TURNOUT AND POWER SHARING

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Abstract. Differences in electoral rules and/or legislative, executive or legal institutions across countries induce different mappings from election outcomes to distributions of power. We explore how these different mappings affect voters' participation in a democracy. Assuming heterogeneity in the cost of voting, the effect of such institutional differences on turnout depends on the distribution of voters' preferences for the parties: when the two parties have similar support, turnout is higher in a winner-take-all system than in a power sharing system; the result is reversed when one side has a larger base. Moreover, the winner-take-all system has higher welfare if and only if the support is uneven. We compare the "size effect" and the "underdog compensation effect" under different systems. All systems induce an underdog compensation which is partial. Namely, unlike other costly voting models, the side with the larger support almost surely wins the majority of the votes. The results obtained in the rational voter model, characterized by the voter free-riding problem, continue to hold in other models of turnout such as ethical voter models and voter mobilization models.

Keywords: Proportional Influence, Winner-Take-All, Underdog Compensation.
JEL classification numbers: D72

Date: 17th January 2010. Columbia University.

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We thank Tilman Borgers, Alessandra Casella, Faruk Gul, John Huber, Navin Kartik, John Morgan, Roger Myerson, Tom Palfrey and Mattias Polborn for important discussions and suggestions. We also thank all the seminar participants at Rochester (Wallis Conference), Caltech, Zurich (ETH), Columbia, Princeton, PUC-Rio, Toronto (CETC meetings) and Boston University (ES Summer Meetings) for useful feedback. The usual disclaimer applies.

1. Introduction

Voters' participation is an essential component of democracy. Yet the positive analysis of turnout is still far from established and many questions remain. Even without taking any normative stand on whether increasing voters' participation is good or bad, we believe that the positive comparative analysis and understanding of what causes variation in turnout across systems has intrinsic value.[1] Is it possible to characterize the influence of institutional systems on turnout, making sure, in addition, that such comparative results are robust to different modeling assumptions? In particular, does turnout depend in any identifiable way on the type of democratic regime, the electoral rules, legislative organization rules, or the degree of separation of powers?

Our idea to make headway on this topic is as follows: different institutional systems impact the mapping from election outcomes (henceforth *vote shares*) to the relative weight of different parties in decision making (henceforth *power shares*).[2] Hence we try to assess in a general way the role of institutions on electoral participation by characterizing how that vote-shares-to-power-shares mapping affects voters' incentives to vote and parties' campaign efforts. The role of individual institutions in determining electoral participation can then be separately evaluated by looking at their impact on that mapping. In sum, the degree of influence on policy for given electoral outcomes is the key exogenous variable for our analysis: we will often refer to this reduced form mapping simply as the "institutional system" ranging from a winner-take-all system to a fully proportional power sharing system.

The results below also depend on another key parameter, namely the expected "winning margin" or "closeness" of the election. Before explaining how the vote-share to power-share

[1]Even though our analysis is mostly positive, there will be opportunities to formulate welfare comparative evaluations as well.

[2]The relative power of the majority party for a given election outcome varies with the degree of separation of powers, the organization of chambers, the assignment of committee chairmanships and institutional rules on agenda setting, allocation of veto powers, and obviously electoral rules. See Lijphart (1999) and Powell (2000) for a comprehensive analysis of the impact of political institutions on what they call degree of proportionality of influence, which is basically our vote-shares to power-shares mapping. Electoral rules determine the mapping from vote shares to seat shares in a legislature, whereas the other institutions determine the subsequent mapping from seat shares to power shares across parties.

mapping and the expected closeness of an election jointly determine turnout, we need to highlight the main modeling choices that we make in the paper.

Since there is no established or canonical model of electoral turnout, we will analyze the variation in electoral turnout in more than one model. The common assumptions to all of the models that we discuss are that (1) the distribution of citizens' preferences over the set of alternatives (candidates or parties or coalitions of parties) is common knowledge; (2) the only relevant decision by each citizen is whether to go to vote or not; (3) each voter is described in a two-dimensional type space, i.e. her preferred party and her cost of voting: the cost of voting for each citizen is drawn from a general continuous distribution with non negative support.

The first model we consider is the standard rational voter model (see e.g. Palfrey and Rosenthal (1985)) under population uncertainty[3] and extending the analysis to the proportional influence or proportional power sharing system, i.e. the system in which power shares are expected to be proportional to vote shares. It is well known that the "point estimates" of turnout based on strategic voting models are too low with respect to observed turnout in large elections, and hence some scholars have turned to mobilization models and ethical voters' models.[4] In a recent paper, Coate, Conlin and Moro (2008) question the ability of the strategic voter model to serve as useful benchmark for elections of any size, not only for very large elections. They show that empirically there is no support for what we call the *full underdog compensation*,[5] i.e., for the theoretical prediction that all electoral races are expected to be close regardless of the distribution of citizens' preferences. Indeed, well known rational voter models predict that in equilibrium the supporters of the underdog party (the party with a lower number of supporters) should turn out more than the supporters of the leader party, making the election always an expected tie, hence fully compensating the initial disadvantage.[6] We believe that the strategic voting model with costly voting should still be considered a key benchmark when comparing institutions. Indeed, we

[3]Viewing the size of the electorate as a random variable (see Myerson 1998 and 2000) has the advantage of simplifying the computations without altering the incentives driving the results. Krishna and Morgan (2009) recently obtained important results in a model similar to ours, but with common values, in which population uncertainty is key.

[4]See Shachar and Nalebuff (1999), Coate and Conlin (2004) and Feddersen and Sandroni (2006).

[5]We will sometimes refer to the "underdog compensation" also as "underdog effect" as Levine and Palfrey (2007).

[6]See for instance Borgers (2004), Krasa and Polborn (2009).

show that the counter-intuitive and empirically falsified full underdog compensation prediction is due to some apparently innocuous assumptions on the distribution of costs and preferences. The property of full underdog compensation identified in earlier papers was due to the assumption that all voters have the same voting cost or have strictly positive voting costs.[7] We allow voting costs of citizens to differ and to take any value in the non negative real numbers. We show that the underdog party supporters turn out in greater numbers proportionally than the favorite party supporters, but this overrepresentation of the underdog party supporters does not fully compensate for their underrepresentation in the electorate and as a consequence, the party with higher ex-ante support is always expected to win, albeit by a smaller margin than the ex-ante support advantage (e.g. the opinion polls) would predict. On the comparative side, we also show that the underdog compensation varies with the vote-share to power-share mapping: the underdog compensation is always larger in a proportional power sharing system. This comparison has welfare consequences: the winner-take-all system yields higher expected total utility due to the lower underdog compensation.[8]

Comparing turnout across systems boils down to comparing the individual benefits of voting across systems. In a proportional power sharing system the expected marginal benefit of a single vote is proportional to the marginal change in the vote share determined by that vote. Whereas in a winner take all system the marginal benefit of a vote is proportional to the probability of that vote being pivotal. Both marginal benefits obviously decrease as the number of voters increases. In large elections the comparison of turnout across systems hence depends on the asymptotic speed with which a larger population reduces the individual benefit of voting, i.e. the magnitude of the "size effect."[9] Quantitatively we show that in a proportional system the benefit of voting decreases asymptotically as $1/N$ when N, the expected size of the electorate, increases; whereas in a winner-take-all system such asymptotic speed is slower when the election is expected to be a tie and much faster otherwise. This fact determines the main conclusion,

[7]This also implies that the expected absolute number of citizens voting remains finite even as the population grows to infinity.

[8]This welfare corollary cannot be established in previous voter models where the underdog compensation effect is full so all elections are tied in expectation

[9]As in Levine and Palfrey (2007) the "size effect" measures how the benefit of voting and hence turnout decrease with the size of the population.

namely that turnout is higher in a proportional system when the election has a clear favorite party while a winner-take-all system induces higher turnout otherwise.

We conduct the bulk of the analysis for the case of two parties, but we show the robustness of all comparisons to changes in the number of parties: in a proportional power sharing system the order of magnitude of the size effect does not depend on the distribution of ex ante support of parties, nor does it depend on the number of parties present in the election. Hence the comparison with the winner take all system is also unaffected by the number of parties. We also show that in a proportional power sharing system turnout increases as the number of parties increases.

Even though the comparative analysis using the rational voter model is well justified given the properties of our model, and even though the number of results we obtain with such a model is significant, our main goal remains to *convince* all readers about the validity of our comparative institutional results, without necessarily taking a methodological strong stand in favor of the rational voter model. Thus, we study the same questions using other well known approaches. In particular, the voter mobilization approach to turnout following the work by Shachar and Nalebuff (1999)[10] and the ethical voter models proposed by Coate and Conlin (2004) and Feddersen and Sandroni (2006) are important alternatives to the strategic voting model because they yield much higher turnout point estimates and have conceptual appeal, as they propose mechanisms through which voters might cooperate and coordinate their actions. Regardless of how the voter free riding problem is overcome, it is important to know whether the comparison across electoral systems depends on it, namely on whether the positive externality of voting is internalized or not.

We show that the alternative models we consider (in which voters cooperate) have the same qualitative properties of the rational model (in which voters do not cooperate).[11] The robustness of this comparative finding depends crucially on a feature common to all models: full underdog compensation does not occur, so an ex-ante uneven election always remains ex-post uneven. Our results suggest the general point that any model of large elections featuring partial (or zero) underdog compensation effect, yields the robust prediction that winner take all system

[10]See also Morton (1987 & 1991) and Uhlaner (1989).

[11]An additional benefit of studying the same questions with these models of mobilization and ethical voting is that they are computationally simpler, and hence we can extend the analysis to *any* power sharing system, whereas in the benchmark strategic model only the two extremes can be compared.

induces higher turnout when the citizens' support for the two main parties or party coalitions is very close, while more proportional systems induce higher turnout when one party has a larger ex-ante support. The intuition is that in the winner-take-all system when preferences are not evenly split the non full underdog compensation preserves the ex-ante leading party as the ex-post leading party in equilibrium, hence preserving a high expected winning margin which discourages participation. In a more proportional power sharing system, a less competitive election (i.e. a higher expected winning margin) does not affect the incentives to vote as much.

The paper is organized as follows. Section 2 contains the complete analysis of a rational voter model of turnout, comparing the properties of proportional power sharing system and winner-take-all system. Section 3 contains the analysis of the ethical voter model and the mobilization model, where even intermediate proportionality levels can be considered, and where we confirm the robustness of our main comparative results across modeling choices. Section 4 will offer some concluding remarks and describe potential paths of future research. All proofs are in the Appendix.

2. Rational Voter Turnout

Consider two parties, A and B, competing for power. Citizens have strict political preferences for one or the other, chosen exogenously by Nature. We denote by $q \in (0,1)$ the preference split, i.e. the chance that any citizen is assigned (by Nature) a preference for party A (thus $1-q$ is the expected fraction of citizens that prefer party B). The indirect utility for a citizen of preference type i, $i = A, B$, is increasing in the share of power that party i has. For normalization purposes, we let the utility from "full power to party i" equal 1 for type i citizens and 0 for the remaining citizens.[12]

Beside partisan preferences, the second dimension along which citizen differ from one another is their cost of voting: each citizen's cost of voting c is drawn from a distribution with infinitely differentiable pdf $f(c)$ over the support $c \in [0, \overline{c}]$, with $\overline{c} > 0$ (we denote the cdf as $F(c)$). The cost of voting and the partisan preferences are two independent dimensions that determine the type of a voter.

[12] This normalization will allow us to match party utility and voters's utilities in a simple way under all the institutional systems that will be considered.

For any vote share V obtained by party A, an institutional system γ determines power shares $P_\gamma^A(V) \in [0,1]$ and $P_\gamma^B(V) = 1 - P_\gamma^A(V)$. Given the above normalization, these are the reduced form "benefit" components of parties' (respectively, voters') utility functions that will determine the incentives to campaign (respectively, vote) in an institutional system. In this section we study the base model in which parties do not campaign nor attempt to coordinate or mobilize voters, hence turnout depends exclusively on voters' comparison between the policy benefits of voting for the preferred candidate and the opportunity costs of voting.

In terms of the size of the electorate, we find it convenient to assume that the population is finite but uncertain. There are n citizens who are able to vote at any given time, but such a number is uncertain and distributed as a Poisson distribution with mean N:

$$n \sim \frac{e^{-N}(N)^n}{n!}$$

Most statements in the paper are made for a large enough population, namely they are true for every N above a given $\overline{N}$.

Citizens have to choose to vote for party A, party B, or abstain. If a share α of A types vote for A and a share β of B types vote for B, the expected turnout T is

$$T = q\alpha + (1-q)\beta$$

Without loss of generality we assume that $q \leq 1/2$, so that the A party is the underdog party (with smaller ex-ante support) and the B party is the leader party (with larger ex-ante support).

We look for a Bayesian equilibrium in which all voters of type A with a cost below a threshold c_α vote for type A and voters of type B with a cost below c_β vote for B. So on aggregate, type A citizens vote for A with chance $\alpha = F(c_\alpha)$ and type B citizens vote for B with chance $\beta = F(c_\beta)$.

In any equilibrium strategy profile (α, β), the expected marginal benefit of voting, B_γ, must be equal to the cutoff cost of voting (indifference condition for the citizen with the highest cost among the equilibrium voters). Hence the equilibrium conditions can be written as

$$B_\gamma^A(\alpha,\beta) = F^{-1}(\alpha), \qquad B_\gamma^B(\alpha,\beta) = F^{-1}(\beta)$$

We compare two systems: a winner-take-all system ($\gamma = M$) and a proportional power sharing system ($\gamma = P$).[13]

2.1. **Winner take all system ($\gamma = M$).** In the M system the expected marginal benefit of voting B_M^A is the chance of being pivotal for a type A citizen, namely

$$B_M^A = \sum_{k=0}^{\infty} \left(\frac{e^{-qN\alpha} (Nq\alpha)^k}{k!} \right) \left(\frac{e^{-(1-q)N\beta} ((1-q) N\beta)^k}{k!} \right) \frac{1}{2} \left(1 + \frac{(1-q) N\beta}{k+1} \right)$$

namely the chance that an A citizen by voting either makes a tie and wins the coin toss or breaks a tie where it would have lost the coin toss. Likewise, for the type B citizens we have

$$B_M^B = \sum_{k=0}^{\infty} \left(\frac{e^{-qN\alpha} (Nq\alpha)^k}{k!} \right) \left(\frac{e^{-(1-q)N\beta} ((1-q) N\beta)^k}{k!} \right) \frac{1}{2} \left(1 + \frac{qN\alpha}{k+1} \right)$$

Equating the benefit side to the cost side we obtain a system of two equations in (α, β) (the M system henceforth). We now show that asymptotically turnout for each party is zero as a percentage of the population, but is infinite in absolute numbers. Moreover, the ratio of turnouts for each party remains finite.

Lemma 1. *Any equilibrium solution (α_N, β_N) to the M system (if it exists) has the following three properties*

$$\lim_{N\to\infty} \alpha_N = \lim_{N\to\infty} \beta_N = 0, \qquad \lim_{N\to\infty} N\alpha_N = \lim_{N\to\infty} N\beta_N = \infty, \qquad \lim_{N\to\infty} \frac{\alpha_N}{\beta_N} \in (0, \infty)$$

The above lemma allows us to use some approximations to show existence and uniqueness of an equilibrium for N large and also the following characterization results.[14]

Proposition 2. *There exists an equilibrium (α, β) in the M system. For uniqueness it suffices that F is weakly concave. The equilibrium has the following properties:*

- Size effect:

$$\frac{dT_M}{dN} < 0$$

[13] Recall that the interpretation is not restricted to electoral rules, as explained in the introduction. Two countries with the same electoral rule can have very different mappings from electoral outcomes to power shares, and this is the summary or reduced form variable that we are interested in and that affects turnout.

[14] We thank John Morgan for pointing out the importance of proving this non trivial lemma for the approximation results and proofs that will follow.

- Partial underdog compensation effect:

$$q < 1/2 \quad \Longrightarrow \quad \alpha > \beta, \quad q\alpha < (1-q)\beta$$

The size effect shows how the benefit of voting declines for larger electorates, although we will show that the rate of decline depends crucially on whether the parties do or do not have the same support ex-ante. The partial underdog compensation shows that the party with less supporters has higher relative expected turnout but lower expected turnout overall. We discuss all these effects in the following section.

2.2. **Discussion of the M System.** The partial underdog compensation arises from the following simple equilibrium relationship between the turnout rates for the two parties (see Appendix)

$$q\alpha \left(F^{-1}(\alpha)\right)^2 = (1-q)\beta \left(F^{-1}(\beta)\right)^2 \tag{1}$$

Since for heterogeneous costs $F^{-1}(\alpha)$ is increasing, then $q < 1/2$ implies an underdog compensation (i.e. $\alpha > \beta$) that must be partial (i.e. $q\alpha < (1-q)\beta$). As a consequence, we have a balanced election with a 50% expectation of victory from each side *only* when $q = 1/2$. With homogeneous costs the result would be different: homogeneous costs mean that $F^{-1}(\alpha) = c = F^{-1}(\beta)$, which implies $q\alpha = (1-q)\beta$, i.e. full underdog compensation and a 50% chance of victory regardless of the ex-ante preference split q.

To understand why the heterogeneity of the cost distribution is so important, assume for instance that $q = 1/3$ so that the leader party has double the ex-ante support than the underdog party. To have an election with a 50-50 chance of victory (i.e. $q\alpha = (1-q)\beta$), the underdog party would have to turn out twice as much as the leader party. We claim that the latter cannot happen unless citizens have homogenous costs. Suppose not; then on the benefit side, in a strategy profile with an ex-ante even outcome, the gross benefit of voting is the same across all voters (as they all individually face the same even environment). On the cost side, since the underdog party has to turn out more, then we must have $\alpha = F(c_\alpha) > \beta = F(c_\beta)$. With heterogenous cost this means that the equilibrium cost thresholds would have to be different $c_\alpha > c_\beta$ which, in turn, implies that the cost thresholds cannot both be equal to the benefit. In other words, the underdog supporters cannot fully rebalance the election because turning out in a higher proportion means that types with a higher cost would have to turn out as well. To have an equilibrium with full underdog compensation (same benefit) we must have $c_\alpha = c_\beta$ (same cost), which happens when F is constant so costs are homogeneous.

Conversely, as the costs become equal, the equilibrium must exhibit full underdog compensation. The intuition is as follows. Suppose, to the contrary, that with homogenous costs a pure strategy equilibrium with partial underdog compensation existed so the ex-ante underdog is expected to lose the election. With such a strategy profile, a supporter of the underdog party who is abstaining, by deviating and going to vote would bring the election closer to a tie, hence he would have a higher benefit than the benefit of his fellow supporters of the underdog party that were voting according to that strategy profile, a contradiction.

Heterogenous costs and their implications are more appealing than homogenous costs not only from a theoretical point of view but also from a normative one and an empirical one: the underdog compensation being just partial guarantees that the party with more ex-ante support is the more likely winner of the election. On the normative side, having the election result be determined by a coin toss as in the homogenous cost full underdog compensation case is clearly unappealing from a welfare perspective. The fact that the 50-50 benchmark result is pervasive in the literature prompted the question of whether it is of any use to have people vote at all as the preferences of the electorate are not reflected in the outcome.[15] On the empirical side, Coate, Conlin and Moro (2008) show how the benchmark homogeneous cost model which predicts election ex-post closeness is at odds with the large winning margins observed in the data.

As for the turnout comparison across different power sharing systems, the distinction between homogenous and heterogeneous costs and hence between full and partial underdog compensation is also key. The different equilibria with different cost assumptions, namely a 50-50 outcome versus a non 50-50 outcome, imply very different overall turnout numbers in large elections. In fact, the benefit of voting and hence the turnout are proportional to

$$B_M \sim \frac{e^{-\left(\sqrt{q\alpha}-\sqrt{(1-q)\beta}\right)^2 N}}{\sqrt{N}}$$

[15]See e.g. Borgers (2004) and Krasa and Polborn (2008). A different line of work that tries to avoid the full compensation undesirable outcome assumes that the preference split q remains unknown to voters: if so, then the compensation effect which rebalances the election and lowers welfare cannot be triggered properly. Hence opinion polls, which reduce uncertainty about q, may be welfare reducing. See Goeree and Grosser (2007) and Taylor and Yilidirm (2009).

In the homogenous cost case, in which $q\alpha = (1-q)\beta$, this implies that turnout declines at the rate $N^{-1/2}$. In the heterogeneous cost case, where $q\alpha \neq (1-q)\beta$ unless $q = 1/2$, turnout declines at the exponential rate e^{-N} for $q \neq 1/2$ and declines at the rate $N^{-1/2}$ when $q = 1/2$.[16]

Even though the nature of our work is primarily positive, we want to conclude this discussion of the M system with a simple welfare corollary:

Corollary 3. *Asymptotically, for the population N going to infinity, neither subsidies nor penalties for voters can improve total expected utility in the M system.*

This could be easily shown by adapting the proof of proposition 5 in Krasa and Polborn (2009), since their model is similar to our model of the M system but with a positive voting cost lower bound $\underline{c} > 0$. They show that in the limit the optimal subsidy to voters converges to $\underline{c}$. Thus, when one considers the same model but with zero as lower bound $\underline{c} = 0$, the optimal subsidy in the limit must be zero.[17] Intuitively, on the one hand introducing a subsidy is unnecessary since asymptotically the party with larger ex-ante support always wins the election in any case. On the other hand, introducing a penalty for voting would bring us back the inefficient lower bound $\underline{c} > 0$ in the voting cost distribution.

2.3. **Proportional Power Sharing System ($\gamma = P$).** With proportional power sharing (P system) the share of power is proportional to the vote share obtained in the election. So if (a, b) are the absolute numbers of votes for each party, the power of parties A and B would be respectively $\left(\frac{a}{a+b}, \frac{b}{a+b}\right)$.[18]

[16]Chamberlain and Rothshild (1981) obtain a similar result on rates of convergence in a model in which two candidates receive votes as binomial random variables. They assume no abstention, so the number of votes can be seen as flips of identical coins with a certain bias q. They show that if you toss an even number n of coins, the chance of obtaining the same number of heads and tails (the chance of a tie) drops asymptotically like $N^{-1/2}$ when the coins are unbiased ($q = 1/2$) and exponentially if the coins are biased ($q < 1/2$).

[17]Krasa and Polborn (2009) obtain the inefficient full compensation result with a non degenerate cost distribution because its support $[\underline{c}, \overline{c}]$ is bounded away from zero. Hence, unlike what we obtain in Lemma 1, only a finite number of voters will vote even when the population N grows large. Asymptotically their model is isomorphic to a homogenous cost model with cost $\underline{c} > 0$.

[18]We assume that if nobody votes, power is shared equally, namely

$$\frac{a}{a+b} = \frac{b}{a+b} = \frac{1}{2} \quad \textit{for} \quad a = b = 0$$

The expected marginal benefit of voting B_P^i for party i is the expected increase in the vote share for the preferred party induced by a single vote, namely

$$B_P^A = \sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\left(\left(\frac{e^{-qN\alpha}\left(qN\alpha\right)^a}{a!}\right)\left(\frac{e^{-(1-q)N\beta}\left(\left(1-q\right)N\beta\right)^b}{b!}\right)\left(\frac{a+1}{a+b+1}-\frac{a}{a+b}\right)\right)$$

$$B_P^B = \sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\left(\left(\frac{e^{-qN\alpha}\left(qN\alpha\right)^a}{a!}\right)\left(\frac{e^{-(1-q)N\beta}\left(\left(1-q\right)N\beta\right)^b}{b!}\right)\left(\frac{b+1}{a+b+1}-\frac{b}{a+b}\right)\right)$$

In this case, unlike in the M system, we have double summations because an A supporter, for instance, has an impact on the electoral outcome not only in the event of a tied election ($a = b$ and $a = b - 1$), but also in all the other cases $a \neq b$. In the P system voters always have some impact on the electoral outcome albeit very small, whereas in the M system voters have a large impact in the very small chance event that $a = b$ and zero impact otherwise. A non obvious quantitative question is to compare how the expected impacts of a voter in the M and in the P systems decline with the electorate size N. Luckily, after some manipulation the double summations above can be expressed in a simple form.

Lemma 4. *The marginal benefit of voting in the P system has the closed form*

$$\begin{aligned} B_P^A &= \frac{(1-q)\,\beta}{NT^2}-\left(\frac{\left((1-q)\,\beta\right)^2-\left(q\alpha\right)^2+(1-q)\,\beta\frac{1}{N}}{2T^2}\right)e^{-NT} \\ B_P^B &= \frac{q\alpha}{NT^2}+\left(\frac{\left((1-q)\,\beta\right)^2-\left(q\alpha\right)^2-q\alpha\frac{1}{N}}{2T^2}\right)e^{-NT} \end{aligned} \tag{2}$$

Similarly to what we obtained for the M model, we now show that asymptotically turnout for each party is zero as a percentage of the population, but is infinite in absolute numbers, moreover the party turnout ratio stays finite.

Lemma 5. *Any solution* (α_N, β_N) *(if it exists) to the P system has the following three properties*

$$\lim_{N\to\infty}\alpha_N = \lim_{N\to\infty}\beta_N = 0, \qquad \lim_{N\to\infty}N\alpha_N = \lim_{N\to\infty}N\beta_N = \infty, \qquad \lim_{N\to\infty}\frac{\alpha_N}{\beta_N}\in(0,\infty)$$

As in the M system, the above lemma allows us to use some approximations to show existence and uniqueness of an equilibrium for N large and also the following characterization results.

Proposition 6. *In the P system there is always a unique equilibrium* (α, β)*. The equilibrium has the following properties:*

- Size effect:
$$\frac{dT_P}{dN} < 0$$
- Partial underdog compensation effect:
$$q < 1/2 \quad \Longrightarrow \quad \alpha > \beta, \quad q\alpha < (1-q)\beta$$

The relation describing quantitatively the underdog compensation under the P system is

$$q\alpha F^{-1}(\alpha) = (1-q)\beta F^{-1}(\beta) \tag{3}$$

which is slightly different from equation (1) describing the underdog compensation under the M system.

2.4. **Comparison.** The size effect and the underdog compensation effect, though qualitatively similar, are quantitatively different across the two institutional systems. We now turn to the implications of these differences and to the comparison of turnout incentives across systems. Turnout is larger in a proportional power sharing system when there is a favorite party, while it is higher in a winner take all system if the election is even.

Proposition 7. .

- *Comparative turnout: for any* $q \in (0,1)$*,* $\exists \bar{N}_q$ *such that for* $N > \bar{N}_q$
$$\begin{aligned} T_M &> T_P && \text{for } q = 1/2 \\ T_P &> T_M && \text{for } q \neq 1/2 \end{aligned}$$
- *Comparative underdog compensation:*
$$\frac{1-q}{q} = \left(\frac{\alpha_P}{\beta_P}\right)^{n+1} = \left(\frac{\alpha_M}{\beta_M}\right)^{2n+1}$$
where $n \geq 1$ *is the lowest integer for which* $\frac{d^n F^{-1}}{dx^n}|_{x=0} \in (0, \infty)$*.*

Regarding the comparative underdog compensation, we have already explained in section 2.2 that with heterogeneous costs full compensation is impossible in equilibrium, and a similar explanation holds for the proportional power sharing system. In both systems the underdog compensation is partial: the ex-ante favorite party obtains the majority of the votes in a large election, but the underdog party has a higher turnout of its supporters. The above proposition shows that the underdog compensation is larger in the P system, namely

$$q < 1/2 \quad \Rightarrow \quad \alpha_P/\beta_P > \alpha_M/\beta_M > 1 \tag{4}$$

This comparative result could also be framed as a result on a higher *relative winning margin* in the M system than in the P for any given preference split q, where the relative winning margin W is defined as

$$W := \frac{|q\alpha - (1-q)\,\beta|}{T}$$

Regarding turnout, the intuition behind the turnout result relies on how fast the marginal benefit of voting decreases in the two models as the electorate gets larger. The M system has two asymptotic regimes: it decreases exponentially for $q \neq 1/2$ and for $q = \frac{1}{2}$ it decreases at the algebraic rate of $N^{-1/2}$. Since we have only partial underdog compensation, then for any $q \neq 1/2$ the majority party is always the more likely side to win. Hence the chance of a tied election, which is what drives rational voters to turn out, is much smaller than in the case $q = 1/2$ for any population size N.[19]

The benefit from voting in the P system drops asymptotically at the intermediate rate of N^{-1}. This rate is independent of q as in the power sharing system the event that a voter is pivotal or the chance of a tied election have no special relevance.

It is perhaps now intuitive that a winner take all system, unlike a proportional power sharing one, should have two quite different rates of convergence regimes (although as we explained this is not the case with a degenerate cost distribution). Be that as it may, only an explicit computation

[19]The two rates of convergence derived above do not depend on the (Poisson) population uncertainty in this model. For instance, Herrera and Martinelli (2006) analyze a majority rule election without population uncertainty. They introduce aggregate uncertainty in a different way, which allows to obtain a closed form for the chance of being pivotal, namely $\frac{(a+b)!}{2^{a+b+1}a!b!}$. As it can be seen using Stirling's approximation, that marginal benefit for large a and b has exactly the square root decline on the diagonal $(a = b)$ and the exponential decline off the diagonal $(a = \omega b,\ \omega \neq 1)$.

could determine that the rate of convergence in the P system is quantitatively *in between* the two rates of convergence in the M system: $N^{-1} \in \left(N^{-1/2}, e^{-N}\right)$.

In order to illustrate the comparison in terms of turnout as well as underdog compensation effects, we now turn to a numerical example.

2.5. **Example.** Consider the cost distribution family ($z > 0$): $F(c) = c^{1/z}$ with $c \in [0,1]$.

This example yields an explicit solution for the P system, i.e.

$$\alpha_P = \left(\frac{1}{N} \frac{(1-q)\, q^{\frac{1}{z+1}} (1-q)^{\frac{1}{z+1}}}{\left(q(1-q)^{\frac{1}{z+1}} + (1-q)\, q^{\frac{1}{z+1}}\right)^2} \right)^{\frac{1}{z+1}} \qquad \beta_P = \left(\frac{1}{N} \frac{q}{1-q} \frac{(1-q)\, q^{\frac{1}{z+1}} (1-q)^{\frac{1}{z+1}}}{\left(q(1-q)^{\frac{1}{z+1}} + (1-q)\, q^{\frac{1}{z+1}}\right)^2} \right)^{\frac{1}{z+1}}$$

The M system equilibrium has no closed form solution, namely (α_M, β_M) jointly solve

$$\beta_M = \left(\frac{q}{1-q}\right)^{\frac{1}{2z+1}} \alpha_M, \qquad \alpha_M^z = \frac{e^{-N\left(\sqrt{(1-q)\beta_M} - \sqrt{q\alpha_M}\right)^2}}{\sqrt{N}} \left(\frac{\sqrt{q\alpha_M} + \sqrt{(1-q)\,\beta_M}}{4\sqrt{\pi}\,\left(q(1-q)\,\alpha_M \beta_M\right)^{1/4}} \right)$$

Setting $N = 3000$ and $z = 5$, the numerical solutions to the M system yield a clear illustration of the comparative result of proposition 7. In the picture below we compare, as the preference split q varies, the turnout T in the M system (continuous line) and in the P system (dashed line).

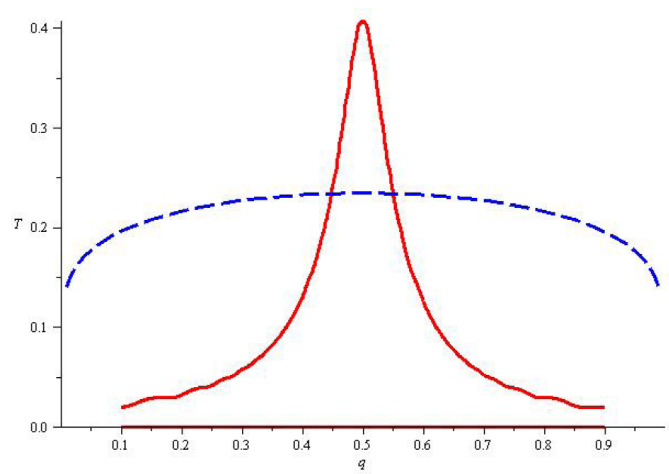


Figure 1: Turnout as a function of q in the M (continuous) and P (dashed) models ($z = 5$, $N = 3000$).

When one party (e.g. party B) has the ex-ante advantage over the other party (A), we have a higher turnout in the P system. Numerically, for instance when $q = 1/3$, we have

$q = 1/3$	α	β	α/β	W	T
P	24.8%	22%	1.27	27.8%	23%
M	7.1%	6.7%	1.06	30.9%	6.8%

Note also in both the M and the P systems the presence of the underdog compensation ($\alpha > \beta$) which is partial ($q\alpha < (1-q)\,\beta$). Moreover, note the higher underdog compensation α/β in the P system and consequently the higher relative winning margin W in the M system.

Whereas when the election is close and no party has an ex ante advantage, i.e. $q = 1/2$, turnout T in the M system surpasses the turnout in the P system

$q = 1/2$	α	β	T
P	23.5%	23.5%	23.5%
M	40.9%	40.9%	40.9%

Finally, note for different q's the much larger variability of turnout numbers T in the M system when compared to the P system.

To compare the underdog compensations in general, the picture below illustrates how the ratio α/β varies with q in the P system (dashed line) and in the M system (continuous line). Contrast these decreasing curves with the steeper one that is obtained in the M system under homogeneous cost (dotted line) when there is full underdog compensation, the election is expected to be tied and the winning margin is zero regardless of the initial preference split.

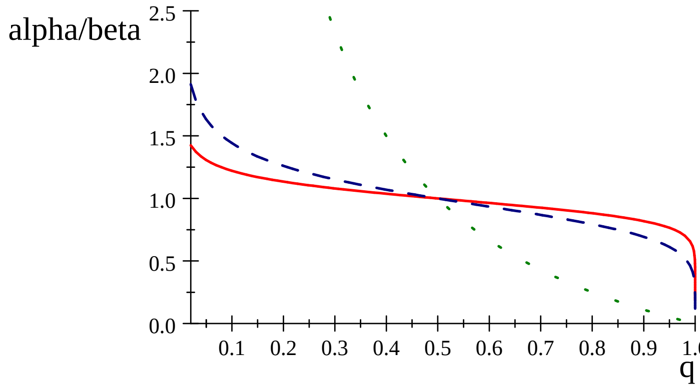


Figure 2: Underdog compensation α/β as a function of q in the P (dashed), M (continuous) and M with homogenous cost (dotted) models ($z = 5$).

In sum, this example illustrates how the underdog compensation is higher in a proportional power sharing system, while the turnout is lower in a proportional power sharing system only when the distribution of party supporters is symmetric.

2.6. **Welfare corollary.** Even though we are primarily interested in the comparative positive analysis of turnout across institutional systems, the results obtained allow us to establish a clear welfare comparison between a winner-take-all system and a proportional power sharing system.

Using total utility as reasonable welfare criterion, it is in line with standard practice to assess one system to be better than another if for every realization of the parameter q the total sum of expected benefits minus total voting costs is higher in such a system. The following corollary establishes the welfare comparison between systems for N large.

Corollary 8. *For $q \neq 1/2$, the winner-take-all system (M) yields higher welfare than a proportional power sharing system (P). For $q = 1/2$ the opposite is true.*

To see how this follows from our results, consider first the gross benefit side. For any $q \neq 1/2$ partial underdog compensation in the M system ensures that the side with the ex-ante majority support always wins the election. Hence, for $q < 1/2$ (wlog) the total expected benefit is $(1 - q)$ as with N large that fraction of citizens obtains the normalized benefit of 1 and the remaining people get zero. In the P system the power is shared between the two sides, so the gross benefit is

$$q\left(\frac{q\alpha}{T}\right)+(1-q)\left(\frac{(1-q)\,\beta}{T}\right)$$

which for $q<1/2$ is strictly less than $(1-q)$.

Having established the comparison in terms of benefits, we just need to add that for any $q\neq 1/2$ the total cost due to voter participation in the M system is lower than in the P system by proposition 7. For $q=1/2$ the welfare is higher in the P system because the benefit is equal to $1/2$ in both systems, but the voting costs are higher in the M system due to higher participation.

2.7. **Extension to many parties.** In this section we show that even when a proportional power sharing system allows for many parties, as one could intuitively expect, the comparative result in terms of turnout is qualitatively analogous to the one obtained above. We explicitly compute the equilibrium in the proportional power sharing system with three parties and show that for more than three parties the derivations are analogous. This allows us to obtain a simple comparative statics result within the proportional power sharing system: turnout increases in the number of parties.

Define

$$A:=\alpha q_A N,\quad B:=\beta q_B N,\quad C:=\gamma q_C N,\quad \text{with: } q_A+q_B+q_C=1$$

The marginal benefit[20] for party A is

$$B_P^A=\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\sum_{c=0}^{\infty}\left(\frac{e^{-A}A^a}{a!}\right)\left(\frac{e^{-B}B^b}{b!}\right)\left(\frac{e^{-C}C^c}{c!}\right)\left(\frac{a+1}{a+b+c+1}-\frac{a}{a+b+c}\right)$$

Lemma 9. *The marginal benefit has the closed form*

$$B_P^A=\left(1-\frac{A}{A+B+C}\right)\frac{1-e^{-(A+B+C)}}{A+B+C}+\left(\frac{A}{A+B+C}-\frac{1}{3}\right)e^{-(A+B+C)}$$

By symmetry the expressions B_P^B and B_P^C for parties B or C are straightforward. We obtain the following comparative result for any number of parties.

Proposition 10. .

[20] Assume again that if nobody votes, power is shared equally, namely

$$\frac{a}{a+b+c}=1/3\quad \text{for } a=b=c=0$$

- *The comparison between turnout in the P system and the M system continues to hold even when there are multiple parties in the P system.*
- *If parties are symmetric, turnout in the P system increases as the number of parties increases.*

The fact that the turnout comparison result remains unchanged is due to the quantitative fact that, regardless of the number of parties involved in the election, the marginal benefit of voting in the P system still declines asymptotically at the intermediate rate $1/N$, as it was the case for the P system with two parties.

Within the $1/N$ order of magnitude of the size effect, turnout increases when there are more symmetric parties. This is consistent with the fact that smaller parties obtain a higher turnout in the P system. The intuition for the latter follows from the following two observations. First, fixing the number of votes z for all other parties, the vote share increase for party A is

$$\left(\frac{a+1}{a+z+1}-\frac{a}{a+z}\right)=\left(\frac{a^2+a}{z}+2a+1+z\right)^{-1}$$

which is larger for smaller values of the random variable a, i.e. the number of votes for party A. Second, for a given a, in the marginal benefit B_P^A (see (2) and expressions above) a smaller party (i.e. a party with a smaller q_A) assigns larger probability weight $\left(\frac{e^{-A}A^a}{a!}\right)$ to small values of a.

3. Ethical Voter and Mobilization Models

Even though we believe that the analysis conducted so far provides *per se* many new insights, we need to extend the analysis to other turnout models for two reasons.

First of all, the rational voter model features the well known free riding problem among voters, which makes turnout in a large election be typically small. Since in large elections the turnout is not always small, the free riding problem seems to be overcome in some way. Economists differ on how this collective action problem is by-passed in an election. Regardless of how this might happen, it is important to know whether the turnout comparisons across electoral systems depend on the presence of this free riding problem, namely on whether the positive externality of voting among supporters of the same party is internalized or not.

Second, in the real world the mapping from vote shares to power shares is not often at the two extremes of proportional power sharing and winner take all, it is more often intermediate.

However, only the comparison of the two extremes was feasible for us to do in the rational voter model analyzed so far.

With this double motivation, we now turn to the comparative analysis of systems with *any* degree of power sharing under two types of alternative models, namely a mobilization model a la Shachar and Nalebuff (1999) (where parties' campaign efforts and spending are able to mobilize and coordinate citizens to go vote) and the ethical voter model (Coate and Conlin (2004) and Feddersen and Sandroni (2006)).

We show that ethical voter models and mobilization voter models can be seen as similar models. They are very similar on the benefit side and may differ slightly on the cost side, leading to partial versus zero underdog compensation, but, crucially, never to full compensation. The non full compensation property of both types of models implies that qualitatively they give the same result in terms of how turnout depends on power sharing and on the preference split. The result and the driving forces behind this result are analogous to what we obtained in the rational voter model where the externality of voting is not eliminated.

The population is a continuum of measure one, divided into q A supporters and $(1-q)$ B supporters. For any voting cost thresholds (c_α, c_β), i.e., given that the voter participation for each side is $(\alpha = F(c_\alpha), \beta = F(c_\beta))$, turnout is again $T = q\alpha + (1-q)\beta$. We assume F is weakly concave.[21]

3.1. **Cost Side for Mobilization Model.** A mobilization model assumes that more campaign spending by a party brings more votes for the party according to an exogenous technology. We assume the cost for a party of mobilizing to the polls all his supporters with voting cost below c is $l(c)$, where $c \in [0, \overline{c}]$ and l is increasing, convex and twice differentiable. We also assume it is infinitely costly for a party to turn out all its supporters: $l(\overline{c}) = \infty$.

3.2. **Cost Side for Ethical Voter Model.** The ethical voter model assumes that citizens are "rule utilitarian" so they act as one. This means that we have to find a party-planner solution on each side A and B. In this solution each planner looks at the total benefit from the outcome of the election considering the total cost of voting incurred by the supporters of his side.[22] The

[21]The same condition was needed to have uniqueness of a solution in the rational voter M-model.

[22]We assume "collectivism" a la Coate and Conlin (2004), so the planner on each side, A and B, only looks at the total cost of voting of the voters on his side. The results would not changed if we assumed "altruism"

cost of turning out the voters for the social planner on side A is the total cost born by all the citizens on side A that vote, namely

$$C(c_\alpha) := q \int_0^{c_\alpha} cf(c)\,dc$$

The citizens with cost below the planner-chosen cost threshold c_α vote because ethical voter models assume citizens get an exogenous benefit D (larger than their private voting cost $c \leq c_\alpha$) for "doing their part" in following the optimal rule established by the planner.

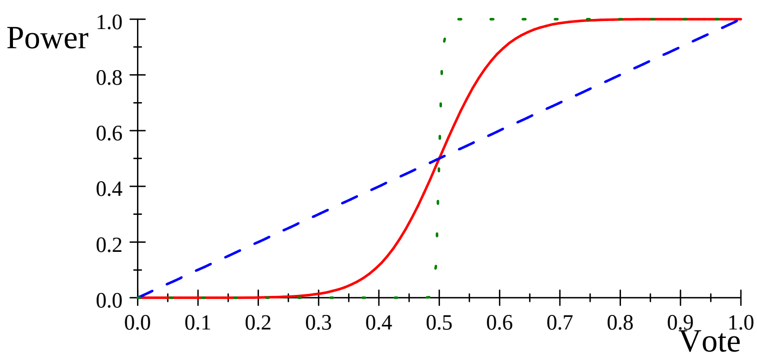
3.3. **Benefit Side under any Power Sharing Rule.** The expected vote shares for party A and B are

$$V = \frac{q\alpha}{T}, \qquad 1 - V = \frac{(1-q)\beta}{T}$$

The expected power share as a function of the vote share is the standard "contest success function"[23]

$$P_\gamma^A(V) = \frac{V^\gamma}{V^\gamma + (1-V)^\gamma}, \qquad P_\gamma^B(V) = \frac{(1-V)^\gamma}{V^\gamma + (1-V)^\gamma}$$

Below we illustrate the power share P_γ^A as a function of the vote share V for various power sharing parameters γ, namely: $\gamma = 1$ (i.e. the P system, dashed line), $\gamma = 5$ (i.e. approaching the M system, continuous line), and $\gamma \to \infty$ (i.e. a pure M system, dotted line).



as in Feddersen and Sandroni (2006): each planner takes into account the cost of voting of all citizens that vote regardless of their side.

[23]See for instance Hirshleifer (1989), among others. When nobody votes ($\alpha = \beta = 0$) assume equal shares ($V = 1/2$).

Figure 3: Power Sharing Functions in the P (dashed), approaching the M system (continuous) and pure M system (dotted).

The marginal benefits with respect to (c_α, c_β) are respectively

$$\frac{\gamma}{V(1-V)}\frac{\left(\frac{V}{1-V}\right)^\gamma}{\left[1+\left(\frac{V}{1-V}\right)^\gamma\right]^2}\left(\frac{(1-q)\beta}{T^2}\right)qf(c_\alpha), \quad \frac{\gamma}{V(1-V)}\frac{\left(\frac{V}{1-V}\right)^\gamma}{\left[1+\left(\frac{V}{1-V}\right)^\gamma\right]^2}\left(\frac{q\alpha}{T^2}\right)(1-q)f(c_\beta)$$

3.4. **First Order Conditions and Underdog Compensation Effects.** For the ethical voter models we have as first order conditions

$$\begin{aligned}\frac{\gamma}{V(1-V)}\frac{\left(\frac{V}{1-V}\right)^\gamma}{\left[1+\left(\frac{V}{1-V}\right)^\gamma\right]^2}\left(\frac{(1-q)\beta}{T^2}\right)qf(c_\alpha) &= qc_\alpha f(c_\alpha)\\ \frac{\gamma}{V(1-V)}\frac{\left(\frac{V}{1-V}\right)^\gamma}{\left[1+\left(\frac{V}{1-V}\right)^\gamma\right]^2}\left(\frac{q\alpha}{T^2}\right)(1-q)f(c_\beta) &= (1-q)c_\beta f(c_\beta)\end{aligned}$$

which gives the condition

$$q\alpha F^{-1}(\alpha) = (1-q)\beta F^{-1}(\beta)$$

The above is a partial underdog compensation condition which happens to be the same as the partial underdog compensation condition (3) obtained in the P system of the rational voter model.

For the mobilization model we have

$$\begin{aligned}\frac{\gamma}{V(1-V)}\frac{\left(\frac{V}{1-V}\right)^\gamma}{\left[1+\left(\frac{V}{1-V}\right)^\gamma\right]^2}\left(\frac{(1-q)\beta}{T^2}\right)qf(c_\alpha) &= l'(c_\alpha)\\ \frac{\gamma}{V(1-V)}\frac{\left(\frac{V}{1-V}\right)^\gamma}{\left[1+\left(\frac{V}{1-V}\right)^\gamma\right]^2}\left(\frac{q\alpha}{T^2}\right)(1-q)f(c_\beta) &= l'(c_\beta)\end{aligned}$$

which yields the following *zero underdog compensation* condition

$$\frac{\alpha l'(c_\alpha)}{f(c_\alpha)} = \frac{\beta l'(c_\beta)}{f(c_\beta)} \quad \Longrightarrow \quad T=\alpha=\beta, \quad c_\alpha = c_\beta$$

that is, both parties turn out the same proportion of their supporters.[24]

[24] We need to assume F weakly concave to guarantee the LHS expressions above are increasing in their argument. The same condition is needed for uniqueness of a solution in the rational voter M model.

3.5. **Solution to the Mobilization Model.** The mobilization model is reduced to one equation in one unknown, equating marginal benefit (MB) and marginal cost

$$MB = \gamma \frac{\left(\frac{V}{1-V}\right)^{\gamma}}{\left[1+\left(\frac{V}{1-V}\right)^{\gamma}\right]^{2}} = \gamma \frac{\left(\frac{q}{1-q}\right)^{\gamma}}{\left[1+\left(\frac{q}{1-q}\right)^{\gamma}\right]^{2}} = G(\alpha) \tag{5}$$

where

$$G(\alpha) := \alpha \frac{l'(c_\alpha)}{f(c_\alpha)}$$

is increasing in α. The solution is hence unique and it exists because $l(\bar{c}) = \infty$. The solution has the following properties:

(1) Turnout $T = \alpha$ increases when the marginal benefit (MB) increases;

(2) As γ goes to infinity (M model) the marginal benefit goes to infinity when $q = 1/2$ and goes to zero otherwise;

(3) When $\gamma = 1$ (P model) the marginal benefit $\frac{\left(\frac{q}{1-q}\right)}{\left[1+\left(\frac{q}{1-q}\right)\right]^{2}}$ is positive for all $q \in (0,1)$ and peaks but stays finite at $q = 1/2$.

The picture below shows the marginal benefit as a function of the closeness of the election q for $\gamma = 1$ (i.e. the P system, dashed line), and for $\gamma = 5$ (i.e. approximating the M system, continuous line).

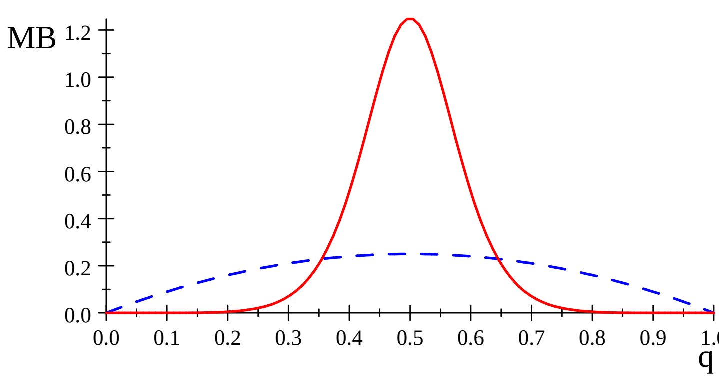


Figure 4: Marginal benefit as a function of q in the P system (dashed) and approaching the M system (continuous).

Turnout $T = \alpha$ can be obtained by a simple rescaling, namely by inverting (5) (i.e.: $T = G^{-1}(MB)$) which preserves the qualitative features of the picture above. Hence, the turnout comparison across systems is analogous to what we already obtained in the rational voter model (see Figure 1).

3.6. **Solution to the Ethical Voter Model.** The solution for the ethical voter model is more complicated, as the underdog compensation is strictly partial (not zero), so $\alpha \neq \beta$ and we maintain the two equations in two unknowns, that is

$$q\alpha F^{-1}(\alpha) = (1-q)\,\beta F^{-1}(\beta) = \gamma \frac{\left(\frac{q\alpha}{(1-q)\beta}\right)^{\gamma}}{\left[1+\left(\frac{q\alpha}{(1-q)\beta}\right)^{\gamma}\right]^{2}}$$

However, given that the underdog compensation is not full the comparative statics is similar to the case of zero compensation obtained in the mobilization model. Namely if a solution (α, β) exists,[25] then α and β, and hence T, increase when the marginal benefit increases. Taking limits, as γ goes to infinity (M model) the marginal benefit on the RHS goes to infinity when $q = 1/2$ and to zero otherwise. When $\gamma = 1$ (P model) the marginal benefit $\frac{\left(\frac{q}{1-q}\right)}{\left[1+\left(\frac{q}{1-q}\right)\right]^{2}}$ is positive for all $q \in (0,1)$ and peaks but stays finite at $q = 1/2$.

Note that if the underdog compensation were full (which happens for instance with homogeneous costs) the marginal benefit becomes

$$MB = \gamma \frac{\left(\frac{q\alpha}{(1-q)\beta}\right)^{\gamma}}{\left[1+\left(\frac{q\alpha}{(1-q)\beta}\right)^{\gamma}\right]^{2}} = \frac{\gamma}{4}$$

so the result would be different: regardless of the initial preference split q, turnout and MB would increase with the intensity of the contest γ. As explained, the rational voter model with homogenous cost gives an equivalent result.

The robustness of the rational voter comparative analysis has been completed, and the reader can probably appreciate the fact that the results at the extremes ($\gamma = 1$ and $\gamma \to \infty$) are not special cases.

In sum, the common feature of all models in this paper is that with heterogeneity of voting costs full underdog compensation is not possible, and hence the majority party is expected to maintain a considerable advantage and winning margin in the election. The small probability of

[25] Coate and Conlin (2004) and Feddersen and Sandroni (2006) provide specific conditions on the voting cost distributions that guarantee existence.

victory for the minority, i.e. the low competitiveness of the electoral race, depresses significantly the incentives to turn out in the winner take all system. Whereas in a power sharing system the incentives to vote or to mobilize voters are affected to a much lesser extent by the competitiveness or the expected closeness of the electoral race.

4. Concluding Remarks and Directions for Future Research

For any distributions of partisan preferences and voting costs, we have shown that turnout (of rational voters as well as of ethical voters and of mobilized voters) depends on the degree of proportionality of influence in the institutional system in a clear way: higher turnout in a winner-take-all system than in a proportional power sharing system when the population is evenly split in terms of partisan preferences, and vice versa when one party's position has a clear majority of support.

In any considered model and in any considered power sharing system, partial (or zero) underdog compensation occurs, which guarantees that the ex-ante favorite party obtains in expectation the higher vote share in the election. From a welfare perspective, the winner take all system, which as a consequence would give all the power to the ex-ante favorite party, is then superior to any other power sharing system.[26]

Even though the number of parties is exogenous in the paper, the fact that the comparative results in terms of turnout *do not* depend on the number of parties under the P system is reassuring, and makes the (hard) extension to endogenous party formation perhaps unnecessary. In light of the robustness results on the number of parties, even the extension to a multistage game in which the parties play some kind of legislative bargaining game after the election is not likely to generate any significant difference in terms of our main comparative results.

Beside the intrinsic value of the theoretical results, the findings of this paper could be useful for future empirical as well as experimental research. Empirically there is plenty of evidence that closeness of elections influences turnout (see e.g. Blais (2000)). However, we are not aware

[26]Of course this point abstracts from fairness or representation considerations, which would instead push any normative statement the other way.

of any empirical work focusing on the interaction effect of expected closeness and the degree of power sharing of the institutional system.[27]

Even though the mapping from vote shares to power shares is only partially determined by the electoral rules (vote shares to seat shares), some empirical and experimental work could be inspired on electoral rules: the empirical evidence on turnout in national elections (see e.g. Powell (1980, 1986), Crewe (1981), Jackman (1987) and Jackman and Miller (1995), Blais and Carthy (1990) and Franklin (1996)) all conclude that, everything else being equal, turnout is lower in plurality and majority elections than under Proportional Representation.[28] On the other hand, experimental evidence (see Schram and Sonnemans (1996)) displays the opposite finding. We have shown that these seemingly inconsistent findings are instead perfectly reconcilable, since the experimental design featured perfect symmetry in the ex-ante supports for the two parties, i.e. in the case in which we have shown that we should expect higher turnout under a winner take all system.

Some of our predictions are experimentally testable. Levine and Palfrey (2007) identified the size effect and underdog compensation effect in a winner take all system experiment. Our paper generalizes the theoretical analysis and allows to compare such effects across systems. Future experimental investigations could employ different treatments, allowing for the possibility of asymmetric distributions of partisan supporters joint with different degrees of power sharing, as well as perhaps the possibility of communication and coordination among voters. Similarly, we believe that the empirical analysis should be extended beyond electoral rules, since there are many other institutional details that affect the degree of proportionality of power as a function of the allocations of seats determined by the vote shares and the electoral formula. Finally, even the prediction that turnout should increase in the number of parties could be tested experimentally as well as on the existing field data.

[27]There will be difficult choices to make in terms of how to measure, proxy or instrument our reduced form variable γ, but the effort could be worthwhile.

[28]The standard caveat is that cross sectional studies are not to be considered conclusive evidence, because of the small sample size and few data points, cultural and idiosyncratic characteristics that are difficult to control for, as emphasized in Acemoglu (2005).

Appendix

Proof of Lemma. 1 We first show that

$$\lim_{N\to\infty} \alpha_N = \lim_{N\to\infty} \beta_N = 0$$

Define the modified Bessel functions of the first kind, see Abramowitz and Stegun (1965), as

$$I_0(z) := \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^k}{k!} \frac{\left(\frac{z}{2}\right)^k}{k!}, \qquad I_1(z) := \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^k}{k!} \frac{\left(\frac{z}{2}\right)^{k+1}}{(k+1)!}$$

Defining

$$x := qN\alpha, \qquad y := (1-q)\,N\beta, \qquad z := 2\sqrt{xy}$$

then the benefits of voting $\left(B_M^A, B_M^B\right)$ can be written as

$$B_M^A = \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{e^{-x}x^k}{k!}\right)\left(\frac{e^{-y}y^k}{k!}\right)\left(1+\frac{y}{k+1}\right) = \frac{e^{-x}e^{-y}}{2}\left(I_0\left(2\sqrt{xy}\right) + \sqrt{\frac{y}{x}}I_1\left(2\sqrt{xy}\right)\right)$$

$$B_M^B = \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{e^{-x}x^k}{k!}\right)\left(\frac{e^{-y}y^k}{k!}\right)\left(1+\frac{x}{k+1}\right) = \frac{e^{-x}e^{-y}}{2}\left(I_0\left(2\sqrt{xy}\right) + \sqrt{\frac{x}{y}}I_1\left(2\sqrt{xy}\right)\right)$$

For large z the modified Bessel functions are asymptotically equivalent and approximate to, see Abramowitz and Stegun (1965)[29]

$$I_0(z) \simeq I_1(z) \simeq \frac{e^z}{2\pi z}$$

For any exogenously fixed $(\alpha,\beta) \in [0,1]^2$ x and y go to infinity as N goes to infinity, so we can approximate the benefits of voting for large N as

$$B_M^A \simeq e^{-x-y+2\sqrt{xy}} \frac{\sqrt{x}+\sqrt{y}}{4\sqrt{\pi}\sqrt{\sqrt{xy}}} \frac{1}{\sqrt{x}}, \qquad B_M^A \simeq e^{-x-y+2\sqrt{xy}} \frac{\sqrt{x}+\sqrt{y}}{4\sqrt{\pi}\sqrt{\sqrt{xy}}} \frac{1}{\sqrt{y}}$$

As a consequence for any given $(\alpha,\beta) \in [0,1]^2$ the benefits of voting vanish as N grows, namely

$$\lim_{N\to\infty} B_M^A(\alpha,\beta) = 0, \qquad \lim_{N\to\infty} B_M^B(\alpha,\beta) = 0$$

Now consider (α,β) as endogenous, i.e. solutions to the system

$$B_M^A(\alpha,\beta) = F^{-1}(\alpha), \qquad B_M^B(\alpha,\beta) = F^{-1}(\beta)$$

---

[29] $X(z) \simeq Y(z)$ means that $\lim_{z\to\infty} \frac{X(z)}{Y(z)} = 1$.

Since F and F^{-1} are increasing and continuous with $F(0) = 0$, then $B_M^A(\alpha, \beta) = F^{-1}(\alpha)$ implies $\lim_{N\to\infty} \alpha_N = 0$. Likewise, we have $\lim_{N\to\infty} \beta_N = 0$.

Next, we show that

$$\lim_{N\to\infty} N\alpha_N = \lim_{N\to\infty} N\beta_N = \infty, \qquad \lim_{N\to\infty} \frac{\alpha_N}{\beta_N} \in (0,\infty)$$

Suppose $\lim_{N\to\infty} N\alpha_N < \infty$ and $\lim_{N\to\infty} N\beta_N < \infty$, then

$$\lim_{N\to\infty} B_M^A(\alpha_N, \beta_N) > 0$$

and any solution to $B_M^A(\alpha, \beta) = F^{-1}(\alpha)$ would imply $\lim_{N\to\infty} \alpha_N > 0$, which contradicts $\lim_{N\to\infty} \alpha_N = 0$.

Suppose $\lim_{N\to\infty} N\alpha_N = \infty$ and $\lim_{N\to\infty} N\beta_N < \infty$, then $\lim_{N\to\infty} \frac{\alpha_N}{\beta_N} = \infty$ which implies (using a Taylor expansion of F^{-1} on the numerator and the denominator around zero) that $\lim_{N\to\infty} \frac{F^{-1}(\alpha_N)}{F^{-1}(\beta_N)} = \infty$.

For all N we have

$$\frac{B_M^A(\alpha_N, \beta_N)}{B_M^B(\alpha_N, \beta_N)} = \frac{F^{-1}(\alpha_N)}{F^{-1}(\beta_N)}$$

Taking the limit on one side we have

$$L := \lim_{N\to\infty} \frac{B_M^A(\alpha_N, \beta_N)}{B_M^B(\alpha_N, \beta_N)} = \lim_{\frac{x}{y}\to\infty} \frac{I_0\left(2\sqrt{xy}\right) + I_1\left(2\sqrt{xy}\right)\sqrt{\frac{y}{x}}}{I_0\left(2\sqrt{xy}\right) + I_1\left(2\sqrt{xy}\right)\sqrt{\frac{x}{y}}} \leq 1$$

In fact, $L \leq 1$ if $\lim_{\frac{x}{y}\to\infty} \frac{I_1\left(2\sqrt{xy}\right)}{I_0\left(2\sqrt{xy}\right)} = 0$ and $L = 0$ if $\lim_{\frac{x}{y}\to\infty} \frac{I_1\left(2\sqrt{xy}\right)}{I_0\left(2\sqrt{xy}\right)} \in (0, +\infty]$. So we have a contradiction as $L \leq 1$ cannot be equal to $\lim_{N\to\infty} \frac{F^{-1}(\alpha_N)}{F^{-1}(\beta_N)} = \infty$. The same argument shows that it cannot be the case that $\lim_{N\to\infty} N\alpha_N < \infty$ and $\lim_{N\to\infty} N\beta_N = \infty$.

The above arguments also imply that we cannot have either

$$\lim_{N\to\infty} \frac{\alpha_N}{\beta_N} = 0, \qquad \lim_{N\to\infty} \frac{\alpha_N}{\beta_N} = \infty$$

$\square$

Proof of Proposition 2. For N large, since $\lim_{N\to\infty} N\alpha_N = \lim_{N\to\infty} N\beta_N = \infty$ we can use the asymptotic expression for the modified Bessel functions, so the system becomes

$$\begin{aligned} B_M^A &\simeq \frac{e^{-N(h-g)^2}}{\sqrt{N}} \frac{g+h}{4\sqrt{\pi}\sqrt{hg}} \frac{1}{g} = F^{-1}(\alpha) \\ B_M^B &\simeq \frac{e^{-N(h-g)^2}}{\sqrt{N}} \frac{g+h}{4\sqrt{\pi}\sqrt{hg}} \frac{1}{h} = F^{-1}(\beta) \end{aligned}$$

where we defined

$$g := \sqrt{q\alpha}, \quad h := \sqrt{(1-q)\,\beta_M(\alpha)}$$

The above system yields

$$\sqrt{q\alpha} F^{-1}(\alpha) = \sqrt{(1-q)\,\beta} F^{-1}(\beta)$$

Since the function $\sqrt{\alpha} F^{-1}(\alpha)$ is increasing we can define the function

$$\beta := \beta_M(\alpha)$$

where $\beta_M : [0,1] \longrightarrow [0,1]$ is an increasing and differentiable function with $\beta_M(0) = 0$. The system is reduced to a single equation

$$B_M^A(\alpha, \beta_M(\alpha)) = F^{-1}(\alpha),$$

We now show existence of a solution to the above equation by showing that the two continuous functions on either side must cross at least once.

Assume wlog $q < 1/2$. We have

$$\alpha \in (0,1] \quad \Longrightarrow \quad g < h$$

and for any fixed N, we have

$$\lim_{\alpha\to 0} \frac{e^{-N(h-g)^2}}{\sqrt{N}} \frac{g+h}{4\sqrt{\pi}\sqrt{hg}} \frac{1}{g} > \lim_{\alpha\to 0} \frac{e^{-N(h-g)^2}}{\sqrt{N}} \frac{2}{4\sqrt{\pi}\sqrt{hg}} = \infty$$

For $\alpha = 1$ we have $h > g = \sqrt{q}$, so for all N above a certain value we have

$$\frac{e^{-N(h-g)^2}}{\sqrt{N}} \left(\frac{g+h}{4\sqrt{\pi}\sqrt{gh}} \frac{1}{g} \right) < 1$$

which proves existence of a solution, because $F^{-1}(\alpha)$ is increasing and $F^{-1}(1) = 1$.

For uniqueness we need to show that the B_M^A is decreasing in α, namely that the following quantity is negative

$$\frac{d}{dg}\left(\frac{e^{-N(h-g)^2}}{\sqrt{N}}\frac{g+h}{4\sqrt{\pi}g\sqrt{hb}}\right)=\frac{e^{-N(h-g)^2}}{\sqrt{N}}\left(-2N\left(h-g\right)\frac{d\left(h-g\right)}{dh}\frac{g+h}{4\sqrt{\pi}g\sqrt{gh}}+\frac{d}{dh}\left(\frac{g+h}{4\sqrt{\pi}g\sqrt{gh}}\right)\right)$$

For large N this derivative will be negative if and only if

$$\frac{d\left(h-g\right)}{da}=\frac{\sqrt{1-q}}{\sqrt{q}}\frac{d\beta'}{d\alpha'}-1>0$$

where we defined

$$\begin{aligned}\beta' &: =\sqrt{\beta},\quad \alpha':=\sqrt{\alpha}\\ G\left(\alpha'\right) &: =\alpha' F^{-1}\left(\left(\alpha'\right)^2\right)=\sqrt{\alpha}F^{-1}\left(\alpha\right)\end{aligned}$$

we have

$$\left(\sqrt{1-q}\right)G\left(\beta'\right)=\left(\sqrt{q}\right)G\left(\alpha'\right)\quad\Longrightarrow\quad\frac{\sqrt{1-q}}{\sqrt{q}}\frac{d\beta'}{d\alpha'}=\frac{G'\left(\alpha'\right)}{G'\left(\beta'\right)}$$

So we need G' to be increasing

$$G'\left(\alpha'\right)=\frac{d}{d\alpha}\left(\sqrt{\alpha}F^{-1}\left(\alpha\right)\right)\frac{d\alpha}{d\alpha'}=2\frac{d}{d\alpha}\left(\alpha F^{-1}\left(\alpha\right)\right)$$

so it suffices for $\alpha F^{-1}\left(\alpha\right)$ to be weakly convex, so it suffices to have $F\left(\alpha\right)$ weakly concave.

As for the size effect, note that the marginal benefit side B_P^A decreases with N for all α while the cost side remains unchanged. Hence by the implicit function theorem as we increase N we have lower α which implies lower β and in turn lower turnout, formally

$$\begin{aligned}0 &= \frac{d\left(B_M^A-F^{-1}\right)}{d\alpha}\frac{d\alpha}{dN}+\frac{d\left(B_M^A-F^{-1}\right)}{dN}\\ \frac{d\alpha}{dN} &= -\frac{\frac{dB_M^A}{dN}}{\frac{d\left(B_M^A-F^{-1}\right)}{d\alpha}}<0\quad\Longrightarrow\quad\frac{d\beta}{dN}<0\Longrightarrow\frac{dT_M}{dN}<0\end{aligned}$$

The underdog compensation is a consequence of F^{-1} being increasing, namely

$$\begin{aligned}q\alpha\left(F^{-1}\left(\alpha\right)\right)^2 &= \left(1-q\right)\beta\left(F^{-1}\left(\beta\right)\right)^2\\ q &< 1/2\quad\Longleftrightarrow\quad\alpha>\beta,\quad q\alpha<\left(1-q\right)\beta\end{aligned}$$

□

Proof of Lemma 4. For given (α, β) call the expected number of voters for each party $R := qN\alpha$, $S := (1-q)\,N\beta$, we have

$$B_P^A = e^{-R-S} \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \left(\frac{R^a}{a!}\right) \left(\frac{S^b}{b!}\right) \left(\frac{a+1}{a+b+1} - \frac{a}{a+b}\right)$$

By differentiating and integrating the summands and inverting the series and integral operators we have

$$\begin{aligned} \sum_{b=0}^{\infty} \frac{S^b}{b!} \frac{a}{a+b} &= \frac{a}{S^a} \sum_{b=0}^{\infty} \int_0^S \frac{d}{dr} \left(\frac{1}{b!} \frac{r^{a+b}}{a+b}\right) dr = \\ &= \frac{a}{S^a} \int_0^S \sum_{b=0}^{\infty} \left(\frac{1}{b!} r^{a+b-1}\right) dr = \begin{cases} \frac{a}{S^a} \int_0^S r^{a-1} e^r dr & \text{for } a \geq 1 \\ 1/2 & \text{for } a = 0 \end{cases} \end{aligned}$$

and

$$\sum_{b=0}^{\infty} \frac{S^b}{b!} \frac{a+1}{a+b+1} = \frac{a+1}{S^{a+1}} \int_0^S r^a e^r dr$$

By inverting the series and integral operators again in the series over a, we have

$$\begin{aligned} B_P^A &= e^{-R-S} \left(\sum_{a=0}^{\infty} \frac{R^a}{a!} \left(\frac{a+1}{S^{a+1}} \int_0^S r^a e^r dr \right) - \sum_{a=1}^{\infty} \frac{R^a}{a!} \left(\frac{a}{S^a} \int_0^S r^{a-1} e^r dr \right) - \frac{1}{2} \right) \\ &= e^{-R-S} \left(\int_0^S \left(\begin{array}{c} \frac{1}{S} \left(\sum_{a=0}^{\infty} \frac{\left(\frac{R}{S} r\right)^a}{a!} + \sum_{a=1}^{\infty} \frac{\left(\frac{R}{S} r\right)^a}{(a-1)!} \right) \\ -\frac{R}{S} \sum_{a=1}^{\infty} \frac{\left(\frac{R}{S} r\right)^{a-1}}{(a-1)!} \end{array} \right) e^r dr - \frac{1}{2} \right) \\ &= e^{-R-S} \left(\frac{1}{S^2} \int_0^S e^{\left(1+\frac{R}{S}\right)r} \left(S - RS + Rr\right) dr - \frac{1}{2} \right) \\ &= e^{-R-S} \left(\frac{1-R}{S} \left(\frac{e^{S+R}-1}{\left(1+\frac{R}{S}\right)} \right) + \frac{R}{S^2 \left(1+\frac{R}{S}\right)^2} \int_0^{S+R} e^r r\, dr - \frac{1}{2} \right) \\ &= \frac{S}{(R+S)^2} - \frac{e^{-(R+S)}}{(R+S)^2} \frac{S^2 - R^2 + S}{2} \end{aligned}$$

and by symmetry

$$B_P^B(R,S) = B_P^A(S,R)$$

$\square$

Proof of Lemma 5. We first show that

$$\lim_{N\to\infty}\alpha_N=\lim_{N\to\infty}\beta_N=0$$

For any fixed $\alpha>0$ and $\beta>0$, by inspection of the closed form expression (2) we see that $\lim_{N\to\infty}B_P^A(\alpha,\beta)=\lim_{N\to\infty}B_P^B(\alpha,\beta)=0$, so the same argument obtained in Lemma (1) for the M system applies.

Next, we show that

$$\lim_{N\to\infty}N\alpha_N=\lim_{N\to\infty}N\beta_N=\infty,\qquad \lim_{N\to\infty}\frac{\alpha_N}{\beta_N}\in(0,\infty)$$

Summing the two P system equations we have

$$\frac{1}{NT}\left(1-\frac{e^{-NT}}{2}\right)=F^{-1}(\alpha)+F^{-1}(\beta)$$

Since the RHS goes to zero the LHS will too, which means that NT must go to infinity so we cannot have both $\lim_{N\to\infty}N\alpha_N<\infty$ and $\lim_{N\to\infty}N\beta_N<\infty$. For N large, since the exponential terms e^{-NT} in (2) vanish faster than the hyperbolic terms, the system approximates to

$$\frac{(1-q)\,\beta}{NT^2}=F^{-1}(\alpha)\,,\qquad \frac{q\alpha}{NT^2}=F^{-1}(\beta)\tag{6}$$

Suppose $\lim_{N\to\infty}N\alpha_N=\infty$ and $\lim_{N\to\infty}N\beta_N<\infty$, then $\lim_{N\to\infty}\frac{\alpha_N}{\beta_N}=\infty$ which implies (using a Taylor expansion of F^{-1} on the numerator and the denominator around zero) that $\lim_{N\to\infty}\frac{F^{-1}(\alpha_N)}{F^{-1}(\beta_N)}=\infty$. From (6) we have

$$\frac{1-q}{q}\frac{\beta_N}{\alpha_N}=\frac{F^{-1}(\alpha_N)}{F^{-1}(\beta_N)}$$

so we reach a contradiction as the above equality cannot hold as $N\to\infty$. The same argument shows that it cannot be the case that $\lim_{N\to\infty}N\alpha_N<\infty$ and $\lim_{N\to\infty}N\beta_N=\infty$.

The above arguments also imply that we cannot have either

$$\lim_{N\to\infty}\frac{\alpha_N}{\beta_N}=0,\qquad \lim_{N\to\infty}\frac{\alpha_N}{\beta_N}=\infty$$

□

Proof of Proposition 6. The approximated system (6) yields

$$\begin{aligned} q\alpha F^{-1}(\alpha) &= (1-q)\,\beta F^{-1}(\beta)\\ q &< 1/2 \iff \alpha>\beta \end{aligned}$$

Since the function $\alpha F^{-1}(\alpha)$ is increasing we can define

$$\beta := \beta_P(\alpha)$$

where $\beta_P(\alpha) : [0,1] \longrightarrow [0,1]$ is an increasing differentiable function with $\beta_P(0) = 0$. We now reduced the P system to one equation

$$B_P^A := \frac{(1-q)\,\beta_P(\alpha)}{NT^2} = F^{-1}(\alpha)$$

which we now show has one and only one solution.

The cost side $F^{-1}(\alpha)$ is increasing from 0 to 1. Uniqueness comes from the fact that the benefit side decreases in α as its derivative is proportional to

$$\begin{aligned}
\frac{\partial B_P^A}{\partial \alpha} &\propto \left[\beta_P'(\alpha)\left(q\alpha + (1-q)\,\beta_P(\alpha)\right) - 2\beta_P(\alpha)\left(q + (1-q)\,\beta_P'(\alpha)\right)\right] \\
&= -\left[\left((1-q)\,\beta - q\alpha\right)\beta_P'(\alpha) + 2q\beta_P(\alpha)\right] < 0
\end{aligned}$$

as

$$\alpha > \beta \quad \Longrightarrow \quad q\alpha < q\alpha \frac{F^{-1}(\alpha)}{F^{-1}(\beta)} = (1-q)\,\beta$$

Existence comes form the fact that for α approaching zero the benefit diverges as for any fixed N we have

$$\lim_{\alpha \to 0} \frac{1}{N} \frac{(1-q)\,\beta_P(\alpha)}{\left(q\alpha + (1-q)\,\beta_P(\alpha)\right)^2} > \lim_{\alpha \to 0} \frac{1}{N} \frac{(1-q)}{\alpha} \frac{\beta_P}{\alpha} = \infty$$

because

$$\lim_{\alpha \to 0} \frac{\beta_P}{\alpha} = \lim_{\alpha \to 0} \frac{q}{1-q} \frac{F^{-1}(\alpha)}{F^{-1}(\beta_P)} > \frac{q}{1-q} > 0$$

and for $\alpha = 1$ we have eventually (i.e. for all N above a certain value),

$$\frac{1}{N}\left(\frac{(1-q)\,\beta_P(1)}{\left(q + (1-q)\,\beta_P(1)\right)^2}\right) < F^{-1}(1) = 1$$

Hence a unique solution $(\alpha_P, \beta_P(\alpha_P))$ exists for the equilibrium problem.

The proofs for the size effect and the underdog compensation effect are analogous to the ones obtained in the M system. $\square$

Proof of Proposition 7. First, we compare turnouts. Assuming the cost side $F^{-1}(\alpha)$ is the same in the two systems, it suffices to show that the benefit sides of the equations determining the equilibrium α are ranked.

For any $q \neq 1/2$ we need to show that eventually (i.e. for any N above a given $\overline{N}$) we have

$$B_M^A(\alpha, \beta_M(\alpha)) < B_P^A(\alpha, \beta_P(\alpha)), \qquad \text{for all } \alpha \in (0,1]$$

namely

$$\frac{e^{-N\left(\sqrt{q\alpha}-\sqrt{(1-q)\beta_M}\right)^2}}{\sqrt{N}}\left(\frac{\sqrt{q\alpha}+\sqrt{(1-q)\beta_M}}{4\sqrt{\pi}\left(q(1-q)\alpha\beta_M\right)^{1/4}}\right)\frac{1}{\sqrt{q\alpha}} < \frac{1}{N}\frac{(1-q)\beta_P}{\left(q\alpha+(1-q)\beta_P\right)^2}$$

Rearranging we have

$$e^{-N\left(\sqrt{q\alpha}-\sqrt{(1-q)\beta_M}\right)^2}\sqrt{N} < \frac{(1-q)\beta_P}{\left(q\alpha+(1-q)\beta_P\right)^2}\left(\frac{\sqrt{q\alpha}+\sqrt{(1-q)\beta_M}}{4\sqrt{\pi}\left(q(1-q)\alpha\beta_M\right)^{1/4}}\frac{1}{\sqrt{q\alpha}}\right)^{-1}$$

which is satisfied as LHS above converges to zero, whereas the RHS is a positive constant for all $\alpha \in (0,1]$ because

$$\begin{aligned} \alpha &\in (0,1] &&\Longrightarrow&& \beta_P \in (0,1], \quad \beta_M \in (0,1] \\ q &\neq 1/2 &&\Longrightarrow&& \sqrt{q\alpha} \neq \sqrt{(1-q)\beta_M(\alpha)} \end{aligned}$$

Hence, for any eventually we have

$$q \neq 1/2 \quad \Longrightarrow \quad \alpha_M < \alpha_P$$

The symmetry property $\beta(q) = \alpha(1-q)$ (which holds in both the M and P systems) implies

$$q \neq 1/2 \quad \Longrightarrow \quad \beta_M < \beta_P$$

hence

$$q \neq 1/2 \quad \Longrightarrow \quad T_M < T_P$$

For $q = 1/2$ we have $\alpha = \beta$ in both P and M systems. We need to show that eventually

$$B_M^A > B_P^A, \qquad \alpha \in (0,1]$$

namely

$$\frac{1}{\sqrt{N}}\left(\frac{2\sqrt{q\alpha}}{4\sqrt{\pi}}\right)\frac{1}{q\alpha} > \frac{1}{N}\left(\frac{q\alpha}{2\left(2q\alpha\right)^2}\right)$$

Rearranging we have

$$\sqrt{N}\left(\frac{1}{2\sqrt{\pi}\sqrt{q\alpha}}\right) > \left(\frac{1}{8q\alpha}\right)$$

which is satisfied as the RHS is a positive constant and the LHS increases to infinity. Hence

$$q = 1/2 \quad \Longrightarrow \quad \alpha_M > \alpha_P \quad \Longrightarrow \quad T_M > T_P$$

Next, we compare underdog compensation effects. Given that for the M system we have

$$q\alpha_M \left(F^{-1}\left(\alpha_M\right)\right)^2 = (1-q)\,\beta_M \left(F^{-1}\left(\beta_M\right)\right)^2$$

and for the P system we have

$$q\alpha_P \left(F^{-1}\left(\alpha_P\right)\right) = (1-q)\,\beta_P \left(F^{-1}\left(\beta_P\right)\right)$$

then

$$\frac{1-q}{q} = \left(\frac{\alpha_P}{\beta_P}\right)^2 \left(\frac{\frac{F^{-1}(\alpha_P)}{\alpha_P}}{\frac{F^{-1}(\beta_P)}{\beta_P}}\right) = \left(\frac{\alpha_M}{\beta_M}\right)^3 \left(\frac{\frac{F^{-1}(\alpha_M)}{\alpha_M}}{\frac{F^{-1}(\beta_M)}{\beta_M}}\right)^2$$

By definition of derivative at zero we have

$$\frac{dF^{-1}}{dx}|_{x=0} = \lim_{x\to 0}\frac{F^{-1}\left(x\right)}{x} \in (0,\infty)$$

For N large, but α and β converge to zero both in the M and in the P system so

$$\lim_{N\to\infty}\left(\frac{\frac{F^{-1}(\alpha)}{\alpha}}{\frac{F^{-1}(\beta)}{\beta}}\right) = 1$$

and the result follows. If $\frac{dF^{-1}}{dx}|_{x=0} \in \{0,\infty\}$ then the above limit is indeterminate and the result need not be true. If the function F^{-1} is infinitely differentiable and n is the lowest integer for which

$$\frac{d^nF^{-1}}{dx^n}|_{x=0} \in (0,\infty)$$

then by iterating the procedure we have

$$\lim_{N\to\infty}\left(\frac{\frac{d^{n-1}F^{-1}(\alpha)}{d\alpha^{n-1}}}{\frac{d^{n-1}F^{-1}(\beta)}{d\beta^{n-1}}}\right) = 1$$

so the underdog compensation comparison generalizes to

$$\frac{1-q}{q} = \left(\frac{\alpha_P}{\beta_P}\right)^{n+1} = \left(\frac{\alpha_M}{\beta_M}\right)^{2n+1}$$

□

Proof of Lemma 9. Express the following series by differentiating and integrating the summands and inverting the series and integral operators

$$\begin{aligned}
\sum_{b=0}^{\infty}\frac{B^b}{b!}\frac{a}{a+b+c} &= \frac{a}{B^{a+c}}\sum_{b=0}^{\infty}\int_0^B \frac{d}{dr}\left(\frac{1}{b!}\frac{r^{a+b+c}}{a+b+c}\right)dr \\
&= \frac{a}{B^{a+c}}\int_0^B \sum_{b=0}^{\infty}\left(\frac{1}{b!}r^{a+b+c-1}\right)dr = \begin{cases} \frac{a}{B^{a+c}}\int_0^B r^{a+c-1}e^r dr & \text{for } a \geq 1 \\ 1/3 & \text{for } a=c=0 \end{cases}
\end{aligned}$$

and likewise

$$\sum_{b=0}^{\infty}\frac{B^b}{b!}\frac{a+1}{a+b+1} = \frac{a+1}{B^{a+c+1}}\int_0^B r^{a+c}e^r dr$$

We compute the marginal benefit for party A by inverting the series and integral operators again over the series over a.

$$\begin{aligned}
B_P^A &= e^{-(A+B+C)}\left(\sum_{c=0}^{\infty}\frac{C^c}{c!}\left(\begin{array}{c} \sum_{a=0}^{\infty}\frac{A^a}{a!}\left(\frac{a+1}{B^{a+c+1}}\int_0^B r^{a+c}e^r dr\right) \\ -\sum_{a=1}^{\infty}\frac{A^a}{a!}\left(\frac{a}{B^{a+c}}\int_0^B r^{a+c-1}e^r dr\right)\end{array}\right)-\frac{1}{3}\right) \\
&= e^{-(A+B+C)}\left(\sum_{c=0}^{\infty}\frac{C^c}{c!}\left(\begin{array}{c} \int_0^B \frac{r^c}{B^{c+1}}\left(\sum_{a=0}^{\infty}\frac{(Ar/B)^a}{(a-1)!}+\sum_{a=0}^{\infty}\frac{(Ar/B)^a}{a!}\right)e^r dr \\ -\int_0^B \frac{r^{c-1}}{B^c}\left(\sum_{a=1}^{\infty}\frac{(Ar/B)^a}{(a-1)!}\right)e^r dr\end{array}\right)-\frac{1}{3}\right) \\
&= e^{-(A+B+C)}\left(\sum_{c=0}^{\infty}\frac{C^c}{c!}\left(\begin{array}{c} \int_0^B \frac{r^c}{B^{c+1}}\left((Ar/B)\,e^{(Ar/B)}+e^{(Ar/B)}\right)e^r dr \\ -\int_0^B \frac{r^{c-1}}{B^c}\left((Ar/B)\,e^{(Ar/B)}\right)e^r dr\end{array}\right)-\frac{1}{3}\right)
\end{aligned}$$

Inverting the series and integral operators again over the series over c.

$$\begin{aligned}
B_P^A &= e^{-(A+B+C)}\left(\begin{array}{c} \int_0^B \left((Ar/B)\,e^{(Ar/B)}+e^{(Ar/B)}\right)\left(\sum_{c=0}^{\infty}\frac{C^c}{c!}\frac{r^c}{B^{c+1}}\right)e^r dr \\ -\int_0^B \left((Ar/B)\,e^{(Ar/B)}\right)\left(\sum_{c=0}^{\infty}\frac{C^c}{c!}\frac{r^{c-1}}{B^c}\right)e^r dr - \frac{1}{3}\end{array}\right) \\
&= e^{-(A+B+C)}\left(\begin{array}{c} \int_0^B \left((Ar/B)\,e^{(Ar/B)}+e^{(Ar/B)}\right)\frac{e^{rC/B}}{B}e^r dr \\ -\int_0^B \left((Ar/B)\,e^{(Ar/B)}\right)\frac{e^{rC/B}}{r}e^r dr - \frac{1}{3}\end{array}\right) \\
&= e^{-(A+B+C)}\left(\int_0^B \left(\left(\frac{A}{B}re^{\frac{A+B+C}{B}r}+e^{\frac{A+B+C}{B}r}\right)\frac{1}{B}-\frac{A}{B}e^{\frac{A+B+C}{B}r}\right)dr-\frac{1}{3}\right)
\end{aligned}$$

Computing the integral and simplifying, we have

$$\begin{aligned}
B_P^A &= e^{-(A+B+C)}\left(\left(\begin{array}{c}\left(AB\left(\frac{1-e^{A+B+C}}{(A+B+C)^2}+\frac{e^{A+B+C}}{A+B+C}\right)+\left(B\frac{e^{A+B+C}-1}{A+B+C}\right)\right)\frac{1}{B}\\ -\frac{A}{B}\left(B\frac{e^{A+B+C}-1}{A+B+C}\right)\end{array}\right)-\frac{1}{3}\right)\\
&= e^{-(A+B+C)}\left(A\frac{1-e^{A+B+C}}{(A+B+C)^2}+\frac{e^{A+B+C}-1}{A+B+C}+\frac{A}{A+B+C}-\frac{1}{3}\right)\\
&= \frac{B+C}{(A+B+C)^2}\left(1-e^{-(A+B+C)}\right)+\left(\frac{A}{A+B+C}-\frac{1}{3}\right)e^{-(A+B+C)}\\
&= \left(1-\frac{A}{A+B+C}\right)\frac{1-e^{-(A+B+C)}}{A+B+C}+\left(\frac{A}{A+B+C}-\frac{1}{3}\right)e^{-(A+B+C)}
\end{aligned}$$

□

Proof of Proposition 10. A similar calculation gives the analogous result for r parties:

$$B_P^A(r)=\left(\begin{array}{c}\left(1-\frac{A}{A+B+C+...+r}\right)\frac{1-e^{-(A+B+C+...+r)}}{A+B+C+...+r}\\ +\left(\frac{A}{A+B+C+...+r}-\frac{1}{r}\right)e^{-(A+B+C+...+r)}\end{array}\right)$$

For large enough N, B_P^A approximates to

$$\begin{aligned}
B_P^A &\simeq \left(1-\frac{A}{A+B+C+...+r}\right)\frac{1}{A+B+C+...+r}\\
&= \left(\frac{\beta q_B+\gamma q_C+..}{(\alpha q_A+\beta q_B+\gamma q_C+..)^2}\right)\frac{1}{N}
\end{aligned}$$

so the benefit still decreases as N^{-1}, which implies a higher turnout than in M except in the case when the two parties in M have the same ex-ante support: $q=1/2$.

For r parties with equal ex-ante support we have

$$q_A=q_B=q_C=...=q_r=1/r \quad\Longrightarrow\quad \alpha=\beta=\gamma=...$$

the first order condition for a party becomes

$$\left(1-\frac{1}{r}\right)\frac{1-e^{-\alpha_r N}}{\alpha_r N}\approx\left(1-\frac{1}{r}\right)\frac{1}{\alpha_r N}=F^{-1}(\alpha_r)$$

so the turnout for that party α_r increases in r. Overall turnout increases too as in this symmetric case we have.

$$T_r=\alpha_r$$

□

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