1}^{m} (\alpha a_k \mu_k + (1-\alpha) b_k \tau_k + \alpha a_k C_i + (1-\alpha) b_k D_i)]^2 + \sum_{k=0}^{\infty} q_k \\ &\leq \sum_{k=0}^{\infty} M_k + \sum_{k=0}^{\infty} q_k < \infty.\end{aligned}$$

Hence $\sum_{k=0}^{\infty} W_k < \infty$.

Therefore, we get, with probability 1, the sequence $\|X_k - y^*\|^2$ converges to some non-negative random variable for every $y^* \in Y(\alpha, \lambda)$. Also with probability 1, we have

$$\sum_{k=0}^{\infty} (\alpha a_k (f(X_k) - f(y^*)) + (1-\alpha) b_k (g(X_k) - g(y^*))) < +\infty,$$

which implies that

$$\begin{aligned}&\sum_{k=0}^{\infty} a_k [(\alpha f + (1-\alpha)\lambda g)(X_k) - (\alpha f + (1-\alpha)\lambda g)(y^*)] \\ &\leq \sum_{k=0}^{\infty} (\alpha a_k (f(X_k) - f(y^*)) + (1-\alpha) b_k (g(X_k) - g(y^*))) + \sum_{k=0}^{\infty} (1-\alpha) |b_k - \lambda a_k| |g(X_k) - g(y^*)| \\ &< +\infty.\end{aligned}$$

For Y is compact, image of g is bounded. Assume $\exists M > 0$ such that $|g(y)| \leq M$ for all $y \in Y$. Then

$$\sum_{k=0}^{\infty} (1-\alpha) |b_k - \lambda a_k| |g(X_k) - g(y^*)| \leq 2M(1-\alpha) \sum_{k=0}^{\infty} |b_k - \lambda a_k| < +\infty.$$

Since $\sum_{k=0}^{\infty} a_k = +\infty$, then

$$\liminf_{k\to\infty} (\alpha f + (1-\alpha)\lambda g)(X_k) = (\alpha f + (1-\alpha)\lambda g)(y^*),$$

with probability 1.

By considering a sample path for which

$$\liminf_{k\to\infty} (\alpha f + (1-\alpha)\lambda g)(X_k) = (\alpha f + (1-\alpha)\lambda g)(y^*)$$

and $\|X_k - y^*\|^2$ converges for any y^*, we conclude that the sample sequence must converge to some y^* in view of continuity of f. Hence, the sequence $\{X_k\}$ converges to some optimal solution in $Y^*(\alpha, \lambda)$ with probability 1. This completes the proof.

5.2 The DA Mechanism with Randomization and Stochastic Noises

Recall that w_k is a random variable taking equiprobable values from the set $\{1, 2, \cdots, m\}$ and w'_k is a random variable taking equiprobable values from the set $\{1, 2, \cdots, n\}$. Also recall that $h_{w_k}(X_k) \in \partial f_{w_k}(X_k)$ and $\ell_{w'_k}(X_k) \in \partial g_{w'_k}(X_k)$, where if w_k takes a value j, then the vector $\partial f_{w_k}(X_k)$ is $\partial f_j(X_k)$, similar for g.

Our sequence $\{X_k\}$ is generated by the DA mechanism with randomization and stochastic noises as below.

Given X_k, let

$$\psi_{k+1} = X_k - a_k(h_{w_k}(X_k) + \epsilon_{w_k,k}), \qquad h_{w_k}(X_k) \in \partial f_{w_k}(X_k)$$

and

$$\varphi_{k+1} = X_k - b_k(\ell_{w'_k}(X_k) + \delta_{w'_k,k}), \qquad \ell_{w'_k}(X_k) \in \partial g_{w'_k}(X_k).$$

And set $X_{k+1} = \alpha\psi_{k+1} + (1-\alpha)\varphi_{k+1}, \quad \alpha \in [0, 1]$. We define $\mathcal{F}_k$ to be the σ-field generated by $X_0, X_1, \cdots, X_k$.

Assumption 5.6. a). The sequence $\{w_k\}(\{w'_k\})$ is a sequence of independent random variables, each unifromly distributed over the set $\{1, 2, \cdots, m\}(\{1, 2, \cdots, n\})$. Furthermore, the two sequences $\{w_k\}$ and $\{w'_k\}$ are independent of the sequence $\{X_k\}$.
b). The two sets of subgradients $\{h_{w_k}(X_k), k = 0, 1, 2, \cdots\}$ and $\{\ell_{w'_k}(X_k), k = 0, 1, 2, \cdots\}$ are bounded. That is, there exist some positive constants C_0 and D_0 such that, with probability 1, $\|h_{w_k}(X_k)\| \le C_0$ and $\|\ell_{w'_k}(X_k)\| \le D_0, \forall k \ge 0$.

Proposition 5.7. *Let Assumptions 3.1, 5.1, 5.4 and 5.6 hold and Y be compact. Let λ be a positive constant such that*

$$\sum_{k=0}^{\infty} \left|\frac{b_k}{n} - \lambda\frac{a_k}{m}\right| < +\infty.$$

Then the sequence $\{X_k\}$ generated by the DA mechanism with randomization and stochastic noises converges to some optimal solution in $Y^*(\alpha, \lambda)$, with probability 1.

Proof. As in the proof of Proposition 5.5, we obtain for all k and $y \in Y^*(\alpha, \lambda)$

$$E[\|X_{k+1} - y\|^2 | \mathcal{F}_k] \quad \le \quad (1 + \alpha a_k \mu_k + (1-\alpha) b_k \tau_k)\|X_k - y\|^2$$

$$-2\alpha\frac{a_k}{m}(f(X_k)-f(y))-2(1-\alpha)\frac{b_k}{n}(g(X_k)-g(y))$$
$$+(\alpha a_k C_0+(1-\alpha)b_k D_0+\alpha a_k\nu_k+(1-\alpha)b_k\sigma_k)^2.$$

Since $\sum_{k=0}^{\infty}(\alpha a_k\mu_k+(1-\alpha)b_k\tau_k)<+\infty$, and we have that

$$\begin{aligned}&\sum_{k=0}^{\infty}(\alpha a_k C_0+(1-\alpha)b_k D_0+\alpha a_k\nu_k+(1-\alpha)b_k\sigma_k)^2\\ \le\;& 4\sum_{k=0}^{\infty}((\alpha a_k C_0)^2+((1-\alpha)b_k D_0)^2+(\alpha a_k\nu_k)^2+((1-\alpha)b_k\sigma_k)^2)<+\infty\end{aligned}$$

Apply Lemma 3.2 in Ram et al. (2009) again. We get, with probability 1, the sequence $\{\|X_k-y^*\|^2\}$ converges to a non-negative random variable and

$$\sum_{k=0}^{\infty}(2\alpha\frac{a_k}{m}(f(X_k)-f(y))+2(1-\alpha)\frac{b_k}{n}(g(X_k)-g(y)))<+\infty.$$

Then

$$\begin{aligned}&\sum_{k=0}^{\infty}2[(\alpha\frac{a_k}{m}f+(1-\alpha)\lambda\frac{a_k}{m}g)(X_k)-(\alpha\frac{a_k}{m}f+(1-\alpha)\lambda\frac{a_k}{m}g)(y)]\\ &\le\sum_{k=0}^{\infty}(2\alpha\frac{a_k}{m}(f(X_k)-f(y))+2(1-\alpha)\frac{b_k}{n}(g(X_k)-g(y)))\\ &+\sum_{k=0}^{\infty}2(1-\alpha)|\frac{b_k}{n}-\lambda\frac{a_k}{m}|\bullet|g(X_k)-g(y)|\\ &<\infty.\end{aligned}$$

Since $\sum_{k=0}^{\infty}|\frac{b_k}{n}-\lambda\frac{a_k}{m}|<+\infty$, Y is compact, the image of g is bounded. Hence

$$\sum_{k=0}^{\infty}2\frac{a_k}{m}[(\alpha f+(1-\alpha)\lambda g)(X_k)-(\alpha f+(1-\alpha)\lambda g)(y)]<+\infty.$$

Then, since $\sum_{k=0}^{\infty}\frac{a_k}{m}=+\infty$ and $(\alpha f+(1-\alpha)\lambda g)(X_k)-(\alpha f+(1-\alpha)\lambda g)(y)\ge 0$, we have that

$$\liminf_{k\to\infty}(\alpha f+(1-\alpha)\lambda g)(X_k)=\inf_{y\in Y}(\alpha f+(1-\alpha)\lambda g)(y).$$

After this step, we use exactly the same argument as in the proofs of Proposition 4.11 and Proposition 5.5. We get the result.

6 Literature and Remarks

In a free-market system, priviate information is successively reflected in the price of a good or an asset in time through individual decision what to buy or sell, as remarked by F.A. Von Hayek in 1937. Without knowing what may be the price at equilibrium for a good, the market with an invisible hand of Adam Smith can in fact reach an equilibrium. Such a view is the foundation for any economic analysis based on equilibrium. The incremental subgradient method in Nedić and Bertsekas (2001) provides a framework how this may be done in theory.[6] Such an approach is especially important for economics since private information is typically unknown publicly while the market equilibrium must contain all relevant private information.

In an exchange market like equity, individuals decide what to buy or sell and their actions are successively accomplished in time through a DA mechanism with bid and ask. Such a decentralized mechanism is far from perfect, as we have shown that both bubbles and crashes for an asset can be produced. Nonetheless we also have shown that it can be efficient in the sense that it can find a market equilibrium in the end without knowing where it is priorly.

The study of competitive efficiency of a DA mechanism starts with experiments for an identical good in Smith (1962, 1965) where an artificial market is created with competitive equilibrium unknown by buyers and sellers who participate the experimental market. A DA mechanism is used for a seller to sell her initial endowed units of an object and for a buyer to buy units of the object in order to realize his valuations. Buyers' valuations and sellers' reservations are private information. In the experiments the DA mechanism converges quickly to a neighbor of the competitive equilibrium, even with a few participants. A great number of experiments have been conducted since then and a similar result has been obtained (see a survey in Friedman (1993) and the edited volume by Friedman and Rust (1993)). In recent years the competitive efficiency of a DA mechanism has been retested in experiments with more complicated environments, which are deliberately designed to be a proxy of an actual exchange market like equity. Smith et al. (1988) find that both bubbles and crashes can be generated by a DA mechanism under these environments. Porter and Smith (2003) provide a survey for bubbles and crashes in the laboratory with DA mechanisms. Therefore, the competitive efficiency of DA mechanisms is a complicated issue and should not be taken for granted. We show that the competitive efficiency of a DA mechanism depends on the relative strength of the stepsizes of bid and ask, together with the buy and sell orders. The efficiency can be jeopardized if the stepsizes of bid and ask are not at equal strength.

Bubbles and Crashes are not limited to the experiments. There was a housing (as well as a mortgage) bubble in year 2007 in the USA. There was an internet bubble in the 90s worldwide. Soft and hard commodities like coffee, sugar, gold and silver, etc. may be in a process of forming a bubble at present. Financial economists are fully aware of these bubbles and crashes as well

[6] See Kibardin (1980) for the earliest contribution about the method. A more detailed reference of the method can be found in Bertsekas (2010) and Ram et al. (2009).

as momentum (see, e.g., Chan and Jegadeesh (1996), Barberis et al. (1998), Shleifer (1999), Lo (2005) and references therein.).[7] Behavioral finance has become quite popular since the 90s in order to understand these financial anomalies. The main conclusion appears to be that investors' sentiment, related to "loss aversion, overconfidence, overreaction, mental accounting, and other behavioral biases" (Lo(2005)), is important for asset pricing. If sentiment affects how investors bid and ask under a DA mechanism, our study provides quantitative evidence how sentiment may affect prices.

The existence of financial anomalies does not imply that the fundamental is not important and the EMH should be abandoned. On the contrary, bubbles, crashes and momentum are all built on the foundation that there exists some fundamental there. Prices under a DA mechanism of this paper do move along with changes in fundamentals in f and g.

For the economy or market studied in this paper, the demand and supply for an agent j equipped with g_j and f_j are related to the subgradients of g_j and f_j, respectively, by the Fenchel duality (Ma and Nie (2003)). The subgradient method widely used to find an optimal solution of problem $\mathcal{P}$ is in fact consistent with the Walrasian hypothesis. Smith (1962, 1965) notice this connection for an economy with a single product.[8] His definition of the economic rent function is just like problem $\mathcal{P}$. This economic rent function is defined by Ma (1998a) for an exchange economy with multiple objects to study the English auction. The connection of the subgradient method and the Walrasian hypothesis has been studied by Ma and Nie (2003).

We have shown that the job matching model of Kelso and Crawford (1982) and its related exchange economy with multiple objects of Bikhchandani and Mamer (1987) are of the form given by problem $\mathcal{P}$. But an optimal solution to problem $\mathcal{P}$, which exists under a broad condition, may not be at a competitive equilibrium. The gross substitutes (GS) condition of Kelso and Crawford (1982) is important.[9] This condition is sufficient for the existence of competitive equilibrium for exchange economies with multiple objects studied by Bikhchandani and Mamer (1987) and Gul and Stacchetti (1999). Kelso and Crawford (1982) define the condition on the demand functions that are set-valued mappings while Gul and Stacchetti (1999) define it on the preference primitive. Loosely speaking, two goods satisfy the GS condition if a good that is in demand and whose price

[7]For counter arguments from the EMH, see studies in Fama (1998), Fama and French (1988), Chordia and Shivakumar(2002), among others. For a joint, see Lo (2005).

[8]He objected the Walrasian hypothesis because it used information on the total demand and supply. This is well justified because the DA mechanism is not a centralized system.

[9]Additional studies and applications of this condition can be found in Fujishige and Yang (2003), Hatfield and Kojima (2010), Hatfield and Milgrom (2005), Mishraa and Parkes (2007), Reijnierse et al. (2002). See Roth and Sotomayor (1990) for the related literature.

is not raised will be still in demand if the price(s) of other good(s) arises. Largely motivated by the spectrum auctions (Milgrom (2000)), many English or ascending price auctions have been studied in the literature with multiple objects.[10] Since prices ascend (with a minimum increment) for objects in excess demand and the market demand $-\sum_{j=1}^{m} D_j(y)$ at $y \in Y$, with a negative sign (see Lemma 2.2), is just a subset of the subdifferential $\sum_{j=1}^{m} \partial g_j(y)$ of the function $\sum_{j=1}^{m} g_j(y)$, the GS condition guarantees that prices are moving closer to the equilibrium, since the function F decreases along such a price path. Since the GS condition is sufficient for the existence of competitive equilibrium, such a process must end up with a competitive equilibrium in finite time, because function F is polehedral so that the speed of descend near equilibrium is also bounded away from zero. A key question is that agents are required to report their demand sets and there are strategic issues of misreporting. This is why it is important to find a Vickrey auction that has the ascending price feature. Note that all these auctions in the literature are not decentralized systems. Moreover, only buyers are active in bidding. Sellers are kept without actions.

Our study of the DA mechanism is motivated by a market where there are potentially a large number of agents and a large number of assets, in spirit of a large DA market studied in Fudenberg et al. (2007). The primary task of our paper is for price discovery. Strategic reporting plays a smaller role. Indeed, the fact that our results are robust with stochastic noises shows that agents may have quite limited gains with misreporting. Nevertheless, our DA mechanism approach may still provide an option to sell multiple objects in an environment where sellers' private information also matters, e.g., the production economy in Gul and Stacchetti (1999).

Our results are closely related to those obtained with the incremental subgradient method in Nedić and Bertsekas (2001). Nonetheless, the DA mechanism, to our best knowledge, has not been studied in the literature along the line of their decentralized approach. Because the problem $\mathcal{P}$ has so many other applications (Bertsekas (2010)), our DA mechanism, as a natural extension of the incremental subgradient method, provides an alternative how an optimal solution can be approached for those environments.

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7 Appendix: Subdifferentials

In this section we introduce some definitions and results in the nonsmooth analysis from Clarke et al (1988), the supermartingale convergence theorem (Nedić, A., and D.P. Bertsekas (2001)), and Lemma 3.2 in Ram et al (2009)), for the sake of completeness. In particular, the class of the Clarke regular functions plays an important role in our analysis.

Let $\varphi : Y \to R \cup \{+\infty\}$ be an extended real-valued mapping on Y where $Y \subset R^n_+$. Define the effective domain of φ by

$$Dom(\varphi) = \{y \in Y : \varphi(y) < \infty\}.$$

It is (locally) Lipschitz near y if for some constant K, for all points y, y' in some neighborhood of y, we have

$$\mid \varphi(y) - \varphi(y') \mid \leq Kd(y, y').$$

The directional direvative of φ at $y \in Y$ in the direction $w \in Y$ is defined as

$$(1) \qquad \varphi'(y; w) = \lim_{t \downarrow 0} \frac{\varphi(y + tw) - \varphi(y)}{t},$$

when the limit exists.

A vector $\eta \in R^n$ is a sub-gradient of φ at $y \in Dom(\varphi)$ if for all directions $w \in Y$ the following holds

$$< \eta, w > \leq \varphi'(y; w).$$

The sub-differentiable of φ at y, denoted by $\partial\varphi(y)$, is the set of all sub-gradients of φ at y (possibly empty), i.e.,

$$\partial\varphi(y) = \{\eta \in R^n :< \eta, w > \leq \varphi'(y; w), \forall w \in Y\}.$$

It is Gâteaux differentiable at y if the limit in (1) exists for all $w \in Y$, and there exists an element $\varphi'(y) \in Y$ (called the Gâteaux derivative) that satisfies

$$\varphi'(y; w) =< \varphi'(y), w >, \forall w \in Y.$$

The Clarke directional direvative $\varphi^0(y; w)$ of φ at y in the direction w is defined as follows

$$\varphi^0(y; w) = \limsup_{\substack{x \to y \\ t \downarrow 0}} \frac{\varphi(x + tw) - f(x)}{t}.$$

A function φ is **regular** at y if it is Lipschitz near y and for all w the directional derivative $\varphi'(y; w)$ exists and $\varphi'(y; w) = \varphi^0(y; w)$.

For a convex function φ that is also Lipschitz near y: Every convex function on a compact set Y is regular Lipschitzian. Convex functions that are Lipschitz near y are regular at y.

If the function φ is convex and locally Lipschitzian at y, we know that $\partial\varphi(y)$ is a nonempty, compact, and convex set. Moreover, it is upper semicontinuous at y in the sense that for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$\| x - y \| < \delta \Longrightarrow \partial\varphi(x) \subset \partial\varphi(y) + \epsilon B$$

where B is the open unit ball in Y.

The sub-differential $\partial\varphi$ can be expressed as

$$\partial\varphi(y) = \{\eta \in R^n : \varphi(w) - \varphi(y) \geq < \eta, w - y >, \forall w \in Y\}.$$

It is known that $\partial\varphi$ is a maximal monotone operator, i.e., $< f_1 - f_2, y_1 - y_2 > \geq 0$ if $f_j \in \partial\varphi(y_j), j = 1, 2$, and there is no other monotone set-valued map whose graph contains strictly the graph of $\partial\varphi$.

If the function φ is Gâteaux differentiable at y, then we know that $\partial\varphi(y) = \{\nabla\varphi(y)\}$.

y minimizes φ on Y iff $0 \in \partial\varphi(y)$.

For any two regular functions φ and ψ at y, the sum $\varphi + \psi$ is regular at y and

$$\partial(\varphi + \psi)(y) = \partial\varphi(y) + \partial\psi(y).$$

7.1 Supermartingale Convergence Theorem and Lemma 3.2

Theorem 3.1 (Nedić, A., and D.P. Bertsekas (2001)): Let X_k, Z_k and W_k, $k = 0, 1, 2, \cdots$, be three sequences of random variables and let $\mathcal{F}_k$, $k = 0, 1, 2, \cdots$, be sets of random variables such that $\mathcal{F}_k \subset \mathcal{F}_{k+1}$ for all k. Suppose that:

(a) The random variables X_k, Z_k, and W_k are nonnegative, and are functions of the random variables in $\mathcal{F}_k$.

(b) For each k, we have $E\{X_{k+1}|\mathcal{F}_k\} \leq X_k - Z_k + W_k$.

(c) There holds $\sum_{k=0}^{\infty} W_k < \infty$.

Then, we have $\sum_{k=0}^{\infty} Z_k < \infty$, and the sequence X_k converges to a nonnegative random varaible X, with probability 1.

Lemma 3.2 (Ram et al. (2009)): Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots$ be a seqenuce of sub σ-fields of $\mathcal{F}$. Let u_k, v_k and w_k, $k = 0, 1, 2, \cdots$, be non-negative $\mathcal{F}_k$-measurable random variables and let $\{q_k\}$ be a deterministic sequence. Assume that $\sum_{k=0}^{\infty} q_k < \infty$, $\sum_{k=0}^{\infty} w_k < \infty$, and

$$E\{u_{k+1}|\mathcal{F}_k\} \leq (1 + q_k)u_k - v_k + w_k$$

hold with probability 1. Then, with probability 1, the sequence $\{u_k\}$ converges to a non-negative random variable and $\sum_{k=0}^{\infty} v_k < \infty$.