Choosing Choices: Agenda Selection with Uncertain Issues[*]

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Abstract

This paper studies *selection rules i.e.* the procedures committees use to choose whether to place an issue on their agenda. The main ingredient of the model is that committee members are uncertain about their final preferences at the *selection stage*: they only know the probability that they will eventually prefer the proposal to the status quo at the *decision stage*. This probability is private information. We find that the more stringent the selection rule, the less voters are inclined to select an issue, so that individual behavior actually reinforces the effect of the rule instead of balancing it. Conditional on the pivotal event, the probability that the issue passes the final stage depends on whether she finally prefers the alternative or the status quo. Increasing the selection quorum increases the probability that the issue passes more if she eventually prefers the alternative than if she eventually prefers the status quo. In order to compensate for that, the agent becomes more selective. The final decision rule has the opposite effect on voters' behavior. Our basic model fits the procedure of the U.S. Supreme Court. The results extend to non-simultaneous selection procedures such as petitions and citizens' initiatives, as well as to selection by subcommittees as in the U.S. Congress. We describe optimal rules when there is a fixed cost of organizing the final election.

Keywords: Selection Rules, Strategic Voting, Agenda Setting, Large Deviations, Petitions, Citizens' Initiative.
JEL classification: D72, D83.

1 Introduction

Before they can be decided according to a majority rule, cases brought to the Supreme Court of the United States need to be approved for selection by at least four of the nine justices. This Rule of Four, which is rather a custom than a constitutional requirement, was used as a defense by the justices when in the mid-1930s the Court came under fire from the president and the

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[*] Preliminary and incomplete.

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Congress. It was accused, among other charges, of "using its discretionary jurisdiction to duck important cases,"[1] to which the justices responded that they use a submajority rule precisely because they prefer "to be at fault in taking jurisdiction rather than to be at fault in rejecting it."[2] The argument of the justices seems obvious at first, it is easier to gather four votes than five. Yet it is not so clear once we take strategic behavior into account: wouldn't the justices offset the effects of the selection rule by adjusting their individual behavior? We show that it is not the case by presenting a model in which rational individual behavior strengthens the effects of the selection rule: voters become more conservative as the rule becomes more stringent.

Selection rules are not specific to the Supreme Court[3]. For instance, any member of the French Assemblée Nationale can place a proposal in the agenda of the parliament as long as the proposed law doesn't increase expenditure for the government. In the United States Congress, bills must be approved by vote in a specialized standing committee before they can be brought to the floor. The agenda of the European Union's main decision-making body, the Council of the European Union, is prepared by the Committee of Permanent Representatives. Citizens' initiatives, which allow a group of citizens to obtain the organization of a referendum by way of petitions, are another form of selection rules. They play a particularly important role in some jurisdictions. For example, the gathering of a sufficient number of signatures famously led to the 2003 California recall election and ultimately to the recall of Governor Gray Davis. In November 2009, a citizens' initiative led to a ban on the construction of minarets in Switzerland creating a controversy across Europe which lead some commentators to question this procedure[4]. A general concern about citizens' initiatives is that they tend to bring too many issues to the agenda. While it doesn't give a general answer to that concern,

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[1] 81 Cong. Rec. 2809-2812 (1939).

[2] Hearings on S. 2176 before the Senate Judiciary Committee, 74th Cong. 1st sess., 9-10 (1935) (statement of Justice Van Devanter). We found a discussion of these events and the citations in Epstein and Knight (1998) p.86 who refer to a memorandum titled "The Rule of Four" that justice Marshall circulated to conference Sept. 21, 1983. For a detailed account of the selection procedure at the Supreme Court, see Perry (1994).

[3] State Supreme Courts also use selection rules. In California, for example, the justices use a supermajority rule of four out of seven justices.

[4] A European Citizens' Initiative is about to come into effect as decided in the Lisbon treaty, but with limited scope as it would only allow a group of citizens to place an issue on the agenda of the European commission. In France, a mixed initiative system between citizens and member of parliaments has been adopted in July 2008.

our study suggests that outcomes may be particularly sensitive to the selection rule that is chosen because of the positive feedback between the direct effect of a change in the rule and the indirect strategic effect. Finally, recruiting committees also use selection rules.

Our model allows us to analyze and compare all these rules in spite of their diversity. To our knowledge, it is the first formal analysis of selection rules in a rational voting framework. Our two working assumption are (i) that voters are uncertain about their preferences at the selection stage: they only know the probability that they will eventually prefer the proposal to the status quo; and (ii) that this probability is private information. There are at least two arguments to support the assumption that voters are uncertain about their final preferences. The first one is that voters are likely to have less information about the issue at the selection stage than at the decision stage. Once an issue is selected, hearings of experts and stakeholders may be organized, public attention and the media may help produce and aggregate information about the issue itself and the preferences of the people which may affect those of their representatives. The second argument is that the process leading to the final proposal is often quite complex and tends to generate uncertainty at the outset about the nature of the final proposal. In parliaments, when a bill is introduced to the floor, it usually goes through long series of amendments that often modify the text of the proposal substantially and unpredictably. Similarly, at the Supreme Court, there is uncertainty about which of the justices will be assigned to write the opinion and about which exact policy relevant points will be raised. Whereas the literature on agenda setting[5] has generally focused on the process leading from the initial proposal to its final version, we are more interested in how initial proposals (issues) are selected and placed on the agenda in the first place. Our approach is to black-box this transformation process and merely assume that it creates uncertainty about what will be voted on in the final stage. The second assumption is a standard private preferences assumption.

We also assume that voters believe the preference parameter (the probability that they prefer the proposal) of other voters to be drawn independently from an identical distribution.

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[5] See the related literature below.

It is arguably more natural to assume private information in a framework with heterogeneous preferences like ours than in the homogeneous preferences framework of the literature on pivotal voting where individuals have private information about a common event. Indeed, while deliberation can be expected to make all the information public in the case of homogeneous preferences, there is no particular reason to assume that it would do so in general in the case of heterogeneous preferences[6].

The basic model is a two-round voting procedure. In the first round, the *selection stage*, committee members vote to select an issue. In the second round, the *decision stage*, they decide whether to adopt a proposal or maintain the status quo. Even though voters' preferences are private, one's expected utility at the selection stage depends indirectly on the preferences, hence on the private information, of other voters since they determine the probability that the proposal would pass the final round if it were to be selected. Therefore, the selection stage aggregates strategically relevant information about the probabilities of different outcomes. Rational voters condition their decision on the event that their vote is pivotal. The exact information conveyed by the pivotal event, however, depends on the selection rule. When a rule requires a higher tally of votes to select an issue, the event that a single vote is pivotal conveys the information that more voters are likely to favor the proposal at the decision stage. Therefore, conditional on being pivotal at the selection stage, a voter who votes to select an issue faces a higher chance that the status quo will be reversed when the selection rule is more stringent. When selecting an issue, however, a voter also keeps the option to vote against change in the second round so this increased probability is not sufficient to explain her behavior. Rather, the voter compares the probability that the proposal passes when she eventually prefers it to when she doesn't. It is the ratio between these two probabilities that determines her strategy. We show that the probability that the proposal passes given that the voter does not support it increases faster with the selection rule than does the same probability when the voter supports the proposal. In order to compensate for that, voters become individually more conservative when the rule

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[6] For an analysis of deliberation that supports this claim see Austen-Smith and Feddersen (2006).

itself is more conservative[7]. Remarkably, this result is completely independent of the particular distribution of preferences. Our formal analysis requires the committee to be large for the result to hold. However, we also conducted numerical calculations of equilibria for different type distributions and committee sizes without ever invalidating the result.

We extend our analysis to selection in subcommittees as in the United States Congress. With a slight generalization of the first theorem of Dekel and Piccione (2000), we also show that the analysis applies to sequential selection procedures such as petitions for citizens' initiatives, regardless of the feedback given to voters about the votes cast by their predecessors.

Finally, these results uncover an interesting general feature of selection rules but have nothing to say about why they should be used, why they exist or which rules are optimal. In order to address these more normative questions, we assume the existence of a fixed cost to organize the second stage elections and derive the efficient rules.

Related Literature. The seminal literature on voting under asymmetric information[8] (Austen-Smith and Banks, 1996; Feddersen and Pesendorfer, 1996, 1997, 1998; Myerson, 1998) focused on the jury model in which agents have common preferences (with possibly heterogeneous intensities) conditional on an unknown state of the world, and private information about this state of the world. An important insight of this literature is that a strategic voter should reason as if her vote were pivotal since it is the only event in which her vote has any effect on the collective decision. Under any majority rule, the pivotal event conveys some information about the votes of others, and therefore about their private information and what it means about the state of the world. In our model, a voter's preferences over alternatives is independent of the information of others. Because of the two-round procedure, however, a voter who is uncertain about his final preferences cares about the preferences of others as they carry information about the chances of each of the alternatives in the final round. To model voters' uncertainty about

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[7] This account leaves some details aside. Indeed the equilibria that we consider (symmetric) are not unique in general and, in standard practice, the exact comparative statics result is that both the minimum and the maximum equilibrium thresholds move as explained with the selection rule.

[8] In a more recent contribution, Laslier and Weibull (2009) generalize existing results with respect to preference heterogeneity.

their own preferences, we take inspiration from the setup of Barbera and Jackson (2004) to which we add asymmetric information.

Several authors have built on the pivotal voting literature to model multiple-round elections. Piketty (2000) analyses a model of two-round elections and common value with asymmetric information, in which the winning policy in the first round of voting faces a new proposal in a second round. Then voters use the first round to communicate their information about the state of the world to other voters. Razin (2003) extends the idea of voting as signaling to a model of elections with only one round but where the information communicated during the elections affects future outcomes. Iaryczower (2008) considers signaling in a bicameral system. Shotts (2006) and Meirowitz and Shotts (2008) consider models of repeated elections with possibly private values and the same signaling motive. By contrast, the signaling channel is completely absent from our two-round model. Hummel (2009a) considers a model of repeated elections with three candidates in which, as in our model, the outcome of earlier rounds is informative about the distribution from which the preferences of other voters are drawn. In his model, however, voters learn their own preferences at the outset.

There is also a rich literature on sequential voting in committees. In these models the individual members of a committee vote sequentially and can observe prior voting history. This literature (Battaglini, 2005; Battaglini, Morton and Palfrey, 2007; Callander, 2007; Ali and Kartik, 2008; Hummel, 2009b) tries to find a way around an equivalence result of Dekel and Piccione (2000) according to which any equilibrium in weakly undominated strategies of a simultaneous election remains an equilibrium of any sequential election process in which voters observe prior history. In order to take into account selection procedures by way of petitions, we generalize this result to all sequential processes in which voters do not necessarily observe the full history of the game when they are called to cast a vote.

Our work is also connected to the literature on agenda setting, foremost because the selection stage of our model is a process of endogenous choice of its agenda by a committee, but also because of the use of sequential election processes in this literature. There are two modeling approaches to agenda setting in this very rich literature. The topic has been treated from the point of view of legislative bargaining (Banks and Duggan, 1998, 2000, 2001; Baron and Ferejohn, 1989; Diermeier and Merlo, 2000; Merlo and Wilson, 1995), and by the literature on sequential agenda (Austen-Smith, 1987; Banks, 1985; Banks and Gasmi, 1987; Bernheim, Rangel and Rayo, 2006; Dutta, Jackson and Le Breton, 2004; Ferejohn, Fiorina and McKelvey, 1987; Romer and Rosenthal, 1978; Shepsle and Weingast, 1984). While this literature aims at modeling the whole process of amendments and modifications of a proposal, we only model the initial decision of placing an issue on the agenda, and merely account for the process between the selection and the decision stage with our assumption that it generates uncertainty at the outset about the final proposal.

On the technical side, this paper makes an intensive use of large deviation techniques. Our equilibrium characterization shows that individual behavior depends on the ratio of two probabilities: (i) the probability for a voter that the alternative will eventually pass conditionally on her being pivotal at the selection stage and on the event that she will eventually prefer the alternative herself; (ii) the same probability conditional on the event that she is pivotal and that she will eventually prefer the status quo. To study this ratio and derive the comparative statics properties of the equilibria, we need to analyze the game asymptotically, taking the size of the committee to infinity. Since these two probabilities are tail probabilities (i.e. they correspond to the probability that a certain random variable takes values in the tail of its distribution), we use large deviation techniques (Dembo and Zeitouni, 1998; Hollander, 2000) and saddlepoint approximations (Jensen, 1995) to analyze their asymptotic behavior. The use of some of these techniques from statistics is, to our knowledge, new to the literature in economic theory[9]. They are a particularly natural tool for the study of large elections and could probably be used more widely.

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[9] [Literature on large deviations in economics here.]

2 The Model

Agents and Preferences. $N = \{1, \ldots, n\}$ is a committee of $n \geq 2$ voters. If an issue is selected, the voters face a pair of alternatives: the *status quo* and the *proposal.* Information about the proposal is incomplete at the outset, so that a voter i only knows her probability $p_i \in [0, 1]$ to be in favor of the proposal. These probabilities are drawn independently across voters from a distribution with density function f on $[0, 1]$. f is assumed to have full support and no atoms, except possibly at the extremities of the support. While the distribution is common knowledge, the realizations are private information. We let $\tilde{p} = \int_0^1 pf(p)dp$ denote the mean of this distribution.

Since there are only two alternatives, we need only keep track of the difference in utilities between them. It is therefore without loss of generality that we normalize the utility for the status quo to 0, whereas the utility of a voter i for the proposal is drawn conditionally on her opinion: if the proposal is adopted, a voter who supports it experiences a utility $u_i^+ > 0$, and a voter in favor of the default policy experiences a utility $-u_i^- < 0$. We assume that these random variables have homogeneous expected values across voters[10] that we denote by u^+ and u^-. At the selection stage of the two-round voting procedure described below, agents only know the probability that they prefer the proposal to the default. When an issue is selected and becomes part of the agenda, more information becomes available to the voters enabling them to form an opinion about the proposal and learn the intensity of their preferences u_i.

Voting Procedure. The voting procedure has two stages, the selection stage and the decision stage. At the selection stage, an issue is placed on the agenda if at least $\lceil Vn \rceil$ committee members select it, where the fraction $V \in [0, 1]$ is the *selection rule.* If the issue is not selected, the default policy is maintained. If it is selected, the agents vote again to decide whether to adopt the proposal. The proposal is adopted if more than $\lceil vn \rceil$ committee members vote in favor, where the fraction $v \in [0, 1]$ is the *decision rule.*We let $n_V = \lceil Vn \rceil$ denote the tally of votes necessary to select an issue, and $n_V^c = n - n_V$ its complement. Similarly, let $n_v = \lceil vn \rceil$

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[10] In fact, we only need the ratio of these expected values to be invariant across voters.

and $n_v^c = n - n_v$. Finally we will also use the fractions $V_n = n_V/n$ and $v_n = n_v/n$.

Equilibrium Definition. A selection strategy of voter i is a function $\sigma_i : [0,1] \rightarrow \{0,1\}$ mapping a probability type p_i to a ballot, where 1 means that i votes in favor of selecting the proposal. For notational simplicity, we do not consider mixing behavior. This is without loss of generality since we show below that all the equilibria feature essentially pure straregies. In the second stage, the voting strategy of the voter may be conditioned on all or any subset of the information that may be available to her at this stage: whether she supports the proposal, the intensity of her preferences, her and other players' voting strategy in the first round. We consider sequential equilibria of this game in weakly undominated strategies. This is a standard way to avoid equilibria in which voters vote for their least preferred policy in binary elections in which no information is aggregated such as our second-round election, and it also rules out equilibria in which all agents vote for or against selection irrespective of their private information.

3 Equilibrium Analysis

Decision Stage. Since we ruled out weakly dominated strategies, no matter what observations a player is allowed to make between rounds, she votes for her preferred policy at the decision stage. Therefore we can take this sincere voting behavior as given and proceed to analyze the first-stage game.

Non Strategic Behaviors. There are two types of non strategic behaviors at the selection stage that can be used as a benchmark. A *naive* voter would just vote for the alternative if its expected payoff $p_i u^+ - (1-p_i)u^-$ is greater than 0. A more sophisticated behavior would be to weigh the payoff of the alternative conditionally on eventually liking it or not by the expected probability that the alternative eventually passes in each of these cases. We call such voters *sophisticated*. Let S be a binomial random variable with parameters $\tilde{p} = \int_0^1 pf(p)dp$, and $n-1$. $\tilde{p}$ is the probability with which any other voter is expected to eventually favor the alternative in the absence of additional information, and S is the random variable a sophisticated voter would use to estimate the tally of votes in favor of the alternative at the selection stage in addition to her own.

Proposition 1 (Naive and Sophisticated Voting). *Irrespective of the particular rule, naive voters use a threshold strategy with the selection threshold* $t_{naive} = \frac{u^-}{u^- + u^+}$ *such that they vote to select the issue if they think they will prefer the proposal to the status quo with a probability greater than* t_{naive}. *A sophisticated voter uses a threshold strategy with the threshold*

$$t_{soph} = \left(1 + \frac{u^+}{u^-} \frac{\Pr(S \geq n_v - 1)}{\Pr(S \geq n_v)}\right)^{-1},$$

which depends on the decision rule, but is independent of the selection rule.

In either case, individual behavior is independent of the selection rule.

Strategic Behavior. Given a profile $p = (p_1, \ldots, p_n)$, a voter i knowing the full profile would expect the following utility if the issue were to be selected in the first stage[11]

$$U_i = p_i u^+ \sum_{\substack{C \subseteq N_i \\ \#C \geq n_v - 1}} \prod_{j \in C} p_j \prod_{l \in N_i \smallsetminus C} (1 - p_l) - (1 - p_i) u^- \sum_{\substack{C \subseteq N_i \\ \#C \geq n_v}} \prod_{j \in C} p_j \prod_{l \in N_i \smallsetminus C} (1 - p_l), \tag{1}$$

where $N_i = N \smallsetminus \{i\}$ is the set of all voters except i. Indeed, with probability p_i, i will support and vote for the proposal in the second stage, winning if a coalition C of at least $n_v - 1$ other players (sincerely) vote likewise, which yields an expected payoff of u^+. With probability $1 - p_i$, she will not support the proposal, and incur the expected loss u^- if a coalition of at least n_v other voters concur against the status quo. If the issue is not selected, the status quo prevails and the expected utility of a voter is 0. Note that we can write $U_i = U(p_i, p_{-i})$, where U is linear and strictly increasing in a voter's own type p_i.

Even though the values of the policies for the voters are private and independent as well as their informational types, the two-round process links a voter's value of selecting an issue

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[11] Note that this function does not satisfy the information smallness assumption of Gerardi and Yariv (2007), hence allowing for deliberation does not necessarily make different selection rules equivalent as to the sets of sequential equilibria in weakly undominated strategies that they generate.

to the types of other voters so that the first round has the analytical features of a common value election. In particular, the first round of this procedure can aggregate some information. This information is not about the quality of the proposal or the status quo, or any other factor that affects the values of the voters for these outcomes. Instead, it concerns the probability distribution of the number of voters who will eventually vote for the proposal.

When making her first stage voting decision, the voter only knows her own probability p_i of favoring the final proposal, and must therefore compute the expected value of (1). If she is rational, she conditions her computation on the event $\mathcal{E}_i \equiv \left\{\sum_{j\in N_i} \sigma_j(p_j) = n_V - 1\right\}$ that her vote is pivotal, and compares it to the null utility that she obtains if the issue is not selected. Because the expression in (1) is linear and strictly increasing in p_i, it is clear that in any equilibrium, player i uses a threshold strategy characterized by a threshold $t_i \in [0,1]$ such that[12] $\sigma_i(p_i) = \begin{cases} 1 & \text{if } p_i > t_i \\ 0 & \text{if } p_i < t_i \end{cases}$.

The following proposition gives a characterization of this threshold in a symmetric equilibrium where all voters use the same threshold. We define $\overline{p}(t) \equiv \int_t^1 zdF(z)/(1-F(z))$ and $\underline{p}(t) \equiv \int_0^t zdF(z)/F(t)$ as the expectation of p conditional on lying above (respectively, below) a threshold t. These functions are strictly increasing and continuously differentiable on $[0,1]$. Let $\overline{X}(t)$ be a generic Bernoulli random variable that takes the value 1 with probability $\overline{p}(t)$. We denote by $\overline{X}_1, \overline{X}_2, \cdots, \overline{X}_k$ an i.i.d. sample of size k of this random variable. Similarly, $\underline{X}(t)$ is a generic Bernoulli random variable with parameter $\underline{p}(t)$.

When other voters are using a threshold t, a voter conditioning on his vote being pivotal knows that $n_V - 1$ voters among the other $n-1$ committee members have a probability to prefer the proposal above t, and that for the rest of them it lies below t. Therefore she estimates that, conditional on what she knows, the tally of votes that will be ultimately cast in favor of the proposal if the issue is selected is given by the random variable

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[12] The prescription of the strategy when $p = t$, which is an event of measure 0, is essentially irrelevant for the analysis.

$$S_n(t) = \overline{X}_1(t) + \cdots + \overline{X}_{n_V - 1}(t) + \underline{X}_1(t) + \cdots + \underline{X}_{n_V^c}(t).$$

Hence the expected utility of a player with type p conditional on being pivotal is given by

$$p \Pr\big(S_n(t) \geq n_v - 1\big) u^+ - (1-p) \Pr\big(S_n(t) \geq n_v\big) u^-.$$

It is clear that the best response of this player to the threshold t used by other players is to use a threshold strategy with the threshold

$$\beta_n(t) = \left(1 + \frac{u^+}{u^-} \frac{\Pr\big(S_n(t) \geq n_v - 1\big)}{\Pr\big(S_n(t) \geq n_v\big)}\right)^{-1}.$$

Symmetric equilibria are therefore threshold equilibria characterized by the fixed points of the function β_n on $[0, 1]$. These results are summarized in the next proposition.

Proposition 2 (Equilibrium Characterization)**.** *In any equilibrium of the game, players use threshold selection strategies such that* $t_i < t_{naive}$*. In particular, equilibrium strategies are essentially pure strategies*[13]*. There exists a symmetric equilibrium of this game, and these equilibria are characterized by the fixed points of* β_n*.*

Proof. See Appendix A. □

Since rational voters use selection thresholds below the naive threshold, they are less conservative than naive voters and accept to select issues even when their expected payoff from the proposal is lower than that from the status quo. This is natural since selecting an issue preserves the option of rejecting the proposal at the decision stage.

The properties of these equilibria are tightly tied to the behavior of the ratio

$$R_n(t) \equiv \frac{\Pr\big(S_n(t) \geq n_v - 1\big)}{\Pr\big(S_n(t) \geq n_v\big)}.$$

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[13] That is, voters may be mixing at the threshold but nowhere else.

This ratio measures the contribution of a voter to the probability that the proposal prevails in the second round, conditional on her being pivotal in the first round, and provided the issue is selected. When $u^- = u^+$, a voter's best response is to set this ratio equal to the ratio measuring her relative likelihood of being in favor of the status quo $(1 - p_i)/p_i$. In the proof of Proposition 2 we derive a closed form expression for this ratio which allows us to study the model numerically very easily once we choose a particular type distribution F. Unfortunately, it is very intractable for the derivation of theoretical properties that apply to general type distributions. This problem can be solved by taking n to the limit. Indeed, large deviation and saddlepoint approximation techniques from statistics[14] provide us with analytical tools to study the limit of this ratio as n goes to infinity.

4 Other Rules

The simultaneous game considered above is relatively simple to analyze, but real world selection procedures often do not have its structure. In this section we show that the equilibria of the simultaneous game remain equilibria in other games that may provide better models for these procedures. A particular case is ballot initiatives or petitions. Many legislatures are legally bound to consider issues that are proposed by a sufficient number of their members. The main difference between the choice to support such an initiative and the voting decision at the selection stage of our basic model is that the process is sequential and agents may be able to observe additional information when they are called to choose such as the number of signatures already gathered. Building on Dekel and Piccione (2000), we show that our initial analysis holds.

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[14] See Hollander (2000) for a general treatment of large deviations, or Dembo and Zeitouni (1998) for a more advanced treatment; for saddlepoint approximation techniques, see Jensen (1995).

4.1 Sequential Procedures: an Equivalence Result

Dekel and Piccione (2000) showed that in symmetric binary elections, the informative symmetric equilibria of the simultaneous voting game are also equilibria in any sequential voting structure in a certain (quite general) class. The selection stage of our game is precisely a symmetric binary election that falls in the scope of applications of the first theorem[15] of Dekel and Piccione (2000). Therefore our equilibrium analysis of the simultaneous selection game applies to any sequential selection procedure in the class considered in their theorem. This class consists of all the games with $T < \infty$ periods in which each voter is called to vote in some period, but voting may be simultaneous in some periods, and where the voters know all the previous history of voting when they are called to vote. Because we are interested in petitions and citizens' initiative we need to consider a larger class of sequential games. Indeed, when someone is asked to sign a petition, she may not be able to observe the full history of choices by individuals who were asked to sign before her. Very often, she only observes the size of the current pool of signatures at the time she is solicited. In what follows, we consider sequential games in which the information available to the individual called to express her choice is any subset of the full history. This extension leaves the essential arguments of Dekel and Piccione (2000) unchanged. In order to state the result precisely, we introduce some straightforward notations.

We consider voting games with $T < \infty$ periods. In each period, some voters are called to vote. There may be some randomness in the way voters are called. When a voter is called she learns some true statement about the current history. For example, if voter 1 and 2 have voted to select the issue and voter 3, 4 and 5 have voted against it, voter 6 may be told that voter 1, Albert, has voted to select the issue, and that at least two other voters have opposed selection. Let M be the set of all possible messages about all possible histories and $m_i \in M$ be the message received by player i when she is called. m_i may or may not inform player i about the other players that have been called in the same period, or about the current period

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[15] That is up to the following detail: for notational convenience, Dekel and Piccione (2000) show their result for a finite type space whereas our type space is the unit interval. The extension of their proof to this case is immediate however. Our extension of their theorem, which covers this case, will be provided in a separate note.

of the game. The only constraint about those messages is that they are truthful and convey information about past events only. A pure strategy of player i in the sequential selection game is now a mapping $s_i : [0,1] \times M \to \{0,1\}$ associating a voter's type and message pair to a vote. Because $U(p_i, p_{-i})$ is linear and increasing, we know that there is no loss of generality in considering pure strategies only, and we can even restrict ourselves to threshold strategies. Hence we identify a strategy s_i to a threshold $t_i \in [0,1]$ above which the voter votes to select the issue. We are not concerned about the particular structure of the game but a player is allowed to make inferences from whatever she knows about this structure. A strategy s_i is irresponsive if it does not depend on the message received by the player but only on her type.

Proposition 3 (Sequential Selection Procedures)**.** *Pick any sequential selection game G in the class described above. The following two statements are equivalent:*

(i) The strategy $\sigma(p) = \mathbb{1}_{p>t^*}$ defines a symmetric equilibrium of the simultaneous selection game.

(ii) The irresponsive strategy $s(p,m) = \mathbb{1}_{p>t^*}$ defines a symmetric equilibrium of G.

The intuition of the result, exactly as in Dekel and Piccione (2000), is that when voters are voting independently of their message and use the same strategy, the event that their vote is pivotal is identical in the simultaneous game and in any of the sequential games.

4.2 Subcommittees

In some committees such as the United States Congress, issues are selected within subcommittees. To describe this procedure we let $C \in [0,1]$ denote the size of the subcommittee, with $V \leq C$. $n_C \equiv \lceil nC \rceil$ is the number of voters in the subcommittee, and $n_C^c \equiv n - n_C$. Finally we let $n_V^C \equiv n_C - n_V$.Making the same assumptions about preferences and information, and considering the voting decision of a member of the selecting committee, it is clear that, conditional on being pivotal, and provided other players are using a threshold t, the random variable that describes the belief of a player about the tally of votes that will finally be cast in favor of the proposal is

$$\tilde{S}_n(t) = \overline{X}_1(t) + \cdots + \overline{X}_{n_V-1}(t) + \underline{X}_1(t) + \cdots + \underline{X}_{n_V^C}(t) + \tilde{X}_1 + \cdots + \tilde{X}_{n_C^c},$$

where $\tilde{X}$ is a generic Bernoulli random variable that takes the value 1 with probability $\tilde{p} = \int_0^1 xf(x)dx$,

We can write the best response function of a voter to all other players playing with a common threshold t.

$$\tilde{\beta}_n(t) = \left(1 + \frac{u^+}{u^-} \frac{\Pr\big(\tilde{S}_n(t) \geq n_v - 1\big)}{\Pr\big(\tilde{S}_n(t) \geq n_v\big)}\right)^{-1}.$$

And we can write the following result that parallels Proposition 2, and hence needs no further proof.

Proposition 4 (Equilibrium Characterization with a Committee)**.** *In any equilibrium of the game with a committee, players use threshold selection strategies such that* $t_i < t_{naive}$*. In particular, equilibrium strategies are essentially pure strategies. There exists a symmetric equilibrium of this game in which all players use the same threshold. The symmetric equilibria of the game are characterized by the fixed points of* $\tilde{\beta}_n$*.*

Clearly, the result on sequential procedures extends to the game with subcommittees so that our analysis of this game holds for sequential voting procedures within the subcommittee.

5 Asymptotic Analysis

As already noted, the best response functions $\beta_n(t)$ depend on the ratio $R_n(t) = \frac{\Pr(S_n(t) \geq n_v - 1)}{\Pr(S_n(t) \geq n_v)}$ and in order to study the asymptotic equilibria of the selection game, it is necessary to understand the asymptotic behavior of this ratio. The law of large numbers implies that both probabilities converge to either 1 or 0. More specifically, letting $m(t) \equiv \lim_{n\to\infty} m_n(t) =$

$V\overline{p}(t) + (1-V)\underline{p}(t)$, where $m_n(t) \equiv \frac{1}{n} E\ S_n(t) = \frac{n_V - 1}{n}\overline{p}(t) + \frac{n_V^c}{n}\underline{p}(t)$, both probabilities converge to 0 if the asymptotic mean of the sequence is less than the second round rule $m(t) < v$, and to 1 if $m(t) \geq v$. Indeed, as the population becomes large, the fraction of the voters who, when conditioning on the pivotal event, eventually support the proposal converges to $m(t)$, and the proposal is rejected if this fraction is below v. Since $m(t)$ is strictly increasing in t, there is a unique, if any, $\tilde{t}$ such that $m(t) < v$ for every $t < \tilde{t}$, and $m(t) > v$ for every $t > \tilde{t}$.

When both probabilities converge to 1, the ratio also converges to 1. When they converge to 0, however, we need to know the speed of convergence of the two probabilities. Large deviation techniques in statistics have been developed precisely to study these tail probabilities, and we can apply Gärtner-Ellis theorem (see for example Hollander, 2000) to show that both probabilities are in the order of e^{-Kn} for some constant K (see Lemma 2). This is not sufficient to conclude and we will need more work to characterize the limit.

This section is technical and a reader who is not interested in this aspect may just read the first subsection to understand our notations and jump to Proposition 5 for the expression of asymptotic best-response functions, and then to the remainder of the paper.

5.1 Notations and Preliminary Results

In order to state these results, we introduce some notations and well known results in statistics (see Jensen, 1995). For the random variable $S_n \in \mathbb{R}$ defined on the probability space $(\Omega, \mathcal{A}, P)$, and a scalar θ, the *Laplace transform* $\varphi_n(\theta)$ of S_n is defined by

$$\varphi_n(\theta) \equiv E\ e^{\theta S_n} = \left(\overline{p}e^{\theta} + 1 - \overline{p}\right)^{n_V - 1} \left(\underline{p}e^{\theta} + 1 - \underline{p}\right)^{n_V^c},$$

and its *cumulant transform* $K_n(\theta)$ by

$$K_n(\theta) \equiv \log \varphi_n(\theta) = (n_V - 1) \log\left(\overline{p}e^{\theta} + 1 - \overline{p}\right) + n_V^c \log\left(\underline{p}e^{\theta} + 1 - \underline{p}\right).$$

The two transforms are defined on $\mathbb{R}$, they are $\mathcal{C}^{\infty}$, and $K_n(.)$ is strictly convex.

The *exponential family* generated by S_n and the original probability measure P consists of the probability measures P_θ given by

$$\frac{dP_\theta}{dP}(\omega) = \varphi_n(\theta)^{-1} e^{\theta S_n(\omega)}. \tag{2}$$

With $\mu_n(\theta) \equiv E_\theta\ S_n$ and $\sigma_n(\theta) \equiv \sqrt{Var_\theta\ S_n}$ denoting the mean and standard deviation, respectively, under the measure P_θ, we have the formulas

$$\mu_n(\theta) = E_\theta\ S_n = K_n'(\theta), \quad \text{and} \quad \sigma_n(\theta) = \sqrt{Var_\theta\ S_n} = \sqrt{K_n''(\theta)}.$$

The log likelihood function for estimating θ in the family $\{P_\theta : \theta \in \mathbb{R}\}$ is $\theta x - K_n(\theta)$, so that the maximum likelihood estimator of θ solves the equation $E_\theta\ S_n = K_n'(\theta) = x$.

Let θ_n be the unique solution of the equation $K_n'(\theta) = n_v$, and θ_n' the unique solution of the equation $K_n'(\theta) = n_v - 1$. In both cases, e^θ is the unique positive root of a second degree polynomial, and it is easy to see that $\lim_{n\to\infty} \theta_n = \lim_{n\to\infty} \theta_n' = \hat{\theta}$. $\hat{\theta}$ is defined by

$$\kappa'(\theta) = \frac{V\overline{p}e^{\hat{\theta}}}{\overline{p}e^{\hat{\theta}} + 1 - \overline{p}} + \frac{(1-V)\underline{p}e^{\hat{\theta}}}{\underline{p}e^{\hat{\theta}} + 1 - \underline{p}} = v, \tag{3}$$

where

$$\kappa(\theta) \equiv \lim_{n\to\infty} \frac{K_n(\theta}{n} = V \log\left(\overline{p}e^{\theta} + 1 - \overline{p}\right) + (1-V)\log\left(\underline{p}e^{\theta} + 1 - \underline{p}\right).$$

$\hat{\theta}$ can be written in closed form by solving for the only positive root of (3) in $e^{\hat{\theta}}$, and we can write $e^{\hat{\theta}} = \Psi(V, 1-V, v)$ where $\Psi(\alpha, \beta, \gamma)$ is defined as the unique positive root[16] of the second degree equation

$$\alpha\frac{\overline{p}X}{\overline{p}X + 1 - \overline{p}} + \beta\frac{\underline{p}X}{\underline{p}X + 1 - \underline{p}} = \gamma, \tag{4}$$

with $\alpha, \beta, \gamma \in (0, 1)$.

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[16] This root exists as long as $\alpha + \beta > \gamma$ which will always be true for the cases we are interested in, at least for n sufficiently high.

The second degree equations solved by e^{θ_n} and $e^{\theta'_n}$ are

$$(V_n - 1/n)\,\frac{\overline{p}X}{\overline{p}X + 1 - \overline{p}} + (1 - V_n)\,\frac{\underline{p}X}{\underline{p}X + 1 - \underline{p}} = v_n, \tag{5}$$

and

$$(V_n - 1/n)\,\frac{\overline{p}X}{\overline{p}X + 1 - \overline{p}} + (1 - V_n)\,\frac{\underline{p}X}{\underline{p}X + 1 - \underline{p}} = v_n - 1/n \tag{6}$$

respectively. Therefore, $e^{\theta_n} = \Psi(V_n - 1/n, 1 - V_n, v_n)$ and $e^{\theta'_n} = \Psi(V_n - 1/n, 1 - V_n, v_n - 1/n)$. With this, we can prove the following lemma which will prove useful in the analysis since we will show that the limit of the ratio R_n is a function of $\hat{\theta}$.

Lemma 1. *The functions $\theta_n(t)$, $\theta'_n(t)$ and $\hat{\theta}(t)$ are all continuous and strictly decreasing in t. $\theta_n(t)$ and $\theta'_n(t)$ converge uniformly to $\hat{\theta}(t)$ in $\mathcal{O}(1/n)$ on $[0, 1]$ if $v \neq V$, and on any compact $K \subseteq (0, 1]$ if $V = v$. Furthermore $\hat{\theta}$ is strictly decreasing in V and strictly increasing in v.*

Proof. See Appendix B ☐

5.2 Asymptotic Equilibria

We start with standard large deviation results about the tail probabilities of interest. The first parts of points (i) and (ii) are just consequences of the law of large numbers, the second parts are consequences of the large deviation principle, and in particular of Gärtner-Ellis Theorem.

Lemma 2.

(i) For every $t < \tilde{t}$, and $\alpha_n \in \{n_v - 1, n_v\}$

$$\lim_{n\to\infty} \Pr\big(S_n(t) \geq \alpha_n\big) = 0.$$

Furthermore

$$\lim_{n\to\infty} \frac{1}{n} \log \Pr\big(S_n(t) \geq \alpha_n\big) = -\Big(v\hat{\theta}(t) - \kappa\big(\hat{\theta}(t)\big)\Big) < 0. \tag{7}$$

(ii) For every $t \geq \tilde{t}$, and $\alpha_n \in \{n_v - 1, n_v\}$,

$$\lim_{n\to\infty} \Pr\big(S_n(t) \geq \alpha_n\big) = 1.$$

Furthermore, for every $t > \tilde{t}$,

$$\lim_{n\to\infty} \frac{1}{n} \log\left(1 - \Pr\big(S_n(t) \geq \alpha_n\big)\right) = -\Big(v\big|\hat{\theta}(t)\big| - \kappa\big(\big|\hat{\theta}(t)\big|\big)\Big) < 0 \tag{8}$$

Proof. See Appendix B □

The lemma implies that the ratio R_n converges to 1 when $t \geq \tilde{t}$. The two probabilities of interest converge to 0 at the same speed in the other case. Although this does not allow us to fully conclude as to the limit of R_n at this stage, it shows that the probabilities on which the voters' equilibrium calculations are based converge exponentially fast to 0 or to 1.

We start by providing new expressions for the tail probabilities of the form $\Pr\big(S_n \geq \alpha_n\big)$ where α_n is a sequence of integers keeping in mind that we will be interested in $\alpha_n = n_v$ and $\alpha_n = n_v - 1$. To obtain these expressions, we use the *exponentially tilted* measures P_θ. The following results are adapted from Jensen (1995, Section 1.4).

Lemma 3. *For any $\alpha_n \in \mathbb{Z}$, and any $\theta > 0$ we have*

$$\Pr\big(S_n \geq \alpha_n\big) = \frac{\varphi_n(\theta)e^{-\theta\alpha_n}}{\sigma_n(\theta)\,(1 - e^{-\theta})} \sum_{z\geq\alpha_n,\ z\in\mathbb{Z}} (1 - e^{-\theta})\sigma_n(\theta)e^{-\theta(z-\alpha_n)} P_\theta\big(S_n = z\big). \tag{9}$$

Proof. See Appendix B. □

We can express the sum in (9) as an inversion integral over the appropriate characteristic function. In order to do that, we need the following inversion formula that can be found in Jensen (1995, theorem 1.2.4), or in Feller (1971, Section XV.3) for a proof.

Lemma 4 (Inversion Formula)**.** *Let X be a lattice distribution concentrated on $\mathbb{Z}$ with maximal step 1. Let*

$$\gamma(s) \equiv E\ e^{isX} = \sum_{x \in \mathbb{Z}} e^{isx} P(X = x),$$

be the characteristic function of X. For any $x \in \mathbb{Z}$ we have the inversion formula

$$P(X = x) = (2\pi)^{-1} \int_{-\pi}^{\pi} e^{-isx} \gamma(s)\ ds. \tag{10}$$

With this, we can prove the following result.

Lemma 5. *The sum in (9) can be written as*

$$(2\pi)^{-1} \int_{I_n(\theta)} \varphi_\theta\left(\frac{s}{\sigma_n(\theta)}\right) J\left(\theta, \frac{s}{\sigma_n(\theta)}\right) \frac{e^{is(\mu_n(\theta) - \alpha_n)/\sigma_n(\theta)}}{1 + \frac{is}{\theta \sigma_n(\theta)}}\ ds \tag{11}$$

where

$$J(\theta, z) \equiv \frac{1 + iz/\theta}{1 + \frac{e^{-\theta}}{1 - e^{-\theta}}(1 - e^{-iz})},$$

$$\varphi_\theta(z) \equiv \frac{\varphi_n(\theta + iz)}{\varphi_n(\theta)} e^{-iz\mu_n(\theta)},$$

and $I_n(\theta) \equiv \big[-\pi\sigma_n(\theta), \pi\sigma_n(\theta)\big]$.

Proof. See Appendix B. ☐

Now using (9) and (11) evaluated at $\theta = \theta'_n$ to express $\Pr(S_n \geq n_v - 1)$, and at $\theta = \theta_n$ to express $\Pr(S_n \geq n_v)$ we obtain the following expression for R_n on $\big[0, \tilde{t}\big]$

$$R_n = \frac{\frac{\varphi_n(\theta'_n) e^{-(n_v - 1)\theta'_n}}{\sigma_n(\theta'_n)(1 - e^{-\theta'_n})}}{\frac{\varphi_n(\theta_n) e^{-n_v \theta_n}}{\sigma_n(\theta_n)(1 - e^{-\theta_n})}} \times \frac{\int_{I_n(\theta'_n)} \varphi_{\theta'_n}\left(\frac{s}{\sigma_n(\theta'_n)}\right) J\left(\theta'_n, \frac{s}{\sigma_n(\theta'_n)}\right) \frac{1}{1 + \frac{is}{\theta'_n \sigma_n(\theta'_n)}} ds}{\int_{I_n(\theta_n)} \varphi_{\theta_n}\left(\frac{s}{\sigma_n(\theta_n)}\right) J\left(\theta_n, \frac{s}{\sigma_n(\theta_n)}\right) \frac{1}{1 + \frac{is}{\theta_n \sigma_n(\theta_n)}} ds}, \tag{12}$$

where we use the identities $\mu_n(\theta_n) = n_v$ and $\mu_n(\theta'_n) = n_v - 1$. Since θ_n and θ'_n both converge to $\hat{\theta}$, it is possible to show that the first fraction converges to $e^{\hat{\theta}}$. This is the easier part of the proof, although we need to show that $\theta_n - \theta'_n$ goes to 0 faster than $1/n$. The technical part of the proof is to show that the second fraction converges to 1. In order to do that, we need to approximate the integrals as n goes to infinity. Consider the integral at the denominator for example. We can approximate $\varphi_{\theta_n}(s/\sigma_n(\theta_n))$, which is the characteristic function of the normalized random variable $(S_n - n_v)/\sigma_n(\theta_n)$ under the exponentially tilted probability P_{θ_n}, by $e^{-s^2/2}$ which is the characteristic function of a standard normal distribution. This is the usual intuition of central limit theorems which say that the distribution of a standardized random variable is asymptotically normal. The term $J(\theta_n, s/\sigma_n(\theta_n))$ can be approximated by 1, and we are left with the approximation

$$e^{-\frac{s^2}{2}} \frac{1}{1 + \frac{is}{\theta_n \sigma_n(\theta_n)}}$$

under the integral sign. Finally, since $\sigma_n(\theta_n) \to \infty$ as $n \to \infty$, we are left with the integral

$$B_0(\lambda) = \int_{-\infty}^{+\infty} e^{-\frac{s^2}{2}} \frac{1}{1 + \frac{is}{\lambda}} ds,$$

with $\lambda = \theta_n \sigma_n(\theta_n)$. $B_0(\lambda)$ is a well studied function that is known to converge to $(2\pi)^{1/2}$ as $\lambda \to \infty$ (see Jensen, 1995, section 2.1). Doing the same calculation for the integral at the numerator gives the result.

The proof follows the general steps of Jensen (1995) with additional difficulties from the fact that S_n is not a sum of identically distributed variables, and that we need to show that the convergence is uniform.

Proposition 5 (Convergence of the Best-Responses)**.** *The best-response functions β_n converge uniformly on $[0,1]$ to*

$$\beta(t) \equiv \left(1 + \frac{u^+}{u^-}\rho(t)\right)^{-1},$$

where

$$\rho(t) \equiv e^{\hat{\theta}(t)} \mathbb{1}_{t<\tilde{t}} + \mathbb{1}_{t \geq \tilde{t}} = \min\left(e^{\hat{\theta}(t)}, 1\right).$$

Furthermore, the function $\beta(t)$ is continuous on $[0,1]$ and strictly increasing in t on $[0,\tilde{t}]$. It is also decreasing in v and increasing in V.

Proof. See Appendix B ☐

The fact that $\beta(.)$ is strictly increasing in t can be interpreted as a certain form of strategic complementarities between voters: when all other players increase their common threshold, a voter best responds by increasing her threshold as well. The uniform convergence is important as it ensures that the fixed points of $\beta(.)$ are indeed the limits of the fixed points of $\beta_n(.)$. The set of asymptotic equilibria is the set of fixed points of $\beta(.)$. Let $T^* = \{t \in [0,1] : \beta(t) = t\}$ be the set of these fixed points. Note that the continuity of $\beta_n(.)$ and $\beta(.)$ implies that T_n^* and T^* are closed sets. Since they are also bounded, we can define the distance to these sets, $d(t,K) \equiv \sup_{t' \in K} \left|t - t'\right|$ for a compact K.

Proposition 6.

(i) If t^* is the limit point of a sequence $\{t_n\}$ such that $t_n \in T_n^*$, then $t \in T^*$.

(ii) For every $\delta > 0$, there exists some N such that for every $n > N$, $t_n \in T_n^*$ implies that $d(t_n, T^*) < \delta$.

Proof. (i) Let $g(t) \equiv \beta(t) - t$ and $g_n(t) \equiv \beta_n(t) - t$. It is clear that g_n converges uniformly to g on $[0,1]$. We can write $\left|g(t_n)\right| = \left|g(t_n) - g_n(t_n)\right|$. Fix some $\varepsilon > 0$. The uniform convergence of g_n on $[0,1]$ implies that for n sufficiently large, we can bound $\left|g(t_n) - g_n(t_n)\right|$ upward by ε. Since $t_n \in T_n *$ we have $g_n(t_n) = 0$, hence $|g(t^*)| = \lim_{n\to\infty} |g(t_n) - g_n(t_n)| \leq \varepsilon$, for every $\varepsilon > 0$, implying that $g(t^*) = 0$.

(ii) Let $B = \{t \in [0,1] : d(t, T^*) \geq \delta\}$. B is closed because the distance function $d(., T^*)$ is continuous, and since B is also clearly bounded, it is a compact set. Let $B^+ = \{t \in B : g(t) \geq 0\}$ and $B^- = \{t \in B : g(t) \leq 0\}$. These two sets are also compact sets by continuity of f, they are closed sets. Then we can define $\varepsilon^+ = \frac{1}{2} \inf_{t \in B^+} g(t)$ and $\varepsilon^- = \frac{1}{2} \inf_{t \in B^-} -g(t)$. These numbers are strictly positive because of the definition of B^+ and B^-. Let $\varepsilon = \min(\varepsilon^-, \varepsilon^+)$. We know that g_n converges uniformly to g on B. Then there exists some N such that for every $n > N$ and every $t \in B$, $\left|f(t) - f_n(t)\right| < \varepsilon$. But then $|g_n(t)| > |g(t)| - \varepsilon > 0$, where the second inequality comes from the fact that either $t \in B^-$ or $t \in B^+$ and from the definition of ε. In particular $T_N^* \subseteq [0,1] \setminus B$.

$\square$

5.3 Uniqueness

Before moving on, we provide a sufficient condition on the distribution f for the asymptotic equilibrium to be unique.

Proposition 7 (Uniqueness)**.** *There is a unique asymptotic equilibrium whenever the distribution f satisfies that for every $t \in [0,1]$*

$$\max\left\{\frac{\underline{p}'(t)}{\underline{p}(t)(1-\underline{p}(t))}, \frac{\overline{p}'(t)}{\overline{p}(t)(1-\overline{p}(t))}\right\} < \frac{1}{t(1-t)}. \tag{13}$$

Proof. See Appendix B $\square$

Note that when the distribution is symmetric, it is sufficient to verify the inequality for only one of the terms in the maximum function on the left-hand side.

Example 1 (The Uniform Distribution)*.* The uniform distribution $f(t) = 1$ satisfies (13) and therefore leads to a unique equilibrium. By symmetry, it is sufficient to show the inequality for $\underline{p}'/\underline{p}(1-\underline{p})$. With the uniform distribution, $\underline{p} = t/2$, and this ratio is equal to $1/t(1-t/2)$, which satisfies the desired inequality.

The next example, however, shows that there can be multiple equilibria. The integrals needed to express the best-response functions were calculated numerically.

Example 2 (Multiple Equilibria)*.* Consider the distribution of preferences f defined by

$$f(x) = \begin{cases} \gamma(1/4 - x + 10^{-5})^{-\frac{10}{11}} & if \quad 0 \leq x \leq 0.25 \\ \gamma(-1/4 + x + 10^{-5})^{-\frac{10}{11}} & if \quad 0.25 < x \leq 0.5 \\ \gamma(3/4 - x + 10^{-5})^{-\frac{10}{11}} & if \quad 0.5 < x \leq 0.75 \\ \gamma(x - 3/4 + 10^{-5})^{-\frac{10}{11}} & if \quad 0.75 < x \leq 1 \end{cases} \tag{14}$$

where γ is chosen so as to make the surface under f equal to 1. In Figure 1, we represented the corresponding asymptotic best-response functions for $V = 0.6$, and $v = 0.65$ showing the multiplicity of equilibria. Note that there are multiple stable equilibria as well.

5.4 Effects of the Rules

The simple form of the asymptotic best-response function enables us to study the effects of the voting rules. In order to do that, we use the following partial order on subsets of $\overline{\mathbb{R}}$: for every $S, T \subset \overline{\mathbb{R}}$, $S < T$ if and only if $\inf S \leq \inf T$ and $\sup S \leq \sup T$, with at least one of the inequalities holding strictly. The following proposition is a corollary of Proposition 5 which says how the best-response functions varies with the rules. We look at the set of equilibria as the image of a function $T^* : [0,1]^2 \to 2^{[0,1]}$ from the set of voting rules to the subsets of $[0,1]$.

Proposition 8 (Effects of the Rules)**.** *$T^*(v, V)$ is increasing in V and decreasing in v. That is the extremal equilibrium thresholds are increasing with the selection rule and decreasing with the decision rule.*

Proof. Given the sense of variation of β with respect to v and V, the result follows from Milgrom and Roberts (1994, Corollary 1) ☐

The proposition says that the equilibrium selection thresholds increase as the selection rule becomes more stringent and decrease as the decision rule becomes more stringent. The latter result is not very surprising: the harder it is for the proposal to pass the second round, the more willing voters are to bring the issue to the ballot. The first statement, however, is more surprising: the more difficult the institution makes it for an issue to be selected, the more selective the voters. In other words, the voters fail to offset the effect of the selection rule, and accentuate it instead. Suppose for example that the voters always play according to the maximal stable equilibrium threshold (this is the threshold they would converge to if they used a collective learning procedure initialized at the naive threshold). Then the proposition says that the fraction of votes cast in favor of selection[17] decreases as the tally of votes needed to select the issue increases. Note that conditional on being pivotal, selecting an issue when the selection rule is more stringent means that the proposal is more likely to pass, and therefore the voters become more selective. But because a voter keeps the option of voting against the proposal when she selects an issue, the driving force is in fact more subtle. What matters to a voter is the difference in likelihood that the proposal eventually passes, conditionally on being pivotal at the selection stage, whether she eventually supports it or not. A stricter selection rule, makes it relatively more likely that the issue passes when the voter eventually doesn't support it compared to when she does. In order to compensate for that increased likelihood ratio, the voter becomes more selective.

With Proposition 6, we can extend the comparative statics to large but finite committees.

Corollary 1. *For every V, V', v, v' with $v \leq v'$ and $V \leq V'$, there exists N such that for $n \geq N$ we have*

$$T^*(v, V') \geq T^*(v, V),$$

and

$$T^*(v', V) \leq T^*(v, V).$$

Example 3 (Comparative Statics with the Uniform Distribution)*.* With the closed form expressions of the best-response function for finite committees obtained in the proof of Proposition 2, we can, for given distributions, study the equilibria of the finite game. In this section, we illustrate our results for the uniform distribution on $[0, 1]$. Figure 2 shows the convergence of the best-response functions; Figure 3 illustrates the comparative statics on the selection rule in the limit; Figure 4 shows the same comparative statics for $n = 9$; finally, Figure 5 shows how the selection threshold and the selection probability vary with the selection rule for $n = 9$.

Even though our comparative statics result is only proved to hold for large committees, numerical analyses suggest that it may hold irrespective of the size of the committee. Figure 4

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[17] This fraction is $1 - F(t^*)$ with a large population.

illustrates this for the uniform distribution, but we have also run the same analysis for many distributions in the classes of Beta and triangular distributions without ever invalidating the result.

5.5 Selection in Subcommittees

It is straightforward (but long) to transpose the asymptotic analysis of the basic model to the case of subcommittees. Let $\tilde{\theta}(t)$ be such that $e^{\tilde{\theta}(t)}$ is the unique solution on $(1, +\infty)$ of the following equation in X

$$\frac{V\overline{p}}{\overline{p}X + 1 - \overline{p}} + \frac{(C - V)\underline{p}}{\underline{p}X + 1 - \overline{p}} + \frac{(1 - C)\tilde{p}X}{\tilde{p}X + 1 - \tilde{p}} = v, \tag{15}$$

Let $\tilde{\tilde{t}}$ be the unique (if any) t that solves $V\overline{p}(t) + (C - V)\underline{p}(t) + (1 - C)\tilde{p} = v$. Then

Proposition 9 (Convergence of the Best-Responses with Committees)**.** *The best-response functions $\tilde{\beta}_n$ converge uniformly on $[0, 1]$ to*

$$\tilde{\beta}(t) = \left(1 + \frac{u^+}{u^-}\tilde{\rho}(t)\right)^{-1},$$

where

$$\tilde{\rho}(t) = e^{\tilde{\theta}(t)}\mathbb{1}_{t<\tilde{\tilde{t}}} + \mathbb{1}_{t\geq\tilde{\tilde{t}}} = \min\left(e^{\tilde{\theta}(t)}, 1\right).$$

Furthermore, the function $\tilde{\beta}(t)$ is continuously on $[0, 1]$ and strictly increasing in t on $[0, \tilde{t}]$. It is also decreasing in v and increasing in V.

And, letting T_C^* denote the set of equilibria with subcommittees.

Proposition 10 (Effect of the Rules with Committees)**.** *$T_C^*(v, V)$ is increasing in V and decreasing in v. That is the extremal equilibrium thresholds are increasing with the selection rule and decreasing withe the decision rule. Furthermore, for any rule (v, V), $T_C^*(v, V) \geq T^*(v, V)$.*

Proof. The only point that needs a proof is the last one. For that we just need to compare (3) and (15), and notice that since for every t, $\underline{p}(t) \leq \tilde{p}$, it must be true that $e^{\tilde{\theta}(t)} \leq e^{-hat\theta(t)}$, and finally that $\tilde{\beta}(t) \geq \beta(t)$ which concludes the proof. $\square$

6 Welfare Analysis

Our analysis so far has focused on understanding the effect of different selection rules on the behavior of strategic voters. This is important because of the more or less implicit but widespread use of these rules. However, we haven't tried to answer the question of why these rules exist, or which rules should be used. In fact, it is easy to answer this question if we do not introduce new elements to the model: selection rules are useless. Indeed, with the current elements of the model, the optimal voting rule from a utilitarian perspective is to allow every possible issue to be selected by choosing $V = 0$, which is equivalent to suppressing the selection stage altogether, and to set $v = u^{-}/(u^{-} + u^{+})$ so that the proposal is adopted if and only if the expected utility gain of its supporters is higher than the expected utility loss of its opponents.

The use of selection rules cannot be justified without positing a cost of making a final decision over an issue (in addition to the cost of selecting it). If the final vote is costly to organize, then there may be some value added in screening issues that cannot make it anyway. This additional cost may be the cost of gathering more information about the issue in order to formulate a proposal, or just the opportunity cost of dealing with an issue rather than an other for an institution with limited time. In the case of citizens' initiatives, it is the cost to organize a referendum. Thus, there are many ways to model this cost and conduct a welfare analysis. In what follows, we proceed in the simplest possible way by assuming a fixed cost c to organize the final election. We assume that the cost of organizing the selection stage is 0 but it is easy to adjust the results to account for a positive cost. We assume a large population and conduct the analysis at the limit. We also choose a particular equilibrium of the selection game: the highest stable equilibrium threshold. A possible justification for selecting this particular equilibrium is that it is the threshold to which a simple collective learning heuristics converges when initiated at the naive threshold. Let t^* denote this equilibrium threshold in what follows. It depends on the voting rules and on the distribution that characterizes the issue at stake.

6.1 Single Issue

We start by assuming that there is a single issue, or equivalently that all the issues that the institution may face are characterized by the same distribution and the same expected payoffs u^+ and u^-.

At the limit, the law of large numbers implies that the fraction of the population that eventually supports (and votes for) the proposal is exactly $\tilde{p}$, and the fraction of the population that votes to select the issue is $1 - F(t^*)$. Then the program of an institution designer with a uniformly weighted utilitarian criterion is

$$\max_{(V,v)} \quad \Big(\mathbb{1}_{\tilde{p}\geq v}\left(\tilde{p}u^+ - (1-\tilde{p})u^-\right) - c\Big)\mathbb{1}_{1-F(t^*_F(V,v))\geq V}. \tag{16}$$

The problem that the selection rule must solve is therefore to screen issues such that $\tilde{p}u^+ - (1-\tilde{p})u^- < c$. If it is successful at doing so, the decision rule can be chosen anywhere in $\left[0, \frac{u^-+c}{u^-+u^+}\right]$. Note that the optimal v in the absence of a selection stage, $v = \frac{u^-}{u^-+u^+}$, lies in that interval. For now, we pick some v anywhere in that interval. An issue is selected if and only if $t^*_F(V,v) \leq F^{-1}(1-V)$. Because the left-hand side is strictly increasing in V and bounded between 0 and $\frac{u^-}{u^-+u^+}$, and the right-hand side is strictly decreasing in V and equal to 1 at $V = 0$ and to 0 at $V = 1$, it is easy to see that there is a unique $\tilde{V}_F(v) \in (0,1)$ such that the issue is always selected when $V \leq \tilde{V}_F(v)$, and never selected otherwise. This leads to the following characterization of optimal rules.

Proposition 11 (Optimal Rules with a Single Issue)**.**

(i) With a single issue such that $\tilde{p}_F \geq \frac{u^-+c}{u^-+u^+}$*, any rule such that* $v \leq \frac{u^-+c}{u^-+u^+}$ *and* $V \leq \tilde{V}_F(v)$ *is optimal.*

(ii) With a single issue such that $\tilde{p}_F \leq \frac{u^-+c}{u^-+u^+}$*, any rule such that* $V > \tilde{V}_F(v)$ *is optimal.*

6.2 Multiple Issues

Suppose now that the committee can face different issues. All the possible issues form a finite set $\mathcal{I} = \{1, \dots, I\}$ indexed by ι. Each issue is characterized by a distribution F_ι and some payoff parameters u_ι^-, u_ι^+, c_ι. Note that the index ι is for the issues and not the voters. We allow the cost of organizing the final vote to depend on the particular issue. Let $\lambda_\iota = \frac{u_\iota^-+c_\iota}{u_\iota^-+u_\iota^+}$. For each issue and each decision rule v there is a unique $\tilde{V}_\iota(v) \equiv \tilde{V}_{F_\iota}(v)$ defined as in the single-issue case such that the issue ι is selected if and only if $V \leq \tilde{V}_\iota(v)$. Finally, let $\tilde{p}_\iota$ denote the mean of F_ι.

With these notations, we can define the sets

$$\mathcal{I}^+ \equiv \{\iota \in \mathcal{I} | \tilde{p}_\iota \geq \lambda_i\},$$

and

$$\mathcal{I}^- \equiv \{\iota \in \mathcal{I} | \tilde{p}_\iota \leq \lambda_i\}.$$

$\mathcal{I}^+$ is the set of issues that are optimally selected and $\mathcal{I}^-$ is the set of issues that are optimally screened. We say that a rule achieves *perfect discrimination* if it selects every issue in $\mathcal{I}^+$ and none other.

And for any decision rule v, we let

$$\tilde{V}^+(v) \equiv \min_{\iota \in \mathcal{I}^+} \tilde{V}_\iota(v),$$

and

$$\tilde{V}^-(v) \equiv \max_{\iota \in \mathcal{I}^-} \tilde{V}_\iota.$$

For a given v, $\tilde{V}^+(v)$ is the highest possible selection rule that selects every issue in $\mathcal{I}^+$, and $\tilde{V}^-(v)$ is the lowest possible selection rule that screens every issue in $\mathcal{I}^-$. The following proposition is a direct consequence of the single-issue case.

Proposition 12 (Perfectly Discriminating Rules with Multiple Issues)**.**

(i) *If there exists some* $v^* \leq \min_{\mathcal{I}^+} \lambda_\iota$ *such that* $\tilde{V}^-(v^*) \leq \tilde{V}^+(v^*)$*, then any voting procedure* (V^*, v^*) *such that* $\tilde{V}^-(v^*) \leq V^* \leq \tilde{V}^+(v^*)$ *achieves perfect discrimination and is therefore optimal.*

(ii) *If for every* $v \leq \min_{\mathcal{I}^+} \lambda_\iota$, $\tilde{V}^-(v^*) > \tilde{V}^+(v^*)$*, there is no voting procedure that achieves perfect discrimination.*

In case (ii), any voting procedure is bound to generate type I and type II errors even though we conducted the analysis at the limit in the size of the committee where there is no uncertainty about which issues should be selected and which issues should be screened. This result suggests an explanation for why certain institutions may use different rules for different types of issues. When this is not possible however, one may still wonder about an optimal rule. In order to answer this question, however, more structure is needed so that type I errors can be weighed against type II errors.

7 Final Remarks

In this paper, we have developed a model of issue selection in committees. Initially, committee members decide whether or not to select a particular issue, through a first vote governed by the *selection* rule, which is the minimal number of favorable votes required to select an issue. Given a selected issue, the committee then decides whether to adopt an alternative policy for the issue or to maintain the status quo, through a second vote. The minimum number of favorable votes required to adopt an alternative is the *decision* rule. Our model predicts that committee members will be more conservative, i.e. less inclined to vote in favor of selecting an issue, as the selection rule increases. The decision rule has the opposite effect. Increasing the selection hence has a double negative impact on the number of issues selected, through the direct effect of the rule and through members' strategies.

Our results rely on assumptions about members' information on their preferences. Before the first vote, committee members have private information about their independent expected gains from a policy governing the issue alternative, relative to the status quo. If the issue is selected, they obtain perfect information about their preferences before the second vote. A natural question would then be: how would members' voting strategy change if their gains from the alternative were not independent? Introducing dependent preferences would broaden the scope of our results, and is left for further research.

Our work has positive applications. Several committees, such as the examples mentioned throughout the paper, use the explicit two-stage procedure we study. Under favorable identification conditions, our results could be tested directly. However, the rules of these institutions rarely change, if at all. For the case of the Supreme Court, as for other major institutions, the continuation of the rules is usually interpreted as a guarantee of credibility. We may also infer from our results that the sensitivity of the number of issues selected to the selection rule may have led institutions with inefficient rules to rapidly appear useless or unmanageable. Finding an identification strategy to test our predictions on "established" committees is a stimulating direction for future research. Our model may also have normative applications. Indeed, we derive efficient selection and decision rules that depend on the costs of organizing elections. As such, we provide a rationale for the choice of an agenda-setting procedure for emerging or established institutions that have no explicit rules, such as the Committee of Permanent representatives of the European Union. Our results can also be used for the choice of rules for citizens' initiatives, a procedure that has recently been introduced or extended in several European countries. The wide variety of committees that use, or could use, selection rules requires to better understand their effect, and offers several potential applications for this research.

Appendix A Proof of the Equilibrium Characterization

Proof of Proposition 2: Equilibrium Characterization. The expected utility of voter i if the issue is selected, conditional on the event $\mathcal{E}_i$ that her vote is pivotal, is given by $E\left(U_i|\mathcal{E}_i\right)$, which is of the form $(A_i+B_i)p_i - B_i$, where

$$A_i = u^+ E\left(\sum_{\substack{C\subseteq N_i\\ \#C\geq n_v-1}} \prod_{j\in C} p_j \prod_{l\in N_i\setminus C}(1-p_l)\middle|\mathcal{E}_i\right) > 0,$$

and

$$B_i = u^- E\left(\sum_{\substack{C\subseteq N_i\\ \#C\geq n_v}} \prod_{j\in C} p_j \prod_{l\in N_i\setminus C}(1-p_l)\middle|\mathcal{E}_i\right) > 0.$$

i selects the issue if this expression is greater than 0, that is if $p_i > t_i = B_i/(A_i+B_i)$. Clearly $B_i/u^- < A_i/u^+$ implying $t_i = B_i/(A_i+B_i) < u^-/(u^-+u^+)$.

In a symmetric equilibrium, all the voters use the same threshold t. Therefore the event $\mathcal{E}_i$ is the event that exactly n_V-1 voters in N_i have a type p above the threshold t. The expected value of the type of these players is then the expectation of p conditional on lying above t, that is $\overline{p}(t)$, while for the n_V^c other voters in N_i, this expectation is $\underline{p}(t)$. Because the types are all independent, A and B can be written as follows, where the subscript i is no longer needed because of the symmetry,

$$A = u^+ \sum_{s=n_v-1}^{n-1} \sum_{\substack{j+l=s\\ j\leq n_V-1\\ l\leq n_V^c}} \binom{n_V-1}{j}\binom{n_V^c}{l} \overline{p}(t)^j(1-\overline{p}(t))^{n_V-1-j}\underline{p}(t)^l(1-\underline{p}(t))^{n_V^c-l} = \Pr\bigl(S_n(t)\geq n_v-1\bigr)u^+,$$

and

$$B = u^- \sum_{s=n_v}^{n-1} \sum_{\substack{j+l=s\\ j\leq n_V-1\\ l\leq n_V^c}} \binom{n_V-1}{j}\binom{n_V^c}{l} \overline{p}(t)^j(1-\overline{p}(t))^{n_V-1-j}\underline{p}(t)^l(1-\underline{p}(t))^{n_V^c-l} = \Pr\bigl(S_n(t)\geq n_v\bigr)u^-.$$

And in the summation term, we can recognize the probability mass function of the random variable $S_n(t)$ defined above as the sum of $n_V - 1$ independent Bernoulli random variables with parameters $\overline{p}(t)$ and n_V^c independent Bernoulli random variables with parameter $\underline{p}(t)$. $S_n(t)$ is the random variable that gives the tally of votes eventually cast by other voters in favor of the proposal given a pivotal voter's information and knowing that other voters use the threshold t. Then, as already argued, the best response function of this voter is given by $\beta_n(t)$, and the symmetric equilibria of the game are exactly characterized by the fixed points of β_n.

The expressions of A and B show the continuity of β_n and, since β_n maps the unit interval to itself, Brouwer's fixed point theorem implies the existence of a symmetric equilibrium. □

Appendix B Proofs for the Asymptotic Analysis

We start by providing the inversion formula for continuous distributions without proof, it is the continuous analog of Lemma 4 and a well known result[18]. Then we prove three additional lemmas which are useful for the main proofs.

Lemma 6 (Inversion Formula for Continuous Distributions)**.** *Let X be a real random variable with a density function $g(x)$ on $\mathbb{R}$. Let*

$$\gamma(s) \equiv E\ e^{isX} = \int_{\mathbb{R}} e^{isx} g(x) dx,$$

be its characteristic function. Then for any $x \in \mathbb{R}$ we have the inversion formula

$$g(x) = (2\pi)^{-1} \int_{-\pi}^{\pi} e^{-isx} \gamma(s) ds.$$

Lemma 7. *There exist positive constants $c_\sigma, C_\sigma > 0$ such that for every n sufficiently large and*

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[18] See Jensen (1995) or Feller (1971)

every $t \in [0,1]$, we have

$$c_\sigma \sqrt{n} \le \min\big\{\sigma_n(\theta_n(t)), \sigma_n(\theta'_n(t))\big\} \le \max\big\{\sigma_n(\theta_n(t)), \sigma_n(\theta'_n(t))\big\} \le C_\sigma \sqrt{n}.$$

Proof. The definition of σ_n implies that for any θ

$$\frac{\sigma_n(\theta,t)}{\sqrt{n}} = e^{\theta/2}\left(\frac{(V_n - 1/n)\,\overline{p}(t)(1-\overline{p}(t))}{\left(\overline{p}(t)e^{\theta} + 1 - \overline{p}(t)\right)^2} + \frac{(1-V_n)\,\underline{p}(t)(1-\underline{p}(t))}{\left(\underline{p}(t)e^{\theta} + 1 - \underline{p}(t)\right)^2}\right)^{1/2}.$$

Because θ_n and θ'_n solve respectively (5) and (6) they must be decreasing in n. We also know that $\underline{p}$ and $\overline{p}$ are increasing in t. Hence we can write that for every $t \in [0,1]$

$$e^{\underline{\theta}/2}\left(\frac{(V_n - 1/n)\,p_e}{\left(1 - p_e + e^{\overline{\theta}}\right)^2}\right)^{1/2} \le \frac{\sigma_n(\theta_n(t),t)}{\sqrt{n}} \le e^{\overline{\theta}/2}\left(\frac{(V_n - 1/n)}{(p_e e^{\underline{\theta}})^2} + \frac{(1-V_n)\,p_e}{(1 - p_e + e^{\underline{\theta}})^2}\right)^{1/2}.$$

Because the right-hand side and the left hand-side both converge to finite and strictly positive real numbers, we can conclude for θ_n. We can write the same for θ'_n. ◻

Lemma 8. *There exist positive constants $C \le c_\sigma\sqrt{n}$ and $\overline{\kappa}$ such that for every $s \in [-C\sqrt{n}, C\sqrt{n}]$, every $t \in [0,1]$, and n sufficiently large we have*

$$\left|\varphi_{\theta_n}\left(\frac{s}{\sigma_n(\theta_n)}\right) - e^{-\frac{s^2}{2}}\right| \le \frac{\overline{\kappa}}{6c_\sigma^3 n^{1/2}}\,|s|^3 \exp\left(-\frac{s^2}{4}\right).$$

Proof. Consider the complex valued function $\kappa_n(s) = \frac{1}{n}\log\varphi_{\theta_n}(s/\sigma_n(\theta_n))$. We will expand it in s to prove the result. For that, we start by writing

$$\kappa_n(s) = (V_n - 1/n)\log\left(\frac{\overline{p}\exp(\theta + is/\sigma) + 1 - \overline{p}}{\overline{p}\exp(\theta_n) + 1 - \overline{p}}\right) + (1 - V_n)\log\left(\frac{\underline{p}\exp(\theta + is/\sigma) + 1 - \underline{p}}{\underline{p}\exp(\theta_n) + 1 - \underline{p}}\right) - i\mu s/\sigma,$$

where we use the notations σ for $\sigma_n(\theta_n)$, θ for θ_n and μ for $\mu_n(\theta_n) = n_v$. Then the first and second derivatives are

$$\kappa_n'(s) = \frac{i}{\sigma}\left((V_n - 1/n)\,\frac{\overline{p}\exp(\theta + is/\sigma)}{\overline{p}\exp(\theta + is/\sigma) + 1 - \overline{p}} + (1 - V_n)\,\frac{\underline{p}\exp(\theta + is/\sigma)}{\underline{p}\exp(\theta + is/\sigma) + 1 - \underline{p}} - \mu\right),$$

and

$$\kappa_n''(s) = \left(\frac{i}{\sigma}\right)^2\left((V_n - 1/n)\,\frac{\overline{p}(1-\overline{p})\exp(\theta + is/\sigma)}{\left(\overline{p}\exp(\theta + is/\sigma) + 1 - \overline{p}\right)^2} + (1 - V_n)\,\frac{\underline{p}(1-\underline{p})\exp(\theta + is/\sigma)}{\left(\underline{p}\exp(\theta + is/\sigma) + 1 - \underline{p}\right)^2}\right).$$

By construction, we have $\kappa_n'(0) = 0$ and $\kappa_n''(0) = -1/n$. An elementary proof by induction shows that for $k \geq 2$ we can write

$$\kappa_n^{(k)}(s) = \left(\frac{i}{\sigma}\right)^k\left((V_n - 1/n)\,\frac{\overline{Q}_k\big(\exp(\theta + is/\sigma)\big)}{\left(\overline{p}\exp(\theta + is/\sigma) + 1 - \overline{p}\right)^k} + (1 - V_n)\,\frac{\underline{Q}_k\big(\exp(\theta + is/\sigma)\big)}{\left(\underline{p}\exp(\theta + is/\sigma) + 1 - \underline{p}\right)^k}\right),$$

where $\overline{Q}_k$ and $\underline{Q}_k$ are polynomials of degree $k-1$ whose coefficients are polynomials in $\overline{p}$ and $\underline{p}$ respectively. For a polynomial $P(X)$, we let $|P|(X)$ be the polynomial whose coefficients are the norms of the coefficients of $P(X)$. Then we can bound $\kappa_n^{(k)}(s)$ upward on any interval $I = [-A\sigma\pi, A\sigma\pi]$ with $A < 1$ by

$$\overline{\kappa}_k(t) = \left(\frac{1}{\sigma}\right)^k\left(\frac{\big|\overline{Q}_k\big|(e^\theta)}{\overline{m}^k} + \frac{\big|\underline{Q}_k\big|(e^\theta)}{\underline{m}^k}\right),$$

where

$$\overline{m} = \min_{z\in[-A\pi,A\pi]}\left|\overline{p}e^{\theta+iz} + (1-\overline{p})\right| > 0,$$

and

$$\underline{m} = \min_{z\in[-A\pi,A\pi]}\left|\underline{p}e^{\theta+iz} + (1-\underline{p})\right| > 0.$$

The dependency of $\overline{\kappa}_k$ on t comes through $\theta = \theta_n(t)$, $\sigma = \sigma_n(\theta_n(t))$, $\overline{p}(t)$ and $\underline{p}(t)$. In particular, we have shown that $\kappa_n^{(3)}(s)$ is Lipschitz-continuous on $I = [-A\sigma\pi, A\sigma\pi]$ since its derivative is uniformly bounded on I. This in turn implies that $\kappa_n^{(3)}(s)$ is absolutely continuous so that for every $s \in I$, by the fundamental theorem of calculus,we can write the following Taylor expansion for $\kappa_n(s)$

$$\kappa_n(s) = -\frac{s^2}{2n} + \int_0^s \frac{\kappa_n^{(3)}(u)}{2}(s-u)^2 du.$$

And this leads to

$$\left|\kappa_n(s) + \frac{s^2}{2n}\right| \leq \frac{\overline{\kappa}_3}{6c_\sigma^3 n^{3/2}} \left|s\right|^3.$$

Remark that $\overline{\kappa}_3$ is continuous in t on $[0,1]$ so that we can define $\overline{\kappa} \equiv \max_{t\in[0,1]} \overline{\kappa}_3$ and replace $\overline{\kappa}_3$ by $\overline{\kappa}$ in the expression above. Hence we can write that

$$\varphi_{\theta_n}\left(\frac{s}{\sigma_n\big(\theta_n(t)\big)}\right) = \exp\left(-\frac{s^2}{2} + \frac{\overline{\kappa}}{6c_\sigma^3 n^{1/2}} \left|s\right|^3 \omega\right),$$

where ω is some complex number with norm less than or equal to 1. Then we can use the following inequality which is a particular case of an inequality from Feller (1971, p.535) and holds for any $\lambda \in \mathbb{C}$

$$\left|e^{\lambda} - 1\right| \leq \left|\lambda\right| e^{\left|\lambda\right|}.$$

With $\lambda = \frac{\overline{\kappa}}{6c_\sigma^3 n^{1/2}} \left|s\right|^3 \omega$, we obtain

$$\left|\varphi_{\theta_n}\left(\frac{s}{\sigma_n(\theta_n)}\right) - e^{-\frac{s^2}{2}}\right| \leq \frac{\overline{\kappa}}{6c_\sigma^3 n^{1/2}} \left|s\right|^3 \exp\left(-\frac{s^2}{4}\right) \exp\left(-\frac{s^2}{4} + \frac{\overline{\kappa}}{6c_\sigma^3 n^{1/2}} \left|s\right|^3\right).$$

And for $|s| \leq cn^{\frac{1}{2}}$ with $c = \frac{3c_\sigma^3}{2\overline{\kappa}}$, the second exponential term is bounded upward by 1. Fix some $A > 1$ and choosing

$$C \equiv \min\left\{c, Ac_\sigma\pi\right\},$$

we have shown that for every s such that $|s| \leq C\sqrt{n}$, every $t \in [0,1]$, and n sufficiently large

$$\left|\varphi_{\theta_n}\left(\frac{s}{\sigma_n(\theta_n)}\right) - e^{-\frac{s^2}{2}}\right| \leq \frac{\overline{\kappa}}{6c_\sigma^3 n^{1/2}} \left|s\right|^3 \exp\left(-\frac{s^2}{4}\right).$$

□

For the next two lemmas let $\gamma(u) \equiv E\ e^{iuS_n}$ be the characteristic function of S_n. Then $\tilde{\gamma}\left(\frac{u}{s_n}\right) \equiv \gamma\left(\frac{u}{s_n}\right) e^{-\frac{iunm_n}{s_n}}$, with $m_n(t) = \frac{1}{n} E\ S_n(t)$ and $s_n(t) = \sqrt{VarS_n(t)}$, is the characteristic function of the standardized random variable $Z_n(t) \equiv \frac{S_n(t)-nm_n(t)}{s_n(t)}$.

Lemma 9. *There exist positive constants c_s, C_s such that for n sufficiently large and every $t \in [0,1]$*

$$c_s < \frac{s_n}{n^{1/2}} < C_s.$$

Proof. This is because

$$\frac{s_n^2}{n} = (V_n - 1/n)\overline{p}(t)(1-\overline{p}(t)) + (1-V_n)\underline{p}(t)(1-\underline{p}(t))$$

converges uniformly on $[0,1]$ to $V\overline{p}(t)(1-\overline{p}(t)) + (1-V)\underline{p}(t)(1-\underline{p}(t)) > 0$. □

Lemma 10. *There exist positive constants $C' \leq c_s\sqrt{n}$ and $\overline{k}$ such that for every $u \in [-C'\sqrt{n}, C'\sqrt{n}]$, every $t \in [0,1]$, and n sufficiently large we have*

$$\left|\tilde{\gamma}\left(\frac{u}{s_n(t)}\right) - e^{-\frac{u^2}{2}}\right| \leq \frac{\overline{k}}{6c_s^3 n^{1/2}} |u|^3 \exp\left(-\frac{u^2}{4}\right).$$

Proof. The proof is essentially the same as for Lemma 8 and we do not write it down to save space. □

Proof of Lemma 1: Convergence of θ_n and θ_n'. The functions $\overline{p}(t)$, $\underline{p}(t)$, $1-\overline{p}(t)$ and $1-\underline{p}(t)$ are all continuous on $[0,1]$ and bounded downward by 0 and upward by 1. Since $\overline{p}$ and $\underline{p}$ are increasing in t, it is easy to see on (4) that $\hat{\theta}$, θ_n and θ_n' are all decreasing in t. Letting $\hat{\psi} = e^{\hat{\theta}}$, and $\psi_n = e^{\theta_n}$ and $\psi_n' = e^{\theta_n'}$, we have that, for every $t \in [0,1]$, $\psi_n(1) \leq \psi_n(t), \psi_n'(t) \leq \psi_n(0)$. Because the function $\Psi(.)$ is continuous, ψ_n and ψ_n' converge pointwise to $\hat{\psi}$, and this implies that for n sufficiently large, $\psi_n(0), \psi_n'(0) \leq 2\hat{\psi}(0)$ and $\psi_n(1), \psi_n'(1) \geq \hat{\psi}(1)/2$. Letting $\underline{\theta} = \log(\hat{\psi}(1)/2)$ and $\overline{\theta} = \log(2\hat{\psi}(0))$, we have shown that for n sufficiently large, the functions $\theta_n(t)$, $\theta_n'(t)$ and $\hat{\theta}(t)$ are uniformly bounded downward and upward by (respectively) $\underline{\theta}$ and $\overline{\theta}$.

Then using the closed form expressions of θ_n and $\hat{\theta}$, and the inequality $|\log x - \log y| \leq \max_{y \leq z \leq x}(z^{-1}) \times |x-y|$, we have, for n sufficiently large and every $t \in [0,1]$,

$$\left|\hat{\theta}(t) - \theta_n(t)\right| \leq e^{-\underline{\theta}} \left|\Psi(V, 1-V, v; t) - \Psi(V_n - 1/n, 1-V_n, v_n; t)\right|$$

Now $\Psi \equiv \Psi(V, 1-V, v; t)$ and $\Psi_n \equiv \Psi(V_n - 1/n, 1-V_n, v_n; t)$ respectively solve the equations

$$a\Psi^2 + b\Psi + c = 0 \tag{17}$$

and

$$a_n\Psi_n^2 + b_n\Psi_n + c_n = 0 \tag{18}$$

with $a = v\underline{p}\overline{p}$, $b = (V-v)\overline{p}(1-\underline{p}) + (1-V-v)\underline{p}(1-\overline{p})$, $c = -v(1-\overline{p})(1-\underline{p})$, $a_n = v_n\underline{p}\overline{p}$, $b = (V_n - 1/n - v_n)\overline{p}(1-\underline{p}) + (1-V_n-v_n)\underline{p}(1-\overline{p})$, and $c = -v_n(1-\overline{p})(1-\underline{p})$. Substracting (18) to (17), we obtain with some algebra

$$|\Psi - \Psi_n| = \frac{|(a_n - a)\Psi_n^2 + (b_n - b)\Psi_n + (c_n - c)|}{|b + a(\Psi + \Psi_n)|}.$$

The term at the numerator is bounded upward uniformly in t by

$$|v_n - v|\, e^{2\overline{\theta}} + 2\big(|V_n - V| + |v - v_n|\big)e^{\overline{\theta}} + |v_n - v|\,.$$

The term at the denominator is bounded downward by

$$M(t) = \max\big(|b| - |a| \cdot |\Psi + \Psi_n| \ \ , \ \ |a| \cdot |\Psi + \Psi_n| - |b|\big).$$

We can write that $|b| - |a| \cdot |\Psi + \Psi_n| \geq |b| - 2\,|a|\,e^{\overline{\theta}}$ and $|a| \cdot |\Psi + \Psi_n| - |b| \geq 2\,|a|\,e^{\underline{\theta}} - |b|$. Hence for every $t \in [0,1]$, $M(t) \geq m(t) = \max\big(|b| - 2\,|a|\,e^{\overline{\theta}} \ , \ 2\,|a|\,e^{\underline{\theta}} - |b|\big)$. But then $m(t)$ is continuous on the compact $[0,1]$ and therefore attains its minimum $m \geq 0$. If $m = 0$, there must exist some t such that $|b(t)| = 2|a(t)|e^{\overline{\theta}} = 2|a(t)|e^{\underline{\theta}}$. This is possible if and only if $a(t) = 0$, that is if $t = 0$, but then we have $b(0) = V - v \neq 0$, a contradiction whenever $v \neq V$. Therefore $m > 0$, and we can write

$$|\Psi - \Psi_n| \leq m^{-1} \Big(|v_n - v|\, e^{2\overline{\theta}} + 2\big(|V_n - V| + |v - v_n|\big)e^{\overline{\theta}} + |v_n - v| \Big) , \tag{19}$$

where the right-hand side converges to 0 in $\mathcal{O}(1/n)$ and is independent of t. If $V = v$, our proof still shows that $m(t)$ attains its lower bound m' on the compact K and that $m' > 0$ with the desired conclusion.

For the sense of variation of $\hat{\theta}(t)$, note that $\overline{p}$ and $\underline{p}$ are both strictly increasing in t, implying that the functions $\frac{1-\overline{p}}{\overline{p}}$ and $\frac{1-\underline{p}}{\underline{p}}$ are strictly decreasing in t. Writing that

$$\frac{V}{e^{-\hat{\theta}} + \frac{1-\overline{p}}{\overline{p}}} + \frac{1-V}{e^{-\hat{\theta}} + \frac{1-\underline{p}}{\underline{p}}} = v, \tag{20}$$

shows that $\hat{\theta}$ must be strictly increasing in t. This also gives us the sense of variation with respect to v. The sense of variation of θ_n and θ'_n is obtained similarly. The continuity of the three functions is seen on their closed form expressions.

For the sense of variation with respect to V, we notice that $\hat{\theta}$ is continuously differentiable with respect to V by looking at its closed form expression, and proceed to differentiate (20) with respect to V yielding

$$dV \left\{ \frac{1}{1 + \frac{\overline{p}}{1-\overline{p}} e^{-\hat{\theta}}} - \frac{1}{1 + \frac{\underline{p}}{1-\underline{p}} e^{-\hat{\theta}}} \right\} + e^{-\hat{\theta}} d\hat{\theta} \left\{ \frac{V}{\left(1 + \frac{\overline{p}}{1-\overline{p}} e^{-\hat{\theta}}\right)^2} + \frac{1-V}{\left(1 + \frac{\underline{p}}{1-\underline{p}} e^{-\hat{\theta}}\right)^2} \right\} = 0,$$

implying that $sign\left(\frac{d\hat{\theta}}{dV}\right) = sign\left(\frac{1}{1+\frac{\underline{p}}{1-\underline{p}}e^{-\hat{\theta}}} - \frac{1}{1+\frac{\overline{p}}{1-\overline{p}}e^{-\hat{\theta}}}\right) = -1$ as $\overline{p} > \underline{p}$. □

Proof of Lemma 2: Convergence of the Tail Probabilities. The first part of point (i) and point (ii) are immediate consequences of the strong law of large numbers which say that for every

$\varepsilon, \delta > 0$, there is some $N_{\varepsilon,\delta}$ such that for every $n > N_{\varepsilon,\delta}$,

$$\Pr\left(\frac{|S_n - m_n|}{n} < \varepsilon\right) > 1 - \delta.$$

Indeed, we can write

$$\Pr\bigl(S_n(t) \geq n_v\bigr) = \Pr\left(\frac{S_n - m_n}{n} \geq \frac{n_v - m_n}{n}\right),$$

and since $m_n/n \to m$ and $n_v/n \to v$, for any $\eta > 0$, there is some N_η such that, for every $n > N_\eta$,

$$v - m - \eta < \frac{n_v - m_n}{n} < v - m + \eta.$$

Then,

$$\Pr\left(\frac{S_n - m_n}{n} > v - m + \eta\right) < \Pr\bigl(S_n(t) \geq n_v\bigr) < \Pr\left(\frac{S_n - m_n}{n} > v - m - \eta\right).$$

If $v > m$, we can choose η such that, for a given small ε, $v - m - \eta > \varepsilon$. But then, for any $\delta > 0$ and $n > \max(N_\eta, N_{\varepsilon,\delta})$,

$$\Pr\bigl(S_n(t) \geq n_v\bigr) < \Pr\left(\frac{S_n - m_n}{n} > v - m - \eta\right) < \Pr\left(\frac{S_n - m_n}{n} > \varepsilon\right) < 1 - \Pr\left(\frac{|S_n - m_n|}{n} < \varepsilon\right) < \delta,$$

which proves that $\Pr\bigl(S_n(t) \geq n_v\bigr) \to 0$ when $m < v$. The arguments for $m \geq v$, and for $\Pr\bigl(S_n(t) \geq n_v - 1\bigr)$ work in the same way. The second parts of point (i) and (ii) result from a direct application of the Gärtner-Ellis Theorem. $\square$

Proof of Lemma 3: Rewriting the Tail Probabilities (1). Using (2) we can write

$$\begin{aligned}
\Pr\bigl(S_n \geq \alpha_n\bigr) &= \int_{z\geq\alpha_n} P(dz) = \int_{z\geq\alpha_n} \frac{dP}{dP_\theta}(z) P_\theta(dz) \\
&= \int_{z\geq\alpha_n} \varphi_n(\theta) e^{-\theta z} P_\theta(dz) = \varphi_n(\theta) e^{-\theta\alpha_n} E_\theta\left(e^{-\theta(S_n-\alpha_n)} \mathbb{1}_{S_n\geq\alpha_n}\right) \\
&= \varphi_n(\theta) e^{-\theta\alpha_n} \sum_{z\geq\alpha_n,\ z\in\mathbb{Z}} e^{-\theta(z-\alpha_n)} P_\theta\bigl(S_n = z\bigr).
\end{aligned}$$

And this proves the lemma since the other terms in (9) cancel each other out. □

Proof of Lemma 5: Rewriting the Tail Probabilities (2). The summation term in (9) is $\sigma_n(\theta)$ times the point probability $P_\theta\left(S_n - \alpha_n - Y = 0\right)$ where Y is independent of S_n and $P_\theta(Y = y) = \left(1 - e^{-\theta}\right) e^{-\theta y}$ for $y = 0, 1, 2, \cdots$ (this works because $\theta > 0$). Then $S_n - \alpha_n - Y$ is concentrated on $\mathbb{Z}$ with maximal step 1 and its characteristic function is

$$\frac{\varphi_n(\theta + is/\sigma_n(\theta))}{\varphi_n(\theta)} e^{is\mu_n(\theta)/\sigma_n(\theta)} e^{is(\mu_n(\theta)-\alpha_n)} \frac{1 - e^{-\theta}}{1 - e^{-\theta-is}}.$$

Using the inversion formula in (10), we obtain (11) after scaling the integrand. □

Proof of Proposition 5: Convergence of the Best-Responses. For some fixed $0 < \alpha < 1/2$, we define the sets

$$I_N^\ell \equiv \left\{t \in [0,1] : \hat{\theta}(t) \geq N^{\alpha-1/2}\right\},$$

$$I_N^m \equiv \left\{t \in [0,1] : \left|\hat{\theta}(t)\right| \leq N^{-\alpha-1/2}\right\},$$

$$I_N^h \equiv \left\{t \in [0,1] : \hat{\theta}(t) \leq -N^{\alpha-1/2}\right\}.$$

Note that, because $\hat{\theta}$ is continuous, strictly decreasing in t and crosses 0 at $\tilde{t}$, I_N^ℓ is of the form $[0, t_0]$ (ℓ stands for *low* t's), I_N^m is of the form $[t_1, t_2]$ with $t_1 < \tilde{t} < t_2$ (m stands for *middle* t's) and I_N^h is of the form $[t_3, 1]$ (h stands for *high* t's). Also for a given N, $t_0 < t_1 < t_2 < t_3$ so that the intervals do not form a partition of $[0, 1]$.

Because θ_n converges uniformly to $\hat{\theta}$ in $\mathcal{O}(1/n)$ (faster than $1/n^{1/2-\alpha}$), it must be true that for n sufficiently large we can bound any $\theta_n(t)$ below on I_N^ℓ by $\underline{\theta}_N \equiv \frac{1}{2n^{1/2-\alpha}}$. For the same reason, we can bound any $\left|\theta_n(t)\right|$ below by the same $\underline{\theta}_n$ on I_N^h.

We divide the proof into six parts, the first five of which prove the uniform convergence. Part I and II prove that

$$\sup_{n\geq N}\ \sup_{t\in I_N^\ell} \left|R_n(t) - \exp(\hat{\theta}(t))\right|$$

converges to 0 as N goes to infinity. Specifically, part I shows that each of the integrals at the numerator and the denominator of the second fraction in (12) converges to $(2\pi)^{1/2}$ at a rate that is independent of t on I_n^ℓ. The second part shows that the first fraction in (12) converges to $\hat{\theta}(t)$ at a rate that does not depend on t on I_N^ℓ. The third part deals with the intervals I_N^h, and the fourth part with the intervals I_N^m. Finally part V puts the pieces together to conclude that the convergence of R_n is uniform on $[0,1]$ and implies the uniform convergence of the best-response functions. Part VI proves all the remaining claims of the proposition.

Part I. First, we look at the interval I_N^ℓ. As we just noted, $\theta_n(t)$ is bounded below by $\underline{\theta}_N$ on I_N^ℓ. We start by decomposing each of the integrals of interest into several terms. We write the decomposition for the integral at the denominator in (12), the convergence proof for the other integral is essentially the same.

$$\underbrace{\int_{-\infty}^{+\infty} e^{-\frac{s^2}{2}} \frac{1}{1+\frac{is}{\theta_n\sigma_n(\theta_n)}}\, ds}_{T_1} - \underbrace{\int_{|s|>\pi\sigma_n(\theta_n)} e^{-\frac{s^2}{2}} \frac{1}{1+\frac{is}{\theta_n\sigma_n(\theta_n)}}\, ds}_{T_2}$$
$$+ \underbrace{\int_{I_n(\theta_n)} e^{-\frac{s^2}{2}} \frac{1}{1+\frac{is}{\theta_n\sigma_n(\theta_n)}} \left(J\left(\theta_n, \frac{s}{\sigma_n(\theta_n)}\right) - 1\right) ds}_{T_3}$$
$$+ \underbrace{\int_{I_n(\theta_n)} \left(\varphi_{\theta_n}\left(\frac{s}{\sigma_n(\theta_n)}\right) - e^{-\frac{s^2}{2}}\right) \frac{1}{1+\frac{is}{\theta_n\sigma_n(\theta_n)}} J\left(\theta_n, \frac{s}{\sigma_n(\theta_n)}\right) ds}_{T_4},$$

where $\gamma_\theta = e^{-\theta}/\left(1-e^{-\theta}\right)$.

First Term $\mathbf{T}_1$. The first term is equal to the function $B_0(\lambda) = 2\pi\lambda e^{\frac{\lambda^2}{2}}\left(1-\mathcal{N}(\lambda)\right)$ where

$\mathcal{N}(.)$ is the standard normal cdf and $\lambda = \theta_n \sigma_n(\theta_n)$. $B_0(\lambda)$ is strictly increasing in λ, and by Lemma 7 we know that $\lambda > \underline{\theta}_N c_\sigma \sqrt{n}$ for every $t \in I_n^\ell$. We also know that $B_0(\lambda)$ converges to $(2\pi)^{1/2}$ when $\lambda \to \infty$, so we can write, that for every $t \in I_n^\ell$

$$0 \leq (2\pi)^{1/2} - B_0\left(\theta_n(t)\sigma_n(\theta_n(t))\right) \leq (2\pi)^{1/2} - B_0\left(\underline{\theta}_N c_\sigma \sqrt{n}\right),$$

and conclude that

$$\sup_{n \geq N} \sup_{t \in I_N^\ell} \left| T_1(n,t) - (2\pi)^{1/2} \right| \xrightarrow[N \to \infty]{} 0.$$

Second Term $\mathbf{T}_2$. For the second term, we can write for every $t \in [0,1]$

$$\begin{aligned} \left| \int\limits_{|s| > \pi\sigma_n(\theta_n)} e^{-\frac{s^2}{2}} \frac{1}{1 + \frac{is}{\theta_n \sigma_n(\theta_n)}} ds \right| &\leq \int\limits_{|s| > \pi c_\sigma \sqrt{n}} \left| e^{-\frac{s^2}{2}} \frac{1}{1 + \frac{is}{\theta_n \sigma_n(\theta_n)}} \right| ds \\ &\leq \int\limits_{|s| > \pi c_\sigma \sqrt{n}} e^{-\frac{s^2}{2}} ds \\ &\leq \int\limits_{|s| > \pi c_\sigma \sqrt{n}} e^{-\frac{|s|}{2}} ds = 4e^{-\frac{\pi c_\sigma \sqrt{n}}{2}}, \end{aligned}$$

so that the second term converges uniformly to 0. In the series of inequalities above, we used the fact that for every $x \in \mathbb{R}$

$$\left| \frac{1}{1 + ix} \right| \leq 1. \tag{21}$$

Third Term $\mathbf{T}_3$. For the third term we start by writing that for any real number z

$$\begin{aligned} |J(\theta, z) - 1| &= \left| \frac{\frac{iz}{\theta} - \gamma_\theta\left(1 - e^{-iz}\right)}{1 + \gamma_\theta\left(1 - e^{-iz}\right)} \right| \\ &\leq (\theta^{-1} + \gamma_\theta)|z|, \end{aligned}$$

where we used the inequalities $|1 - e^{-iz}| \leq |z|$ and

$$\left|1 + \gamma_\theta\left(1 - e^{-iz}\right)\right| = \sqrt{(1 + \gamma_\theta(1 - \cos(-z)))^2 + (\gamma_\theta \sin(-z))^2} \geq 1. \tag{22}$$

Using (21) as well, we conclude that, for every $t \in I_N^\ell$, we can bound above the absolute value of the third term by

$$\frac{\underline{\theta}_N^{-1} + \gamma_{\underline{\theta}_N}}{c_\sigma \sqrt{n}} \int_{-\infty}^{+\infty} e^{-\frac{s^2}{2}} |s| ds = 2 \left(\frac{\underline{\theta}_N^{-1} + \gamma_{\underline{\theta}_N}}{c_\sigma \sqrt{n}} \right).$$

Because $\underline{\theta}_N^{-1}$ and $\gamma_{\underline{\theta}_N}$ are $\mathcal{O}(N^{1/2-\alpha})$, we can conclude that

$$\sup_{n \geq N} \sup_{t \in I_N^\ell} \left| T_3(n,t) \right| \xrightarrow[N \to \infty]{} 0.$$

Fourth Term $\mathbf{T_4}$. From Lemma 8 we obtain that for $|s| \leq C\sqrt{n}$

$$\left| \varphi_{\theta_n} \left(\frac{s}{\sigma_n(\theta_n)} \right) - e^{-\frac{s^2}{2}} \right| \leq \frac{\overline{\kappa}}{6 c_\sigma^3 n^{1/2}} |s|^3 \exp\left(-\frac{s^2}{4} \right).$$

From (22) and Lemma 7, we have for every $t \in I_N^\ell$

$$\left| J \left(\theta_n, \frac{s}{\sigma_n(\theta_n)} \right) \right| \leq 1 + \frac{|s|}{\underline{\theta}_N c_\sigma n^{1/2}}.$$

From (21), we deduce that the norm of T_4 is bounded upward by

$$\underbrace{\int_{|s| \leq C\sqrt{n}} \frac{\overline{\kappa}_3}{6 c_\sigma^3 n^{1/2}} |s|^3 \exp\left(-\frac{s^2}{4} \right) \left(1 + \frac{|s|}{\underline{\theta}_N c_\sigma n^{1/2}} \right) ds +}_{T_{4.1}} \underbrace{\int_{|s| \geq C\sqrt{n}} \exp\left(-\frac{s^2}{2} \right) \left(1 + \frac{|s|}{\underline{\theta}_N c_\sigma n^{1/2}} \right) ds}_{T_{4.2}}$$
$$+ \underbrace{\int_{C\sqrt{n} \leq |s| \leq \pi \sigma_n(\theta_n)} \left| \varphi_{\theta_n} \left(\frac{s}{\sigma_n(\theta_n)} \right) \right| \left(1 + \frac{|s|}{\underline{\theta}_N c_\sigma n^{1/2}} \right) ds}_{T_{4.3}}$$

(i) $T_{4.1}$ is bounded upward by

$$2 \left(1 + \frac{C}{\underline{\theta}_N c_\sigma} \right) \frac{\overline{\kappa}_3}{6 c_\sigma^3 n^{1/2}} \int_0^{+\infty} s^3 e^{-\frac{s^2}{4}} = 8 \left(1 + \frac{C}{\underline{\theta}_N c_\sigma} \right) \frac{\overline{\kappa}_3}{6 c_\sigma^3 n^{1/2}},$$

where the right-hand side is obtained by integration by part. It is immediate to conclude that

$$\sup_{n\geq N}\ \sup_{t\in I_N^\ell}\ \left|T_{4.1}(n,t)\right| \xrightarrow[N\to\infty]{} 0.$$

(ii) $T_{4.2}$ is equal to

$$\begin{aligned}2\left(\int_{C\sqrt{n}}^{\infty} e^{-\frac{s^2}{2}}ds + \frac{1}{\underline{\theta}_N c_\sigma n^{1/2}}\int_{C\sqrt{n}}^{\infty} se^{-\frac{s^2}{2}}ds\right) &\leq 2\left(\int_{C\sqrt{n}}^{\infty} e^{-\frac{s}{2}}ds + \frac{1}{\underline{\theta}_N c_\sigma n^{1/2}}\int_{C\sqrt{n}}^{\infty} se^{-\frac{s^2}{2}}ds\right)\\ &\leq 2\left(2e^{-\frac{Cn^{1/2}}{2}} + \frac{1}{\underline{\theta}_N c_\sigma n^{1/2}}e^{-\frac{C^2n}{2}}\right),\end{aligned}$$

where we used the fact that $e^{-\frac{s^2}{2}} \leq e^{-\frac{s}{2}}$ for positive and sufficiently large s. This proves that

$$\sup_{n\geq N}\ \sup_{t\in I_N^\ell}\ \left|T_{4.2}(n,t)\right| \xrightarrow[N\to\infty]{} 0.$$

(iii) For the last term, first note that $C \leq \pi c_\sigma \leq \pi\sigma_n(\theta_n)$ by construction. We need to go back to the definition of $\varphi_\theta(.)$ in Lemma 5 to get the expression

$$\left|\varphi_{\theta_n}\left(\frac{s}{\sigma_n(\theta_n)}\right)\right| = \left|\frac{\overline{p}e^{is/\sigma_n(\theta_n)} + (1-\overline{p})\,e^{-\theta_n}}{\overline{p} + (1-\overline{p})\,e^{-\theta_n}}\right|^{n_V-1}\left|\frac{\underline{p}e^{is/\sigma_n(\theta_n)} + \left(1-\underline{p}\right)e^{-\theta_n}}{\underline{p} + \left(1-\underline{p}\right)e^{-\theta_n}}\right|^{n_V^c}. \tag{23}$$

At this point we use the fact that for any real number $z \in [-\pi,\pi]$ and any real numbers a and b,

$$\left|ae^{iz} + b\right| \leq |a+b|,$$

with a strict inequality if $z \neq 0$. This inequality and the fact that the function $\left|\frac{\overline{p}e^{iz}+(1-\overline{p})e^{-\theta_n}}{\overline{p}+(1-\overline{p})e^{-\theta_n}}\right|$ is continuous in t and z imply together that we can define the following quantity

$$\overline{\delta} \equiv \max_{C/c_\sigma\leq z\leq\pi}\ \max_{t\in[0,1]}\ \left|\frac{\overline{p}e^{iz} + (1-\overline{p})\,e^{-\theta_n}}{\overline{p} + (1-\overline{p})\,e^{-\theta_n}}\right| < 1.$$

And similarly

$$\underline{\delta} \equiv \max_{C/c_\sigma \le z \le \pi} \max_{t\in[0,1]} \left| \frac{\underline{p}e^{iz} + \left(1-\underline{p}\right) e^{-\theta_n}}{\underline{p} + \left(1-\underline{p}\right) e^{-\theta_n}} \right| < 1.$$

Then the first term in (23) is bounded upward by $\overline{\delta}^{n_V}$ and the second term by $\underline{\delta}^{n_V^c}$. Finally, letting $\delta = \max(\overline{\delta}, \underline{\delta})$, we have obtained that

$$\left| \varphi_{\theta_n} \left(\frac{s}{\sigma_n(\theta_n)} \right) \right| \le \delta^{n-1},$$

whenever $C\sqrt{n} \le |s| \le \pi\sigma_n(\theta_n)$ with $\delta < 1$. Therefore we can bound $T_{4.3}$ upward by

$$2\delta^{n-1} \int_{C\sqrt{n}}^{\pi C_\sigma \sqrt{n}} \left(1 + \frac{s}{\underline{\theta}_N c_\sigma n^{1/2}} \right) ds \le \delta^{n-1} \left(\pi C_\sigma - C \right) \sqrt{n} \left(1 + \frac{\pi C_\sigma}{\underline{\theta}_N c_\sigma} \right).$$

Therefore

$$\sup_{n\ge N} \sup_{t\in I_N^\ell} \left| T_{4.3}(n,t) \right| \xrightarrow[N\to\infty]{} 0.$$

To sum up, we have shown each of the integrals in the second fraction in (12) converges to $(2\pi)^{1/2}$, call them $J^1(n,t)$ and $J^2(n,t)$ (say J^1 is at the numerator), both satisfy

$$\sup_{n\ge N} \sup_{t\in I_N^\ell} \left| J^k(n,t) - (2\pi)^{1/2} \right| \xrightarrow[N\to\infty]{} 0.$$

This implies

$$\sup_{n\ge N} \sup_{t\in I_N^\ell} \left| \frac{J^1(n,t)}{J^2(n,t)} - 1 \right| \xrightarrow[N\to\infty]{} 0.$$

Part II. Now we consider the first fraction in (12). By Lemma 1, we know that θ_n and θ_n' converge uniformly to $\hat{\theta}$ on $[0,1]$ in $\mathcal{O}(1/n)$. Then for the ratio $\frac{1-e^{-\theta_n}}{1-e^{-\theta_n'}}$ we can write

$$\left| \frac{1-e^{-\theta_n}}{1-e^{-\theta_n'}} - 1 \right| = \frac{\left| e^{-\theta_n'} - e^{-\theta_n} \right|}{\left| 1 - e^{-\theta_n'} \right|},$$

and the numerator of the last term is in $\mathcal{O}(1/n)$ while the denominator is minimized on I_N^l at

$\underline{\theta}_N$ and is therefore in $\mathcal{O}(1/N^{\alpha-1/2})$ so that the term goes to 0, implying that

$$\sup_{n\geq N}\ \sup_{t\in I_N^\ell}\left|\frac{1-e^{-\theta_n(t)}}{1-e^{-\theta_n'(t)}}-1\right|\xrightarrow[N\to\infty]{}0.$$

Consider now the ratio of the standard deviations

$$\frac{\sigma_n(\theta_n)}{\sigma_n(\theta_n')}=\left(e^{\theta_n-\theta_n'}\frac{\frac{(V_n-1/n)\overline{p}(1-\overline{p})}{\left(\overline{p}e^{\theta_n}+1-\overline{p}\right)^2}+\frac{(1-V_n)\underline{p}(1-\underline{p})}{\left(\underline{p}e^{\theta_n}+1-\underline{p}\right)^2}}{\frac{(V_n-1/n)\overline{p}(1-\overline{p})}{\left(\overline{p}e^{\theta_n'}+1-\overline{p}\right)^2}+\frac{(1-V_n)\underline{p}(1-\underline{p})}{\left(\underline{p}e^{\theta_n'}+1-\underline{p}\right)^2}}\right)^{1/2}.$$

It is clear that $e^{\theta_n-\theta_n'}$ converges to 1 uniformly on $[0,1]$, as for the second fraction, it is easy to show that the numerator and the denominator both converge uniformly on $[0,1]$ to

$$\frac{V\overline{p}(1-\overline{p})}{\left(\overline{p}e^{\hat{\theta}}+1-\overline{p}\right)^2}+\frac{(1-V)\,\underline{p}(1-\underline{p})}{\left(\underline{p}e^{\hat{\theta}}+1-\underline{p}\right)^2}>0,$$

implying that the fraction converges to 1 uniformly on $[0,1]$ as well as the ratio of the standard deviations.

Now, note that by definition of θ_n and θ_n', we have $K_n'(\theta_n)-K'(\theta_n')=1$. Since K' is continuously differentiable, there exists some $\tilde{\theta}_n$ between θ_n and θ_n' such that $K_n'(\theta_n)-K'(\theta_n')=(\theta_n-\theta_n')K''(\tilde{\theta}_n)$. Since by definition $K_n''(\theta)=\sigma_n^2(\theta)$, we can write

$$\theta_n-\theta_n'=\frac{1}{\sigma_n^2(\tilde{\theta}_n)}.$$

And since $\tilde{\theta}_n$ is between θ_n and θ_n', it converges uniformly to $\hat{\theta}$. This implies that

$$\frac{\sigma_n^2(\tilde{\theta}_n)}{n^2}=e^{\tilde{\theta}_n}\left(\frac{(V_n-1/n)\,\overline{p}(1-\overline{p})}{\left(\overline{p}e^{\tilde{\theta}_n}+1-\overline{p}\right)^2}+\frac{(1-V_n)\,\underline{p}(1-\underline{p})}{\left(\underline{p}e^{\tilde{\theta}_n}+1-\underline{p}\right)^2}\right)$$

converges uniformly to the finite valued function of t

$$e^{\hat{\theta}}\left(\frac{V\overline{p}(1-\overline{p})}{\left(\overline{p}e^{\hat{\theta}}+1-\overline{p}\right)^2}+\frac{(1-V)\,\underline{p}(1-\underline{p})}{\left(\underline{p}e^{\hat{\theta}}+1-\underline{p}\right)^2}\right)>0.$$

Therefore we can write that

$$e^{n_v(\theta_n-\theta_n')}=\exp\left(\frac{1}{n}\cdot\frac{n^2}{\sigma_n^2(\tilde{\theta}_n)}v_n\right),$$

converges uniformly to 1 on $[0,1]$.

Now consider the ratio

$$\frac{\varphi_n(\theta_n')}{\varphi_n(\theta_n)}=\exp\left(K_n(\theta_n')-K_n(\theta_n)\right)=\exp\left(K_n'(\dot{\theta}_n)(\theta_n-\theta_n')\right),$$

where $\dot{\theta}_n$ is between θ_n and θ_n'. Since K' is increasing, the definitions of θ_n and θ_n' imply that $n_v-1\leq K_n'(\dot{\theta}_n)\leq n_v$ and therefore

$$\exp\bigl((n_v-1)(\theta_n-\theta_n')\bigr)\leq\frac{\varphi_n(\theta_n')}{\varphi_n(\theta_n)}\leq\exp\bigl(n_v(\theta_n-\theta_n')\bigr).$$

We have already argued that the upper bound converges uniformly to 1, and the same argument obviously extends to the lower bound, hence the ratio itself converges to 1 uniformly on $[0,1]$.

To sum up, Part I and II show together that

$$\sup_{n\geq N}\sup_{t\in I_N^\ell}\left|R_n(t)-1\right|\xrightarrow[N\to\infty]{}0.$$

Part III. We want to show the same on I_N^h. For that, we use (8) in Lemma 2. It implies that for every $t\in I_N^h$, and n sufficiently large

$$\frac{1}{n}\log\left(1-\Pr\bigl(S_n(t)\geq n_v-1\bigr)\right)\leq-\frac{1}{2}\left(v|\hat{\theta}(t)|-\kappa(|\hat{\theta}(t)|)\right).$$

Then, by taking the minimum of $\left(v|\hat{\theta}(t)| - \kappa(|\hat{\theta}(t)|)\right)$ over $t \in I_N^h$ and remembering that $v > \kappa'(\theta)$ for $\theta > 0$, we have for every $t \in I_N^h$

$$\frac{1}{n} \log\left(1 - \Pr\big(S_n(t) \geq n_v - 1\big)\right) \leq -K_N,$$

with $K_n = \frac{1}{2}\left(v|N^{\alpha-1/2}| - \kappa(|N^{\alpha-1/2}|)\right) > 0$. In particular, K_N is in $\mathcal{O}(N^{\alpha-1/2})$. Noticing that $R_n \geq 1$, we can write that for n sufficiently large and for every $t \in I$

$$1 \geq R_n(t) \geq 1 - e^{-nK_N},$$

and this proves that

$$\sup_{n \geq N} \sup_{t \in I_N^h} \left|R_n(t) - 1\right| \xrightarrow[N \to \infty]{} 0.$$

Part IV. To prove the result on the intervals I_N^m, we will use the same type of approximations as in Part I, but this time we work with $S_n(t)$ in the original probability rather than with the tilted probability. The idea is basically to use a central limit theorem to approximate the distribution of the standardized[19] $Z_n(t) \equiv \frac{S_n(t) - nm_n(t)}{s_n(t)}$, where $s_n(t) = \sqrt{VarS_n(t)}$ around 0 by a normal distribution. However we need the approximation to work uniformly for all t in a shrinking neighborhood of $\tilde{t}$, making the direct application of any of the usual central limit theorems useless for our purpose.

Let $\gamma(u) \equiv E\ e^{iuS_n}$ be the characteristic function of S_n. By Lemma 4, we can write for any $k \in \{0, \ldots, n\}$

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[19] These notations were already introduced in the paragraph preceding Lemma 9.

$$\begin{aligned}\Pr\left(S_n \geq \alpha_n\right) &= \sum_{z=0}^{n-\alpha_n} (2\pi)^{-1} \int_{-\pi}^{\pi} \exp\left(-iu\alpha_n\right) e^{-iuz} \gamma(u) du \\ &= \sum_{z=0}^{n-\alpha_n} (2\pi s_n)^{-1} \int_{-\pi s_n}^{\pi s_n} \exp\left(-\frac{is(\alpha_n - nm_n)}{s_n}\right) e^{-iz\frac{s}{s_n}} \gamma\left(\frac{s}{s_n}\right) e^{-ism_n/s_n} ds \\ &= \sum_{z=0}^{n-\alpha_n} (2\pi s_n)^{-1} \int_{-\pi s_n}^{\pi s_n} \exp\left(-\frac{is(\alpha_n + z - nm_n)}{s_n}\right) \tilde{\gamma}\left(\frac{s}{s_n}\right) ds, \end{aligned} \tag{24}$$

where $\tilde{\gamma}\left(\frac{s}{s_n}\right) \equiv \gamma\left(\frac{s}{s_n}\right) e^{-\frac{isnm_n}{s_n}}$ is the characteristic function of Z_n.

We show that the each of these integrals for $\alpha_n \in \{n_v, n_v - 1\}$ (respectively at the numerator and the denominator) satisfies

$$\sup_{n \geq N} \sup_{t \in I_N^m} \left|\Pr\left(S_n \geq \alpha_n\right) - 1/2\right| \xrightarrow[N \to \infty]{} 0,$$

thus implying that

$$c \sup_{n \geq N} \sup_{t \in I_N^m} \left|R_n(t) - 1\right| \xrightarrow[N \to \infty]{} 0.$$

Before starting, note that since $m_\infty(\tilde{t}) = v$, we can write

$$\begin{aligned}\left|\frac{n_v - nm_n(t)}{s_n(t)}\right| &= \left|n\frac{v_n - m_n(t)}{s_n(t)}\right| \\ &\leq c_s^{-1} n^{1/2}\Big(\left|v_n - v\right| + \left|m_\infty(\tilde{t}) - m_\infty(t)\right| + \left|m_n(t) - m_\infty(t)\right|\Big).\end{aligned}$$

The first term on the right-hand side is bounded upward by $1/n$, the last term is equal to $\left|(V_n - 1/n - V)\overline{p}(t) + (V - V_n)\underline{p}(t)\right| \leq 3/n$ for every $t \in [0,1]$. The second term is bounded upward by

$$V\overline{p}(t)\left|\frac{\overline{p}(t)e^{\hat{\theta}(t)}}{\overline{p}(t) + (1 - \overline{p}(t))e^{\hat{\theta}(t)}} - 1\right| + (1 - V)\underline{p}(t\left|\frac{\underline{p}(t)e^{\hat{\theta}(t)}}{\underline{p}(t) + (1 - \underline{p}(t))e^{\hat{\theta}(t)}} - 1\right|,$$

which in turn can be bounded upward by

$$(V\overline{p}(t) + (1-V)\underline{p}(t))\left|e^{\hat{\theta}(t)} - 1\right| \leq \left|e^{\hat{\theta}(t)} - 1\right|.$$

Because there exists a neighborhood $\mathcal{V}$ of 0 such that $\left|e^{\theta} - 1\right| \leq 2\theta$ for $\theta \in \mathcal{V}$, it must be true that for n sufficiently large, we can write

$$\sup_{t \in I_N^m} \left|e^{\hat{\theta}(t)} - 1\right| \leq 2N^{-(\alpha+1/2)}.$$

Noticing that a similar reasoning can be made by replacing n_v by $n_v - 1/n$, these calculations lead to the following result.

Remark 1. *For $\alpha_n \in \{n_v, n_v - 1/n\}$, we have*

$$\sup_{n \geq N} \sup_{t \in I_N^m} \left| \frac{\alpha_n - nm_n(t)}{s_n(t)} \right| \xrightarrow[N\to\infty]{} 0.$$

Now going back to the main argument, we decompose (24) as follows

$$\begin{aligned}
&(2\pi s_n)^{-1} \sum_{z=0}^{n-\alpha_n} \int_{-\infty}^{+\infty} \exp\left(-\frac{is(\alpha_n + z - nm_n)}{s_n}\right) e^{-\frac{s^2}{2}} ds \\
&\quad -(2\pi s_n)^{-1} \sum_{z=0}^{n-\alpha_n} \int\limits_{|s|>\pi s_n} \exp\left(-\frac{is(\alpha_n + z - nm_n)}{s_n}\right) e^{-\frac{s^2}{2}} ds \\
&\quad\quad +(2\pi s_n)^{-1} \sum_{z=0}^{n-\alpha_n} \int_{-\pi s_n}^{\pi s_n} \exp\left(-\frac{is(\alpha_n + z - nm_n)}{s_n}\right) \left(\tilde{\gamma}\left(\frac{s}{s_n}\right) - e^{\frac{s^2}{2}}\right) ds. \qquad (25)
\end{aligned}$$

We proceed term by term.

(i) The integral in the first term is well defined and it is the inversion formula for the characteristic function $e^{-\frac{s^2}{2}}$ of the standard normal distribution, hence by Lemma 6 it is equal to $2\pi\phi\left(\frac{\alpha_n+z-nm_n}{s_n}\right)$, where $\phi(x) = (2\pi)^{-1/2}e^{-\frac{x^2}{2}}$ is the pdf of the standard normal distribution. Therefore the first term is equal to

$$\sum_{z=0}^{n-\alpha_n} s_n^{-1}\phi\left(\frac{\alpha_n+z-nm_n}{s_n}\right). \tag{26}$$

A Taylor expansion of the cdf of the standard normal distribution Φ yields for every z

$$s_n^{-1}\phi\left(\frac{\alpha_n+z-nm_n}{s_n}\right) = \\ \Phi\left(\frac{\alpha_n-nm_n+z+1}{s_n}\right)-\Phi\left(\frac{\alpha_n-nm_n+z}{s_n}\right)-s_n^{-2}\phi'\left(\frac{\alpha_n-nm_n+\zeta(z)}{s_n}\right),$$

where $\zeta(z)\in[0,1]$. Hence (26) is equal to

$$\Phi\left(\frac{n(1-m_n)}{s_n}\right)-\Phi\left(\frac{\alpha_n-nm_n}{s_n}\right)-s_n^{-2}\sum_{z=0}^{n-\alpha_n}\phi'\left(\frac{\alpha_n-nm_n}{s_n}+\frac{\zeta(z)}{s_n}\right).$$

Because $m_n(t)$ converges to $V\overline{p}(t)+(1-V)\underline{p}(t)$ which is uniformly (in t) bounded upward by $V+(1-V)p^e<1$, and because $s_n(t)$ is uniformly bounded upward by $C_s\sqrt{n}$, the first term of this equation converges to 1 uniformly for $t\in[0,1]$. By Remark 1, we also have

$$\sup_{n\geq N}\sup_{t\in I_N^m}\left|\Phi\left(\frac{\alpha_n-nm_n}{s_n}\right)-\Phi(0)\right|\xrightarrow[N\to\infty]{}0.$$

Finally the last term is bounded upward in absolute value by

$$(2\pi)^{-1/2}c_s^{-2}n^{-1}(n-\alpha_n)\left(\left|\frac{\alpha_n-nm_n}{s_n}\right|+\left|\frac{1}{s_n}\right|\right),$$

and by Remark 1 and Lemma 9,

$$\sup_{n\geq N}\sup_{t\in I_N^m}(2\pi)^{-1/2}c_s^{-2}n^{-1}(n-\alpha_n)\left(\left|\frac{\alpha_n-nm_n}{s_n}\right|+\left|\frac{1}{s_n}\right|\right)\xrightarrow[N\to\infty]{}0.$$

(ii) For n sufficiently large, the absolute value of the integral in the second term of (25) is bounded upward by

$$2\int_{s>\pi s_n} e^{-\frac{s}{2}} \leq 4e^{-\frac{c_s\sqrt{n}}{2}}.$$

Therefore, using Lemma 9, the absolute value of the second term of (25) is bounded upward by $\frac{2n^{1/2}}{\pi c_s}e^{-\frac{c_s\sqrt{n}}{2}}$, which converges to 0 uniformly for $t \in [0,1]$.

(iii) For the last term of (25), we start by switching the integral and the sum signs, which can be done since the sum is finite and the integral well defined. After scaling the integrand by π, we have

$$(2s_n)^{-1}\int\limits_{-s_n}^{s_n} \exp\left(-\frac{is\pi(\alpha_n - nm_n)}{s_n}\right) H\left(\frac{\pi s}{s_n}, \alpha_n\right)\left(\tilde{\gamma}\left(\frac{\pi s}{s_n}\right) - e^{-\frac{(\pi s)^2}{2}}\right) ds,$$

where $H(u,\alpha_n) \equiv \sum_{z=0}^{n-\alpha_n} e^{-iuz}$. For $s \neq 0$ we have

$$H\left(\frac{\pi s}{s_n}, \alpha_n\right) = \begin{cases} 1 & \text{if } n-\alpha_n \text{ is even} \\ 1+e^{-\frac{i\pi s}{s_n}} & \text{if } n-\alpha_n \text{ is odd} \end{cases},$$

and $H(0,n_v) = 1+n-\alpha_n$. That is, the absolute value of $H\left(\frac{\pi s}{s_n}, \alpha_n\right)$ is bounded upward by 2 on every compact set that excludes 0, and by n on any compact neighborhood of 0. Therefore, the last term of (25) is bounded upward in absolute value by

$$(\pi s_n)^{-1}\int\limits_{-\pi s_n}^{\pi s_n} \left|\tilde{\gamma}\left(\frac{s}{s_n}\right) - e^{-\frac{s^2}{2}}\right| ds + (2\pi s_n)^{-1}\int\limits_{-1/n^2}^{1/n^2} n\left(\left|\tilde{\gamma}\left(\frac{s}{s_n}\right)\right| + \left|e^{-\frac{s^2}{2}}\right|\right) ds. \qquad (27)$$

Both $\left|\tilde{\gamma}\left(\frac{s}{s_n}\right)\right|$ and $\left|e^{-\frac{s^2}{2}}\right|$ are bounded upward by 1, and with Lemma 9, we can conclude that the second term in (27) is bounded upward by $2(\pi c_s)^{-1}n^{-3/2}$ which goes to 0 independently of t as n goes to infinity.

For the first term of (27), we use Lemma 10 which implies that for $|u| \leq C'\sqrt{n}$

$$\left|\tilde{\gamma}\left(\frac{u}{s_n(t)}\right) - e^{-\frac{u^2}{2}}\right| \leq \frac{\overline{k}}{6c_s^3 n^{1/2}} |u|^3 \exp\left(-\frac{u^2}{4}\right).$$

Hence the absolute value of the first term in (27) is bounded upward by

$$\underbrace{(\pi s_n)^{-1} \int\limits_{|u|\leq C'\sqrt{n}} \frac{\overline{k}}{6c_s^3 n^{1/2}} |u|^3 \exp\left(-\frac{u^2}{4}\right) du}_{T_1}$$

$$+ \underbrace{(\pi s_n)^{-1} \int\limits_{|u|>C'\sqrt{n}} e^{-\frac{u^2}{2}} du}_{T_2}$$

$$+ \underbrace{(\pi s_n)^{-1} \int\limits_{C'\sqrt{n}\leq|u|\leq\pi s_n} \left|\tilde{\gamma}\left(\frac{u}{s_n}\right)\right| du}_{T_3}.$$

We have

$$T_1 \leq 2(\pi s_n)^{-1} \frac{\overline{k}}{6c_s^3 n^{1/2}} \int_0^\infty s^3 e^{-\frac{s^2}{4}} ds \leq \frac{4\overline{k}}{3\pi c_s^4 n},$$

where the last term is obtained by Lemma 9 and integration by part. Hence T_1 goes to 0 independently of t as n goes to infinity.

T_2 is bounded upward by $2(\pi c_s\sqrt{n})^{-1} e^{-\frac{C'\sqrt{n}}{2}}$ which goes to 0 independently of t as n goes to infinity.

Finally, for T_3, we start by noting that $C' \leq \pi c_s \leq \pi s_n$ by construction (see Lemma 10). By definition of $\tilde{\gamma}$, we have

$$\left|\tilde{\gamma}\left(\frac{s}{s_n}\right)\right| = \left|\overline{p}e^{i\frac{s}{s_n}} + 1 - \overline{p}\right|^{n_V - 1} \left|\underline{p}e^{i\frac{s}{s_n}} + 1 - \underline{p}\right|^{n_V^c},$$

and at this point we use the fact that for any real number $z \in [\pi, \pi]$ and any real numbers

a and b,

$$\left|ae^{iz}+b\right| \leq |a+b|,$$

with a strict inequality if $z \neq 0$. This inequality and the fact that the function $|\overline{p}e^{iz}+1-\overline{p}|$ is continuous in t imply together that we can define the following quantity

$$\overline{\delta} \equiv \max_{C/c_s \leq z \leq \pi} \max_{t\in[0,1]} \left|\overline{p}e^{iz}+1-\overline{p}\right| < 1.$$

And similarly

$$\underline{\delta} \equiv \max_{C/c_s \leq z \leq \pi} \max_{t\in[0,1]} \left|\underline{p}e^{iz}+1-\underline{p}\right| < 1.$$

Then the term under the integral in t_3 is bounded upward by $\overline{\delta}^{n_V-1}\underline{\delta}^{n_V^c} \leq \delta^{n-1}$ where $\delta \equiv \max\{\underline{\delta},\overline{\delta}\} < 1$. Finally, this shows that t_3 is bounded upward by $(\pi c_s)^{-1}\delta^{n-1}(\pi C_s - C')$ which goes to 0 independently of t as n goes to infinity.

All this shows that the last term in (25) goes to 0 uniformly on $[0,1]$.

To conclude, since $1-\Phi(0)=1/2$ we have proved that for $\alpha_n \in \{n_v, n_v-1\}$,

$$\sup_{n\geq N} \sup_{t\in I_N^m} \left|\Pr\left(S_n \geq \alpha_n\right) - 1/2\right| \xrightarrow[N\to\infty]{} 0,$$

thus implying that

$$\sup_{n\geq N} \sup_{t\in I_N^m} \left|R_n(t)-1\right| \xrightarrow[N\to\infty]{} 0.$$

To finish the proof, we need to show that

$$\sup_{n\geq N} \sup_{t\in I_N^m} \left|R_n(t)-\rho(t)\right| \xrightarrow[N\to\infty]{} 0.$$

For that, we just write that

$$\sup_{n\geq N} \sup_{t\in I_N^m} \left|R_n(t)-\rho(t)\right| \leq \sup_{n\geq N} \sup_{t\in I_N^m} \left|R_n(t)-1\right| + \sup_{n\geq N} \sup_{t\in I_N^m} \left|\rho(t)-1\right|.$$

The first term converges to 0 as we just proved. We know that ρ is continuous, increasing and bounded upward by 1. Hence the second term is bounded upward by $1-\rho(\min\{t \in I_N^m\})$,which by definition converges to $1-\rho(\tilde{t})=0$.

Part V. Fix some $\varepsilon > 0$. We just proved that there exists N_ℓ, N_m and N_h such that for every $k \in \{\ell, m, h\}$, and for every $N \geq N_k$

$$\sup_{n \geq N} \sup_{t \in I_N^k} \left|R_n(t) - 1\right| < \varepsilon.$$

Fix N_m and choose N_ℓ' and N_h' such that $\Big(\max\{N_\ell', N_h'\}\Big)^{\alpha-1/2} < N_m^{-\alpha-1/2}$, so that

$$I_{N_\ell'}^\ell \cup I_{N_m}^m \cup I_{N_h'}^h = [0,1].$$

Then it is clear that for every $n > \max\{N_m, N_h', N_\ell'\}$ we have

$$\sup_{t \in [0,1]} \left|R_n(t) - \rho(t)\right| < \varepsilon,$$

which proves that $R_n(t)$ converges uniformly on $[0,1]$.

Part VI. The continuity of β at every $t \neq \tilde{t}$ can be deduced from the continuity of $\hat{\theta}(t)$ which is implied by the continuity of $\overline{p}$ and $\underline{p}$ and the continuity of $\hat{\theta}$ in $\overline{p}$ and $\underline{p}$. For the continuity at $\tilde{t}$, it is implied by the fact that the solution of (3) is $e^{\hat{\theta}} = 1$ if and only if $V\overline{p} + (1-V)\underline{p} = v$, that is if and only if $t = \tilde{t}$. Therefore

$$\lim_{\substack{t \to \tilde{t} \\ t < \tilde{t}}} e^{\hat{\theta}(t)} = 1,$$

which implies the continuity of β at $\tilde{t}$.

The sense of variation of β with respect to t, v and V can be deduced from that of $\hat{\theta}$ which was analyzed in Lemma 1. □

Proof of Proposition 7 Uniqueness of the Asymptotic Equilibrium. If the function $\beta(t) - t$ is crosses 0 only once, the equilibrium is unique. Some algebra shows that it has the same sign as the function $g(t) \equiv \frac{1-t}{t\rho(t)} - \frac{u^+}{u^-}$, hence one of them crosses 0 only once if and only if it is true of the other as well. A sufficient condition for this is if g is strictly decreasing. This is true when, for every t

$$\frac{\rho'(t)}{\rho(t)} + \frac{1}{t(1-t)} > 0.$$

By differentiating (3), we obtain the following expression for the first term

$$\frac{\rho'}{\rho} = -\frac{\frac{V}{(1+\overline{r}\rho)^2}\overline{r}' + \frac{1-V}{(1+\underline{r}'\rho)^2}\underline{r}'}{\frac{V}{(1+\overline{r}\rho)^2}\overline{r} + \frac{1-V}{(1+\underline{r}'\rho)^2}\underline{r}}.$$

Because the denominator of this function is positive, g' has the same sign as

$$\frac{V}{(1+\overline{r}\rho)^2}\left(\frac{\overline{r}}{t(1-t)} - \overline{r}'\right) + \frac{V}{(1+\underline{r}\rho)^2}\left(\frac{\underline{r}}{t(1-t)} - \underline{r}'\right).$$

A quick calculation shows that (13) implies that each of the terms in parenthesis is strictly negative and is therefore a sufficient condition for uniqueness. □

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