# Do You Think About What I Think you Think? Finite Belief Hierarchies in Games*

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December 28, 2009

**Abstract**

This paper models players with limited depths of reasoning. It does so by constructing finite belief hierarchies. A key feature is that players' language is too coarse to conceive of higher levels than their own. The type space I construct embeds the universal type space with infinite hierarchies. As in the standard framework, a type corresponds to a belief over other players' types. However, players with limited depth of reasoning have a coarser language to "talk" about other players' types than more sophisticated players. Unlike in models of cognitive hierarchies or $k$-level reasoning, a player can believe that another player is at least as sophisticated as she is.

*Many thanks to Adam Brandenburger—in all honesty, this paper could not have been written without his input. I'm also grateful to Songzi Du, Yossi Feinberg, Aviad Heifetz, Joe Halpern, Andrés Perea, and Elias Tsakas for comments and stimulating discussions. This research was carried out during a visit to Stanford University. I thank Stanford for its hospitality.

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*Information travels at the speed of logic, genuine knowledge only travels at the speed of cognition and inference.*
Barwise (1988)

# 1 Introduction

How do people reason about others in strategic settings and how does that affect their behavior? These questions have been at the forefront of game theory since its inception in the first half of the twentieth century. Traditionally, the focus has been on the question how "rational" players behave. More recently, the literature in behavioral game theory has investigated various deviations of perfect rationality (Camerer, 2003). As already observed by von Neumann and Morgenstern (1944, 4.1.2), however, the question how rational players should behave cannot be separated from the question how non-rational players behave. Even if one is concerned only with rational behavior, the interactive nature of the problem makes that one has to deal with *all* possible types of players: What is optimal for a rational player depends on what he expects his opponents to do, and these opponents may be boundedly rational. It is therefore desirable to have a theory of behavior in strategic settings that encompasses both perfect rationality and forms of bounded rationality.

A natural assumption is that players may not reason about everything they could potentially reason about. In particular, players may not form beliefs of arbitrarily high order, i.e., beliefs about others beliefs about their beliefs... about their beliefs about some event $E$. While natural,[1] this assumption is very much at odds with the Bayesian approach, which assumes that players have (subjective) beliefs about all relevant uncertainty, and therefore also about the beliefs of other players, the beliefs of other players about their beliefs, and so on (Tan and Werlang, 1988). The Bayesian approach naturally leads to infinite hierarchies of beliefs (e.g., Mertens and Zamir, 1985; Brandenburger and Dekel, 1993). This paper takes a different perspective, based on the idea that players may stop reasoning at some point, and thus do not further "refine" their view of the world.

To understand the main idea, it is instructive to consider the following setting, loosely based on an example by Savage (1954, pp. 13–15). Bob walks into the kitchen, seeing that Ann, his wife, has broken five good eggs into a bowl to make an omelet. A sixth egg, which must either be used for the omelet or wasted, lies beside the bowl, unbroken. Bob needs to decide whether to break the egg into the bowl, break it into a separate saucer for inspection, or throw it away without inspection. According to Savage, there are two states of the world: "The egg is good" and "The egg is rotten". If Bob assigns (subjective) probability 0.9 to the

[1] The hypothesis that players only have a limited depth of reasoning has indeed received some empirical support; see the discussion below.

egg being good, he may well decide to break the egg into the bowl. However, a question that Bob *could* have asked is why Ann did not break the egg into the bowl. That is, while he has a first-order belief on the state of the egg, he does not have a second-order belief, i.e., a belief about Ann's belief about the egg. Maybe Ann knows that the sixth egg is old, and should be thrown away. Perhaps she also believes that Bob will understand that that is the reason why she did not use the egg for the omelet. Had Bob reasoned about Ann's beliefs about the egg, and about Ann's beliefs about his beliefs about her beliefs, he might not have wasted five good eggs by adding the sixth, rotten, one to the bowl.

This example suggests that the state space of Savage can be further refined by considering Ann's and Bob's higher-order beliefs about each other. Indeed, taking this to the logical extreme leads to infinite belief hierarchies, such as those constructed by Mertens and Zamir (1985), Brandenburger and Dekel (1993) and others. However, this example also suggests that this refinement of the state space is based on the extent to which Bob takes Ann's perspective—on the state of the egg, Ann's beliefs about the egg, and so on. However, the extent to which players can take each other's perspective seems to be limited by cognitive constraints. Hence, it is natural to assume that players will have limited depth of reasoning, i.e., have *finite belief hierarchies*.

What do these finite hierarchies look like? First, as the example suggests, a player with a finite hierarchy does not "reason" about certain higher-order beliefs, and that means that she can only "talk" of an opponent having limited depth of reasoning. In the example, Ann has a belief about the egg (the egg is bad) and about Bob's beliefs about the egg (Bob believes the egg is bad), about Bob's beliefs about her beliefs about the egg (Bob believes that Ann believes the egg is bad), but she may not have higher-order beliefs. That means she can only "talk" about a Bob who reasons about her beliefs about the egg, not about a Bob who reasons to any higher orders, e.g., about her beliefs about her beliefs about the egg. Second, by a similar argument, Ann can only talk about herself reasoning about Bob's beliefs about the egg. Third, while Ann is certain in the current example that Bob is at least as sophisticated as she is (in her language, they can both talk about a player who has beliefs about the other player's beliefs about the egg), it is also possible that Ann is unsure how sophisticated Bob is. In the example, Ann is certain that Bob would think about her beliefs about the egg (even though he did not), but if she considers the possibility that he doesn't, she might throw away the egg herself, just in case.

I formalize these ideas as follows. A player with a greater depth of reasoning has a "finer" language to talk about higher-order beliefs than a player of shallower depth: If Ann has a greater depth of reasoning than Bob, she can distinguish more events than he does. Specifically, I construct the belief hierarchies of players using a similar approach as Mertens and Zamir (1985), but with the important difference that players have a coarser $\sigma$-algebra than players who are more sophisticated in their reasoning. In the approach of Mertens and Zamir, each player has a probability measure on the Borel $\sigma$-algebra about each level of uncertainty. Given the topological assumptions Mertens and Zamir make, this means that at some level $k$, each player can assign a probability to some (singleton) element of a given basic space of uncertainty $S$ (e.g., the space consisting of the two states "The egg is good", "The egg is bad"), the beliefs (probability measures) of all players about $S$ (their first-order beliefs), the beliefs of all players about the beliefs of all players about $S$ (their second-order beliefs), $\ldots$, and their $(k-1)$th-order beliefs. That is, at level $k$, each player can distinguish the singleton belief hierarchies up to level $k-1$. By a recursive construction, Mertens and Zamir arrive at infinite belief hierarchies, meaning that each player can assign a probability to the belief hierarchies that coincide for the first $k$ levels, for any finite $k$. In fact, they show that such an infinite belief hierarchy or type is equivalent to a Borel probability measure on $S$ and the set of infinite belief hierarchies. Hence, a player with a Mertens-Zamir type can assign a probability to each individual belief hierarchy or type.

By contrast, in the current construction, a player may stop reasoning at a certain depth $\Delta$. Technically, that means that her $\sigma$-algebra—i.e., her language to talk about $S$ and about (higher-order) beliefs—is not refined above $\Delta$. While a player who reasons up to level $\Delta$ can distinguish the belief hierarchies that differ at some level lower than $\Delta$, she lumps together all belief hierarchies that only differ at higher levels. I show that this implies that a player of depth $\Delta$ can assign a probability to the event that another player has depth $k$ for any $k < \Delta - 1$, but not for any $k \geq \Delta - 1$; rather, she can only assign a probability to the event that another player has depth *at least* $\Delta - 1$. Indeed, it is possible that a player is certain that a player has depth at least $\Delta - 1$, that is, the greatest depth she can imagine. Similarly, she knows that her own depth is at least $\Delta - 1$, but reasoning about it further would require her to have greater depth.

It is important to note that the current construction does not directly endow a player with a belief about her opponents' depth of reasoning, given her own depth. Rather, this follows from the construction: If Ann has a given depth $\Delta$, her language is such that she can assign a probability to Bob having depth $k$ for any $k < \Delta - 1$, or to the event that Bob has depth at least $\Delta - 1$, i.e., is at least as sophisticated as she is. In this, the current framework differs markedly from the approach taken by the behavioral economics literature on finite-level reasoning (e.g., Nagel, 1995; Stahl and Wilson, 1995; Camerer, 2003; Camerer et al., 2004; Costa-Gomes and Crawford, 2006).[2] This literature assumes that each player is endowed with

[2]See Strzalecki (2009) for a model that unifies different approaches and a construction of a space of cognitive types in the spirit of Harsanyi (1967–1968).

a so-called cognitive type; a player with a cognitive type $k$ must assign probability 1 to other players having depth at most $k-1$. One feature of this setup that is somewhat awkward is that if Ann and Bob have cognitive types $k^a$ and $k^b$ respectively, then the beliefs of one of them must be wrong, and this is common belief. The framework proposed here does not have this feature, since players can believe that others are at least as sophisticated as they are. Indeed, it seems natural in some settings that a player knows that his opponent is at least as sophisticated as she is, even if she cannot "say" what his depth of reasoning is exactly—that is something she cannot reason about. This opens up the way to exploring new solution concepts, where players can try to outguess the other player, but realize that the other may outguess them instead.

A further advantage of the current model is that it allows both for perfect rationality and bounded rationality, something which is not possible in models of $k$-level reasoning and cognitive hierarchies, where players can have any finite depth, but not infinite depth.[3] That is, the type space I construct contains both finite and infinite belief hierarchies. In fact, I show that the Mertens-Zamir universal type space can be embedded in the current type space as a belief-closed subset.

Finally, the current approach provides a natural framework to study belief revision. If Ann has a given depth, and she observes a move by Bob that she cannot immediately rationalize, she may realize that he is more sophisticated than she is. This allows her to refine her language, thus becoming more sophisticated herself. By contrast, in the literature on cognitive hierarchies and $k$-level reasoning, players can only become more sophisticated by conditioning on probability-0 events, which seems less natural.

A distinct feature of the current approach is that it explicitly models players' reasoning processes and beliefs, as in epistemic program in game theory (see e.g. Brandenburger, 2007, for an overview of recent results). However, the literature on epistemic game theory typically studies the implications of rationality in settings where players are perfect reasoners, focusing on decision-theoretic criteria, such as dominance and admissibility.[4] By contrast, the current model explicitly allows for players who are boundedly rational. More generally, the current paper can be viewed as an attempt to use the epistemic language to directly model players' cognition.

The outline of this paper is as follows. Section 2 contains some examples to illustrate some

[3]Of course, a type with infinite depth of reasoning could be added to these models, but it is not clear whether such a type would be the limit of finite belief hierarchies with increasing depth.

[4]An exception is the literature on unawareness (e.g., Fagin and Halpern, 1988; Modica and Rustichini, 1994; Dekel et al., 1998; Heifetz et al., 2006; Feinberg, 2009). While one could say that a player with a limited depth of reasoning is unaware that he could reason further, this is a different form of unawareness than generally considered in the economics literature. See the discussion in Section 5.4

modeling considerations. Belief hierarchies and the type space are constructed in Section 3. Section 4 discusses various notions of beliefs in the current setting. Section 5 contains a discussion of a number of technical issues and open questions. Section 6 concludes. Proofs not included in the main text can be found in the appendices.

# 2 Examples

There are two players, Ann and Bob, who are uncertain about some set $S = \{s_1, s_2\}$. Given a space of uncertainty $X$, a player either has a belief $\mu \in \mathcal{M}(X)$ about $X$, where $\mathcal{M}(X)$ is the set of Borel probability measures on $X$, or she has "no beliefs" about $X$. For now, I will not define formally what is meant by "no beliefs". I will be similarly vague when using terms such as "reason" and "think".

**Example 2.1 (Depth of reasoning)**
Suppose that Ann assigns probability $p_1^a$ to $s_1$, and probability $1 - p_1^a$ to $s_1$. Ann also has beliefs about Bob's beliefs about $S$: she assigns probability $p_2^a$ to Bob assigning probability $\tilde{p}_1^b = 1$ to $s_1$ and probability $1 - p_2^a$ to Bob assigning probability $\tilde{\tilde{p}}_1^b = 1$ to $s_2$. Finally, she has beliefs about Bob's beliefs about her beliefs about $S$: she assigns probability $p_3^a$ to Bob placing probability $\tilde{p}_2^b$ on her assigning probability 1 to $s_1$ and $1 - \tilde{p}_2^b$ on her assigning probability 1 to $s_2$ and probability $1 - p_a^3$ to Bob placing probability $\tilde{\tilde{p}}_2^b = 1$ on her assigning probability 1 to $s_1$. Ann has no beliefs of higher order: she has no beliefs about Bob's beliefs about her beliefs about his beliefs about $S$, no beliefs about Bob's beliefs about her beliefs about his beliefs about her beliefs about $S$, and so on. What does Ann "think" about Bob's higher-order beliefs? In a sense, she only "reasons" about a Bob who has beliefs about $S$ and beliefs about her beliefs about $S$; she does not "reason" about a Bob who has beliefs about her beliefs about his beliefs about $S$. ◁

This example illustrates some important points. First, if Ann has depth $k$, she can only "reason" about a Bob who is less sophisticated than she is. Similarly, Ann can only "talk" about a Bob who can only "talk" about an Ann who is less sophisticated than she is, and so on. Finally, if Ann cannot "reason" about Bob's $k$th-order beliefs, she cannot reason about his $(k+1)$th-order beliefs. Without this, it is possible that Ann does not have beliefs at level $k$, while she does does have beliefs about Bob's $k$th-level beliefs, or even about his beliefs about her beliefs at level $k$. That is, Ann can "reason" about $k$-level beliefs, though she may not have a $k$-level belief. As we will see, this condition naturally arises when players' beliefs are required to be coherent in the sense that higher-order beliefs do not contradict their beliefs at lower order.

The next example suggests that there are also some natural constraints on players' reasoning about others when there is uncertainty about others' depth of reasoning.

**Example 2.2 (Uncertainty about others' depth of reasoning)**
Suppose Ann assigns probability $p_1^a$ to $s_1$ and probability $1 - p_1^a$ to $s_2$. Also, assume that she assigns probability $p_2^a \in (0,1)$ to Bob having no beliefs about $S$. Then, if Ann has beliefs about Bob's beliefs about her beliefs about $S$, it seems reasonable to require that she assigns probability at least $p_2^a$ to Bob having no beliefs about her beliefs about $S$: a player (i.e., Bob) who has no beliefs at level $k$ has no beliefs at level $k+1$. $\triangleleft$

We have looked at how players reason about others' depth of reasoning. But, what can a player "know" about her own cognitive depth? That is, suppose Ann has a belief about $S$, a belief about Bob's beliefs about $S$, but no belief about Bob's beliefs about her beliefs about $S$, and so on. What does she believe about *her* beliefs about Bob's beliefs about her beliefs about $S$? One possible answer that I focus on here is that she has no beliefs about her beliefs about Bob's beliefs about her beliefs about $S$: Ann simply cannot "speak" about her beliefs at that order. An alternative answer that I briefly discuss in Section 5 is that she *does* know that she does not know: she assigns probability 1 to the event that she has no beliefs about her beliefs about Bob's beliefs about her beliefs about $S$, she assigns probability one to the event that she assigns probability one to the event that she does not have beliefs about Bob's beliefs about her beliefs about $S$, and so on.

# 3 Finite belief hierarchies and type space

## 3.1 Preliminaries

Given a metric space $X$, denote by $\mathcal{M}(X)$ the set of probability measures on the Borel $\sigma$-algebra $\mathscr{B}(X)$ in $X$, metrized by the Prohorov metric. If $X$ is compact metric (and thus Polish) then $\mathcal{M}(X)$ is compact metric (e.g., Kechris, 1995, Thm. 17.23). Let $X$ be a topological space, and let $\mu$ be a probability measure on $(X, \mathscr{B}(X))$. Then, a support of $\mu$, if it exists, is a closed set, denoted $\text{supp}(\mu)$, such that $\mu(X \setminus \text{supp}(\mu)) = 0$ and $\mu(G \cap \text{supp}(\mu)) > 0$ for any open set $G$ such that $G \cap \text{supp}(\mu) \neq \emptyset$. If $\mu$ has a support, it is unique, and $\mu(\text{supp}(\mu)) = 1$. If $X$ is second countable or if $\mu$ is tight, then $\text{supp}(\mu)$ exists (Aliprantis and Border, 2005, Thm. 12.14).[5] Also, given a topological space $X$, let $\nu_X$ be the probability measure on the trivial measurable space $(X, \{X, \emptyset\})$.

[5] A Polish space is second countable (Dudley, 2002, Prop. 2.1.4). Also, it is not hard to see that a subspace of a second-countable space is second countable.

The Cartesian product of topological spaces is endowed with the product topology. Given a Cartesian product $Z = \times_\ell Z_\ell$, the projection mapping from $U \subseteq Z$ to $Z_\ell$ is denoted $\pi^U_{Z_\ell}$. Given a probability measure $\mu$ on a subset $Z$ of a product space $Z_1 \times Z_2$, the marginal of $\mu$ on $Z_\ell$ "inherits" the $\sigma$-algebra of $\mu$ in the following sense. If $\mu$ is defined on the $\sigma$-algebra $\Sigma$, then the $\sigma$-algebra for $\mu \circ (\pi^Z_{Z_\ell})^{-1}$ is given by $\{\pi^Z_{Z_\ell}(B) : B \in \Sigma\}$. If $Z = A_1 \times \cdots \times A_1 \times A_2 \times \cdots A_2 \times \cdots \times A_m$ for spaces $A_1, \ldots, A_m$, denote by $(A_\ell)^i$ the $i$th copy of $A_\ell$, $\ell = 1, \ldots, m$. If $X = X^1 \times X^2 \times \cdots$ is a (finite or infinite) product set and $X^j = \emptyset$ for some $j$, then $X^i = \emptyset$ for all $i$.

## 3.2 Belief hierarchies

There are two players, Ann (denoted by 1) and Bob (denoted by 2), and a common uncertainty space $S$, assumed to be a compact metric space.[6] To avoid trivialities, I assume that $S$ has at least two elements. I construct players' belief hierarchies in a bottom-up fashion (cf. Mertens and Zamir, 1985).

Define

$$Y_0 := S,$$

and

$$\mathcal{M}^+(Y_0) := \mathcal{M}(Y_0) \cup \{\nu_{Y_0}\}.$$

Also, let

$$Y_1 := Y_0 \times \mathcal{M}^+(Y_0) \times \mathcal{M}^+(Y_0).$$

For $k = 2, 3, \ldots$, let

$$\begin{aligned} Y_k := \Big\{ & y_k \in Y_{k-1} \times \mathcal{M}^+(Y_{k-1}) \times \mathcal{M}^+(Y_{k-1}) : \text{ for } i = 1, 2, \\ & (1)\ \big(\pi^{Z_k}_{(\mathcal{M}^+(Y_{k-1}))^i}(y_k)\big) \circ \big(\pi^{Y_{k-1}}_{Y_{k-2}}\big)^{-1} = \pi^{Y_{k-1}}_{(\mathcal{M}^+(Y_{k-2}))^i}\big(\pi^{Z_k}_{Y_{k-1}}(y_k)\big); \\ & (2)\ \text{if } \pi^{Z_k}_{(\mathcal{M}^+(Y_{k-1}))^i}(y_k) \in \mathcal{M}(Y_{k-1}), \text{ then} \\ & \quad \pi^{Z_k}_{(\mathcal{M}^+(Y_{k-1}))^i}(y_k) \circ \left(\pi^{Y_{k-1}}_{(\mathcal{M}^+(Y_{k-2}))^i}\right)^{-1} = \delta_{\pi^{Y_{k-1}}_{(\mathcal{M}^+(Y_{k-2}))^i}(\pi^{Z_k}_{Y_{k-1}}(y_k))} \Big\}, \end{aligned} \tag{3.1}$$

where $Z_k := Y_{k-1} \times \mathcal{M}^+(Y_{k-1}) \times \mathcal{M}^+(Y_{k-1})$, and

$$\mathcal{M}^+(Y_k) := \mathcal{M}(Y_k) \cup \left( \bigcup_{\ell=0}^{k-1} \mathcal{M}_\ell(Y_k) \right) \cup \{\nu_{Y_k}\} \tag{3.2}$$

with $\mathcal{M}_\ell(Y_k)$ the collection of probability measures on $Y_k$ with $\sigma$-algebra

$$\Sigma^k_\ell := \big\{ \big(\pi^{Y_k}_{Y_\ell}\big)^{-1}(B) : B \in \mathscr{B}(Y_\ell) \big\},$$

[6]It may be possible to weaken some of the topological assumptions. The results extend to any finite number players in a straightforward way.

and $\delta_u$ is the point mass at $u$. By continuity of $\pi^{Y_k}_{Y_\ell}$, it is immediate that $\Sigma^k_\ell \subseteq \mathscr{B}(Y_k)$.

As is standard, the elements of the spaces $Y_0, Y_1, \ldots$ specify an element of $S$ as well as players' (higher-order) beliefs about $S$. Given $y = (y_0, y_1, \ldots) \in \times_{k=0}^{\infty} Y_k$ such that (1) and (2) in (3.1) are satisfied, refer to $h^i(y) := (\pi^{Y_1}_{(\mathcal{M}^+(Y_0))^i}(y_1), \pi^{Y_2}_{(\mathcal{M}^+(Y_1))^i}(y_2), \ldots) \in \times_k \mathcal{M}^+(Y_{k-1})$ as the *belief hierarchy* of player $i$ generated by $y$. There are three possible cases. First, it is possible to have $\pi^{Y_k}_{(\mathcal{M}^+(Y_{k-1}))^1}(y_k) \in \mathcal{M}(Y_{k-1})$ for all $k$, i.e., at every level, Ann has a belief (probability measure) defined on the Borel $\sigma$-algebra. In that case, we say that she has an *infinite (belief) hierarchy.* A second possibility is that there exists $\Delta$ such that $\pi^{Y_k}_{(\mathcal{M}^+(Y_{k-1}))^1}(y_k) \in \mathcal{M}(Y_{k-1})$ if $k-1 \le \Delta$, and $\pi^{Y_k}_{(\mathcal{M}^+(Y_{k-1}))^1}(y_k) \in \mathcal{M}_\Delta(Y_{k-1})$ for $k-1 > \Delta$. In that case, Ann has a *finite (belief) hierarchy.* Finally, it is possible that $\pi^{Y_k}_{(\mathcal{M}^+(Y_{k-1}))^1}(y_k) = \nu_{Y_{k-1}}$ for all $k$: Ann has *trivial beliefs.* Intuitively, if Ann has an infinite hierarchy, she has the finest possible language to talk about beliefs, while if she has a finite belief hierarchy, she can only distinguish between hierarchies that differ at lower levels. If Ann has trivial beliefs, she cannot "talk" about beliefs—her own or others'—at all.

We can now interpret the conditions in the definition of $Y_k$. Condition (1) requires that beliefs be coherent, to the extent that a player can "talk" about her beliefs. That is, if Ann has an infinite belief hierarchy, (1) is identical to the coherency condition of Mertens and Zamir (1985). Given $y = (y_0, y_1, \ldots) \in \times_{\ell=0}^{\infty} Y_k$ such that (1) and (2) are satisfied, suppose that Ann has a finite hierarchy at $y$: there exists $\Delta$ such that $\pi^{Y_k}_{(\mathcal{M}^+(Y_{k-1}))^1}(y_k) \in \mathcal{M}(Y_{k-1})$ if $k \le \Delta$, and $\pi^{Y_k}_{(\mathcal{M}^+(Y_{k-1}))^1}(y_k) \in \mathcal{M}_\Delta(Y_{k-1})$ otherwise. In that case, condition (1) requires that Ann's beliefs about $Y_k$ for $k > \Delta$ coincide with $\pi^{Y_\Delta}_{(\mathcal{M}^+(Y_{\Delta-1}))^1}(y_\Delta)$ regarding anything that concerns $Y_{\Delta-1}$, but there are no other restrictions on her beliefs at that level. Similarly, if Ann has trivial beliefs, there are no conditions on what she can believe. In the latter two cases, are Ann's beliefs coherent at the levels she cannot "reason" about? First note that while Ann cannot have higher-order beliefs that contradict the beliefs at the levels she can "talk" about, the higher-order beliefs that are consistent with her lower-level beliefs may contradict each other. However, Ann cannot distinguish between these different higher-order beliefs: she assigns positive probability to a sequence of sets, not to a sequence of individual higher-order beliefs.[7]

Condition (1) does more than requiring that belief hierarchies are coherent as in Mertens and Zamir (1985): it also ensures that players cannot have a "finer language" at higher levels than at lower levels. Intuitively, if Ann cannot "speak" of certain events at a given level, then she cannot "reason" about them at higher levels. Technically, (1) puts constraints on how players' $\sigma$-algebras can change from level to level. To see this, suppose that Ann has belief

[7] A similar issue seems to arise in the construction of higher-order beliefs in the presence of ambiguity, though in an entirely different framework (Ahn, 2007).

$\nu_{Y_{k-2}}$ about $Y_{k-2}$. What possible beliefs can she have about $Y_{k-1}$? Condition (1) requires that the marginal of her belief about $Y_{k-1}$ on $Y_{k-2}$ has $\sigma$-algebra $\{Y_{k-2}, \emptyset\}$. Given the set of possible measures $\mathcal{M}^+(Y_{k-1})$, she can only have belief $\nu_{Y_{k-1}}$ about $Y_{k-1}$. Similarly, suppose Ann's belief about $Y_{k-2}$ is given by some measure in $\mathcal{M}_\ell(Y_{k-2})$ for some $\ell < k-2$. Then her belief $\mu_{k-1}$ about $Y_{k-1}$ cannot be a member of $\mathcal{M}(Y_{k-1})$ or $\mathcal{M}_m(Y_{k-1})$ for some $m > \ell$: if that were the case, the $\sigma$-algebra of the marginal of $\mu_{k-1}$ on $Y_{k-2}$ would be too fine; conversely, if $\mu_{k-1}$ were defined on $\Sigma^h_{k-1}$ for $h < \ell$ or on $\{Y_{k-1}, \emptyset\}$, then the marginal of $\mu_{k-1}$ would be defined on too coarse a $\sigma$-algebra. What happens if Ann's belief about $Y_{k-2}$ is given by a measure defined on the Borel $\sigma$-algebra $\mathscr{B}(Y_{k-2})$? Then there are two options: either her belief about $Y_{k-1}$ is defined on the Borel $\sigma$-algebra $\mathscr{B}(Y_{k-1})$, or it is defined on $\Sigma^{k-1}_{k-2}$. In the latter case, one could say that she stops "reasoning": her $\sigma$-algebra on $Y_{k-1}$ does not distinguish between elements that coincide up to $Y_{k-2}$.

Condition (2) requires Ann to know her own lower-order beliefs at $k$ *provided* her belief about $Y_{k-1}$ is defined on the full $\sigma$-algebra $\mathscr{B}(Y_{k-1})$. Why only require that Ann knows her own lower-order beliefs if her $k$-level belief is defined on the full $\sigma$-algebra? The reason is simple: if a player does not have $k$-level beliefs defined on the full $\sigma$-algebra, she cannot "speak" of her own higher-order beliefs. Hence, she cannot know them. I discuss an alternative specification in Section 5.

## 3.3 Universal beliefs space and type space

Now that we have constructed players' belief hierarchies, we can construct the state space and type spaces. I start with some preliminary results.

**Proposition 3.1** *For each $k$, $Y_k$ is nonempty and compact metric, and thus Polish.*

Let $G := \times_{k=0}^{\infty} Y_k$, and let $Y$ be the subset of $G$ that consists of all $y$ that satisfy

$$\pi^G_{Y_k}(y) = \pi^{Y_m}_{Y_k}\left(\pi^G_{Y_m}(y)\right) \tag{3.3}$$

for all $k, m = 0, 1, \ldots$ such that $k \leq m$. Let $\tau$ be the weakest topology on $Y$ such that $\pi^G_{Y_k}$ is continuous for each $k$. That is, $(Y, \tau)$ is the *inverse limit* of the spaces $Y_0, Y_1, \ldots$ with their respective topologies. The restriction of $\pi^G_{Y_k}$ to $Y$ is the *canonical mapping* from $Y$ into $Y_k$, and is denoted by $\pi_{Y_k}$.

**Theorem 3.2** *The inverse limit $Y$ is nonempty and compact metric.*

**Proof.** I first show that $\pi^{Y_k}_{Y_{k-1}}(Y_k) = Y_{k-1}$. Clearly, $\pi^{Y_k}_{Y_{k-1}}(Y_k) \subseteq Y_{k-1}$ for all $k$. Hence, it is sufficient to show that the reverse inclusion holds for all $k$. The inclusion holds by definition for $k = 1$. Suppose that $\pi^{Y_k}_{Y_{k-1}}(Y_k) \supseteq Y_{k-1}$, and consider $y_k \in Y_k$. We need to show that there exist $\mu^1_k, \mu^2_k \in \mathcal{M}^+(Y_k)$ such that $(y_k, \mu^1_k, \mu^2_k) \in Y_{k+1}$. If $\pi^{Y_k}_{(\mathcal{M}^+(Y_{k-1}))^i}(y_k) = \nu_{Y_{k-1}}$, set $\mu^i_k = \nu_{Y_k}$. If $\pi^{Y_k}_{(\mathcal{M}^+(Y_{k-1}))^i}(y_k) \in \mathcal{M}_\ell(Y_{k-1})$, take $\mu^i_k \in \mathcal{M}_\ell(Y_{k-1})$ such that it coincides with $\pi^{Y_k}_{(\mathcal{M}^+(Y_{k-1}))^i}(y_k)$ on all elements of $\mathscr{B}(Y_\ell)$. If we have thus constructed a probability measure $\mu^i_k$ for both players, we are done: $(y_k, \mu^1_k, \mu^2_k) \in Y_{k+1}$. If this procedure gives us $\mu^1_k$, then by Proposition 2.10 of Mertens and Zamir (1985), there exists $\tilde{\mu}^2_k \in \mathcal{M}(Y_k)$ such that $(y_k, \mu^1_k, \tilde{\mu}^2_k) \in Y_{k+1}$. Otherwise, the result follows directly from Proposition 2.10 of Mertens and Zamir (1985).

Using this result and Proposition 3.1, we can now apply Proposition 9.6.8 of Bourbaki (1998) which establishes that $Y$ is nonempty and compact. To prove that $Y$ is metric, note that the topology $\tau$ on $Y$ is equivalent to the topology on $Y$ induced by the product topology on $G$ (Bourbaki, 1998, p. 48). Since $Y_k$ is metric for all $k$ (Proposition 3.1), $G$ is metric. It follows that $G$ is Polish, and thus Hausdorff. Hence, as $Y$ is a compact subspace of $G$ in the relative topology, it is a closed subspace of $G$ and therefore compact metric (Prop. 3.3 Kechris, 1995). $\square$

The space $Y$ is the analogue of the *universal beliefs space* of Mertens and Zamir (1985). Let $y = (y_0, y_1, \dots) \in Y$ and $i = 1, 2$. Recall that $h^i(y) := (\pi^{Y_1}_{(\mathcal{M}^+(Y_0))^i}(y_1), \pi^{Y_2}_{(\mathcal{M}^+(Y_1))^i}(y_2), \dots) \in \times_k \mathcal{M}^+(Y_{k-1})$ is the belief hierarchy of player $i$ generated by $y$. I show that each belief hierarchy generated by some element of $Y$ defines a unique probability measure on $Y$. I treat the different cases—infinite hierarchies, finite hierarchies, and trivial beliefs— in turn; throughout, I use the notation $\mu^i_{k-1}(y) := \pi^{Y_k}_{(\mathcal{M}^+(Y_{k-1}))^i}(y_k)$.

**Case 1: Infinite hierarchies** Let $y \in Y$ and let $i$ be a player such that $\mu^i_{k-1}(y) \in \mathcal{M}(Y_{k-1})$ for all $k$. Let $f$ be a real-valued continuous function that depends on a finitely many coordinates of $y \in Y$, and consider $(\int_Y f d\mu^i_{k-1}(y))_{k \in \mathbb{N}}$. By condition (1) in (3.1), and since $\pi^Y_{Y_k}(Y) = Y_k$ for all $k$, this sequence of integrals is well defined and constant for $k$ sufficiently large. Hence, the sequence $(\mu^i_0(y), \mu^i_1(y), \dots)$ defines a positive linear functional $M^i(y)$ on the Riesz space $E$ of real-valued continuous functions on $Y$ that depend on finitely many coordinates:

$$\forall f \in E : \qquad M^i(y)(f) = \lim_{k \to \infty} \int_Y f d\mu^i_k(y),$$

and $M^i(y)$ has norm $\|M^i(y)\| = 1$.[8] It is immediate that the set $E$ forms an algebra in $C(Y)$, separates points in $Y$, and contains the constant functions. Hence, by the Stone-Weierstrass theorem (e.g., Aliprantis and Border, 2005, Thm. 9.13), $E$ is dense in the space $C(Y)$ of

[8]Recalling that $Y$ is a compact space (and that each $f \in E$ is continuous), we endow $E$ with the supremum norm; also, $\mathbb{R}$ is endowed with its usual Euclidean norm.

continuous real-valued functions on $Y$ in the uniform topology. That is, the closure of $E$ is $C(Y)$. By Theorem 8.32 of Aliprantis and Border (2005), $M^i(y)$ extends to a positive linear functional $\widetilde{M}^i(y)$ on $C(Y)$. Because $M^i(y)$ is bounded, the functional is continuous (on $E$) (Aliprantis and Border, 2005, Lemma 6.4), so that $\|\widetilde{M}^i(y)\| = 1$ (recall that for every $f \in C(Y)$, there exists a sequence in $E$ that converges to $f$ in the uniform topology). Then, by a version of the Riesz representation theorem (Aliprantis and Border, 2005, Thm. 14.12), there exists a unique regular Borel probability measure $t^i(y)$ on $Y$ that represents $\widetilde{M}^i(y)$, that is, for all $g \in C(Y)$,

$$\widetilde{M}^i(y)(g) = \int_Y g dt^i(y).$$

Let $T_\infty \subseteq \mathcal{M}(Y)$ be the collection of Borel probability measures on $Y$ defined in this way.

What is the relation between the probability measure $t^i(y)$ and the hierarchy $h^i(y) = (\mu_0^i(y), \mu_1^i(y), \ldots)$? For each $y \in Y$, the belief $t^i(y)$ on $Y$ agrees with the belief $\mu_k^i(y)$ on $Y_k$ in the following way: for every $B_k \in \mathscr{B}(Y_k)$,

$$t^i(y)\left(\left(\pi_{Y_k}\right)^{-1}(B_k)\right) = \mu_k^i(y)(B_k).$$

To see this, first note that $\pi_{Y_k}$ is continuous, and thus Borel measurable, so that indeed $(\pi_{Y_k})^{-1}(B_k) \in \mathscr{B}(Y)$. Let $f : Y \to \mathbb{R}$ be defined by:

$$\forall y \in Y: \qquad f(y) = \begin{cases} 1 & \text{if } y \in \left(\pi_{Y_k}\right)^{-1}(B_k); \\ 0 & \text{otherwise.} \end{cases}$$

Then the indicator function $f$ depends only on finitely many coordinates, and is continuous, i.e., $f \in E$, so that

$$\begin{aligned} \widetilde{M}^i(y)(f) &= \int_Y f dt^i(y) \\ &= t^i(y)\left(\left(\pi_{Y_k}\right)^{-1}(B_k)\right) \\ &= \lim_{n\to\infty} \int_Y f d\mu_n^i(y) \\ &= \mu_k^i(y)(B_k), \end{aligned}$$

where I have used that $\widetilde{M}^i(y)(g) = M^i(y)(g)$ for $g \in E$.

**Case 2: Finite hierarchies** Fix a player $i$ and $y \in Y$, and suppose $h^i(y) = (\mu_0^i(y), \mu_1^i(y), \ldots)$ is such that $\mu_k^i(y) \in \mathcal{M}(Y_k)$ for $k \leq \Delta$, and $\mu_k^i(y) \in \mathcal{M}_\Delta(Y_k)$ otherwise. That is, the hierarchy $h^i(y)$ has depth $\Delta$. Then, $h^i(y)$ can be represented by the unique probability measure $t^i(y)$ on the measurable space $(Y, \Sigma_\Delta^Y)$, with

$$\Sigma_\Delta^Y := \left\{\left(\pi_{Y_\Delta}\right)^{-1}(B) : B \in \mathscr{B}(Y_\Delta)\right\},$$

such that for all $E \in \Sigma_\Delta^Y$,

$$t^i(y)(E) = \mu_\Delta^i\left(\pi_{Y_\Delta}(E)\right).$$

It can be checked that the collection of probability measures on $(Y, \Sigma_\Delta^Y)$ is homeomorphic to $\mathcal{M}(Y_\Delta)$. For $\Delta = 0, 1, \ldots$, let $T_\Delta$ be the subset of probability measures on $(Y, \Sigma_\Delta^Y)$ defined in this way.

In this case, $t^i(y)$ is consistent with $h^i(y)$, but only to the extent that a player can "speak" of her beliefs. That is, suppose $k \leq \Delta$, and let $B_k \in \mathscr{B}(Y_k)$. Then,

$$\begin{aligned} t^i(y)\left(\left(\pi_{Y_k}\right)^{-1}(B_k)\right) &= \mu_\Delta^i\left(\pi_\Delta\left(\{y \in Y : \pi_{Y_k}^{Y_\Delta}\left(\pi_{Y_\Delta}(y)\right) \in B_k\}\right)\right) \\ &= \mu_\Delta^i\left(\left(\pi_{Y_k}^{Y_\Delta}\right)^{-1}(B_k)\right) \\ &= \mu_k^i(y)(B_k), \end{aligned}$$

where in the first equality I have used that $\pi_{Y_k} = \pi_{Y_k}^{Y_\Delta} \circ \pi_{Y_\Delta}$, and in the last that beliefs are coherent. What about $k > \Delta$? Player $i$ can only "talk" about elements of the set

$$\left\{\left(\pi_{Y_\Delta}\right)^{-1}(E_\Delta) : E_\Delta \in \mathscr{B}(Y_\Delta)\right\} = \left\{\left(\pi_{Y_k}\right)^{-1}\left(\left(\pi_{Y_\Delta}^{Y_k}\right)^{-1}(E_\Delta)\right) : E_\Delta \in \mathscr{B}(Y_\Delta)\right\}.$$

Hence, $i$'s vocabulary to talk about subsets of $Y_k$ is limited. But for the subsets $i$ can talk about, her beliefs $t^i(y)$ coincide with $\mu_\Delta^i(y)$. Suppose $B_k = (\pi_{Y_\Delta}^{Y_k})^{-1}(E_\Delta) \in \mathscr{B}(Y_k)$ for some $E_\Delta \in \mathscr{B}(Y_\Delta)$. Then,

$$\begin{aligned} t^i(y)\left(\left(\pi_{Y_k}\right)^{-1}(B_k)\right) &= \mu_k^i(y)\left(\pi_{Y_\Delta}^{Y_k}\left(\pi_{Y_k}\left((\pi_{Y_k})^{-1}(B_k)\right)\right)\right) \\ &= \mu_\Delta^i(y)(E_\Delta). \end{aligned}$$

**Case 3: Trivial belief** The hierarchy $h^i(y) = (\nu_{Y_0}, \nu_{Y_1}, \ldots)$ can be represented by $t^i(y) = \nu_Y$, where $\nu_Y$ is the probability measure on $(Y, \{Y, \emptyset\})$. A player with such a hierarchy cannot reason about any proper subset of $Y$, and there is no relation between $t^i(y)$ and $\mu_k^i(y)$ for any $k$.

Summarizing, each belief hierarchy $h^i(y)$—finite, infinite or trivial—-defines a belief on $Y$. While a player with an infinite hierarchy can distinguish between all individual hierarchies,[9] a player with depth $\Delta$ at $y \in Y$ can only distinguish between hierarchies that differ in their beliefs on $Y_{\Delta-1}$, but not between hierarchies that differ at higher levels: the finest events in her "language" are sets of the form

$$\{y \in Y : \pi_{Y_\Delta}(y) = y_\Delta\},$$

[9]Recall that singletons are closed in Hausdorff spaces, and that the Borel $\sigma$-algebra contains the closed sets.

where $y_\Delta \in Y_\Delta$. A player with trivial beliefs, finally, cannot distinguish any hierarchies: she cannot reason about any proper subset of $Y$.

Define

$$T := T_\infty \cup \left( \bigcup_{\Delta=0}^{\infty} T_\Delta \right) \cup \{\nu_Y\}.$$

It will be helpful to define a metric on $T$. As $Y$ is compact metrizable, so is $\mathcal{M}(Y)$ (Aliprantis and Border, 2005, Thm. 15.11, 15.15), and its topology can be metrized by the Prohorov metric $\rho$ (Dudley, 2002, Thm. 11.3.3). Also, let $\rho_\Delta$ be the Prohorov metric on $\mathcal{M}(Y_\Delta)$. Define the function $\rho^+ : T \times T \to \mathbb{R}$ by:

$$\forall t, t' \in T : \qquad \rho^+(t, t') = \begin{cases} \rho(t, t') & \text{if } t, t' \in T_\infty, \\ \rho_\Delta(t \circ (\pi_{Y_\Delta})^{-1}, t' \circ (\pi_{Y_\Delta})^{-1}) & \text{if } t, t' \in T_\Delta, \\ 0 & \text{if } t = t' = \nu_Y, \\ 2 & \text{otherwise,} \end{cases}$$

where it can be checked that $t \circ (\pi_{Y_\Delta})^{-1} \in \mathcal{M}(Y_\Delta)$ if $t \in T_\Delta$. It can be verified that $\rho^+$ is a metric on $T$, so that $T$ is Hausdorff.

The space $T$ is the analogue of the universal type space of Mertens and Zamir (1985). I will refer to the elements of $T$ as *types* and to $T$ as the *type space*. However, note that types here need not correspond to infinite hierarchies of beliefs, as is standard. Also, it is not clear whether $T$ is universal; see the discussion in Section 5.3.

**Proposition 3.3** *There exists a homeomorphism $\phi$ from $Y$ to $S \times T \times T$.*

**Proof.** I first construct a mapping from $Y$ to $S \times T \times T$ and show that it is a bijection. Let $\phi : Y \to S \times T \times T$ be the mapping defined by:

$$\forall y \in Y : \qquad \phi(y) = (\pi_{Y_0}(y), t^1(y), t^2(y)),$$

where $t^1(y), t^2(y)$ are probability measures on $Y$ generated by the belief hierarchies $h^1(y), h^2(y)$, respectively, as described above. By construction, $z = \phi(y)$ and $z' = \phi(y)$ implies $z' = z$. That is, $\phi$ is a function. Conversely, let $(s, t^1, t^2) \in S \times T \times T$. For $i = 1, 2$, $t^i$ is a probability measure on $Y$ that is derived from a belief hierarchy $h^i = (\mu_0^i, \mu_1^i, \ldots)$ such that for all $k$, $\mu_k^i \in \mathcal{M}^+(Y_k)$ and conditions (1) and (2) in (3.1) are satisfied. Hence, it is possible to associate with each $(s, t^1, t^2)$ a unique $\chi((s, t^1, t^2)) \in Y$ such that $y_0 = s$ and $(\pi^{Y_1}_{(\mathcal{M}^+(Y_0))^i}(y_1), \pi^{Y_2}_{(\mathcal{M}^+(Y_1))^i}(y_2), \ldots) = h^i$ for $i = 1, 2$. That is, $\chi$ defines a function from $S \times T \times T$ to $Y$, and it can be checked that $\chi$ is the inverse of $\phi$. Consequently, each element of $Y$ corresponds to a unique element of $S \times T \times T$, and vice versa, so that $\phi$ is a bijection.

Since a continuous bijection from a compact space to a Hausdorff space is a homeomorphism (Aliprantis and Border, 2005, Thm. 2.36), it suffices to show that $\phi$ is continuous and that $S \times T \times T$ is Hausdorff. (Recall that $Y$ is compact (Theorem 3.2).) First note that because $T$ is a metric space, the product topology on $S \times T \times T$ is metrizable (Aliprantis and Border, 2005, Thm. 3.36) and thus Hausdorff. Second, we need to show that $\phi$ is continuous. First observe that $y_n \to y$ in $(Y, \tau)$ only if for all $k$, it holds that $\pi^{Y_k}_{(\mathcal{M}^+(Y_{k-1}))^i}(y_n) \to \pi^{Y_k}_{(\mathcal{M}^+(Y_{k-1}))^i}(y)$. In particular, eventually $\pi^{Y_k}_{(\mathcal{M}^+(Y_{k-1}))^i}(y_n)$ needs to be defined on the same $\sigma$-algebra as $\pi^{Y_k}_{(\mathcal{M}^+(Y_{k-1}))^i}(y)$ for all $k$. Let $i = 1, 2$ and $y \in Y$. Suppose $t^i(y) \in T_\infty$, and let $\{y_n\}$ be a sequence in $(Y, \tau)$ that converges to $y$. Recall that $Y$ is compact metric, so that by the Stone-Weierstrass theorem, the collection $\mathcal{C}$ of continuous functions on $Y_k$, $k = 0, 1, \ldots$, is dense in the space $C(Y)$ of continuous functions on $Y$ (endowed with the uniform topology). Then for all $f \in \mathcal{C}$, $M^i(y_n)(f) \to M^i(y)(f)$, so that $t^i(y_n) \to t^i(y)$ by Theorem 15.3 of Aliprantis and Border (2005). If $t^i(y) \in T_\Delta$ or $t^i(y) = \nu_Y$, then $y_n \to y$ directly implies that $t^i(y_n) \to t^i(y)$. □

For future reference, note that it follows from the proof of Proposition 3.3 that $T$ is compact metric.

### 3.4 Beliefs about the other's type

In models of infinite belief hierarchies, an infinite belief hierarchy or type defines an unambiguous belief about other players' types. Is that still true when one allows for finite belief hierarchies? It will be instructive to consider a player's belief about her own beliefs first. In the case that players can only have an infinite belief hierarchy, each player knows her own belief hierarchy (Mertens and Zamir, 1985, Lemma 2.14). When players have finite hierarchies, this no longer holds. Nevertheless, the following result shows that a player knows her own beliefs, to the extent that she can reason about them:

**Lemma 3.4** *Let $y = (y_0, y_1, \ldots) \in Y$ and $i = 1, 2$.*

*(a) If $t^i(y) \in T_\infty$, then*

$$y' \in supp(t^i(y)) \implies t^i(y') = t^i(y).$$

*(b) If $t^i(y) \in T_\Delta$, then*

$$y' \in supp\left(t^i(y) \circ \left(\pi_{Y_\Delta}\right)^{-1}\right) \implies$$
$$(t^i(y') \circ (\pi_{Y_0})^{-1}, \ldots, t^i(y') \circ (\pi_{Y_{\Delta-1}})^{-1}) = (t^i(y) \circ (\pi_{Y_0})^{-1}, \ldots, t^i(y) \circ (\pi_{Y_{\Delta-1}})^{-1})$$

*Moreover, for $k > \Delta$,*

$$\left(t^i(y') \circ \left(\pi_{Y_k}\right)^{-1}\right) \left(\left(\pi^{Y_k}_{(\mathcal{M}^+(Y_{\Delta-1}))^i}\right)^{-1} \left(\left\{t^i(y) \circ \left(\pi_{Y_{\Delta-1}}\right)^{-1}\right\}\right)\right) = 1.$$

**Proof.** **(a)** The proof follows the proof of Lemma 2.14 of Mertens and Zamir (1985), and is only included to facilitate comparison between (a) and (b). If $y' \in \text{supp}(t^i(y))$, then for $k = 0, 1, \ldots,$

$$\left(t^i(y') \circ \left(\pi_{Y_0}\right)^{-1}, \ldots, t^i(y') \circ \left(\pi_{Y_{k-1}}\right)^{-1}\right) \in \text{supp}\left(t^i(y) \circ \left(\pi_{\times_{\ell=0}^{k-1}(\mathcal{M}^+(Y_\ell))^i}\right)^{-1}\right).$$

But since $y \in Y$, we can use (1) and (2) in (3.1) repeatedly to find that $t^i(y)$ assigns probability 1 to $(t^i(y) \circ (\pi_{Y_0})^{-1}, \ldots, t^i(y) \circ (\pi_{Y_{k-1}})^{-1})$. Hence, $(t^i(y') \circ (\pi_{Y_0})^{-1}, \ldots, t^i(y') \circ (\pi_{Y_{k-1}})^{-1}) = (t^i(y) \circ (\pi_{Y_0})^{-1}, \ldots, t^i(y) \circ (\pi_{Y_{k-1}})^{-1})$ for all $k$, so that $t^i(y') = t^i(y)$.

**(b)** First consider the first claim. By a similar argument as for part (a), it is possible to show that if $y' \in \text{supp}(t^i(y))$, then for $k = 0, 1, \ldots, \Delta$,

$$\left(t^i(y') \circ \left(\pi_{Y_0}\right)^{-1}, \ldots, t^i(y') \circ \left(\pi_{Y_{k-1}}\right)^{-1}\right) \in \text{supp}\left(t^i(y) \circ \left(\pi_{\times_{\ell=0}^{k-1}(\mathcal{M}^+(Y_\ell))^i}\right)^{-1}\right).$$

Using that $y \in Y$, and by repeatedly applying (1) and (2) in (3.1), it follows that $t^i(y)$ assigns probability 1 to $(t^i(y) \circ (\pi_{Y_0})^{-1}, \ldots, t^i(y) \circ (\pi_{Y_{\Delta-1}})^{-1})$. This establishes the first claim.

Turning to the second claim, let $k > \Delta$. First I show that

$$\left(\pi^{Y_k}_{(\mathcal{M}^+(Y_{\Delta-1}))^i}\right)^{-1}\left(\left\{t^i(y) \circ \left(\pi_{Y_{\Delta-1}}\right)^{-1}\right\}\right) \in \left\{\left(\pi^{Y_k}_{Y_\Delta}\right)^{-1}(B) : B \in \mathscr{B}(Y_\Delta)\right\},$$

i.e., that the claim is well defined. Let

$$B := \left(\pi^{Y_\Delta}_{(\mathcal{M}^+(Y_{\Delta-1}))^i}\right)^{-1}\left(\left\{t^i(y) \circ \left(\pi_{Y_{\Delta-1}}\right)^{-1}\right\}\right).$$

Since the singletons are closed sets in a Hausdorff space, and by continuity of $\pi^{Y_\Delta}_{(\mathcal{M}^+(Y_{\Delta-1}))^i}$, $B$ is a closed subset of $Y_\Delta$, so that $B \in \mathscr{B}(Y_\Delta)$. Furthermore, it is not hard to check that $(\pi^{Y_k}_{Y_\Delta})^{-1}(B) = (\pi^{Y_k}_{(\mathcal{M}^+(Y_{\Delta-1}))^i})^{-1}(\{t^i(y) \circ \left(\pi_{Y_{\Delta-1}}\right)^{-1}\})$, so that the claim is indeed well defined. The result now follows by noticing that

$$Z_\Delta := \left(\pi^{Y_\Delta}_{(\mathcal{M}^+(Y_{\Delta-1}))^i}\right)^{-1}\left(\left\{t^i(y) \circ \left(\pi_{Y_{\Delta-1}}\right)^{-1}\right\}\right)$$

is the subset of $Y_\Delta$ in which $i$'s marginal on $Y_{\Delta-1}$ is consistent with $t^i(y)$. Hence, by the coherency condition (1) in (3.1), player $i$ assigns probability 1 to $(\pi^{Y_k}_{Y_\Delta})^{-1}(Z_\Delta) \subseteq Y_k$. □

Lemma 3.4 tells us something about what a player "knows" at a given state about her own higher-order beliefs. Part (a) and the first claim in (b) say that in all states to which a player assigns positive probability in a given state $y$, her relevant higher-order beliefs are equal to her true beliefs at $y$, as given by her type $t^i(y)$. When she has an infinite belief hierarchy, she in fact knows her own type. If she has a hierarchy of depth $\Delta < \infty$, she only knows her beliefs up to level $\Delta - 1$. The second part of (b) shows two things. First, a player does not know her beliefs at any higher level than $\Delta$. On the other hand, she does not "loose" the ability to "speak of" her lower-level beliefs at higher orders: all her higher order beliefs are consistent with her true lower-order beliefs. Importantly, it does *not* follow from 3.4(b) that $t^i(y') \in T_\Delta$: the result only implies that $t^i(y') \in T_\Lambda$ for some $\Lambda \geq \Delta$. That is, a player cannot distinguish between hierarchies that coincide regarding her own beliefs at the levels she can reason about, even if they have greater depth than her own.

What about a player's belief about other players' types? Intuitively, if Ann has type $\nu_Y$, she cannot distinguish between any (proper) subsets of $S$ or $T$. If she has a type in $T_0$, she knows her own beliefs about $S$, but cannot discriminate any beliefs of Bob. If she has a type in $T_1$, Ann can "talk" about $S$, her own beliefs about $S$ and Bob's beliefs about $S$, but not about Bob's beliefs about her beliefs about $S$. And so on. That is, Ann can make a finer distinction among Bob's types if she has a greater depth. To formalize this idea, we construct a homeomorphism between the type space $T$ and a collection of beliefs over $S \times T$, where different types may have beliefs defined on different $\sigma$-algebras. More precisely, the $\sigma$-algebras of different types are nested, with types of greater depth having finer $\sigma$-algebras than types of shallower depth. This implies that certain types are able to make finer distinctions among types than others, depending on their level of sophistication.

Let $i = 1, 2$, and for $\Delta = 0, 1, \ldots$, let $\Xi^i_\Delta$ be the $\sigma$-algebra

$$\left\{\pi^{S\times T\times T}_{S\times(T)^j} \circ \phi \circ \left(\pi_{Y_\Delta}\right)^{-1}(B) : B \in \mathscr{B}(Y_\Delta)\right\}.$$

in $S \times T$, where $\phi$ is the homeomorphism from $Y$ to $S \times T \times T$ constructed in the proof of Proposition 3.3, and $j \neq i$. There is a natural ordering among these different $\sigma$-algebras in $S \times T$:

**Proposition 3.5** *Let $i = 1, 2$. For any $\Delta, \Delta' = 0, 1, \ldots$ such that $\Delta' > \Delta$, $\Xi^i_\Delta \subseteq \Xi^i_{\Delta'}$. Also, for any $\Delta$, $\Xi^i_\Delta \subseteq \mathscr{B}(S \times T)$.*

**Proof.** I first prove the first claim. Let $\Delta, \Delta' = 0, 1, \ldots$ such that $\Delta' > \Delta$. Let $B_\Delta \in \mathscr{B}(Y_\Delta)$, so that $\pi^{S\times T\times T}_{S\times(T)^j} \circ \phi \circ \left(\pi_{Y_\Delta}\right)^{-1}(B) \in \Xi^i_\Delta$. Then, using that $Y$ is the inverse limit of $Y_0, Y_1, \ldots$,

$$\pi^{S\times T\times T}_{S\times(T)^j}\left(\phi\left(\left(\pi_{Y_\Delta}\right)^{-1}(B_\Delta)\right)\right) = \pi^{S\times T\times T}_{S\times(T)^j}\left(\phi\left(\left(\pi_{Y_{\Delta'}}\right)^{-1} \circ \left(\pi^{Y_{\Delta'}}_{Y_\Delta}\right)^{-1}(B_\Delta)\right)\right).$$

But, by continuity of $\pi^{Y_{\Delta'}}_{Y_\Delta}$, it holds that $(\pi^{Y_{\Delta'}}_{Y_\Delta})^{-1}(B_\Delta) \in \mathscr{B}(Y_{\Delta'})$, so $\pi^{S\times T\times T}_{S\times(T)^j}(\phi(\left(\pi_{Y_\Delta}\right)^{-1}(B))) \in \Xi^i_{\Delta'}$. The proof of the second claim is similar. Again, let $B_\Delta \in \mathscr{B}(Y_\Delta)$. By continuity of $\pi_{Y_\Delta}$, $(\pi_{Y_\Delta})^{-1}(B_\Delta) \in \mathscr{B}(Y)$, so that $\phi \circ (\pi_{Y_\Delta})^{-1}(B_\Delta) \in \mathscr{B}(S\times T\times T)$, and $\pi^{S\times T\times T}_{S\times(T)^j} \circ \phi \circ (\pi_{Y_\Delta})^{-1}(B_\Delta) \in \mathscr{B}(S \times T)$. $\square$

For $i = 1, 2$, $j \neq i$, let $\mathcal{P}_\Delta(S \times (T)^j)$ be the set of probability measures on $(S \times T, \Xi^i_\Delta)$, and let

$$\mathcal{P}(S \times (T)^j) := \mathcal{M}(S \times (T)^j) \cup \left( \bigcup_{\Delta=0}^{\infty} \mathcal{P}_\Delta(S \times (T)^j) \right) \cup \{\nu_{S \times (T)^j}\}.$$

I first define a topology in $\mathcal{P}(S \times T)$, using the following lemma:

**Lemma 3.6** *Let $i = 1, 2$, $j \neq i$ and $\Delta = 0, 1, \ldots$. Then,*

$$\left\{ \pi^{S \times T \times T}_{S \times \times_{\ell=0}^{\Delta-1} (\mathcal{M}^+(Y_\ell))^j} \circ \phi \circ \left(\pi_{Y_\Delta}\right)^{-1}(B) : B \in \mathscr{B}(Y_\Delta) \right\} = \mathscr{B}(S \times \times_{\ell=0}^{\Delta-1} \mathcal{M}^+(Y_\ell)).$$

Define the function $\rho^{S \times T} : \mathcal{P}(S \times T) \times \mathcal{P}(S \times T) \to \mathbb{R}$ as follows. For all $\mu, \mu' \in \mathcal{P}(S \times T)$,

$$\rho^{S \times T}(\mu, \mu') = \begin{cases} \rho^{S \times T}_\infty(\mu, \mu') & \text{if } \mu, \mu' \in \mathcal{M}(S \times T), \\ \rho^{S \times T}_\Delta\Big(\mu \circ \big(\pi^{S \times T}_{S \times \times_{\ell=0}^{\Delta-1} \mathcal{M}^+(Y_\ell)}\big)^{-1}, \mu' \circ \big(\pi^{S \times T}_{S \times \times_{\ell=0}^{\Delta-1} \mathcal{M}^+(Y_\ell)}\big)^{-1}\Big) & \text{if } \mu, \mu' \in \mathcal{P}_\Delta(S \times T), \\ 0 & \text{if } \mu = \mu' = \nu_{S \times T}, \\ 2 & \text{otherwise,} \end{cases}$$

where $\rho^{S \times T}_\infty$ is the Prohorov metric on $\mathcal{M}(S \times T)$, and $\rho^{S \times T}_\Delta$ is the Prohorov metric on $\mathcal{M}(S \times \times_{\ell=0}^{\Delta-1} \mathcal{M}^+(Y_\ell))$. (Note that by Lemma 3.6, $\mu \circ \big(\pi^{S \times T}_{S \times \times_{\ell=0}^{\Delta-1} \mathcal{M}^+(Y_\ell)}\big)^{-1} \in \mathcal{M}(S \times \times_{\ell=0}^{\Delta-1} \mathcal{M}^+(Y_\ell))$ if $\mu \in \mathcal{P}_\Delta(S \times T)$.) It can be verified that $\rho^{S \times T}$ is a metric on $\mathcal{P}(S \times T)$. Throughout this note, the topology on $\mathcal{P}(S \times T)$ is the topology induced by $\rho^{S \times T}$.

**Theorem 3.7** *(a) There exists a homeomorphism $\psi_\infty : T_\infty \to \mathcal{M}(S \times T)$.*

*(b) For $\Delta = 0, 1, \ldots$, there exists a homeomorphism $\psi_\Delta : T_\Delta \to \mathcal{P}_\Delta(S \times T)$.*

**Proof.** Let $i = 1, 2$, and let $t^i$ be a type of $i$.

**(a)** The proof follows Mertens and Zamir (1985), and is only included for completeness. By Lemma 3.4(a), if $y \in Y$ lies in the support of $t^i$, then $t^i(y) = t^i$. Also, by definition, $t^i \in \mathcal{M}(Y)$, so that by Proposition 3.3, $t^i$ corresponds to a unique $r^i \in \mathcal{M}(S \times T \times T)$. Hence, it is natural to set

$$\psi_\infty(t^i) = r^i \circ \big(\pi^{S \times T \times T}_{S \times (T)^j}\big)^{-1},$$

where $j \neq i$. That is, each $t^i \in T_\infty$ is mapped into a unique $\psi_\infty(t^i) \in \mathcal{M}(S \times T)$. Also, by the continuity of $\pi^{S \times T \times T}_{S \times (T)^j}$, the image measure $\psi_\infty$ is continuous.

Now let $\mu \in \mathcal{M}(S \times T)$. We want to show that there exists $t^i \in T_\infty \subseteq \mathcal{M}(Y)$ such that

1. $r^i \circ \big(\pi^{S \times T \times T}_{(T)^i}\big)^{-1} = \delta_{t^i}$,

2. $r^i \circ \big(\pi^{S \times T \times T}_{S \times (T)^j}\big)^{-1} = \mu$,

where $r^i$ is the unique element of $\mathcal{M}(S \times T \times T)$ that corresponds to $t^i$ (Proposition 3.3), and $\delta$ is the delta function. To show this, we construct a sequence $(t_0^i, t_1^i, \ldots)$ on $Y_0, Y_1, \ldots$ that satisfy conditions (1) and (2) in (3.1) for all $k$ and which defines an element $t^i \in \mathcal{M}(Y)$ such that the probability measure $r^i$ on $(S \times T \times T, \mathscr{B}(S \times T \times T))$ that corresponds to $t^i$ (defined on $(Y, \mathscr{B}(Y))$) has the desired marginal distributions. For $k = 1, 2, \ldots$, define

$$\mu_k := \mu \circ \left(\pi^{S\times T}_{S\times(\times_{\ell=0}^{k-1}(\mathcal{M}^+(Y_\ell))^j)}\right)^{-1}$$

be the marginal of $\mu$ on $S \times (\times_{\ell=0}^{k-1}(\mathcal{M}^+(Y_\ell))^j)$. That is, $\mu_k$ is defined on the measurable space $(S \times \times_{\ell=0}^{k-1}\mathcal{M}^+(Y_\ell), \mathscr{B}(S \times \times_{\ell=0}^{k-1}\mathcal{M}^+(Y_\ell))$. Let $t_0^i := \mu \circ (\pi_S^{S\times T})^{-1}$, so that $t_0^i \in \mathcal{M}(S) = \mathcal{M}(Y_0)$, and for $k = 1, 2, \ldots$, define inductively

$$t_k^i := \mu_k \times \delta_{(t_0^i, \ldots, t_{k-1}^i)}.$$

That is, $t_k^i$ is a probability measure on

$$(Y_0 \times \times_{\ell=0}^{k-1}\mathcal{M}^+(Y_\ell) \times \times_{\ell=0}^{k-1}\mathcal{M}^+(Y_\ell), \mathscr{B}(Y_0 \times \times_{\ell=0}^{k-1}\mathcal{M}^+(Y_\ell) \times \times_{\ell=0}^{k-1}\mathcal{M}^+(Y_\ell)))$$

which satisfies (1) and (2) in (3.1) by construction. By the construction in Section 3, this defines a probability measure $t^i \in T_\infty \subseteq \mathcal{M}(Y)$.

It can be verified that the function from $\mathcal{M}(S \times T)$ to $T_\infty$ we constructed in this way is the inverse of $\psi_\infty$. Furthermore, $T_\infty$ is a closed subspace of the compact space $T$, and therefore compact, and $\mathcal{M}(S \times T)$ is Hausdorff. Hence, $\psi_\infty$ is a homeomorphism (Aliprantis and Border, 2005, Thm. 2.36).

**(b):** For $t^i \in T_\Delta$, set

$$\psi_\Delta(t^i) := r^i \circ \left(\pi^{S\times T\times T}_{S\times(T)^j}\right)^{-1},$$

where $r^i$ is the probability measure on the measurable space $(S \times T \times T, \{\phi \circ (\pi_{Y_\Delta})^{-1}(B) : B \in \mathscr{B}(Y_\Delta)\})$ that corresponds to $t^i$ (Proposition 3.3). Then $\psi_\Delta(t^i)$ is a probability measure on $(S \times T, \{\pi^{S\times T\times T}_{S\times(T)^j} \circ \phi \circ (\pi_{Y_\Delta})^{-1}(B) : B \in \mathscr{B}(Y_\Delta)\})$, i.e., $\psi_\Delta(t^i) \in \mathcal{P}_\Delta(S \times T)$. Clearly, $\psi_\Delta$ is a function on $T_\Delta$. To show that $\psi_\Delta$ is continuous, consider a sequence $\{t_k\}$ in $T_\Delta$ that converges to $t \in T_\Delta$, and let $\{r_k\}, r$ be the corresponding probability measures on $(S \times T \times T, \{\phi \circ (\pi_{Y_\Delta})^{-1}(B) : B \in \mathscr{B}(Y_\Delta)\})$. Then (Dudley, 2002, Thm. 11.3.3),

$$\int_{S\times T\times T} f \circ \pi_{Y_\Delta} \circ \phi^{-1} dr_k \to \int_{S\times T\times T} f \circ \pi_{Y_\Delta} \circ \phi^{-1} dr$$

for any continuous function $f : Y_\Delta \to \mathbb{R}$. As for any continuous function $g : S \times T \times T \to \mathbb{R}$, there exists a continuous function $f : Y_\Delta \to \mathbb{R}$ such that $g = f \circ \pi_{Y_\Delta} \circ \phi^{-1}$, it follows that

$$\int_{S\times T\times T} g dr_k \to \int_{S\times T\times T} g dr \tag{3.4}$$

for any continuous $g : S\times T\times T \to \mathbb{R}$. Fix any continuous function $\gamma : S\times \times_{\ell=0}^{\Delta-1}\mathcal{M}^+(Y_\ell) \to \mathbb{R}$. Then, using (3.4),

$$\int_{S\times \times_{\ell=0}^{\Delta-1}\mathcal{M}^+(Y_\ell)} \gamma d\left(r_k \circ \left(\pi^{S\times T\times T}_{S\times \times_{\ell=0}^{\Delta-1}(\mathcal{M}^+(Y_\ell))^j}\right)^{-1}\right) = \int_{S\times T\times T} \gamma\circ \pi^{S\times T\times T}_{S\times \times_{\ell=0}^{\Delta-1}(\mathcal{M}^+(Y_\ell))^j} dr_k \to$$
$$\int_{S\times T\times T} \gamma\circ \pi^{S\times T\times T}_{S\times \times_{\ell=0}^{\Delta-1}(\mathcal{M}^+(Y_\ell))^j} dr = \int_{S\times \times_{\ell=0}^{\Delta-1}\mathcal{M}^+(Y_\ell)} \gamma d\left(r \circ \left(\pi^{S\times T\times T}_{S\times \times_{\ell=0}^{\Delta-1}(\mathcal{M}^+(Y_\ell))^j}\right)^{-1}\right),$$

so that, again by using Theorem 11.3.3 of Dudley (2002), $\psi_\Delta(t_k)$ converges to $\psi_\Delta(t)$, and it follows that $\psi_\Delta$ is continuous.

Conversely, let $\mu \in \mathcal{P}_\Delta(S\times T)$. First consider the case $\Delta = 0$. Notice that $\mu$ is defined on $\Xi^i_0 = \mathscr{B}(S) \otimes \{T, \emptyset\}$. The interpretation is that $i$ cannot "talk" about the other player's type. We map $\mu$ into a type $t \in T_0$ such that for all $B \in \Sigma^Y_0$,

$$t(B) = \mu\left(\pi_{Y_0}(B) \times T\right),$$

and this mapping from $\mathcal{P}_0(S\times T)$ to $T_0$ is the inverse of $\psi_0$.

Now let $\Delta = 1, 2, \ldots$. We want to map $\mu$ to some type $t^i \in T_\Delta$ of $i$ such that

(a) $r^i \circ \left(\pi^{S\times T\times T}_{(T)^i}\right)^{-1}\left(\left\{t \in T : t\circ\left(\pi_{\times_{\ell=0}^{\Delta-1}(\mathcal{M}^+(Y_\ell))^i}\right)^{-1} = t^i \circ \left(\pi_{\times_{\ell=0}^{\Delta-1}(\mathcal{M}^+(Y_\ell))^i}\right)^{-1}\right\}\right) = 1;$

(b) $r^i \circ \left(\pi^{S\times T\times T}_{S\times (T)^j}\right)^{-1} = \mu,$

where $r^i$ is probability measure on $(S\times T\times T, \{\phi\circ(\pi_{Y_\Delta})^{-1}(B) : B \in \mathscr{B}(Y_\Delta)\})$ that corresponds to $t^i$ (Proposition 3.3). It can be verified that (a) and (b) are well defined, i.e., $\{t \in T : t\circ(\pi_{\times_{\ell=0}^{\Delta-1}(\mathcal{M}^+(Y_\ell))^i})^{-1} = t^i\circ(\pi_{\times_{\ell=0}^{\Delta-1}(\mathcal{M}^+(Y_\ell))^i})^{-1}\}$ is a measurable set, and $\mu$ and $r^i \circ \left(\pi^{S\times T\times T}_{S\times(T)^j}\right)^{-1}$ are defined on the same $\sigma$-algebra. We construct a sequence $(t^i_0, t^i_1, \ldots)$ on $Y_0, Y_1, \ldots$ that satisfies conditions (1) and (2) in (3.1) for all $k$ and which defines an element $t^i \in T_\Delta$ such that the probability measure $r^i$ that corresponds to $t^i$ satisfies (a) and (b). For $k = 1, 2, \ldots, \Delta-1$, let

$$\mu_k := \mu \circ \left(\pi^{S\times T}_{S\times(\times_{\ell=0}^{k-1}(\mathcal{M}^+(Y_\ell))^j)}\right)^{-1}$$

be the marginal of $\mu$ on $S\times(\times_{\ell=0}^{k-1}(\mathcal{M}^+(Y_\ell))^j)$. As before, $\mu_k \in \mathcal{M}(S\times \times_{\ell=0}^{k-1}\mathcal{M}^+(Y_\ell))$. Let $t^i_0 := \mu\circ(\pi^{S\times T}_S)^{-1}$, so that $t^i_0 \in \mathcal{M}(S) = \mathcal{M}(Y_0)$, and for $k = 1, \ldots, \Delta$, let

$$t^i_k := \mu_k \times \delta_{(t^i_0,\ldots,t^i_{k-1})}.$$

That is, $t^i_k \in \mathcal{M}(Y_0 \times \times_{\ell=0}^{k-1}\mathcal{M}^+(Y_\ell) \times \times_{\ell=0}^{k-1}\mathcal{M}^+(Y_\ell))$. Note that by construction, $\mathrm{supp}(t^i_k) \subseteq Y_k$.[10] For $k = \Delta+1, \Delta+2, \ldots$, define $t^i_k \in \mathcal{M}_\Delta(Y_k)$ by:

$$\forall E \in \Sigma^k_\Delta : \qquad t^i_k(E) = t^i_\Delta\left(\pi^{Y_k}_{Y_\Delta}(E)\right).$$

[10]Since $Y_0 \times \times_{\ell=0}^{k-1}\mathcal{M}^+(Y_\ell) \times \times_{\ell=0}^{k-1}\mathcal{M}^+(Y_\ell)$ is Polish, it follows from Theorem 12.7 of Aliprantis and Border (2005) that the support of $t^i_k$ exists.

Again, conditions (1) and (2) in (3.1) are satisfied by construction. Using the construction in Section 3, the sequence $(t_0^i, t_1^i, \ldots)$ defines a probability measure $t^i \in T_\Delta$. We have thus constructed a function from $\mathcal{P}_\Delta(S \times T)$ to $T_\Delta$, and it can be checked that it is the inverse of $\psi_\Delta$. Finally, since $T_\Delta$ is a closed subspace of $T$ and thus compact, and $\mathcal{P}_\Delta(S \times T)$ is Hausdorff, $\psi_\Delta$ is a homeomorphism (Aliprantis and Border, 2005, Thm. 2.36). □

**Corollary 3.8** *There exists a homeomorphism $\psi : T \to \mathcal{P}(S \times T)$.*

**Proof.** It is easy to see that the trivial mapping $\psi_\emptyset : \{\nu_Y\} \to \{\nu_{S\times T}\}$ is a homeomorphism. Now define $\psi : T \to \mathcal{P}(S \times T)$ by:

$$\forall t \in T : \psi(t) = \begin{cases} \psi_\infty(t) & \text{if } t \in T_\infty; \\ \psi_\Delta(t) & \text{if } t \in T_\Delta; \\ \psi_\emptyset(t) & \text{if } t = \nu_Y. \end{cases}$$

It is immediate that $\psi$ is a bijection. It remains to verify that $\psi$ is continuous and has a continuous inverse. To see that $\psi$ is continuous, note that $t_k \to t$ if and only if there exists $N$ such that $t_n$ and $t$ are defined on the same $\sigma$-algebra for all $n > N$. Continuity of $\psi$ then follows from the continuity of $\psi_\infty$, $\psi_\emptyset$, and $\psi_0, \psi_1, \ldots$. Also, from the constructions for the various cases in the proof of Theorem 3.7, it is immediate that $\psi$ has an inverse; denote this inverse by $\psi^{-1}$. Again, from the topology on $\mathcal{P}(S \times T)$, it follows that a sequence $\{\mu_k\}$ in $\mathcal{P}(S\times T)$ converges to $\mu \in \mathcal{P}(S\times T)$ if and only if $\mu_n$ and $\mu$ are defined on the same $\sigma$-algebra for $n$ sufficiently large, so that continuity of $\psi^{-1}$ follows from the continuity of the inverses of $\psi_\infty$, $\psi_\emptyset$ and $\psi_0, \psi_1, \ldots$. □

These results have an intuitive interpretation. The probability measures in $\mathcal{P}(S \times (T)^2)$ represent Ann's beliefs about $S$ and Bob's type, given her own type. By Proposition 3.5, the probability measures in $\mathcal{P}_{\Delta'}(S \times (T)^2)$ are defined on a finer $\sigma$-algebra than those in $\mathcal{P}_\Delta(S \times (T)^2)$ if $\Delta' > \Delta$, and the measures in $\mathcal{M}(S \times (T)^2)$ have the finest $\sigma$-algebra. Indeed, if Ann's belief is represented by a probability measure in $\mathcal{M}(S \times (T)^2)$, she can distinguish the singletons of $S$ and $T$ and can talk about her belief that Bob has a given type $t \in T$.[11] On the other hand, if her belief is represented by a probability measure in $\mathcal{P}_\Delta(S \times (T)^2)$, she can distinguish the singletons of $S$, but she cannot talk about the singletons of $T$ (Lemma 3.6). Rather, the subsets of $T$ she can speak of are sets of the form $\pi^{S\times T\times T}_{S\times(T)^2} \circ \phi \circ (\pi_{Y_\Delta})^{-1}(B)$, where $B \in \mathscr{B}(Y_\Delta)$. That is, Ann can reason about the event that Bob's beliefs about $Y_{\Delta-1}$ are given by a particular probability measure, but she cannot distinguish between types for Bob

[11]Note that $\mathscr{B}(S\times T) = \mathscr{B}(S)\otimes\mathscr{B}(T)$ (Aliprantis and Border, 2005, Thm. 4.44), and $S$ and $T$ are Hausdorff.

that coincide in their beliefs up to $Y_{\Delta-1}$ but differ in their beliefs at higher levels—those are levels she cannot reason about. Finally, $\nu_{S\times(T)^j}$ is of course defined on the coarsest possible $\sigma$-algebra: $\{S \times (T)^j, \emptyset\} \subseteq \Xi^i_\Delta$ for all $\Delta$. If Ann's beliefs are represented by $\nu_{S\times(T)^2}$, she cannot speak of the space of uncertainty or Bob's type at all.

## 3.5 Reasoning about others' reasoning abilities

A notable feature of the current model is that players may differ in their reasoning abilities, and that, additionally, they may reason about each others' depth of reasoning. Here I take up the question what players can believe about each others' level of sophistication. Specifically, I ask whether a player of a certain depth can reason about the depth of others, i.e., whether the set of states in which her opponent has a given depth is an event, given her own depth.

Recall that each type $t_\infty \in T_\infty$ corresponds to a belief $\mu_\infty \in \mathcal{M}(S \times T)$, and each $t_\Delta \in T_\Delta$ corresponds to a belief $\mu_\Delta \in \mathcal{P}_\Delta(S \times T)$. First consider the type $t_\infty$ associated with an infinite belief hierarchy. Such a type can discriminate all the singletons of $T$, and since $T_\infty$, $T_\Delta$ ($\Delta = 0, 1, \ldots$), and $\{\nu_Y\}$ are all closed subsets of $T$, these sets are all Borel-measurable, so that type $t_\infty$ can assign probabilities to each individual type, as well as to the sets $T_\infty$, $T_0, T_1, \ldots$, and $\{\nu_Y\}$. On the other hand, a player of type $\nu_Y$ obviously cannot assign probabilities to any proper subset of $T$.

What about players with a type $t_\Delta \in T_\Delta$? If $\Delta = 0$, a player with a type $t_\Delta$ cannot think about another player's beliefs, even at the first level, so there is no proper subset of $T$ to which a player with such a type can assign a probability. How about $\Delta \geq 1$? It is not hard to verify that the singleton $\{\nu_Y\}$ is measurable for any such type, so that a player of type $t_\Delta$ for $\Delta \geq 1$ can assign a probability to the other player having trivial beliefs. Is the set $T_k$ measurable for $t_\Delta$? First suppose that $k < \Delta - 1$, and define

$$E_k := \left\{ y_\Delta \in Y_\Delta : \pi^{Y_\Delta}_{\times_{\ell=0}^{k}(\mathcal{M}^+(Y_\ell))^j}(y_\Delta) \in \times_{\ell=0}^{k} \mathcal{M}(Y_\ell) \text{ and } \pi^{Y_\Delta}_{\times_{\ell=k+1}^{\Delta-1}(\mathcal{M}^+(Y_\ell))^j}(y_\Delta) \in \times_{\ell=k+1}^{\Delta-1} \mathcal{M}_k(Y_\ell) \right\}.$$

Since $\times_{\ell=n_1}^{n_2} \mathcal{M}^+(Y_\ell)$ and $\times_{\ell=n_1}^{n_2} \mathcal{M}_k(Y_\ell)$ are closed in $\times_{\ell=n_1}^{n_2} \mathcal{M}^+(Y_\ell)$, it follows from the continuity of $\pi^{Y_\Delta}_{\times_{\ell=n_1}^{n_2}(\mathcal{M}^+(Y_\ell))^j}$ that $E_k \in \mathscr{B}(Y_\Delta)$. Furthermore, $\pi^{S\times T\times T}_{(T)^j} \circ \phi \circ (\pi_{Y_\Delta})^{-1}(E_k) = T_k$. Hence, a player of type $t_\Delta$ can assign a probability to the event that her opponent has a type in $T_k$ for any $k < \Delta - 1$. Of course, this implies directly that she can assign a probability to the event that her opponent has depth at least $\Delta - 1$.[12]

But can she assign a probability to the event that her opponent has depth $\Delta - 1$, or a depth equal to $k$ for any $k > \Delta - 1$? The answer is no; the reason is intuitive: a player cannot

[12] It can also be checked that the singletons in $T_k$ are measurable for a player with a type in $T_\Delta$ if and only if $k < \Delta - 1$.

distinguish among types of depth at least $\Delta - 1$. That is, for a player to be able to assign a probability to some $T_k$ for $k > \Delta - 1$, she has to be able to "talk" about her beliefs about $Y_\Delta, \ldots, k+1$–but she cannot. In other words, if a player with a type in $T_\Delta$ assigns positive probability to the other player having a type in $T_{\Delta-1}$, then she cannot rule out that he has a type in $T_k$ for any $k \geq \Delta - 1$ or in $T_\infty$: just as she does not "realize" that she cannot reason beyond $Y_{\Delta-1}$, she cannot "think" about the question whether her opponent reasons beyond $Y_{\Delta-1}$. This seems natural: When you believe that your opponent is less sophisticated than you are, then you can have a clear idea of the reasoning patterns he might employ. On the other hand, when you believe that someone is at least as sophisticated as you are, you cannot imagine how he might reason.

These results can be related to the literature on cognitive hierarchies and $k$-level reasoning discussed in Section 1. In these literatures, players are endowed with a cognitive type, and they believe that others are less sophisticated than they are. Of course, the current framework does not require in any sense that a player assigns probability zero to other players being more sophisticated than she is: a player with a finite hierarchy simply does not "reason" about beliefs at sufficiently high order. However, it is possible that a player with a type in $T_\Delta$ assigns probability 0 to her opponent having depth $\Delta - 1$ or higher. In that sense, the current model, though entirely different in nature, is in line with models of cognitive hierarchies and $k$-level reasoning. Of course, the issue what players believe about others' reasoning abilities is also important when one considers an epistemic condition such as rationality and $m$th-order belief of rationality.

# 4 Belief and Confidence

In the standard framework, a player with a given type believes an event if she assigns probability 1 to it. In the current model, this notion needs to be amended, since types may have different $\sigma$-algebras, so that a subset that is an event for one type may not be an event for another. Also, it may be natural to consider other notions of beliefs, especially when considering higher-order "beliefs". Here I discuss two notions of "belief" for the current model, the first one arguably closest to the standard notion (and thus referred to as belief), the other one different from the standard notion in that it allows players to "believe" subsets that they cannot "talk" about (i.e., that are not in their $\sigma$-algebra). This notion is called confidence.

## 4.1 Belief

Let $i = 1, 2$, and let $j \neq i$. For $E \in \mathscr{B}(S \times (T)^j)$, let

$$B^i(E) := \{t^i \in T : E \in \Sigma(t^i) \text{ and } \psi(t^i)(E) = 1\},$$

where $\Sigma(t^i)$ is the $\sigma$-algebra of $\psi(t^i)$. That is, $B^i(E)$ is the set of types of $i$ that can "talk" about $E$ and that assign probability 1 to $E$. If $t^i \in B^i(E)$, say that $t^i$ *believes* $E$. Since in the standard setting, all types have the same $\sigma$-algebra in $S \times T$, and belief is only defined for events in that $\sigma$-algebra, this definition seems to be the direct extension of the standard notion of belief.

As a simple application of this notion, define $U_0^i := T_\infty$, and for $k = 1, 2, \ldots$, let

$$U_k^i := B^i(S \times U_{k-1}^i).$$

Then, $U := \bigcap_{k=0}^{\infty} U_k$ is the set of types that represent infinite hierarchies and assign probability 1 to the other player having an infinite belief hierarchy and assigning probability 1 to the other player having and infinite belief hierarchy and assigning probability 1 to.... That is, at a state $(s^1, t^1, s^2, t^2) \in S \times U \times S \times U$, there is *full sophistication and common belief of full sophistication*. Let $T_{MZ}$ be the universal type space of Mertens and Zamir (1985) generated from a compact metric space $S$. The following result is immediate:

**Proposition 4.1** *There exists a homeomorphism* $\eta : T_{MZ} \to U$*.*

That is, there is *no* homeomorphism from $T_{MZ}$ to $T_\infty$ (it is easy to verify that $U$ is a strict subset of $T_\infty$). In the setting of Mertens and Zamir, players not only have infinite belief hierarchies, they also believe that others have infinite hierarchies, that others believe that their opponents have infinite hierarchies, and so on: The universal type space of Mertens and Zamir and any subset thereof satisfy full sophistication and common belief of full sophistication.

An open question is what the conditions are on an event (subset of $S \times T$ or $Y$) such that there can be common belief in that event if players have finite hierarchies. Also, it is unclear what the belief-closed subsets of $T$ are (Battigalli and Siniscalchi, 1999); see the discussion in Section 5.3.

## 4.2 Confidence

The notion of belief, as defined in the previous subsection, may be fairly restrictive when one is interested in higher-order "belief" in the current setting. If players only have a limited depth of reasoning, it seems that there cannot be higher-order belief, while it may be natural to assume that if a player cannot "reason" about higher orders, he "trusts" that what he believes at lower levels will hold at higher levels as well. This motivates the following definition: for $i = 1, 2$ and $E \in \mathscr{B}(S \times T)$, let

$$\tilde{B}^i(E) := \{t^i \in T : E \notin \Sigma(t^i) \text{ or } \psi(t^i)(E) = 1\}.$$

If $t^i \in \tilde{B}^i(E)$, we say that $t^i$ is *confident* that $E$. When Ann is confident that $E$, then either she believes $E$, or she cannot "talk" about $E$.

As an application, suppose $S = S^1 \times S^2$, where $S^i$ is the set of actions of player $i$, and suppose each player $i$ is endowed with a utility function $u_i : S \to \mathbb{R}$. Say that an action-type pair $(s^i, t^i) \in S^i \times T \setminus \{\nu_Y\}$ is *rational* if $s^i$ maximizes $u_i$ given $\text{marg}_{S^j}\psi(t^i)$. Let $R^i_1$ be the set of rational action type pairs of player $i$, and for $k = 2, 3, \ldots$, define

$$R^i_k := R^i_{k-1} \cap \left[S \times \tilde{B}^i(R^j_{k-1})\right],$$

where $j \neq i$, and let $R_k := R^1_k \times R^2_k$. Note that $R_k$ is not the set of states at which there is rationality and $k$th order belief of rationality (Tan and Werlang, 1988). Then $R := \bigcap_{k=1}^{\infty} R_k$ is the set of states (action-type pairs for each player) such that there is *rationality and common confidence in rationality*. It seems possible to establish a relation between strategy profiles that are iteratively undominated and states with finite hierarchies at which there is rationality and common confidence in rationality, much in the spirit of earlier results for rationality and common belief of rationality.

# 5 Discussion

## 5.1 Knowing that you don't know

In the current framework, a player may simply stop reasoning at a certain level, also about her own beliefs. As suggested in Section 2, there is an alternative modeling approach: one could also assume that a player knows that she does not know, and knows that she knows that she does not know, and so on.[13] Technically, if Ann has depth $\Delta$, her belief about $Y_{\Delta+1}$ assigns probability 1 to the event that her belief about $Y_\Delta$ is defined on the Borel $\sigma$-algebra $\mathscr{B}(Y_\Delta)$, her belief about $Y_{\Delta+2}$ assigns probability 1 to the event that her beliefs about $Y_{\Delta+1}$ are defined on a coarser $\sigma$-algebra than $\mathscr{B}(Y_{\Delta+1})$ (she cannot reason about Bob's beliefs about $Y_{\Delta+1}$), and so on. This second approach seems to be the extension of the standard assumption that a player knows her own lower-order beliefs. How would that work out in a context where players are allowed to have finite belief hierarchies and may be uncertain about others' depth of reasoning, formally and conceptually?

[13] See Fagin et al. (1991) for a related approach in the context of knowledge structures.

Formally, the key issue is to replace the collection $\mathcal{M}_\Delta(Y_k)$, $k > \Delta$, of probability measures on $(Y_k, \{(\pi^{Y_k}_{Y_\Delta})^{-1}(B) : B \in \mathscr{B}(Y_\Delta)\})$ by a collection of probability measures defined on a finer $\sigma$-algebra. Roughly speaking, it is possible to replace (2) in the definition (3.1) of $Y_k$ by its unconditional version, denoted (2'), and inductively define $\mathcal{N}_\Delta(Y_k)$ for $k > \Delta$ to be the collection of probability measures on $Y_k$ endowed with the $\sigma$-algebra generated by the collections of sets

$$\{(\pi^{Y_k}_{Y_\Delta})^{-1}(B) : B \in \mathscr{B}(Y_\Delta)\}$$

and

$$\left\{\{y_k \in Y_k : \pi^{Y_k}_{(\mathcal{N}(Y_{k-1}))^i}(y_k) = \delta_{\mu_{k-1}}\} : \mu_{k-1} \in \mathcal{N}_\Delta(Y_{k-1})\right\},$$

where $\mathcal{N}(Y_{k-1})$ is defined in a similar way as $\mathcal{M}^+(Y_{k-1})$, but with (2) in (3.1) replaced by (2'), and the collection $\mathcal{M}_\Delta(Y_{k-1})$ of probability measures in (3.2) replaced by the collection $\mathcal{N}_\Delta(Y_{k-1})$ defined on the finer $\sigma$-algebra.

What are the conceptual implications? It seems that this alternative approach models a situation where a player simply stops reasoning about the other players at a certain level, but is fully "aware" of her own limitations in reasoning about the other player. Indeed, even when a player has a finite hierarchy, she assigns positive probability to her true beliefs at all levels, just like a player with an infinite belief hierarchy. Arguably, this alternative framework is perhaps not so much a model of limited reasoning ability—a player can reason about her own beliefs at all orders—, but may be more suitable for situations where players can "think" or "reason" about certain things, but may be unable to formulate a clear opinion about them, much in the spirit of ambiguity (though to model ambiguity, it would make sense to allow for further refinements of the $\sigma$-algebras).[14] Finally, note that this alternative model does seem to be closer to the models employed in the literature on cognitive hierarchies and $k$-step reasoning, where a player's cognitive type is private information.

## 5.2 Top-down construction

The current construction is a bottom-up one: we start with a common space of uncertainty, and explicitly delineate which higher-order beliefs are allowed (cf. Mertens and Zamir, 1985). Could we have used a top-down approach, as in Brandenburger and Dekel (1993)? Under such a construction, first all possible belief hierarchies are constructed, also belief hierarchies that are not coherent. One then discards all belief hierarchies that are not coherent, or assign positive probability to belief hierarchies that are not coherent, or assign positive probability to

[14] In this context, it is interesting to note that there is some connection between multiple-prior models which are commonly applied to model ambiguity in beliefs, and models in which there are various restrictions on the $\sigma$-algebras (Halpern, 2003, Thm. 2.3.3).

belief hierarchies that are not coherent, and so on. Somewhat surprisingly, a direct analogue of the top-down approach does not work for the current setting. The reason is that the coherency requirement (cf. condition (1) in (3.1)) plays a dual role here: The coherency condition does not only ensure that a given event is given the same probability at different levels (as in Brandenburger and Dekel (1993)), but also that the $\sigma$-algebras match across levels. This means that any analogue of the "coherency and common belief of coherency"-condition of Brandenburger and Dekel will have to reflect these two aspects of coherency: At the outset, a wide range of $\sigma$-algebras should be thought possible by the players, which can then be successively trimmed at different stages of the iterative procedure, as in Brandenburger and Dekel (1993, p. 193). However, apart from the difficulty of defining (and finding an appropriate topology for) such a huge space of beliefs at each level, a player's belief now needs to refer to his opponent's possible set of $\sigma$-algebras, depending on his own $\sigma$-algebra. This seems to be a very difficult problem, suggesting that there is no gain from going to a top-down approach in the current setting.

## 5.3 Harsanyi type spaces

In Section 3, the point of departure was a common space of uncertainty. Via the construction of belief hierarchies, this gave rise to a universal belief space $Y$ and a type space $T$. It was shown that each type corresponds to a belief about the other player's type and the basic space of uncertainty $S$. Alternatively, one may want to start with some set $\mathcal{T}^i$ for each player and associate with each element of $\mathcal{T}^i$ a belief about $S$ and $\mathcal{T}^j$, $j \neq i$, in the vein of Harsanyi (1967–1968).

In the standard setting, any Harsanyi-type space can be embedded in the universal type space as a belief-closed subset (Mertens and Zamir, 1985; Battigalli and Siniscalchi, 1999). It is not obvious that the Harsanyi-approach can be extended to the current setting, and whether each such set $\mathcal{T}^i$ can be embedded in the type space $T$. Remember that players with limited reasoning abilities (i.e., those with a type in $T_\Delta$ for some $\Delta < \infty$ or with type $\nu_Y$) do not rule out a large class of belief hierarchies of their opponents, simply because they do not reason about certain higher orders. When working with the universal beliefs space $Y$, this class of belief hierarchies is indeed very large. Now consider a set $\mathcal{T}^i$ that includes the trivial type $\nu_Y$. Then it seems that the only belief-closed subset that includes $\mathcal{T}^i$ is the universal space $T$! This suggests that not all sets $\mathcal{T}^i$ can be embedded as belief-closed subsets in $T$. Alternatively, one could require that players do not "rule out" belief hierarchies that are somehow consistent with the types in $\mathcal{T}^1, \mathcal{T}^2$, but do rule out other hierarchies. However, this seems to amount to assuming that the sets $\mathcal{T}^i$ are common belief, which was just asserted to be problematic when players only reason through a finite number of steps.[15]

In addition its intrinsic interest and to the insights it provides to the current construction, investigating the possibilities of working with Harsanyi-like type spaces in the current setting would also be worthwhile because it would force us to model explicitly what it means for a player to stop reasoning at some level. In the construction in Section 3, this was done through the definition of $\mathcal{M}^+(Y_k)$. Generating belief hierarchies from type spaces may yield complementary insights. Also, if we know what the belief-closed subsets of $T$ are, it may be possible to investigate more deeply what the implications are of allowing for uncertainty about the other players' cognitive depth.

## 5.4 State space models

It is well known that standard type space models can be directly related to models in propositional modal logic (Fagin et al., 1995; Aumann, 1999a,b). It is not clear what the connection is when belief hierarchies are allowed to be finite and when there can be uncertainty about other players' depth of reasoning. In the present setting, a type still captures everything what a player beliefs about the space of basic uncertainty and about the others' (higher-order) beliefs, but belief now has two dimensions: belief refers to what a player deems possible, and at the same time entails a restriction on a player's language, and these two aspects cannot be separated. Therefore, to gain a better insight into these issues, it would be worthwhile to try to construct a state space model starting from the universal beliefs space.

It may be possible to relate the current framework to a modal logic that allows for unawareness and reasoning about knowledge of unawareness (Halpern and Rego, 2009). The type of unawareness in the current model is qualitatively different from the types of unawareness considered previously in the economics literature. In particular, unlike in the models considered in that literature, awareness is not generated by primitive propositions; rather, unawareness in the current framework concerns the modal operators. Indeed, a key feature of the current approach is that a player may think it possible that another player is as least as sophisticated as she is, something which cannot be captured if awareness is generated by primitive propositions (Halpern, 2001).

[15]Note though that common belief can arise in various ways: for instance through an iterative process, co-presence, or via the observation of some public event (e.g. Lewis, 1969; Clark and Marshall, 1978; Barwise, 1988). The second avenue for obtaining common belief does not seem to depend on players' reasoning abilities. However, it is hard to see how there can be common belief in $\mathcal{T}^i$, $i = 1, 2$, in that way.

# 6 Conclusions

Two major questions are left unanswered in the current work; in fact, they are not even touched upon explicitly. The first is how various forms of evidence on perception and reasoning processes should be translated into game-theoretic models and epistemic conditions. The focus of the current work is a very basic idea: Individuals do not always reason about all they can reason about; that is, logical omniscience fails in a very particular way. The game-theoretic literature on unawareness considers settings where individuals may not be aware of all possible courses of action (e.g., Heifetz et al., 2006; Feinberg, 2009).[16] Both seem plausible assumptions, but so far they are not well-founded in direct cognitive evidence.[17] Also, it may be worthwhile to consider other aspects of individuals' mental models of strategic situations. For instance, individuals may have very different representations of a "game" than the modeler, and this is likely to affect the way they reason about the game.[18]

A second question is how one could derive plausible (epistemic) conditions from stylized models of reasoning processes, which can then be used to motivate solution concepts, i.e., testable predictions of behavior. While solution concepts have been developed for the case where individuals may not be aware of certain moves or the existence of some players (e.g., Heifetz et al., 2006; Feinberg, 2009), and standard solution concepts have been adapted in the literature on cognitive hierarchies and $k$-level reasoning (where players are certain that other players are less sophisticated than they are (e.g., Camerer et al., 2004; Costa-Gomes and Crawford, 2006)), the question is wide open for the current setting. The experimental literature has provided some evidence on behavior and higher-order beliefs. The hope is that this can be used as an inspiration for future research, which then can inspire novel experiments.

# Appendix A  Proof of Proposition 3.1

To prove that $Y_k$ is compact metric (and thus Polish), I first show that $\mathcal{M}^+(Y_k)$ is compact metric, provided that $Y_k$ is compact metric.

[16] Fagin and Halpern (1988) and Halpern and Rego (2009) provide a more general model of unawareness that may be able to capture the idea of limited depth of reasoning to some extent, but they do not explore the consequences of this type of unawareness.

[17] While the literature on Theory of Mind does address the question to what extent humans are capable of taking the perspective of another player, there seems to be little work on "higher-order" perspective taking. See Hedden and Zhang (2002) for an experimental study of the effect of limited depth of reasoning on behavior in games from a Theory-of-Mind perspective.

[18] The literature on unawareness cited above addresses this question to a certain extent, but only allows for a limited set of representations.

**Lemma A.1** *For each $k$, if $Y_k$ is compact metric, $\mathcal{M}^+(Y_k)$ is compact metric.*

**Proof.** Fix $k = 0, 1, \ldots$ and suppose $Y_k$ is compact metric. Then $\mathcal{M}(Y_k)$ is compact metric in the topology $\tau_k$ induced by the Prohorov metric $\rho_k$. Define $\rho_k^+ : \mathcal{M}^+(Y_k) \times \mathcal{M}^+(Y_k) \to \mathbb{R}$ by:

$$\forall \mu, \mu' \in \mathcal{M}^+(Y_k): \qquad \rho_k^+(\mu, \mu') = \begin{cases} \rho_k(\mu, \mu') & \text{if } \mu, \mu' \in \mathcal{M}(Y_k), \\ \rho_\ell(\mu \circ (\pi_{Y_\ell}^{Y_k})^{-1}, \mu' \circ (\pi_{Y_\ell}^{Y_k})^{-1}) & \text{if } \mu, \mu' \in \mathcal{M}_\ell(Y_k), \\ 0 & \text{if } \mu = \mu' = \nu_{Y_k}, \\ 2 & \text{otherwise,} \end{cases}$$

where we note that $\mu \circ (\pi_{Y_\ell}^{Y_k})^{-1} \in \mathcal{M}(Y_\ell)$ if $\mu \in \mathcal{M}_\ell(Y_k)$. It can be verified that $\rho_k^+$ is a metric; denote the induced topology on $\mathcal{M}^+(Y_k)$ by $\tau_k^+$. Notice that the topology on $\mathcal{M}(Y_k)$ induced by $\tau_k^+$ is just $\tau_k$.

Clearly $\{\nu_{Y_k}\}$ is a compact subset of $(\mathcal{M}^+(Y_k), \tau_+^k)$. Also, using the identity mapping from $\mathcal{M}(Y_k)$ to $\mathcal{M}^+(Y_k)$, one can show that $\mathcal{M}(Y_k)$ is a compact subset of $(\mathcal{M}^+(Y_k), \tau_+^k)$. If we show that $\mathcal{M}_\ell(Y_k)$ is a compact subset of $(\mathcal{M}^+(Y_k), \tau_k^+)$ for every $\ell = 0, \ldots, k$, it thus follows that $\mathcal{M}^+(Y_k)$ is compact, because a finite union of compact sets is compact.

To show that $\mathcal{M}_\ell(Y_k)$ is a compact subset of $(\mathcal{M}^+(Y_k), \tau_k^+)$ for a given $\ell$, define the function $f_\ell^k : \mathcal{M}(Y_\ell) \to \mathcal{M}^+(Y_k)$ by

$$\forall \mu_\ell \in \mathcal{M}(Y_\ell): \quad f_\ell^k(\mu_\ell) = \mu_k,$$

where $\mu_k$ is the unique probability measure in $\mathcal{M}_\ell(Y_k) \subseteq \mathcal{M}^+(Y_k)$ that satisfies

$$\mu_k\left( (\pi_{Y_\ell}^{Y_k})^{-1}(B) \right) = \mu_\ell(B)$$

for every $B \in \mathscr{B}(Y_\ell)$. It can be verified that $f_\ell^k(\mathcal{M}(Y_\ell)) = \mathcal{M}_\ell(Y_k)$. Since $(\mathcal{M}(Y_\ell), \tau_\ell)$ is compact, and because continuous functions carry compact sets into compact sets, it is sufficient to show that $f_\ell^k$ is continuous. Because the open balls form a basis for $(\mathcal{M}^+(Y_k), \tau_k^+)$, $f_\ell^k$ is continuous if for all $r \in \mathcal{M}^+(Y_k)$ and $\varepsilon > 0$,

$$F_k(r, \varepsilon) := (f_\ell^k)^{-1}\left( \{ y \in \mathcal{M}^+(Y_k) : \rho_k^+(r, y) < \varepsilon \} \right)$$

is open in $(\mathcal{M}(Y_\ell), \tau_\ell)$. Fix $r \in \mathcal{M}^+(Y_k)$ and $\varepsilon > 0$. Suppose $\varepsilon > 2$. Then it is immediate that $F_k(r, \varepsilon) = \mathcal{M}(Y_\ell)$. So suppose $\varepsilon \leq 2$. If $r \notin \mathcal{M}_\ell(X_k)$, it is easy to see that $F_k(r, \varepsilon) = \emptyset$. If $r \in \mathcal{M}_\ell(Y_k)$,

$$F_k(r, \varepsilon) = \{ y \in \mathcal{M}(Y_\ell) : \rho_\ell(r, y) < \varepsilon \}.$$

Hence, $F_k(r, \varepsilon)$ is open in $(\mathcal{M}(Y_\ell, \tau_\ell)$ for all $r \in \mathcal{M}^+(Y_k)$ and $\varepsilon > 0$, and it follows that $(\mathcal{M}^+(Y_k), \tau_k^+)$ is compact. Since the space $(\mathcal{M}^+(Y_k), \tau_k^+)$ is metrizable and compact, it is complete and totally bounded (e.g., Aliprantis and Border, 2005, Thm. 3.28), and thus separable (e.g., Aliprantis and Border, 2005, Lemma 3.26). Hence, $(\mathcal{M}^+(Y_0), \tau_k^+)$ is compact Polish. □

I now prove that $Y_k$ is compact metric for all $k$. The space $Y_0$ is compact metric by assumption, so that $\mathcal{M}^+(Y_0)$ is compact metric by Lemma A.1. Since the product of compact metric spaces is compact metric, $Y_1$ is compact metric.

Let $k = 2, 3, \ldots$, and suppose that $Y_{k-1}$ is compact metric (so that by Lemma A.1, $\mathcal{M}^+(Y_{k-1})$ is compact metric). I first show that $Y_k$ is compact. Because closed subsets of compact spaces are compact (in the relative topology), it is sufficient to show that $Y_k$ is a closed subset of $Y_{k-1} \times \mathcal{M}^+(Y_{k-1}) \times \mathcal{M}^+(Y_{k-1})$. Define

$$
\begin{aligned}
Q_k \;&:=\; \Big\{ y_k \in Y_{k-1} \times \mathcal{M}^+(Y_{k-1}) \times \mathcal{M}^+(Y_{k-1}) : \text{ for } i = 1, 2, \\
&\qquad \big(\pi^{Z_k}_{(\mathcal{M}^+(Y_{k-1}))^i}(y_k)\big) \circ \big(\pi^{Y_{k-1}}_{Y_{k-2}}\big)^{-1} = \pi^{Y_{k-1}}_{(\mathcal{M}^+(Y_{k-2}))^i}\big(\pi^{Z_k}_{Y_{k-1}}(y_k)\big) \Big\}, \\
R_k \;&:=\; \Big\{ y_k \in Y_{k-1} \times \mathcal{M}^+(Y_{k-1}) \times \mathcal{M}^+(Y_{k-1}) : \text{ for } i = 1, 2, \\
&\qquad \text{if } \pi^{Z_k}_{(\mathcal{M}^+(Y_{k-1})^i)}(y_k) \in \mathcal{M}(Y_{k-1}), \text{ then} \\
&\qquad\quad \pi^{Z_k}_{(\mathcal{M}^+(Y_{k-1}))^i}(y_k) \circ \Big(\pi^{Y_{k-1}}_{(\mathcal{M}^+(Y_{k-2}))^i}\Big)^{-1} = \delta_{\pi^{Y_{k-1}}_{(\mathcal{M}^+(Y_{k-2}))^i}(\pi^{Z_k}_{Y_{k-1}}(y_k))} \Big\}, \\
U_k \;&:=\; \Big\{ y_k \in Y_{k-1} \times \mathcal{M}^+(Y_{k-1}) \times \mathcal{M}^+(Y_{k-1}) : \text{ for } i = 1, 2, \\
&\qquad \text{if } \pi^{Y_{k-1}}_{(\mathcal{M}^+(Y_{k-2}))^i}\big(\pi^{Z_k}_{Y_{k-1}}(y_k)\big) \in \mathcal{M}_\ell(Y_{k-2}), \text{ then} \\
&\qquad\quad \pi^{Z_k}_{(\mathcal{M}^+(Y_{k-1}))^i}(y_k) \in \mathcal{M}_\ell(Y_{k-1}) \Big\},
\end{aligned}
$$

where $Z_k := Y_{k-1} \times \mathcal{M}^+(Y_{k-1}) \times \mathcal{M}^+(Y_{k-1})$. If we show that $Q_k$, $R_k$ and $U_k$ are closed, we are done, since $Y_k = Q_k \cap R_k \cap U_k$.

Consider a sequence $\{q_k^n\}_{n\in\mathbb{N}}$ in $Q_k$ that converges to $q_k \in Y_{k-1} \times \mathcal{M}^+(Y_{k-1}) \times \mathcal{M}^+(Y_{k-1})$. I show that $q_k \in Q_k$, so that $Q_k$ is closed. Since $Y_{k-1} \times \mathcal{M}^+(Y_{k-1}) \times \mathcal{M}^+(Y_{k-1})$ is endowed with the product topology, for $i = 1, 2$,

$$\big(\pi^{Z_k}_{(\mathcal{M}^+(Y_{k-1}))^i}(q_k^n)\big) \circ \big(\pi^{Y_{k-1}}_{Y_{k-2}}\big)^{-1} \to \big(\pi^{Z_k}_{(\mathcal{M}^+(Y_{k-1}))^i}(q_k)\big) \circ \big(\pi^{Y_{k-1}}_{Y_{k-2}}\big)^{-1}$$

and

$$\pi^{Y_{k-1}}_{(\mathcal{M}^+(Y_{k-2}))^i}\big(\pi^{Z_k}_{Y_{k-1}}(q_k^n)\big) \to \pi^{Y_{k-1}}_{(\mathcal{M}^+(Y_{k-2}))^i}\big(\pi^{Z_k}_{Y_{k-1}}(q_k)\big)$$

Since $q_k^n \in Q_k$ for all $\ell$,

$$\big(\pi^{Z_k}_{(\mathcal{M}^+(Y_{k-1}))^i}(q_k)\big) \circ \big(\pi^{Y_{k-1}}_{Y_{k-2}}\big)^{-1} = \pi^{Y_{k-1}}_{(\mathcal{M}^+(Y_{k-2}))^i}\big(\pi^{Z_k}_{Y_{k-1}}(q_k)\big)$$

so $q_k \in Q_k$.

Next, let $\{r_k^n\}_{n\in\mathbb{N}}$ be a sequence in $R_k$ that converges to $r_k \in Y_{k-1}\times\mathcal{M}^+(Y_{k-1})\times\mathcal{M}^+(Y_{k-1})$. Since $r_k^n \to r_k$, for $n$ sufficiently large, either $r_k^n, r_k \in \mathcal{M}(Y_k)$, $r_k^n, r_k \in \mathcal{M}_m(Y_k)$ for some $m$, or $r_k^n = r_k = \nu_{X_k}$. In the latter two cases, $R_k = Y_{k-1} \times \mathcal{M}^+(Y_{k-1}) \times \mathcal{M}^+(Y_{k-1})$, and we are done. So suppose $r_k^n, r_k \in \mathcal{M}(Y_k)$. Clearly,

$$\pi^{Z_k}_{(\mathcal{M}^+(Y_{k-1}))^i}(r_k^n) \circ \left(\pi^{Y_{k-1}}_{(\mathcal{M}^+(Y_{k-2}))^i}\right)^{-1} \to \pi^{Z_k}_{(\mathcal{M}^+(Y_{k-1}))^i}(r_k) \circ \left(\pi^{Y_{k-1}}_{(\mathcal{M}^+(Y_{k-2}))^i}\right)^{-1}$$

Also,

$$\delta_{\pi^{Y_{k-1}}_{(\mathcal{M}^+(Y_{k-2}))^i}(\pi^{Z_k}_{Y_{k-1}}(r_k^n))} \to \delta_{\pi^{Y_{k-1}}_{(\mathcal{M}^+(Y_{k-2}))^i}(\pi^{Z_k}_{Y_{k-1}}(r_k))}$$

under the Prohorov metric if and only if

$$\pi^{Y_{k-1}}_{(\mathcal{M}^+(Y_{k-2}))^i}(\pi^{Z_k}_{Y_{k-1}}(r_k^n)) \to \pi^{Y_{k-1}}_{(\mathcal{M}^+(Y_{k-2}))^i}(\pi^{Z_k}_{Y_{k-1}}(r_k)).$$

But this is immediate from the continuity of the projection operators, so that $R_k$ is closed.

Finally, consider a sequence $\{u_k^n\}_{n\in\mathbb{N}}$ in $u_k$ that converges to $u_k \in Y_{k-1} \times \mathcal{M}^+(Y_{k-1}) \times \mathcal{M}^+(Y_{k-1})$. Again, for $n$ sufficiently large, either $u_k^n, u_k \in \mathcal{M}(Y_k)$, $u_k^n = u_k = \nu_{X_k}$, or $u_k^n, u_k \in \mathcal{M}_m(Y_k)$ for some $m$. In the first two cases, $U_k = Y_{k-1} \times \mathcal{M}^+(Y_{k-1}) \times \mathcal{M}^+(Y_{k-1})$, and we are done. If $u_k^n, u_k \in \mathcal{M}_m(Y_k)$, $u_k^n \to u_k$ implies $\pi^{Z_k}_{(\mathcal{M}^+(Y_{k-1}))^i}(u_k^n) \to \pi^{Z_k}_{(\mathcal{M}^+(Y_{k-1}))^i}(u_k)$, so that $u_k \in U_k$.

It follows that $Y_k$ is compact. Hence, by Lemma A.1, $\mathcal{M}^+(Y_k)$ is compact metric. To see that $Y_k$ is nonempty for $k = 0, 1, \ldots$, let $y_0 := s$ for some $s \in S$, and for $k \geq 1$, set $y_1 := \nu_{Y_k}$. Then $y_k \in Q_k \cap R_k \cap U_k$ for all $k$, so that $Y_k$ is nonempty.

# Appendix B Proof of Lemma 3.6

Let $\Delta = 0, 1, \ldots$, and let $\mathcal{G}(Y_\Delta)$ be the collection of open sets in $Y_\Delta$. Recall that the open sets generate the Borel $\sigma$-algebra. If we show that

$$\left\{\pi^{S\times T\times T}_{S\times\times_{\ell=0}^{\Delta-1}(\mathcal{M}^+(Y_\ell))^j} \circ \phi \circ \left(\pi_{Y_\Delta}\right)^{-1}(B) : B \in \mathcal{G}(Y_\Delta)\right\}$$

generates the $\sigma$-algebra

$$\left\{\pi^{S\times T\times T}_{S\times\times_{\ell=0}^{\Delta-1}(\mathcal{M}^+(Y_\ell))^j} \circ \phi \circ \left(\pi_{Y_\Delta}\right)^{-1}(B) : B \in \mathscr{B}(Y_\Delta)\right\},$$

and that

$$\{G \subseteq S : G \text{ open}\} \subseteq \left\{\pi^{S\times T\times T}_{S\times\times_{\ell=0}^{\Delta-1}(\mathcal{M}^+(Y_\ell))^j} \circ \phi \circ \left(\pi_{Y_\Delta}\right)^{-1}(G) : G \in \mathcal{G}(Y_\Delta)\right\} \subseteq \mathscr{B}(S),$$

then we are done (e.g., Billingsley, 1995, Ex. 2.5).

The first claim follows directly from Lemma 4.23 of Aliprantis and Border (2005):

$$\sigma\left(\left\{\pi^{S\times T\times T}_{S\times\times_{\ell=0}^{\Delta-1}(\mathcal{M}^+(Y_\ell))^j}\circ\phi\circ\left(\pi_{Y_\Delta}\right)^{-1}(G) : G\in\mathcal{G}(Y_\Delta)\right\}\right) = \left\{\pi^{S\times T\times T}_{S\times\times_{\ell=0}^{\Delta-1}(\mathcal{M}^+(Y_\ell))^j}\circ\phi\circ\left(\pi_{Y_\Delta}\right)^{-1}(B) : B\in\mathscr{B}(Y_\Delta)\right\},$$

where $\sigma(\mathcal{C})$ for a nonempty collection $\mathcal{C}$ of subsets denotes the $\sigma$-algebra generated by $\mathcal{C}$.

Turning to the second claim, I now show that

$$\{G\subseteq S : G \text{ open}\}\subseteq\left\{\pi^{S\times T\times T}_{S\times\times_{\ell=0}^{\Delta-1}(\mathcal{M}^+(Y_\ell))^j}\circ\phi\circ\left(\pi_{Y_\Delta}\right)^{-1}(G) : G\in\mathcal{G}(Y_\Delta)\right\}.$$

Let $G$ be an open set in $S\times\times_{\ell=0}^{\Delta-1}(\mathcal{M}^+(Y_\ell))^j$. Then, by continuity of $\pi^{S\times T\times T}_{S\times\times_{\ell=0}^{\Delta-1}(\mathcal{M}^+(Y_\ell))^j}$, the set $(\pi^{S\times T\times T}_{S\times\times_{\ell=0}^{\Delta-1}(\mathcal{M}^+(Y_\ell))^j})^{-1}(G)$ is open in $S\times T\times T$. Also, since the inverse of a homeomorphism is a homeomorphism, and homeomorphisms are open mappings, $\phi^{-1}((\pi^{S\times T\times T}_{S\times\times_{\ell=0}^{\Delta-1}(\mathcal{M}^+(Y_\ell))^j})^{-1}(G))$ is open in $Y$. Finally, since projections are open mappings, $\pi_{Y_\Delta}(\phi^{-1}((\pi^{S\times T\times T}_{S\times\times_{\ell=0}^{\Delta-1}(\mathcal{M}^+(Y_\ell))^j})^{-1}(G)))$ is open in $Y_\Delta$, and thus belongs to $\mathscr{B}(Y_\Delta)$. Hence,

$$G\in\left\{\pi^{S\times T\times T}_{S\times\times_{\ell=0}^{\Delta-1}(\mathcal{M}^+(Y_\ell))^j}\circ\phi\circ\left(\pi_{Y_\Delta}\right)^{-1}(B) : B\in\mathcal{G}(Y_\Delta)\right\}.$$

It remains to show that

$$\left\{\pi^{S\times T\times T}_{S\times\times_{\ell=0}^{\Delta-1}(\mathcal{M}^+(Y_\ell))^j}\circ\phi\circ\left(\pi_{Y_\Delta}\right)^{-1}(G) : G\in\mathcal{G}(Y_\Delta)\right\}\subseteq\mathscr{B}(S).$$

That is, we need to show that, given an open set $G_\Delta$ in $Y_\Delta$, there exists $B\in\mathscr{B}(S)$ such that $\pi^{S\times T\times T}_{S\times\times_{\ell=0}^{\Delta-1}(\mathcal{M}^+(Y_\ell))^j}\circ\phi\circ\left(\pi_{Y_\Delta}\right)^{-1}(G_\Delta) = B$. By similar reasoning as above, it follows that the set $\pi^{S\times T\times T}_{S\times\times_{\ell=0}^{\Delta-1}(\mathcal{M}^+(Y_\ell))^j}\circ\phi\circ\left(\pi_{Y_\Delta}\right)^{-1}(G_\Delta)$ is open in $S$, so that it is an element of $\mathscr{B}(S)$.

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