Context-Dependent Forward Induction Reasoning*

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Abstract

Battigalli-Siniscalchi [7, 2002] formalize the idea of forward induction reasoning as "rationality and common strong belief of rationality" (RCSBR). Here, we study the behavioral implications of RCSBR across all type structures—we argue that, in so doing, we study the behavioral implications of context-dependent forward induction. Formally, we show that RCSBR is characterized by a solution concept we call Extensive Form Best Response Sets (EFBRS's). It turns out that the EFBRS concept is equivalent to a concept already proposed in the literature, namely Directed Rationalizability [8, 2003]. We conclude by applying the EFBRS concept to games of interest.

1 Introduction

Forward induction is a basic concept in game theory. It reflects the idea that players rationalize their opponents' behavior, whenever possible. In particular, players form an assessment about the future play of the game, given the information about the past play and the presumption that their opponents are strategic. This affects the players' choices.

Formalizing forward induction reasoning requires an epistemic apparatus: To express the idea that a player rationalizes their opponents' past behavior, we need a language that explicitly describes what a player believes about the strategies her opponents play and the beliefs they hold, at each information set. An (extensive-form based) epistemic type structure gives such a language.

Within this framework, Battigalli-Siniscalchi [7, 2002] formalize forward induction reasoning using the idea of "strong belief." (See also Stalnaker [28, 1998].) A player **strongly believes** an

*We are indebted to Adam Brandenburger, John Nachbar, and Marciano Siniscalchi for many helpful conversations. Jeff Ely and three referees provided important input—much thanks. We also thank Ethan Bueno de Mesquita, Alfredo Di Tillio, Alejandro Manelli, Elena Manzoni, Andres Perea, Larry Samuelson, Adam Szeidl, and seminar participants at Bocconi University, Boston University, Maastricht University, New York University, Northwestern University, Toulouse, UC Berkeley, the 2009 Southwest Economic Theory Conference, the 2009 North American Econometric Society Meetings, the Kansas Economic Theory Conference, the 2009 SAET conference, and the 2009 European Econometric Society Meetings for important input. Battigalli thanks MIUR and Bocconi University. Friedenberg thanks the W.P. Carey School of Business and the Olin Business School. cdfi-04-10-10.rap

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event E if he assigns probability one to E, so long as E is consistent with the information set he has reached. With this, the conditions that each player is rational, strongly believes that "each (other) player is rational," strongly believes "each (other) player is rational and strongly believes others are rational," etc. formally capture the idea of forward induction reasoning. The collection of these assumptions is called **rationality and common strong belief of rationality** (**RCSBR**).

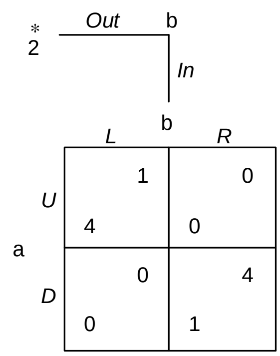

Figure 1.1

To illustrate the concept, consider the game of Battle of the Sexes (BoS) with an Outside Option (between Bob and Ann), as in Figure 1.1. Append to it a "largest" type structure, i.e., a structure that contains all possible extensive-form systems of beliefs. In this structure, the strategy *In-Right* is consistent with rationality—there is a type of Bob that assigns probability greater than $\frac{2}{3}$ to Ann's playing *Down* and, for such a type, *In-Right* is a sequential best response (i.e., it is optimal at each information set). On the other hand, the strategy *In-Left* is inconsistent with rationality—the strategy *Out* dominates *In-Left* at the beginning of the tree.[1] So, if Ann strongly believes Bob is rational, she must assign probability one to Bob playing *In-Right*, if her information set is reached. With this, she should play *Down*. Now, if Bob begins the game understanding that Ann is rational and rationalizes past behavior—i.e., that Ann is rational and strongly believes Bob is rational—Bob should begin the game assigning probability one to *Down* and should indeed play *In-Right*. This is what is viewed as the standard forward-induction outcome. This argument is in the spirit of Kohlberg-Mertens [20, 1986].[2]

[1] Note, we often conflate a strategy with its associated plan of action. No confusion should result.

[2] See, e.g., Hillas-Kohlberg [18, 2002; Section 11], Cho-Kreps [14, 1987], Govindan-Wilson [16, 2008], Man [21, 2009]. These papers analyze and/or define forward induction, so that it is a refinement of the Nash equilibrium concept. We instead follow Battigalli-Siniscalchi [7, 2002] and use the epistemic approach. This approach explicitly specifies what Ann believes about Bob's beliefs about her play—as such, it provides an explicit language within which we can specify the idea that a player rationalizes past play. See [7, 2002], [8, 2003], and [9, 2007] on the relationship between the two approaches.

Note, RCSBR (and so forward induction reasoning) depends, in somewhat subtle ways, on the particular (extensive-form) epistemic type structure studied. (See Section 3.4 in [7, 2002].) To see this, let's again consider BoS with an Outside Option, now played in a society that has come to form a "lady's choice convention." Loosely: Everyone in society thinks that, if the lady gets to move in a BoS-like situation, she makes choices that can lead to her "best payoff," i.e., she will play *Up*, hoping to get a payoff of 4. And, moreover, it is "transparent" that everyone thinks this. That is, this convention restricts the players' beliefs—it restricts what beliefs players do vs. do not consider possible. (There is no explicit restriction on which strategies players can vs. cannot play.) In particular, the convention corresponds to a type structure, where each type of Bob assigns probability one to Ann's playing *Up*. (In Section 3, we formally describe the type structure corresponding to the lady's choice convention.)

Under this convention, a rational Bob plays *Out*, thinking that Ann will play *Up*. Now, if Ann is given the opportunity to move, she can no longer rationalize Bob's behavior—after all, it is transparent that Bob believes she would play *Up* and, given this, a rational Bob should have played *Out*. Thus, conditional upon her information set being reached, Ann must forgo the hypothesis that Bob is rational and so may very well think that Bob is playing *In-Right*. In this case, she may make the choice that allows her "best payoff." That is, she may indeed choose to play *Up* over *Down*.

A formal treatment of RCSBR under the lady's choice convention will be given in Section 4. For now we note that the lady's choice convention leads us to study an epistemic type structure that does not contain all possible beliefs. The key is that, because Ann does not consider the possibility that "Bob thinks she may not go for her best outcome," Ann cannot rationalize Bob's past behavior when her information set is reached. As a result, the RCSBR analysis based on this smaller structure can lead to an outcome that is precluded by the RCSBR analysis on a larger structure.

This shows that the RCSBR analysis of the largest structure does not suffice to capture the predictions of forward induction reasoning that hold across all type structures. We must perform a separate forward induction analysis for different type structures. Is it of interest to do so? Put differently, is it of interest to study the implications of forward induction reasoning on some arbitrary structure that need not contain all conceivable beliefs?

We would argue yes—that the application may drive the analyst to study such a type structure (as it did in the case of the lady's choice convention). We take a Savage small worlds [27, 1972] view of games: In practice, we study a snapshot of the strategic situation. As a result, there is a context to the strategic situation studied—e.g., players come to the game with social conventions, a history, etc. . . —and this context influences what beliefs players do vs. do not consider possible. (For instance, in the US, it may be transparent that "drivers think that all others drive on the right side of the road, irrespective of whether they are driving north or south.") If this is the case, it may be of interest to study a given game relative to different type structures, depending on the context within which the game is played. (For instance, it may be of interest to study a driving-coordination game under different contexts depending on "who the drivers are," i.e., if they are US drivers, UK drivers, or a mix.) That is, different type structures reflect different contexts. The implication is that—because forward induction reasoning is type structure-dependent—forward induction is context-dependent.

Given a game and an epistemic type structure, we (the analysts) can identify the strategies consistent with RCSBR. But, often times, the analyst does not know the particular beliefs that the players' do vs. do not consider possible. That is, often times the analyst does not know the players' type structure. In this case, a question arises: Are there observable implications of RCSBR that hold across all type structures? The answer will be yes if we can identify the strategies consistent with context-dependent forward induction reasoning (across all contexts), by looking only at the game tree. This is the main question we ask here: Can we identify the sets of strategies consistent with RCSBR (across all type structures)?

We show that RCSBR is captured by a solution concept we call **extensive-form best response set** (**EFBRS**). (See Theorem 5.1.) The extensive-form rationalizable strategy set (Pearce [24, 1984]) is one EFBRS. But, in general, there may be other EFBRS's for a given game. Which EFBRS obtains depends on the given type structure (i.e., the given context). While the EFBRS definition is new, we will see that it is equivalent to one already proposed in the literature, namely, the Directed Rationalizability concept. This solution concept is due to Battigalli-Siniscalchi [8, 2003], who refer to it as Δ-rationalizability. We will discuss the connection in Section 9a below. We will see that, in some ways, the questions raised here can be viewed as a follow-up to the questions raised in [8, 2003].

To sum up: We began with the observation that the behavior allowed by forward induction reasoning depends on the type structure within which it is analyzed. Our interpretation of this fact is that forward induction reasoning is context dependent. We characterize forward induction reasoning across all type structures—this gives the EFBRS concept. In practice, the analyst should apply the EFBRS concept, if he is interested in studying the implications of forward induction reasoning but does not know the context within which the players are playing the game, i.e., does not know what beliefs players do vs. do not consider possible. (Contrast this with extensive-form rationalizability: The analyst should apply the extensive-form rationalizability concept, if he is interested in forward induction reasoning and understands that the players consider all possible beliefs. This is the implication of Proposition 6 in [7, 2002].)

The paper proceeds as follows. The game and epistemic structure are defined in Sections 2-3. Rationality and strong belief are defined in Section 4. Section 5 gives the main theorem, a characterization of RCSBR in terms of EFBRSs. Section 6 gives an alternate characterization theorem, in terms of Directed Rationalizability. We then turn to applications, in Sections 7-8. Finally, in Section 9, we conclude by discussing certain conceptual and technical aspects of the paper.

2 The Game

We consider finite extensive form games of perfect recall. We write Γ for such a game. The definition we consider is similar to that in Osborne-Rubinstein [23, 1994; Definition 200.1]. In particular, it allows for simultaneous moves.[3]

There are two players, namely a (Ann) and b (Bob).[4] Let C_a and C_b be **choice** or **action sets** for Ann and Bob. A history for the game consists of (possibly empty) sequences of simultaneous choices for Ann and Bob. More formally, a **history** is either (i) the empty sequence, written ϕ, or (ii) a sequence of choice pairs $\left(c^1, \ldots, c^K\right)$, where $c^k = (c_a^k, c_b^k) \in C_a \times C_b$. Histories have the property that, if $\left(c^1, \ldots, c^K\right)$ is a history, then so is $\left(c^1, \ldots, c^L\right)$ for each $L \leq K$. Each history can be viewed as a node in the tree and so we will interchangeably use the terms "node" and "history."

Write x for a history of the game and let $C(x) = \{c \in C_a \times C_b : (x, c) \text{ is a history for the game}\}$. Write $C_a(x) = \text{proj}_{C_a} C(x)$ and $C_b(x) = \text{proj}_{C_b} C(x)$. By assumption, these sets have the property that $C(x) = C_a(x) \times C_b(x)$. The interpretation is that $C_a(x)$ is the set of **choices available to** a **at history** x. If $|C_a(x)| \geq 2$, say a **moves at history** x or a **is active at** x. (If $|C_a(x)| \leq 1$, a is inactive at history x.) Call x a **terminal history** of the game if $C(x) = \emptyset$. (Terminal histories can be viewed either as **terminal nodes** or **paths** for the game.)

Let H_a (resp. H_b) be a partition of the set of all nodes at which a (resp. b) is active plus the initial node ϕ. The partition H_a (resp. H_b) has the property that if x, x' are contained in the same partition member, viz. h in H_a (resp. H_b), then $C_a(x) = C_a(x')$ (resp. $C_b(x) = C_b(x')$). The interpretation is that H_a (resp. H_b) is the family of **information sets** for a (resp. b). (Notice that $\{\phi\} \in H_a \cap H_b$. Perfect recall imposes further requirements on H_a and H_b. See Osborne-Rubinstein [23, 1994; Definition 203.3].) Write $H = H_a \cup H_b$.

Let Z be the set of terminal histories of the game, and let z be an arbitrary element of Z. **Extensive-form payoff functions** are given by $\Pi_a : Z \to \mathbb{R}$ and $\Pi_b : Z \to \mathbb{R}$.

We abuse notation and write $C_a(h)$ for the set of choices available to a at information set $h \in H_a$. With this, the set of **strategies** for player a is given by $S_a = \prod_{h \in H_a} C_a(h)$. Define S_b analogously. Each pair of strategies (s_a, s_b) induces a path through the tree. Let $\zeta : S_a \times S_b \to Z$ map each strategy profile into the induced path. **Strategic-form payoff functions** are given by $\pi_a = \Pi_a \circ \zeta$ and $\pi_b = \Pi_b \circ \zeta$. Given a profile (s_a, s_b), write $\pi(s_a, s_b) = (\pi_a(s_a, s_b), \pi_b(s_a, s_b))$ and refer to this payoff vector as an **outcome** of the game. Two strategy profiles, (s_a, s_b) and (r_a, r_b), are **outcome equivalent** if $\pi(s_a, s_b) = \pi(r_a, r_b)$. (Of course, if (s_a, s_b) and (r_a, r_b) induce the same path (i.e., if $\zeta(s_a, s_b) = \zeta(r_a, r_b)$), they are outcome equivalent. But, they may be outcome equivalent even if they do not.)

For each information set $h \in H$, write $S_a(h)$ (resp. $S_b(h)$) for the set of strategies for a (resp. b) that allow h. (That is, $s_a \in S_a(h)$ if there is some $s_b \in S_b$ so that the path induced by (s_a, s_b) passes through h.) Let $\mathcal{S}_a$ (resp. $\mathcal{S}_b$) be the collection of all $S_a(h)$ (resp. $S_b(h)$) for $h \in H_b$ (resp.

[3] This definition incorporates repeated games. Our analysis does not depend on the specific definition used.
[4] The analysis extends to n-player games, up to issues of correlation. See Section 9b.

$h \in H_a$). Thus, $\mathcal{S}_a$ represents the information structure of b about the strategy of a. In particular, at each of b's information sets, he will have a belief about a that assigns probability one to the set of a's strategies consistent with the information set being reached.

3 The Type Structure

This section defines an epistemic type structure. There are two ingredients: First, for each player, there are type sets T_a and T_b. Informally, each player "knows" his own type, but faces uncertainty about the strategy the other player will choose and the type of the other player. So, each type $t_a \in T_a$ is associated with a belief on $S_b \times T_b$. Of course, we want to specify a belief at each information set. Therefore, we map each type into a conditional probability system (CPS) on $S_b \times T_b$, where the conditioning events correspond to the information sets in the game-tree. That is, for each type, there is an array of probability measures on $S_b \times T_b$, one for each information set, and this array satisfies the rules of conditional probability when possible.

We now give the formal definitions. These closely follow the definitions in Battigalli-Siniscalchi [7, 2002]. Throughout, let Ω be a separable metrizable space and let $\mathcal{B}(\Omega)$ the Borel σ-algebra on Ω. We endow the product of separable metrizable spaces with the product topology, and a subset of a separable metrizable space with the relative topology. Write $\mathcal{P}(\Omega)$ for the set of Borel probability measures on Ω, and endow $\mathcal{P}(\Omega)$ with the topology of weak convergence.

Definition 3.1 (Renyi [26, 1955]) *Fix a separable metrizable space Ω and a non-empty collection of events $\mathcal{E} \subseteq \mathcal{B}(\Omega)$. A **conditional probability system (CPS)** on $(\Omega, \mathcal{E})$ is a mapping $\mu(\cdot|\cdot) : \mathcal{B}(\Omega) \times \mathcal{E} \to [0,1]$ such that, for any $E \in \mathcal{B}(\Omega)$ and $F, G \in \mathcal{E}$,*

(i) $\mu(F|F) = 1$,

(ii) $\mu(\cdot|F) \in \mathcal{P}(\Omega)$, *and*

(iii) $E \subseteq F \subseteq G$ *implies* $\mu(E|G) = \mu(E|F)\,\mu(F|G)$.

*Call $\mathcal{E}$, with $\emptyset \neq \mathcal{E} \subseteq \mathcal{B}(\Omega)$, a **collection of conditioning events** for Ω.*

When it is clear that $\mu(\cdot|\cdot)$ is a CPS on $(\Omega, \mathcal{E})$, we omit reference to its arguments simply writing μ instead of $\mu(\cdot|\cdot)$.

Write $\mathcal{C}(\Omega, \mathcal{E})$ for the set of conditional probability systems on $(\Omega, \mathcal{E})$. Note, $\mathcal{C}(\Omega, \mathcal{E})$ can be viewed as a subset of $[\mathcal{P}(\Omega)]^{|\mathcal{E}|}$. We endow $[\mathcal{P}(\Omega)]^{|\mathcal{E}|}$ with the product topology and, then, $\mathcal{C}(\Omega, \mathcal{E})$ with the relative topology. When $\mathcal{E}$ is countable, $\mathcal{C}(\Omega, \mathcal{E})$ is separable metrizable. When it is clear from the context what the set of conditioning events are, we omit reference to $\mathcal{E}$, simply writing $\mathcal{C}(\Omega)$.

We will often be interested in product sets. We adopt the convention that if $\Omega_1 \times \Omega_2 = \emptyset$ then both $\Omega_1 = \emptyset$ and $\Omega_2 = \emptyset$. Fix some $\mathcal{E} \subseteq \mathcal{B}(\Omega_1)$, and write $\mathcal{E} \otimes \Omega_2$ for the set of all $E \times \Omega_2$ where $E \in \mathcal{E}$. Of course, $\mathcal{E} \otimes \Omega_2 \subseteq \mathcal{B}(\Omega_1 \times \Omega_2)$.

Consider a CPS $\mu(\cdot|\cdot)$ on $(\Omega_1 \times \Omega_2, \mathcal{E} \otimes \Omega_2)$, where $\mathcal{E} \subseteq \mathcal{B}(\Omega_1)$. Define $\nu(\cdot|\cdot) : \mathcal{B}(\Omega_1) \times \mathcal{E} \to [0,1]$ so that $\nu(E|F) = \mu(E \times \Omega_2 | F \times \Omega_2)$ for all $E \in \mathcal{B}(\Omega_1)$ and $F \in \mathcal{E}$. Then ν is a conditional probability system on $(\Omega_1, \mathcal{E})$. When $\nu(\cdot|\cdot)$ is defined in this way, write $\nu(\cdot|\cdot) = \text{marg}_{\Omega_1} \mu(\cdot|\cdot)$. No confusion should result.

Definition 3.2 *Fix an extensive-form game Γ. A **Γ-based type structure** is a collection*

$$\langle S_a, S_b; \mathcal{S}_a, \mathcal{S}_b; T_a, T_b; \beta_a, \beta_b \rangle ,$$

*where T_a (resp. T_b) is a nonempty separable metrizable space and $\beta_a : T_a \to \mathcal{C}(S_b \times T_b, \mathcal{S}_b \otimes T_b)$ (resp. $\beta_b : T_b \to \mathcal{C}(S_a \times T_a, \mathcal{S}_a \otimes T_a)$) is a measurable **belief map**. Members of T_a (resp. T_b) are called **types**. Members of $S_a \times T_a \times S_b \times T_b$ are called **states**.*

Let's use this framework to model the lady's choice convention, based on the game in Figure 1.1.

Example 3.1 *The lady's choice convention is modelled by a type structure $\langle S_a, S_b; \mathcal{S}_a, \mathcal{S}_b; T_a, T_b; \beta_a, \beta_b \rangle$ based on the game in Figure 1.1. The type structure satisfies the following conditions: For each CPS on $S_b \times T_b$, there is a type of Ann, viz. t_a, so that $\beta_a(t_a)$ is exactly that CPS. (That is, β_a is onto.) Each type t_b of Bob is mapped to a CPS on $S_a \times T_a$ that assigns probability one to $\{Up\} \times T_a$ at each information set. Moreover, for each such CPS, there is a type of Bob, viz. t_b, so that $\beta_b(t_b)$ is exactly that CPS. (See [3, 2009] on how to construct such a structure.)*

Why does the type structure in Example 3.1 capture the lady's choice convention? Note, at each information set, each type of Bob assigns probability one to the event "Ann plays *Up*," i.e., to Ann trying to achieve her "best payoff." Likewise, at each information set, each type of Ann assigns probability one to the event "at each information set, Bob assigns probability one to the event 'Ann plays *Up*.'" And so on. In this sense, it is "transparent" that Bob thinks that, if Ann gets to move, she will play *Up*.

4 Rationality and Strong Belief

We now turn to the main epistemic definitions, all of which have counterparts with a and b reversed. Begin by extending $\pi_a(\cdot,\cdot)$ to $S_a \times \mathcal{P}(S_b)$ in the usual way, i.e., $\pi_a(s_a, \varpi_a) = \sum_{s_b \in S_b} \pi_a(s_a, s_b) \varpi_a(s_b)$. Since the measure ϖ_a on S_b reflects a belief by a about b, we write $\varpi_a \in \mathcal{P}(S_b)$.

Definition 4.1 *Fix $X_a \subseteq S_a$ and $s_a \in X_a$. Say s_a is **optimal under** $\varpi_a \in \mathcal{P}(S_b)$ **given** X_a if $\pi_a(s_a, \varpi_a) \geq \pi_a(r_a, \varpi_a)$ for all $r_a \in X_a$.*

Definition 4.2 *Say $s_a \in S_a$ is **sequentially optimal under** $\mu_a(\cdot|\cdot) : \mathcal{B}(S_b) \times \mathcal{S}_b \to [0,1]$ if, for all h with $s_a \in S_a(h)$, s_a is optimal under $\mu_a(\cdot|S_b(h))$ given $S_a(h)$. Say $s_a \in S_a$ is **sequentially justifiable** if there exists $\mu_a(\cdot|\cdot) : \mathcal{B}(S_b) \times \mathcal{S}_b \to [0,1]$ so that s_a is sequentially optimal under $\mu_a(\cdot|\cdot)$.*

Definition 4.3 *Say* (s_a, t_a) *is **rational** if* s_a *is sequentially optimal under* $\text{marg}_{S_b} \beta_a(t_a)$.

Let R_a be the set of strategy-type pairs, viz. (s_a, t_a), at which a is rational.

Definition 4.4 (Battigalli-Siniscalchi [7, 2002]) *Fix a CPS* $\mu(\cdot|\cdot) : \mathcal{B}(\Omega) \times \mathcal{E} \to [0,1]$ *and an event* $E \in \mathcal{B}(\Omega)$. *Say* μ ***strongly believes*** E *if*

(i) there exists $F \in \mathcal{E}$ *so that* $E \cap F \neq \emptyset$, *and*

(ii) for each $F \in \mathcal{E}$, $E \cap F \neq \emptyset$ *implies* $\mu(E|F) = 1$.

If a CPS μ strongly believes E and $\Omega \in \mathcal{E}$, then $\mu(E|\Omega) = 1$. In our application, we will, in general, have $\Omega \in \mathcal{E}$. Now, we point out two general properties about strong belief.

Property 4.1 (Conjunction) *Fix a CPS on* $(\Omega, \mathcal{E})$, *viz.* μ, *and a finite or countable collection of events* $E_1, E_2, \ldots$. *If* μ *strongly believes* $E_1, E_2, \ldots$ *then* μ *strongly believes* $\bigcap_m E_m$.

Property 4.2 (Marginalization) *Fix a CPS* μ *on* $(\Omega_1 \times \Omega_2, \mathcal{E} \otimes \Omega_2)$, *where* $\mathcal{E} \subseteq \mathcal{B}(\Omega_1)$. *If* μ *strongly believes* $E \in \mathcal{B}(\Omega_1 \times \Omega_2)$ *and* $\text{proj}_{\Omega_1} E$ *is Borel, then* $\text{marg}_{\Omega_1} \mu$ *strongly believes* $\text{proj}_{\Omega_1} E$.

Definition 4.5 *Say* $t_a \in T_a$ ***strongly believes*** $E_b \in \mathcal{B}(S_b \times T_b)$ *if* $\beta_a(t_a)$ *strongly believes* E_b.

Let $\text{SB}_a(E_b)$ be the set of strategy-types pairs (s_a, t_a) such that t_a strongly believe event E_b. That is, $\text{SB}_a(E_b)$ is the event that "Ann strongly believes E_b."

Now, we inductively define the set of states at which there is rationality and m^{th}-order strong belief of rationality. Set $R_a^1 = R_a$ (resp. $R_b^1 = R_b$). The event that Ann is rational and Ann strongly believes "Bob is rational" is then

$$R_a^2 = R_a^1 \cap \text{SB}_a\left(R_b^1\right).$$

And, the event that Ann is rational, Ann strongly believes "Bob is rational," and strongly believes "Bob is rational and strongly believes 'I am rational'" is

$$R_a^3 = R_a \cap \text{SB}_a(R_b) \cap \text{SB}_a(R_b \cap \text{SB}_b(R_a)) = R_a^2 \cap \text{SB}_a\left(R_b^2\right).$$

More generally, define R_a^m (resp. R_b^m), so that $R_a^{m+1} = R_a^m \cap \text{SB}_a(R_b^m)$ (resp. $R_b^{m+1} = R_b^m \cap \text{SB}_b(R_a^m)$).

Definition 4.6 *Say there is **rationality and common strong belief of rationality (RCSBR) at state*** (s_a, t_a, s_b, t_b) *if* $(s_a, t_a, s_b, t_b) \in \bigcap_m R_a^m \times \bigcap_m R_b^m$.

Notice, for a given type structure, it may well be the case that $\bigcap_m R_a^m = \emptyset$ and $\bigcap_m R_b^m = \emptyset$. For example, in a structure where each type of Ann initially assigns positive probability to a strictly dominated strategy of Bob, we have $\text{SB}_a\left(R_b^1\right) = \emptyset$, hence $R_a^2 = \emptyset$. It follows that $\text{SB}_b(R_a^2) = \emptyset$, hence $R_b^3 = \emptyset$.

Refer to Figure 4.1. There $Q_a \times Q_b$ is the set of strategies played under RCSBR, i.e., the projection of $\bigcap_m R_a^m \times \bigcap_m R_b^m$ on $S_a \times S_b$. This is the prediction of RCSBR for a given game and epistemic type structure.

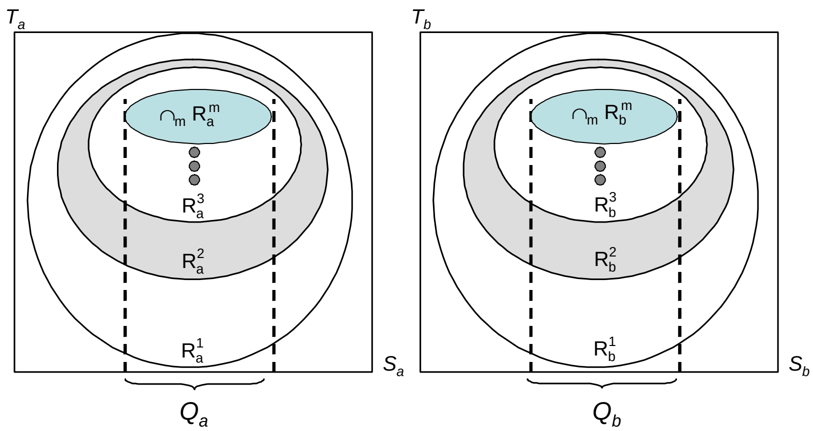

Figure 4.1

Example 4.1 *Return to Example 3.1, i.e., the BoS with an outside option game and the type structure associated with the lady's choice convention. For each m,* $\text{proj}_{S_a} R_a^m \times \text{proj}_{S_b} R_b^m$ *is* $\{Up, Down\} \times \{Out\}$.

We are interested in characterizing the strategies played under RCSBR across all type structures—i.e., to obtain, for each type structure, sets of the form $Q_a \times Q_b$ as in Figure 4.1. This is the subject of the next two sections.

5 Characterization Theorem: EFBRS's

We now turn to characterizing RCSBR. For this it will be useful to introduce a **best reply correspondence**, viz. $\rho_a : \mathcal{C}(S_b) \to 2^{S_a}$, where $\rho_a(\mu_a)$ is the set of strategies that are sequentially optimal under μ_a. We begin with extensive-form best response sets.

Definition 5.1 *Call $Q_a \times Q_b \subseteq S_a \times S_b$ an **extensive-form best response set (EFBRS)** if, for each $s_a \in Q_a$ there is a CPS $\mu_a \in \mathcal{C}(S_b)$ so that:*

(i) $s_a \in \rho_a(\mu_a)$,

(ii) μ_a *strongly believes* Q_b, *and*

(iii) $\rho_a(\mu_a) \subseteq Q_a$.

And similarly with a and b reversed.

Example 5.1 *Return to BoS with the outside option, in Figure 1.1. There are three EFBRS's: $\{Up, Down\} \times \{Out\}$, $\{Up\} \times \{Out\}$, and $\{Down\} \times \{In\text{-}Right\}$. The first of these is the set of strategies consistent with RCSBR when we append to the game the type structure associated with the lady's choice convention. (See Example 4.1.) The latter of these is the set of strategies consistent with RCSBR when we append to the game a type structure that contains all beliefs, i.e., where β_a and β_b are onto. (See Proposition 6 in [7, 2002].)*

Why is the EFBRS definition "right" for characterizing RCSBR? To see this, refer back to Figure 4.1. Fix some $(s_a, t_a) \in \bigcap R_a^m$. We can immediately identify the first two properties of Definition 5.1. For the first: Recall, s_a is optimal under the CPS associated with t_a, namely $\beta_a(t_a)$. It follows that s_a is optimal under the marginal of $\beta_a(t_a)$ on S_b (a CPS on Bob's strategies). For the second: Recall, t_a strongly believes the events R_b^1, R_b^2, R_b^3, etc. So, by the conjunction property of strong belief, t_a strongly believes the event $\bigcap R_b^m$. It then follows from a marginalization property of strong belief that the marginal of $\beta_a(t_a)$ on S_b strongly believes Q_b (i.e., the projection of $\bigcap R_b^m$ onto S_b). Thus, $Q_a \times Q_b$ satisfies both conditions (i)-(ii) of an EFBRS for (s_a, μ_a), where we take μ_a to be the marginal of $\beta_a(t_a)$ on S_b.

But, conditions (i)-(ii) do not suffice to characterize RCSBR: We can have a set $Q_a \times Q_b$ that satisfies conditions (i)-(ii) but which is inconsistent with RCSBR (for every type structure). This is illustrated by the next example.

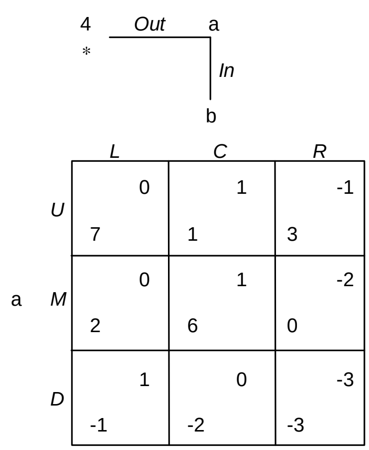

Figure 5.1

Example 5.2 *Consider the game in Figure 5.1, and the set $Q_a \times Q_b = \{Out\} \times \{Left, Center\}$. We* *will see: The set* $Q_a \times Q_b$ *satisfies conditions (i)-(ii) of Definition 5.1. But, for each type structure,* $\text{proj}_{S_a} \bigcap_m R_a^m \cap \{Out\} = \emptyset$. *That is, for each type structure, Out is inconsistent with RCSBR.*

To see that $Q_a \times Q_b$ *satisfies conditions (i)-(ii) of Definition 5.1: Begin with Ann and consider the CPS that assigns probability* $\frac{1}{2} : \frac{1}{2}$ *to* $Left : Center$*, at each information set. The strategy Out is sequentially optimal under this CPS. Of course, this CPS strongly believes* Q_b. *Turning to Bob, consider a CPS that assigns probability one to Out at the initial node and probability* $\frac{1}{4} : \frac{1}{4} : \frac{1}{2}$ *to* $In\text{-}Up : In\text{-}Middle : In\text{-}Down$ *conditional upon Bob's subgame being reached. The strategies* $Left$ *and* $Center$ *are sequentially optimal under this CPS and this CPS strongly believes* Q_a. *So, conditions (i)-(ii) are satisfied for* $Q_a \times Q_b$.

To see that, for each type structure, $\text{proj}_{S_a} \bigcap_m R_a^m \cap \{Out\} = \emptyset$*: Suppose, contra hypothesis, that there exists some type structure and some type* t_a *so that* $(Out, t_a) \in \bigcap_m R_a^m$. *Certainly, (Out,*$t_a$*) is rational, and* t_a *strongly believes each* R_b^m. *Since each pair in* $\{Right\} \times T_b$ *is irrational and* t_a *strongly believes "Bob is rational," the type* t_a *is associated with a CPS that (at each node) assigns probability one to* $\{Left, Center\} \times T_b$. *Now, since (Out,* t_a*) is rational, the CPS associated with* t_a *must assign probability* $\frac{1}{2} : \frac{1}{2}$ *to* $\{Left\} \times T_b : \{Center\} \times T_b$*, at each node. With this, (In-Up,* t_a*) and (In-Middle,* t_a*) are also rational. Indeed, since* t_a *strongly believes each of the* R_b^m *sets, both (In-Up,* t_a*) and (In-Middle,* t_a*) must be contained in* $\bigcap_m R_a^m$. *Now, consider some* $(s_b, t_b) \in \bigcap_m R_b^m$. *Conditional upon Bob's information set being reached,* t_b *must assign probability one to* $\{In\text{-}Up, In\text{-}Middle\} \times T_a$. *(To see this, note that this event contains rational strategy-type pairs, while the event* $\{In\text{-}Down\} \times T_a$ *does not contain any rational strategy-type pairs.) Since* (s_b, t_b) *is rational,* $s_b = Center$. *Thus,* $\bigcap_m R_b^m \subseteq \{Center\} \times T_b$. *But, now notice that the CPS associated with* t_a *does not strongly believe the event* $\bigcap_m R_b^m$. *By the conjunction property of strong belief, this implies that* t_a *does not strongly believe some* R_m^b*, a contradiction.*

What went wrong in this example? We began with a set $Q_a \times Q_b$ satisfying conditions (i)-(ii). In particular, we had a strategy $s_a \in Q_a$ for which there was a unique CPS $\mu_a(s_a)$, so that s_a and $\mu_a(s_a)$ satisfy conditions (i)-(ii). But, there was also a strategy $r_a \in S_a \backslash Q_a$ that was sequentially optimal under $\mu_a(s_a)$. (Actually, there were two such strategies.) As a result, if (s_a, t_a) is consistent with RCSBR, then (r_a, t_a) must also be consistent with RCSBR. Thus, Q_a may exclude some strategy of Ann consistent with RCSBR. If so we may be able to find an s_b and a CPS $\mu_b(s_b)$ (on S_a) so that s_b and $\mu_b(s_b)$ satisfy conditions (i)-(ii), despite the fact that s_b is not optimal under any CPS (on $S_a \times T_a$) that strongly believes the RCSBR strategy-type pairs for Ann.

This suggests that we need to add a maximality criterion to conditions (i)-(ii) of Definition 5.1. Indeed, this is what condition (iii) achieves.

Theorem 5.1 *Fix an extensive-form game* Γ.

(i) For any Γ*-based type structure,* $\text{proj}_{S_a} \bigcap_m R_a^m \times \text{proj}_{S_b} \bigcap_m R_b^m$ *is an EFBRS.*

(ii) Fix a nonempty EFBRS $Q_a \times Q_b$. *There exists a* Γ*-based type structure, so that* $Q_a \times Q_b = \text{proj}_{S_a} \bigcap_m R_a^m \times \text{proj}_{S_b} \bigcap_m R_b^m$.

Proof. Begin by showing part (i) of the theorem. Fix a Γ-based type structure. If $\bigcap_m R_a^m \times \bigcap_m R_b^m = \emptyset$ then the result is immediate. So, suppose $\bigcap_m R_a^m \times \bigcap_m R_b^m \neq \emptyset$.

Fix $(s_a, s_b) \in \text{proj}_{S_a} \bigcap_m R_a^m \times \text{proj}_{S_b} \bigcap_m R_b^m$. Then there exists (t_a, t_b) such that

$$(s_a, t_a, s_b, t_b) \in \bigcap_m R_a^m \times \bigcap_m R_b^m.$$

We will show that the CPS $\text{marg}_{S_b} \beta_a(t_a)$ satisfies conditions (i)-(iii) of an EFBRS, for the strategy s_a. A similar argument holds for s_b.

First note,

$$(s_a, t_a) \in \rho_a(\text{marg}_{S_b} \beta_a(t_a)) \times \{t_a\} \subseteq R_a.$$

Now use the fact that t_a strongly believes each R_b^m to get that

$$\rho_a(\text{marg}_{S_b} \beta_a(t_a)) \times \{t_a\} \subseteq \bigcap_m R_a^m.$$

So, $s_a \in \rho_a(\text{marg}_{S_b} \beta_a(t_a)) \subseteq \text{proj}_{S_a} \bigcap_m R_a^m$, establishing conditions (i) and (iii) of an EFBRS. Next note that, using the Conjunction Property of strong belief (Property 4.1), $\beta_a(t_a)$ strongly believes $\bigcap_m R_b^m$. Using the Marginalization Property (Property 4.2), $\text{marg}_{S_a} \beta_a(t_a)$ strongly believes $\text{proj}_{S_b} \bigcap_m R_b^m$. This establishes condition (ii) of an EFBRS.

Now turn to part (ii) of the Theorem. Fix an EFBRS $Q_a \times Q_b \neq \emptyset$. Let $T_a = Q_a$ and $T_b = Q_b$. Fix a type $t_a \in T_a = Q_a$. There is a CPS $\mu_a(t_a) \in \mathcal{C}(S_b)$ satisfying conditions (i)-(iii) of an EFBRS. Now construct a CPS $\beta_a(t_a) \in \mathcal{C}(S_b \times T_b, \mathcal{S}_b \otimes T_b)$ as follows. If $Q_b \cap S_b(h) \neq \emptyset$, set $\beta_a(t_a)((t_b, t_b)|S_b(h) \times T_b) = \mu_a(t_a)(t_b|S_b(h))$ for each $t_b \in Q_b = T_b$. Next, fix some arbitrary element $t_b^* \in T_b$. If $Q_b \cap S_b(h) = \emptyset$, set $\beta_a(t_a)((s_b, t_b^*)|S_b(h) \times T_b) = \mu_a(t_a)(s_b|S_b(h))$ for each $s_b \in S_b$. (Note, t_b^* is the same for each information set with $Q_b \cap S_b(h) = \emptyset$.)

Indeed, each $\beta_a(t_a)$ is a CPS on $\mathcal{S}_b \otimes T_b$. Note that conditions (i)-(ii) of a CPS are immediate. For condition (iii), fix an event E_b and two information sets $h, i \in H_a$ with $E_b \subseteq S_b(h) \times T_b \subseteq S_b(i) \times T_b$. First, consider the case where $Q_b \cap S_b(h) \neq \emptyset$. In this case, $Q_b \cap S_b(i) \neq \emptyset$. So,

$$\begin{aligned}
\beta_a(t_a)(E_b|S_b(i) \times T_b) &= \mu_a(t_a)(\{t_b \in Q_b : (t_b, t_b) \in E_b\}|S_b(i)) \\
&= \mu_a(t_a)(\{t_b \in Q_b : (t_b, t_b) \in E_b\}|S_b(h)) \times \mu_a(t_a)(S_b(h)|S_b(i)) \\
&= \mu_a(t_a)(\{t_b \in Q_b : (t_b, t_b) \in E_b\}|S_b(h)) \times \mu_a(t_a)(Q_b \cap S_b(h)|S_b(i)) \\
&= \beta_a(t_a)(E_b|S_b(h) \times T_b) \times \beta_a(t_a)(S_b(h) \times T_b|S_b(i) \times T_b),
\end{aligned}$$

where the first and fourth lines follow from the construction, the second follows from the fact that $\mu_a(t_a)$ is a CPS, and the third line follows from the fact that $\mu_a(t_a)(Q_b|S_b(h)) = 1$ (since $Q_b \cap S_b(h) \neq \emptyset$ and $\mu_a(t_a)$ strongly believes Q_b). This establishes condition (iii) of a CPS when $Q_b \cap S_b(h) \neq \emptyset$. So, suppose $Q_b \cap S_b(h) = \emptyset$ and recall $E_b \subseteq S_b(h) \times T_b$. If $Q_b \cap S_b(i) \neq \emptyset$, then $\mu_a(t_a)(\text{proj}_{S_b} E_b|S_b(i)) = 0$ and $\mu_a(t_a)(S_b(h)|S_b(i)) = 0$. (This uses the fact that

$\mu_a(t_a)(Q_b|S_b(i)) = 1$, which follows from strong belief.) So, here too,

$$\begin{aligned}\beta_a(t_a)(E_b|S_b(i)\times T_b) &= \beta_a(t_a)(E_b|S_b(h)\times T_b)\times\beta_a(t_a)(S_b(h)\times T_b|S_b(i)\times T_b)\\ &= 0.\end{aligned}$$

Finally, suppose $Q_b\cap S_b(i) = \emptyset$. Here,

$$\begin{aligned}\beta_a(t_a)(E_b|S_b(i)\times T_b) &= \mu_a(t_a)(\{s_b:(s_b,t_b^*)\in E_b\}|S_b(i))\\ &= \mu_a(t_a)(\{s_b:(s_b,t_b^*)\in E_b\}|S_b(h))\times\mu_a(t_a)(S_b(h)|S_b(i))\\ &= \beta_a(t_a)(E_b|S_b(h)\times T_b)\times\beta_a(t_a)(S_b(h)\times\{t_b^*\}|S_b(i)\times T_b)\\ &= \beta_a(t_a)(E_b|S_b(h)\times T_b)\times\beta_a(t_a)(S_b(h)\times T_b|S_b(i)\times T_b),\end{aligned}$$

as required.

We will conclude the proof by showing

$$Q_a = \bigcup_{t_a\in T_a}[\rho_a(\mathrm{marg}_{S_b}\beta_a(t_a))] \tag{5.1}$$

$$R_a^m = \bigcup_{t_a\in T_a}[\rho_a(\mathrm{marg}_{S_b}\beta_a(t_a))\times\{t_a\}] \quad \text{for each } m, \tag{5.2}$$

and likewise with a and b interchanged. Taken together, they give the desired result.

To show Equation 5.1: Recall, for each $t_a\in T_a = Q_a$, $\mu_a(t_a) = \mathrm{marg}_{S_b}\beta_a(t_a)$. So, it is immediate from the construction that $Q_a\subseteq\bigcup_{t_a\in T_a}\rho_a(\mathrm{marg}_{S_b}\beta_a(t_a))$. Conversely, fix any strategy s_a in $\bigcup_{t_a\in T_a}\rho_a(\mathrm{marg}_{S_b}\beta_a(t_a))$. Then, there is a type $t_a\in T_a = Q_a$ so that s_a is sequentially optimal under $\mu_a(t_a)(\cdot|\cdot)$. It follows from part (iii) of the definition of an EFBRS that $s_a\in Q_a$.

To show Equation 5.2: The proof is by induction on m. The Equation is immediate for $m=1$. Assume the result holds for m. In order to show that it holds for $m+1$, it suffices to show that each $t_a\in T_a$ strongly believes R_b^m. For this, fix an information set h such that $R_b^m\cap[S_b(h)\times T_b]\neq\emptyset$. Observe that

$$\begin{aligned}[\mathrm{proj}_{S_b}R_b^m]\cap S_b(h) &= [\textstyle\bigcup_{t_b\in T_b}\rho_b(\mathrm{marg}_{S_a}\beta_b(t_b))]\cap S_b(h)\\ &= Q_b\cap S_b(h).\end{aligned}$$

(The first equality follows from the induction hypothesis for b. The second equality follows from Equation 5.1.) Since $R_b^m\cap[S_b(h)\times T_b]\neq\emptyset$, it follows that $Q_b\cap S_b(h)\neq\emptyset$, and so $\mu_a(t_a)(Q_b|S_b(h)) = 1$. (Here, we use part (ii) of the definition of an EFBRS.) So, by construction, $\beta_a(t_a)(R_b^m|S_b(h)\times T_b) = 1$, as required. ■

Part (i) of Theorem 5.1 says that the projection of the RCSBR event on $S_a\times S_b$ is an EFBRS. But, this may form an empty EFBRS. That said, there is a non-empty EFBRS.

Remark 5.1 *For any game, there exists a non-empty EFBRS—namely, the set of extensive-form rationalizable strategy profiles.*

Proposition 6 in Battigalli-Siniscalchi [7, 2002] implies that there exists some type structure (one that contains all possible beliefs) so that the projection of the RCSBR event onto $S_a \times S_b$ is the set of extensive-form rationalizable strategies. So, using Theorem 5.1(i), this set is an EFBRS. The fact that it is non-empty is shown as Corollary 1 in Battigalli [2, 1997].

6 Alternate Characterization Theorem: Directed Rationalizability

Return to the "lady's choice convention" example—i.e., Figure 1.1 plus the type structure in Example 3.1. There, each type of Bob was associated with some CPS that assigned probability one to $\{Up\} \times T_a$. This gives a restriction on Bob's first-order beliefs, i.e., his beliefs about what Ann will choose. Let Δ_b represent this restriction on first-order beliefs. So, Δ_b is a subset of the CPS's on S_a and, in our example, Δ_b only contains the CPS that assigns probability one to Up. We did not have a restriction on Ann's first-order beliefs. So, we will write Δ_a for the set of all CPS's on S_b.

With $\Delta = \Delta_a \times \Delta_b$ in hand, we can take an iterative approach to analyzing the game tree—much like a "typical rationalizability" procedure. On round one, we eliminate $In\text{-}Left$ and $In\text{-}Right$ for Bob, since these strategies are not sequentially optimal under the CPS in Δ_b. We do not eliminate any of Ann's strategies, since they are each sequentially optimal under some CPS (in Δ_a). So, on round one, we are left with the set $\{Up, Down\} \times \{Out\}$. Turning to round two, Out is sequentially optimal under the CPS in Δ_b and that CPS strongly believes $\{Up, Down\}$. Thus, we cannot eliminate Out on round two. Likewise, Up (resp. $Down$) is sequentially optimal under a CPS that assigns probability one to Out at the initial node, and probability one to $Left$ (resp. $Right$) at Bob's subgame. This CPS is contained in Δ_a and strongly believes $\{Out\}$. So, we also get $\{Up, Down\} \times \{Out\}$ on round two. Indeed, a standard induction argument gives that $\{Up, Down\} \times \{Out\}$ is the outcome of the procedure. Of course, this was the EFBRS we identified in Section 4.

The procedure used above is called **Δ-rationalizability**, due to Battigalli-Siniscalchi [8, 2003].[5] More generally, let Δ_a (resp. Δ_b) be a non-empty subset of $\mathcal{C}(S_b)$ (resp. $\mathcal{C}(S_a)$), i.e. a set of first-order beliefs of Ann (resp. Bob). Call $\Delta = \Delta_a \times \Delta_b$ a **set of first-order beliefs**. Set $S_a^{\Delta,0} = S_a$ and $S_b^{\Delta,0} = S_b$. Inductively define $S_a^{\Delta,m}$ and $S_b^{\Delta,m}$ as follows: Let $S_a^{\Delta,m+1}$ be the set of all $s_a \in S_a^{\Delta,m}$ so that, there is some CPS $\mu_a \in \Delta_a$ with (i) $s_a \in \rho_a(\mu_a)$ and (ii) μ_a strongly believes $S_b^{\Delta,m}$. And, likewise, with a and b interchanged.

[5] Battigalli-Sinsicalchi [8, 2003] introduced the concept to study a different problem from the one studied here. In their problem, the set Δ is given to the analyst. In our problem, Δ may be unknown to the analyst and we obtain a characterization across all Δ's. See Section 9a.

Definition 6.1 (Battigalli-Siniscalchi [8, 2003]) *Call* $S_a^{\Delta} = \bigcap_{m \geq 0} S_a^{\Delta,m}$ *(resp.* $S_b^{\Delta} = \bigcap_{m \geq 0} S_b^{\Delta,m}$*) the* ***Δ-rationalizable strategies of Ann*** *(resp.* ***Bob****). Call* $S_a^{\Delta} \times S_b^{\Delta}$ *the* ***Δ-rationalizable strategy set****.*

Since the sets $S_a^{\Delta,m} \times S_b^{\Delta,m}$ form a decreasing sequence and $S_a \times S_b$ is finite, there is some (finite) M so that $S_a^{\Delta} \times S_b^{\Delta} = S_a^{\Delta,M} \times S_b^{\Delta,M}$.

Note, there may be many Δ-rationalizable sets—each of which is obtained by beginning the procedure with a different set of first-order beliefs $\Delta = \Delta_a \times \Delta_b$. We use the phrase **Directed Rationalizability** to refer to the set of all $S_a^{\Delta} \times S_b^{\Delta}$. So, for a given game Γ, the Directed Rationalizability concept gives $\{S_a^{\Delta} \times S_b^{\Delta} : \Delta = \Delta_a \times \Delta_b \subset \mathcal{C}(S_b) \times \mathcal{C}(S_b)\}$.

Beginning from the lady's choice example, we can use the type structure to construct an associated set of first-order beliefs Δ and this set of first-order beliefs Δ can be used to perform Δ-rationalizability. The output is the EFBRS we identified earlier. But, the lady's choice convention had a particular feature: it was a restriction on first-order beliefs and a requirement that the restriction be "transparent" to the players. So, the only restriction on second-order beliefs (i.e., beliefs about the first-order beliefs and strategy the other player chooses) was the requirement that, at each information set, Ann must believe that Bob believes she will play *Up*. And so on. It was this transparency of (only) first-order restrictions that allowed us to directly compute the associated Directed Rationalizability set.

More generally, when we begin from a give type structure, we impose substantive assumptions about which beliefs players do versus do not consider possible. These assumptions may correspond to restrictions (only) on players' first-order beliefs which are transparent to the players. But, they need not—they may involve further restrictions on higher-order beliefs. And, if they do, the procedure outlined above fails. To see this, let's take an example.

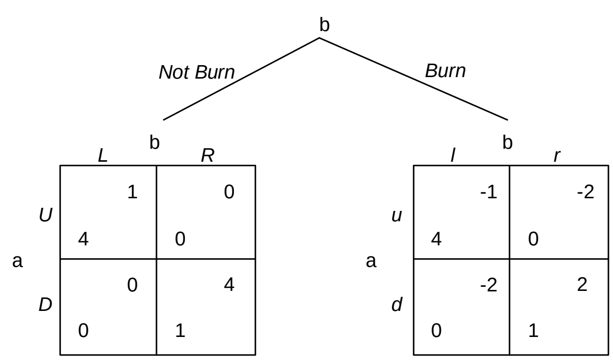

Figure 6.1

Example 6.1 *Figure 6.1 is a game of Battle of the Sexes preceded by an observed "money burning" move by Bob. (See Ben Porath-Dekel [11, 1992].) Here, Ann and Bob are playing a BoS game. However, prior to the game, Bob has the option of burning (B) or not burning (NB)* \$2.

Let $\langle S_a, S_b; \mathcal{S}_a, \mathcal{S}_b; T_a, T_b; \beta_a, \beta_b \rangle$ *be a type structure based on Figure 6.1. Now,* β_b *is onto but* β_a *is not. Formally: Write* $[Up]_a$ *for the event "Ann plays Up, if Bob does not burn," i.e.,* $[Up]_a = \{Up\text{-}down, Up\text{-}up\} \times T_a$*, and write* $[NB]_b$ *for the event "Bob does not burn," i.e.,* $[NB]_b = \{NB\text{-}Left, NB\text{-}Right\} \times T_b$*. Let* U_b *be the set of types* $t_b \in T_b$ *with* $\beta_b(t_b)([Up]_a|S_a \times T_a) = 1$*, i.e., the set of types of Bob that assign probability one to the event "Ann plays Up, when Bob chooses not to burn." Then, for each type* $t_a \in T_a$*,*

$$\beta_a(t_a)(S_b \times U_b | [NB]_b) = 1,$$

i.e., conditional upon Bob choosing not to burn, each type of Ann assigns probability one to the event that "Bob believes that 'Ann plays Up, when Bob does not burn.'" For any belief μ_a *of Ann with* $\mu_a(S_b \times U_b | [NB]_b) = 1$*, there is a type* t_a *so that* $\beta_a(t_a) = \mu_a$*.*[6]

This type structure models a modified version of the lady's choice convention. Now, there are no restrictions on players' first-order beliefs. (So, in particular, there are types of Bob that think Ann does not go for her best payoff.) But, there is a restriction on Ann's second-order beliefs. Specifically, conditional upon observing so-called "normal" behavior, i.e., a decision to not burn, Ann thinks that Bob thinks she goes for her best payoff and chooses Up. (There is no restriction on Ann's second-order belief conditional upon observing "strange" behavior, i.e., upon observing a decision to burn. Likewise, there are no restrictions on Bob's second-order beliefs.)

The set of first-order beliefs induced by this type structure is $\Delta = \mathcal{C}(S_b) \times \mathcal{C}(S_a)$*. The* Δ*-rationalizable set is* $\{(Down\text{-}down, NB\text{-}Right)\}$*. (This is also the set of extensive form rationalizable strategies.) It is obtained as follows:*

$$\begin{aligned}
S_a^{\Delta,1} \times S_b^{\Delta,1} &= S_a \times \{NB\text{-}Left, NB\text{-}Right, B\text{-}right\} \\
S_a^{\Delta,2} \times S_b^{\Delta,2} &= \{Up\text{-}down, Down\text{-}down\} \times S_b^{\Delta,1} \\
S_a^{\Delta,3} \times S_b^{\Delta,3} &= S_a^{\Delta,2} \times \{NB\text{-}Right, B\text{-}right\} \\
S_a^{\Delta,4} \times S_b^{\Delta,4} &= \{Down\text{-}down\} \times S_b^{\Delta,3} \\
S_a^{\Delta,5} \times S_b^{\Delta,5} &= \{Down\text{-}down\} \times \{NB\text{-}Right\}.
\end{aligned}$$

But, the projection of event RCSBR onto $S_a \times S_b$ *is* $\{(Up\text{-}down, B\text{-}right)\}$*. It is obtained as follows:*

$$\begin{aligned}
\text{proj}_{S_a} R_a^1 \times \text{proj}_{S_b} R_b^1 &= S_a \times \{NB\text{-}Left, NB\text{-}Right, B\text{-}right\} \\
\text{proj}_{S_a} R_a^2 \times \text{proj}_{S_b} R_b^2 &= \{Up\text{-}down\} \times \text{proj}_{S_b} R_b^1 \\
\text{proj}_{S_a} R_a^3 \times \text{proj}_{S_b} R_b^3 &= \{Up\text{-}down\} \times \{B\text{-}right\}
\end{aligned}$$

[6] See Appendix A in [3, 2009] on how to construct such a type structure.

Why the difference?

We began with an epistemic structure and used the structure itself to form the set of first-order beliefs $\Delta = \mathcal{C}(S_b) \times \mathcal{C}(S_a)$. *(So, for each* $\mu_a \in \Delta_a = \mathcal{C}(S_b)$ *there is type* $t_a \in T_a$ *such that the marginal of* $\beta_a(t_a)$ *on* S_b *is* μ_a*; and likewise for b.) With this set of first-order beliefs, the strategies that survive one round of* Δ*-rationalizability are exactly the strategies that are consistent with rationality. But, on the next round, we lose the equivalence: If* $\beta_a(t_a)$ *strongly believes* R_b^1*, then the marginal of* $\beta_a(t_a)$ *must strongly believe* $S_b^1 = \mathrm{proj}_{S_b} R_b^1$*. (Here, we use the marginalization property of strong belief.) Thus* $\mathrm{proj}_{S_a} R_a^2 \subseteq S_a^2$*. But, the converse does not hold. We have Down-down* $\in S_a^2$*, but Down-down* $\notin \mathrm{proj}_{S_a} R_a^2$*. The reason is that, conditional upon Bob choosing* NB*, each* $\beta_a(t_a)$ *assigns probability one to the event "Bob assigns probability one to* $[Up]_a$*." So, if Bob does not burn, Ann can only maintain a hypothesis that Bob is rational, if she assigns probability one to Bob's playing* NB*-Left, in which case the choice Down is not a best response. With this,* $S_a^2 = \{$*Up-down, Down-down*$\}$ *and* $\mathrm{proj}_{S_a} R_a^2 = \{$*Up-down*$\}$*. As a result,* $S_b^3 = \{$*NB-Right, B-right*$\}$ *and* $\mathrm{proj}_{S_b} R_b^3 = \{$*B-right*$\}$*. It follows that* $S_a^4 = \{$*Down-down*$\}$*, despite the fact that* $\mathrm{proj}_{S_a} R_a^4 = \{$*Up-down*$\}$*. The key to this last step is that Up-down is optimal under a CPS that strongly believes* $\mathrm{proj}_{S_b} R_b^3 \subsetneq S_b^3$*, but not optimal under a CPS that strongly believes* S_b^3*. This can occur because strong belief fails a monotonicity requirement.*

In Example 6.1, we began with an epistemic structure and used the structure itself to form a set of first-order beliefs $\Delta_a \times \Delta_b = \mathcal{C}(S_b) \times \mathcal{C}(S_a)$. We saw that this $\Delta_a \times \Delta_b$-rationalizable outcome is different from the RCSBR outcome in the given structure.

But there is another route. Instead of using the type structure to form a set of first-order beliefs, we can use the EFBRS properties. Refer back to Example 6.1 and consider the EFBRS $\{$*Up-down*$\} \times \{$*B-right*$\}$. Note, there is a CPS μ_b that satisfies conditions (i)-(ii)-(iii) of Definition 5.1 for *B-right*, i.e., the CPS that assigns probability one to *Up-down* at each information set. Take Δ_b to be the singleton that contains this CPS. Construct Δ_a similarly. With this $\Delta = \Delta_a \times \Delta_b$, we do have an equivalence between the RCSBR strategies and the Δ-rationalizable strategies.

This procedure can be done more generally. We can use the EFBRS properties to construct some set of first-order beliefs so that the associated Δ-rationalizable strategy set "gives back" the original EFBRS. With this, we get an equivalence between the EFBRS concept and Directed Rationalizability.

Proposition 6.1 *Fix an extensive-form game* Γ*.*

(i) Given an EFBRS, viz. $Q_a \times Q_b$*, there exists a set of first-order beliefs, viz.* $\Delta = \Delta_a \times \Delta_b$*, so that* $S_a^\Delta \times S_b^\Delta = Q_a \times Q_b$*.*

(ii) Given a set of first-order beliefs, viz. $\Delta = \Delta_a \times \Delta_b$*,* $S_a^\Delta \times S_b^\Delta$ *is an EFBRS.*

Thus, in conjunction with Theorem 5.1, we have the following alternate Characterization Theorem.

Corollary 6.1 *Fix an extensive-form game* Γ*.*

(i) For any Γ*-based type structure, there exists a set of first-order beliefs, viz.* $\Delta = \Delta_a \times \Delta_b$*, so that* $S_a^\Delta \times S_b^\Delta = \text{proj}_{S_a} \bigcap_m R_a^m \times \text{proj}_{S_b} \bigcap_m R_b^m$*.*

(ii) Fix a set of first-order beliefs, viz. $\Delta_a \times \Delta_b$*. Then there exists a* Γ*-based structure, so that* $S_a^\Delta \times S_b^\Delta = \text{proj}_{S_a} \bigcap_m R_a^m \times \text{proj}_{S_b} \bigcap_m R_b^m$*.*

Proof of Proposition 6.1. Begin with part (i). Fix an EFBRS set $Q_a \times Q_b$. For each $s_a \in Q_a$, there exists a corresponding CPS $\mu_a(s_a) \in \mathcal{C}(S_b)$ satisfying conditions (i)-(iii) of an EFBRS for $Q_a \times Q_b$. Take Δ_a so that, for each $s_a \in Q_a$, Δ_a contains exactly one such CPS $\mu_a(s_a)$. There are no other CPS's in Δ_a. Define Δ_b analogously. We will show that, for each $m \geq 1$, $S_a^{\Delta,m} \times S_b^{\Delta,m} = Q_a \times Q_b$. This will establish the result.

The proof is by induction. Begin with $m = 1$. Certainly $Q_a \subseteq S_a^{\Delta,1}$. Fix $s_a \in S_a^{\Delta,1}$. Then there exists some $\mu_a \in \Delta_a$ so that s_a is sequentially optimal under μ_a. This CPS μ_a is associated with some $r_a \in Q_a$, i.e., so that r_a and μ_a jointly satisfy conditions (i)-(iii) of an EFBRS. Now apply condition (iii) of an EFBRS to get that $s_a \in Q_a$. And, likewise, for b.

Now assume $S_a^{\Delta,m} \times S_b^{\Delta,m} = Q_a \times Q_b$ for $m \geq 2$. We will show it also holds for $m+1$. Fix $s_a \in Q_a = S_a^{\Delta,m}$. Then, using the construction of Δ_a, there exists some $\mu_a \in \Delta_a$ satisfying conditions (i)-(ii) of an EFBRS for $Q_a \times Q_b$, so that $s_a \in \rho_a(\mu_a)$ and μ_a strongly believes $Q_b = S_b^{\Delta,m}$. So, certainly, $Q_a \subseteq S_a^{\Delta,m+1}$. Conversely, fix some $s_a \in S_a^{\Delta,m+1}$. Then, there exists a CPS $\mu_a \in \Delta_a$ so that $s_a \in \rho_a(\mu_a)$ and μ_a strongly believes $S_b^{\Delta,m}$. Again, since each element of Δ_a satisfies conditions (i)-(iii) of an EFBRS for some $r_a \in Q_a$, it follows that $\rho_a(\mu_a) \subseteq Q_a$, and so $s_a \in Q_a$.

Now turn to part (ii) of the proposition. Fix some set of first-order beliefs, viz. $\Delta = \Delta_a \times \Delta_b$. There exists some M with $S_a^\Delta \times S_b^\Delta = S_a^{\Delta,M} \times S_b^{\Delta,M}$. Fix $s_a \in S_a^\Delta \times S_b^\Delta$. By Lemma B1 in Appendix B, we can find a CPS μ_a so that $s_a \in \rho_a(\mu_a)$ and μ_a strongly believes each $S_b^{\Delta,m}$ for $m \leq M$. Thus, s_a satisfies conditions (i)-(ii) of an EFBRS for $Q_a \times Q_b = S_a^\Delta \times S_b^\Delta$. Moreover, if $r_a \in \rho_a(\mu_a)$, then r_a is optimal under a CPS that strongly believes each $S_b^{\Delta,m}$, for $m \leq M$. As such, $r_a \in S_a^{\Delta,m}$ for each $m \leq M$, establishing that $r_a \in S_a^\Delta$. Therefore condition (iii) of an EFBRS is also satisfied. A similar argument applies to b. Therefore $S_a^\Delta \times S_b^\Delta$ is an EFBRS. ■

The proof of Proposition 6.1 gives an ancillary result: Begin with some finite set of first-order beliefs, viz. $\Delta = \Delta_a \times \Delta_b$. Proposition 6.1(ii) says that $S_a^\Delta \times S_b^\Delta$ is an EFBRS. Conversely, begin with some EFBRS. The proof of Proposition 6.1(i) says that we can find a finite set of first-order beliefs, viz. $\Delta = \Delta_a \times \Delta_b$, so that $S_a^\Delta \times S_b^\Delta$ is this EFBRS.

Remark 6.1 *Fix a game tree* Γ*. The Directed Rationalizability set is*

$$\{S_a^\Delta \times S_b^\Delta : \Delta = \Delta_a \times \Delta_b \subseteq \mathcal{C}(S_b) \times \mathcal{C}(S_b)\} = \{S_a^\Delta \times S_b^\Delta : \Delta = \Delta_a \times \Delta_b \textit{ is finite}\}.$$

Thus, using the EFBRS properties, we can see that we only need to compute the Δ-rationalizable sets for finite sets of first-order beliefs. Of course, much as is the case with EFBRS's, the Δ-

rationalizable strategy set may be empty. When $\Delta = \mathcal{C}(S_a) \times \mathcal{C}(S_b)$, $S_a^{\Delta} \times S_b^{\Delta}$ is the extensive-form rationalizable strategy set. So, in keeping with Remark 5.1, there always exists a non-empty Δ-rationalizable strategy set.

While the EFBRS and Directed Rationalizability concepts are equivalent, it will often be useful to focus on the former definition. The reason is that properties (i), (ii), and (iii) of an EFBRS give some immediate implications in terms of behavior. In Sections 7-8, we will discuss the consequences of context-dependent forward reasoning for some specific games. There, the EFBRS properties will play an important role, much in the same way that the properties of a self-admissible set (Brandenburger-Friedenberg-Keisler [13, 2008]) play an important role in analyzing games. Indeed, we will see that these properties help to analyze games such as centipede, the finitely repeated prisoner's dilemma, and perfect information games.

7 Analyzing Games

In Section 5, we used the EFBRS concept to analyze Battle of the Sexes with an Outside Option. Now, we turn to analyze what the EFBRS does vs. does not give in games of interest. The approach will be to make use of Properties (i)-(iii) of the EFBRS definition, and not the equivalent Directed Rationalizability definition.

Example 7.1 *Consider the three-legged Centipede game, given in Figure 7.1 below.*

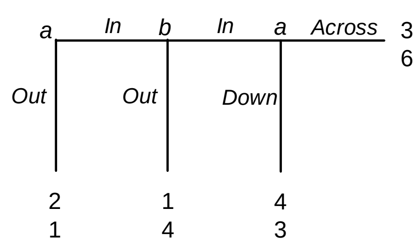

Figure 7.1

Here, the EFBRS's are $\{Out\} \times \{Down\}$ *and* $\{Out\} \times \{Down, Across\}$. *In particular, there is no EFBRS where Ann plays In at the first node. To see this, suppose otherwise, i.e., there exists an EFBRS* $Q_a \times Q_b$ *and a strategy* $s_a \in Q_a$ *where* s_a *plays In at the first node. By condition (i) of an EFBRS, we must have that* $Q_a \subseteq \{Out, In\text{-}Down\}$, *so that* $s_a = In\text{-}Down$. *Now, fix* $s_b \in Q_b$ *and recall that* s_b *must be sequentially optimal under a CPS that strongly believes* Q_a. *Then, at Bob's information set, this CPS must assign probability one to In-Down. Since* s_b *is sequentially optimal under this CPS,* $s_b = Down$. *So, we have that* $Q_b = \{Down\}$. *But, then, In-Down cannot simultaneously satisfy conditions (i)-(ii) of an EFBRS.*

The argument we have presented for the three-legged Centipede is more general. In particular, fix an EFBRS for an n-legged Centipede game. Under the EFBRS, the first player chooses *Out*. This will be a consequence of Proposition 8.1(i) to come.

Example 7.2 *Figure 7.2 gives the Prisoner's Dilemma. Consider the* 3*-repeated version of the game.*

a \ b	C	D
C	c, c	e, d
D	d, e	0, 0

$d > c > 0 > e$

Figure 7.2

Let $Q_a \times Q_b$ be a nonempty EFBRS. Then each $(s_a, s_b) \in Q_a \times Q_b$ results in the Defect-Defect path.[7]

Let us give an intuition: By condition (i) of an EFBRS, each strategy $s_a \in Q_a$ (resp. $s_b \in Q_b$) is sequentially justifiable. As such, s_a (resp. s_b) plays Defect in the last period, at each history allowed by s_a (resp. s_b). Now, consider a second period information set h, where $s_a \in S_a(h)$ and $Q_b \cap S_b(h) \neq \emptyset$. By conditions (i)-(ii) of an EFBRS, s_a must be sequentially optimal under a CPS $\mu_a(s_a)$ with $\mu_a(s_a)(Q_b|S_b(h)) = 1$. Then, conditional on h, $\mu_a(s_a)$ assigns probability one to Bob defecting in the third period, irrespective of Ann's play. As such, s_a plays D at h. And, likewise, with a and b reversed.

Turn to the first period, and suppose, contra hypothesis, there is some $s_a \in Q_a$ so that s_a initially chooses C. For each $s_a \in Q_b$, (s_a, s_b) results in the Defect-Defect path in periods two and three. So, Ann's expected payoffs from s_a corresponds to her first period expected payoffs from playing s_a. With this, the Defect-always strategy yields a strictly higher expected payoff in the first period and an expected payoff of at least zero in subsequent periods. This contradicts s_a being optimal under $\mu_a(s_a)(\cdot|S_b)$.

An analogous result holds for the N-repeated Prisoner's Dilemma, for N finite. The proof is given in Appendix C.

Let us take stock of the examples above. First, in Battle of the Sexes with the Outside Option, we get that either (i) Bob plays *Out* or (ii) Bob plays *In-Right* and Ann plays *Down*. Each of these were subgame perfect paths of play. In Centipede, we get the backward induction path (but not

[7] In the once or twice repeated Prisoner's Dilemma we have a stronger result: If (s_a, s_b) is contained in an EFBRS, then each of s_a and s_b specify *Defect* at each information set.

necessarily the backward induction strategies). And, likewise, in the Finitely Repeated Prisoner's Dilemma, we get the unique Nash (and so subgame perfect) path, where each player *Defects* in all periods.

In each of these cases, the outcomes allowed by an EFBRS coincide with the outcomes allowed by some subgame perfect equilibrium (SPE). This raises the question: what is the relationship between the EFBRS concept and the SPE concept? Are the two concepts equivalent? If so, then we have a good idea what the EFBRS concept delivers (in games of interest), since we have a good idea about what SPE delivers.

We will see that, in a particular class of games, any pure-strategy SPE corresponds to some EFBRS. Each of the examples we mentioned is contained in this class of games. But the EFBRS and SPE concepts are not equivalent.

Definition 7.1 *Say a game Γ has **observable actions** if each information set is a singleton.*

Given distinct terminal (histories) nodes z and z', we can write $z = \left(x, c^1, \ldots, c^K\right)$ and $z' = \left(x, d^1, \ldots, d^L\right)$, where x is the last common predecessor of z and z', i.e., $c^1 \neq d^1$. (Recall, $c^k = \left(c_a^k, c_b^k\right)$ and $d^l = \left(d_a^l, d_b^l\right)$.) Now:

Definition 7.2 *Fix two distinct terminal nodes $z = \left(x, c^1, \ldots, c^K\right)$ and $z' = \left(x, d^1, \ldots, d^L\right)$. Say* **a** ***is decisive for*** $(\mathbf{z}, \mathbf{z}')$ *if a moves at x, $c_a^1 \neq d_a^1$, and $c_b^1 = d_b^1$. And, likewise, with a and b interchanged.*

Definition 7.3 (Battigalli [2, 1997]) *A game satisfies **no relevant ties (NRT)** if whenever a (resp. b) is decisive for (z, z'), $\Pi_a(z) \neq \Pi_a(z')$.*

A game with no ties satisfies NRT, but the converse does not hold. Reny's [25, 1993; Figure 1] Take-It-Or-Leave-It game is one such example.

Fix a strategy s_a and write $[s_a]$ for the set of all r_a that induce the same plan of action as s_a, i.e., the set of all r_a so that $\zeta(r_a, \cdot) = \zeta(s_a, \cdot)$. And, likewise, define $[s_b]$.

Proposition 7.1 *Fix a game Γ with observable actions and a pure-strategy SPE, viz. (s_a, s_b).*

(i) There is an EFBRS, viz. $Q_a \times Q_b$, so that $[s_a] \times [s_b] \subseteq Q_a \times Q_b$.

(ii) If Γ satisfies NRT, then $[s_a] \times [s_b]$ is an EFBRS.

Appendix C proves a result somewhat more general than Proposition 7.1. Note, each of the examples we have seen satisfies both observable actions and NRT. In each of these examples, any pure-strategy subgame perfect equilibrium (s_a, s_b) belongs to an EFBRS, where the EFBRS only allows the terminal node $\zeta(s_a, s_b)$. This fits with part (ii) of the Proposition. Part (i) says that, even if the game fails NRT, (s_a, s_b) will still be contained in some EFBRS. Example C1 in Appendix C shows that, if the game fails NRT, an EFBRS that contains (s_a, s_b) may also allow other paths.

It is important to note that Proposition 7.1 does not say that the pure-strategy SPE concept and the EFBRS concept are equivalent. For a given game, there may be a pure-strategy subgame perfect equilibrium whose outcome is precluded by any EFBRS. (Of course, per Proposition 7.1, this can only occur in games that do not have observable actions.) And, conversely, a given EFBRS may allow outcomes which are precluded by any (even randomized) subgame perfect equilibrium. (This can happen even in a game with observable actions and NRT.) The next examples demonstrate these points.

Example 7.3 *The game in Figure 7.3 satisfies NRT but fails the observable actions condition. It is obtained from the game in Figure 5.1 by two transformations. First, the simultaneous move subgame is transformed into one where Ann moves first and then Bob moves not knowing Ann's choice. Second, two of Ann's decision nodes are coalesced.*

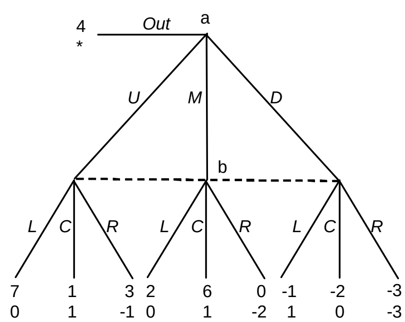

Figure 7.3

Here, $(Out, Right)$ *is a pure strategy subgame perfect equilibrium. But, applying the argument in Section 5, Out is not contained in any EFBRS.*[8]

Example 7.4 *The game in Figure 7.4 satisfies both NRT and the observable actions condition.*

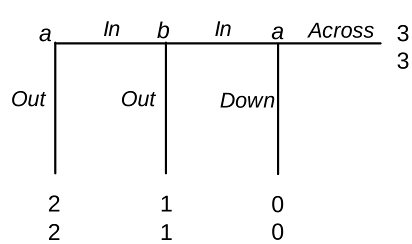

Figure 7.4

[8]Unlike the subgame perfect concept, the EFBRS concept is invariant to coalescing decision nodes.

The unique subgame perfect equilibrium is (In-Across, Across), which results in the (3, 3) outcome. Indeed, this profile induces an EFBRS, viz. {In-Across} × {Across}. But, there are two EFBRS's that give the (2, 2) outcome, namely {Out} × {Down} and {Out} × {Down, Across}.

Taken together with the Main Theorem (Theorem 5.1), Example 7.4 says that a non-backward induction outcome, namely (2, 2), is consistent with RCSBR. To understand this better, notice that *Out* is the unique best response for Ann, under a CPS that assigns probability one to the event "Bob plays *Down*." So, if each type of Ann assigns probability one to $\{Down\} \times T_b$, then conditional upon Bob's node being reached, he must conclude that Ann is irrational. In this case, Bob may very well believe that Ann is playing *In-Down*; if so, *Down* is a unique (sequential) best response for Bob.

8 Perfect Information Games

Example 7.3 shows that, in games without observable actions, the SPE concept allows for outcomes excluded by every EFBRS. On the other hand, Proposition 7.1 and Example 7.4 show that, in games with observable actions, the SPE concept is a strict refinement of the EFBRS concept. Thus, even in these games, we cannot use the SPE concept to analyze the consequences of context-dependent forward induction reasoning.

Now we turn to a particular class of games with observable actions—namely, perfect information games (i.e., games with observable actions and with at most one active player at each information set). We've seen some examples of perfect-information games, e.g., Examples 7.1 and 7.4. In the former case, each EFBRS yields the backward induction path (and so the backward induction outcome). Of course, for that game, the Nash and backward induction paths coincide. On the other hand, in Example 7.4, one EFBRS corresponds to backward induction, but others do not. However, there we do get that the EFBRS paths correspond (exactly) to the Nash paths (and so Nash outcomes) of the game.

The examples suggest there may be a connection between EFBRS's and Nash outcomes, at least for perfect-information (PI) games. (Of course, for non-PI games, an EFBRS may give non-Nash outcomes.) Indeed, there will be a connection, for PI games satisfying a "no ties" condition.

Definition 8.1 (Marx-Swinkels [22, 1997]) *A game satisfies **transference of decision-maker indifference (TDI)** if $\pi_a(s_a, s_b) = \pi_a(r_a, s_b)$ implies $\pi_b(s_a, s_b) = \pi_b(r_a, s_b)$; and likewise with a and b interchanged.*

If a game satisfies NRT, it also satisfies TDI. Yet, many games of interest satisfy TDI, but fail NRT. For example, zero sum games satisfy TDI but may fail NRT.

Proposition 8.1

(i) Fix a PI game Γ satisfying TDI. If $Q_a \times Q_b$ is an EFBRS then, there exists a pure-strategy Nash equilibrium, viz. (s_a, s_b), so that each profile in $Q_a \times Q_b$ is outcome equivalent to (s_a, s_b).

(ii) Fix a PI game Γ satisfying NRT. If (s_a, s_b) is a pure-strategy Nash equilibrium in sequentially justifiable strategies, then there is an EFBRS, viz. $Q_a \times Q_b$, so that $(s_a, s_b) \in Q_a \times Q_b$.

The proof can be found in Appendix D. Taken together Theorem 5.1 and Proposition 8.1 give:

Corollary 8.1

(i) Fix a PI game Γ satisfying TDI, and an epistemic type structure. If there is RCSBR at the state (s_a, t_a, s_b, t_b), then (s_a, s_b) is outcome equivalent to a pure-strategy Nash equilibrium.

(ii) Fix a PI game Γ satisfying NRT, and a pure-strategy Nash equilibrium, viz. (s_a, s_b), in sequentially justifiable strategies. Then, there exists an epistemic structure and a state thereof, viz. (r_a, t_a, r_b, t_b), at which there is RCSBR and $(r_a, r_b) = (s_a, s_b)$.

Why the connection between EFBRS's and Nash equilibria? Recall, if each player is "rational" (i.e., maximizes subjective expected utility) and places probability one on the actual strategy choices by the other player, then the strategy choices constitute a Nash equilibrium. In a PI game satisfying TDI, RCSBR imposes a form of correct beliefs about the actual outcomes that will obtain. Let us recast this at the level of the solution concept: In a PI game satisfying TDI, each strategy profile in a given EFBRS is outcome equivalent. (This will be Lemma D2 in Appendix D.) So, along the path of play, the associated CPS('s) must assign probability one to a particular outcome—the outcome associated with the EFBRS, i.e., the "correct" outcome. (This uses condition (ii) of an EFBRS.) With this, we get a Nash outcome (but not necessarily the Nash strategies).[9]

This was the intuition for part (i) of Corollary 8.1. The proof follows the proof of Proposition 6.1a in Brandenburger-Friedenberg [12, 2010], though now making use of the EFBRS properties. (The proof in [12, 2010] makes use of properties of self-admissible sets. See 9c below.) Indeed, we only use properties (i)-(ii) of Definition 5.1.

The converse, i.e., part (ii), is novel. (In particular, both the 'no ties' condition and the proof are quite different from the analysis in [12, 2010].) A Nash equilibrium in sequentially justifiable strategies will, in general, satisfy conditions (i)-(ii) of an EFBRS. However, it may fail the maximality criterion. Indeed, the proof makes use of all three properties of Definition 5.1. See Appendix D.

The no ties conditions are important for both directions of Proposition 8.1. Appendix D explains why, by way of examples. Also, notice that there is a gap between parts (i)-(ii) of Proposition 8.1. In particular, part (i) says that starting from an EFBRS we can get a pure Nash outcome, while part (ii) says that starting from a sequentially justifiable pure-strategy Nash equilibrium, we can get an EFBRS.

[9] Ben Porath [10, 1997] is another epistemic analysis of perfect information games. His analysis is based on "rationality and common initial belief of rationality" plus a grain of truth assumption. It also gives Nash outcomes.

We cannot improve part (ii) to say that, starting from any pure Nash equilibrium, we get an EFBRS. (This is because a Nash equilibrium may not be sequentially justifiable. See Appendix D.) We do not know if we can improve part (i) to say that, starting from an EFBRS, we get a pure-strategy Nash equilibrium in sequentially justifiable strategies. (Appendix D elaborates on the issue.) However, we will see that, starting from an EFBRS, we can get a mixed-strategy Nash equilibrium that satisfies a "sequential justifiability" condition. (We'll make the condition precise below.)

Consider a pure strategy profile (s_a, s_b) and a mixed strategy profile $(\varpi_a, \varpi_b) \in \mathcal{P}(S_a) \times \mathcal{P}(S_b)$. Call (s_a, s_b) and (ϖ_a, ϖ_b) **outcome equivalent** if $\pi(s_a, s_b) = \pi(\varpi_a, \varpi_b)$. Likewise, call $Q_a \times Q_b \subseteq S_a \times S_b$ and $(\varpi_a, \varpi_b) \in \mathcal{P}(S_a) \times \mathcal{P}(S_b)$ **outcome equivalent** if each $(s_a, s_b) \in Q_a \times Q_b$ is outcome equivalent to (ϖ_a, ϖ_b). Then:

Proposition 8.2 *Fix a PI game satisfying TDI. If $Q_a \times Q_b$ is an EFBRS, then there exists a mixed-strategy Nash equilibrium, viz. (σ_a, σ_b), so that:*

(i) $Q_a \times Q_b$ is outcome equivalent to (σ_a, σ_b), and

(ii) each $s_a \in \mathrm{Supp}\, \sigma_a$ (resp. $s_b \in \mathrm{Supp}\, \sigma_b$) is sequentially justifiable.

Proposition 8.2 gives that, if we begin with an EFBRS, we can construct an equivalent mixed-strategy Nash equilibrium. The Nash equilibrium has the property that each strategy in its support is sequentially justifiable. But, it is important to note that this does not necessarily give that the mixed-strategy itself is sequentially justifiable.[10] More to the point: Given a PI game satisfying TDI and some mixed-strategy Nash equilibrium, viz. (σ_a, σ_b), does there exist some pure-strategy Nash equilibrium, viz. (s_a, s_b), so that s_a (resp. s_b) is contained in the support of σ_a (resp. σ_b)? If so, using Proposition 8.2, we get that starting from an EFBRS, there is a pure-strategy Nash equilibrium in sequentially justifiable strategies. But, this too is not known.

9 Discussion

In this section, we discuss some conceptual aspects of the paper, as well as some extensions.

a. Context-Dependent Forward Induction: We characterized the behavioral implications of forward induction reasoning across all type structures. In Section 1, we explained the desire to have such a characterization theorem. In particular, our view is that there may be a context to the particular strategic situation studied and this context may lead to restrictions on players' beliefs that are "transparent" to the players. But, the analyst himself may not know the context, i.e., may

[10] In non-PI games, we can construct a mixed-strategy Nash equilibrium, viz. (σ_a, σ_b), where each strategy in the support of σ_a and σ_b is sequentially justifiable, but σ_a is itself not sequentially justifiable. The question remains whether or not the same can occur in PI games.

not know which beliefs are vs. are not "transparent" to the players. If this is the case, the analyst will want to understand the behavioral implications of forward induction reasoning across all type structures.

Notice that we have implicitly equated analyzing forward induction reasoning across all "transparent restrictions on players beliefs" with analyzing forward induction reasoning across all type structures. We can make this step precise: First, formalize the idea that certain (events about) beliefs are "transparent" to the players. For this, begin with Battigalli-Siniscalchi's [6, 1999] canonical construction of a type structure; this type structure contains all hierarchies of conditional beliefs (satisfying coherency and common belief of coherency).[11] Let us look at the self-evident events within this structure. Loosely, we look at events $S_a \times E_a \times S_b \times E_b \in \mathcal{B}\left(S_a \times T_a \times S_b \times T_b\right)$, where $E = S_a \times E_a \times S_b \times E_b$ obtains and, at each information set, each player assigns probability one to E, each player assigns probability one to the other player assigning probability one to E, etc. These self-evident events represent "transparent" restrictions on players' beliefs: Each type structure can be mapped into the canonical construction and, in a certain sense, each type structure forms a self-evident event in the canonical construction, i.e., under this mapping.[12] Furthermore, each such self-evident event in the canonical type structure corresponds to a "smaller" type structure. Forward induction reasoning is preserved under these mappings. (See [3, 2009] for the formal statement.)

There is a special type of "transparent" restriction on beliefs: those generated only by restrictions on first-order beliefs. (For instance, in the lady's choice convention, we restricted Bob's first-order beliefs requiring that he assign probability one to Ann playing *Up*. The only restriction on Ann's second-order beliefs is to require that Ann assigns probability one to the event that "Bob assigns probability one to Ann playing *Up*.") These restrictions on first-order beliefs, viz. Δ, generate a particular type of self-evident event. Here, analyzing RCSBR within the associated type structure leads to the Δ-rationalizable strategy set. Indeed, this is related to Battigalli-Siniscalchi's [8, 2003] motivation in defining Directed Rationalizability.[13]

In light of the above discussion, it is not conceptually difficult to see that each Δ-rationalizable strategy set is an EFBRS and so each Δ-rationalizable strategy set is consistent with RCSBR in some structure. Although, we note, the proof is not trivial. The converse is conceptually challenging. Referring back to Section 6, if we begin with an arbitrary structure and take Δ to be the set of first-order beliefs induced by that structure, the Δ-rationalizable strategy set may be distinct from the strategies consistent with RCSBR. Nonetheless, we have seen that we can find some other set of first-order beliefs, viz. $\bar{\Delta}$, so that the $\bar{\Delta}$-rationalizable strategy set is the set of strategies consistent with RCSBR. This step is not obvious *ex ante*, but it is easy to prove building on Theorem 5.1, i.e.,

[11] Note, Battigalli-Siniscalchi's [6, 1999] canonical construction is a type structure in the sense of Definition 3.2. Specifically, in the case of a game tree, the basic conditioning events are clopen and so [6, 1999] get T_a and T_b to be Polish, as an output. Here, we don't require Polishness.

[12] This statement presumes that the image of the type set (under the mapping to the canonical construction) is measurable.

[13] The treatment here is due to Battigalli-Prestipino [4, 2009]. It is related to, by somewhat different from, the motivation in Battigalli-Siniscalchi [8, 2003]-[9, 2007].

once we already have the definition of an EFBRS in hand.

b. Two vs. Three Player Games: Here, we have focused on two player games. The main results (Theorem 5.1 and Corollary 6.1) extend to games with three or more players, up to issues of correlation. Specifically, if we allow for correlated assessments in Definition 4.6, then we must also allow for correlated assessments in Definition 5.1. A similar statement holds for the case of independence—though, of course, care is needed in defining independence for CPS's. The central issue is that Charlie's belief about Bob should not change after Charlie learns information only about Ann. (The idea dates back to Hammond [17, 1987] and is related to the "do not signal what you do not know" condition of Fudenberg-Tirole [15, 1991]. See Battigalli [1, 1996] for a formalization of the idea and a discussion of [15, 1991].)

There is an additional issue that arises in the three player case: Should we require that Ann strongly believes "Bob and Charlie are rational"? Or should we instead require that Ann strongly believes "Bob is rational" and strongly believes "Charlie is rational"? Arguably, in the case of independence, we should require the latter.

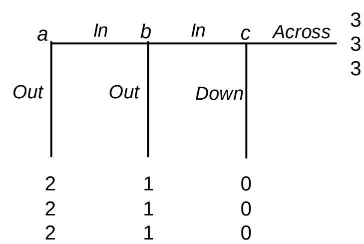

Figure 9.1

How does this affect our analysis of games? Amend Figure 7.4, so that it is a three-player game, as in Figure 9.1. Consider a state at which there is RCSBR in the sense explained above (i.e., Bob has an independent assessment and strongly believes both "Ann is rational" and "Charlie is rational"). Let's ask which strategies can be played. Of course, using rationality, Charlie must play *Across* (at this state). Next, we require that a type of Bob strongly believe "Ann is rational" and also "Charlie is rational." So, conditional upon Bob's information set being reached, this type must maintain a hypothesis that Charlie is rational, and so that Charlie plays *Across*. In this case, Bob's unique best response is to play *In*. Turning to Ann, we see that under an RCSBR analysis she will choose *In*. So, we only get the backward induction outcome. (Battigalli-Siniscalchi [5, 1999] provide a "context free" epistemic analysis of this notion of independent rationalization.)

This example also shows that, in the case of independence, Proposition 8.1(ii) does not hold. If we instead consider the case of correlation, then it may also be natural to instead require that Bob strongly believe "Ann and Charlie are rational" (i.e., as opposed to strong belief of "Ann is rational" and strong belief of "Charlie is rational"). Of course, it may be the case that, when Bob's node is reached, he must forgo the hypothesis that "Ann and Charlie are rational." Thus, in this case, we do have an analogue of Proposition 8.1(ii). Indeed, both parts (i)-(ii) of Proposition 8.1 hold for the case of correlation.

c. Properties of EFBRS's: Refer back to Sections 7-8. To analyze games of interest, we made use of the three properties of an EFBRS. Many of these arguments drew from Brandenburger-Friedenberg's [12, 2010] analysis of self-admissible sets: They began with properties of self-admissible sets (SAS's) and, analogously, used these properties to draw implications in terms of behavior in games.

While there is a close connection between the EFBRS properties and the SAS properties, there are also important points of difference. Indeed, the concepts are distinct. For an SAS, viz. $Q_a \times Q_b$, each $s_a \in Q_a$ must be admissible (i.e., not weakly dominated) in both the matrices $S_a \times S_b$ and $S_a \times Q_b$. For an EFBRS, we only require that each $s_a \in Q_a$ must be sequentially optimal under a CPS that strongly believes Q_b. If s_a meets the former criterion, it meets the latter criterion, but the converse need not hold. So, in this sense, it is harder to meet the SAS criterion vs. the EFBRS criterion. On the other hand, SAS also has a maximality criterion, and it is easier to meet the SAS maximality criterion vs. the EFBRS maximality criterion.

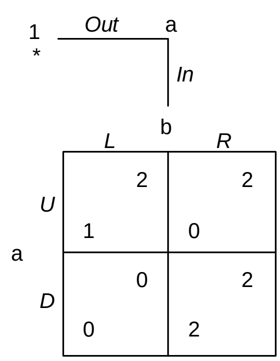

Figure 9.2

Putting these considerations together, we can have an EFBRS that is not an SAS, and an SAS that is not an EFBRS. To see that an EFBRS need not be an SAS, refer to Figure 9.2. There, $\{Out\} \times \{Left, Right\}$ is an EFBRS, but the only SAS is $\{In\text{-}Down\} \times \{Right\}$. (Here, we use the admissibility criteria of SAS's.) To see that an SAS need not be an EFBRS, refer to Figure 5.1. There, $\{Out\} \times \{Left, Center\}$ is an SAS, but the only EFBRS is $\{In\text{-}Middle\} \times \{Center\}$.

(Here, we use the fact that it is easier to meet the maximality criteria for SAS's vs. EFBRS's.)

Appendix A Proofs for Section 4

Proof of Property 4.1. Fix an event $F \in \mathcal{E}$ with $F \cap \bigcap_m E_m \neq \emptyset$. Then $F \cap E_m \neq \emptyset$ for all m. So, for each m, $\mu(E_m|F) = 1$. (This is because μ strongly believes each E_m.) But then $\mu(\bigcap_m E_m|F) = 1$. ■

Proof of Property 4.2. Fix an event $F \in \mathcal{E}$ with $F \cap \text{proj}_{\Omega_1} E \neq \emptyset$. Then $(F \times \Omega_2) \cap E \neq \emptyset$. Note that $\text{marg}_{\Omega_1} \mu \left(\text{proj}_{\Omega_1} E|F\right)$ is well defined because $\text{proj}_{\Omega_1} E$ is Borel by assumption. Since μ strongly believes E, $\mu(E|F \times \Omega_2) = 1$. Then $(\text{marg}_{\Omega_1} \mu)\left(\text{proj}_{\Omega_1} E|F\right) = 1$, as required. ■

Appendix B Proofs for Section 6

In what follows, we fix a set of first-order beliefs $\Delta = \Delta_a \times \Delta_b$, with $\Delta_a \subseteq \mathcal{C}(S_b)$, $\Delta_b \subseteq \mathcal{C}(S_a)$.

Lemma B1 *Fix $s_a \in S_a^{\Delta,m+1}$, for $m \geq 0$. There exists a CPS μ_a so that $s_a \in \rho_a(\mu_a)$ and μ_a strongly believes each $S_b^{\Delta,n}$ for $n \leq m$.*

Proof. Fix $s_a \in S_a^{\Delta,m+1}$. Then, for each $n \leq m$, there exists a CPS μ_a^n so that $s_a \in \rho_a(\mu_a^n)$ and μ_a^n strongly believes $S_b^{\Delta,n}$. We will show that there exists a CPS μ_a so that $s_a \in \rho_a(\mu_a)$ and μ_a strongly believes each $S_b^{\Delta,n}$ for each $n \leq m$.

Begin by constructing a CPS μ_a: For each information set $h \in H_a$, let $n(h) = \max\{n : S_b(h) \cap S_b^{\Delta,n} \neq \emptyset\}$ and set $\mu_a(\cdot|S_b(h)) = \mu_a^{n(h)}(\cdot|S_b(h))$. To see that μ_a is indeed a CPS, first note that conditions (i)-(ii) are immediate from the construction. For condition (iii), fix events $E \subseteq S_b(h) \subseteq S_b(i)$. Note that $n(h) \leq n(i)$. We have to show that

$$\mu_a^{n(i)}(E|S_b(i)) = \mu_a^{n(h)}(E|S_b(h))\, \mu_a^{n(i)}(S_b(h)|S_b(i)). \tag{B1}$$

First, suppose that $S_b(h) \cap S_b^{\Delta,n(i)} \neq \emptyset$. Then $n(h) = n(i)$, so (B1) follows from condition (iii) for CPS $\mu_a^{n(i)}$. Next, suppose that $S_b(h) \cap S_b^{\Delta,n(i)} = \emptyset$. Since $\mu_a^{n(i)}$ strongly believes $S_b^{\Delta,n(i)}$ and $E \subseteq S_b(h)$,

$$\mu_a^{n(i)}(E|S_b(i)) = \mu_a^{n(i)}(S_b(h)|S_b(i)) = 0,$$

from which (B1) follows.

It is immediate from the construction that s_a is sequentially optimal under μ_a. (Use the fact that s_a is sequentially optimal under each μ_a^n.) Moreover, μ_a strongly believes each $S_b^{\Delta,n}$ for $n \leq m$. (Use the fact that $\mu_a^{n(h)}(S_b^{\Delta,n}|S_b(h)) = 1$ for each $n \leq n(h)$, i.e. each n with $S_b^{\Delta,n} \cap S_b(h) \neq \emptyset$.) ■

Appendix C Examples and Proofs for Section 7

We begin by showing that, for the finitely repeated Prisoner's Dilemma, any EFBRS results in the *Defect-Defect* path of play. To show this, we will need to make use of certain properties of EFBRS's. We will again make use of these properties in Appendix D. We begin with the best response property.

Definition C1 *Say $Q_a \times Q_b \subseteq S_a \times S_b$ satisfies the* ***best response property*** *if, for each $s_a \in Q_a$ there is a CPS $\mu_a \in \mathcal{C}(S_b)$, so that $s_a \in \rho_a(\mu_a)$ and μ_a strongly believes Q_b. And similarly for b.*

An EFBRS satisfies the best response property. But the converse need not hold, i.e., $Q_a \times Q_b$ may satisfy the best response property, but fail to be an EFBRS because it violates the maximality condition. (See the example in Section 5.)

Let us introduce some notation, to relate the whole game to its parts. Fix a game Γ and a subgame Σ. Write H_a^{Σ} for the set of information sets that are contained in Σ. We will abuse notation and write $S_a(\Sigma)$ for the set of strategies of Γ that allow Σ. We also write $S_a^{\Sigma} = \prod_{h \in H_a^{\Sigma}} C_a(h)$ for the set of strategies of a in the subgame Σ. Note, each strategy $s_a^{\Sigma} \in S_a^{\Sigma}$ can be viewed as the projection of a strategy $s_a \in S_a(\Sigma)$ into S_a^{Σ}. Given a set $E_a \subseteq S_a$, write E_a^{Σ} for the set of strategies $s_a^{\Sigma} \in S_a^{\Sigma}$ so that there is some $s_a \in E_a \cap S_a(\Sigma)$ whose projection into S_a^{Σ} is s_a^{Σ}. We will write π_a^{Σ} and π_b^{Σ} for the payoff functions associated with the subtree Σ. So, if (s_a, s_b) allows Σ, then $\pi^{\Sigma}(s_a^{\Sigma}, s_b^{\Sigma}) = \pi(s_a, s_b)$.

Lemma C1 *Fix a game Γ and a subgame Σ. If $Q_a \times Q_b$ satisfies the best response property for the game Γ, then $Q_a^{\Sigma} \times Q_b^{\Sigma}$ satisfies the best response property for the subgame Σ.*

Proof. If $Q_a^{\Sigma} \times Q_b^{\Sigma} = \emptyset$ (if no profile in $Q_a \times Q_b$ allows Σ), then it is immediate that $Q_a^{\Sigma} \times Q_b^{\Sigma}$ satisfies the best response property. So, we will suppose $Q_a^{\Sigma} \times Q_b^{\Sigma} \neq \emptyset$.

Fix a strategy $s_a^{\Sigma} \in Q_a^{\Sigma}$. Then there exists a strategy $s_a \in Q_a \cap S_a(\Sigma)$ whose projection into $\prod_{h \in H_a^{\Sigma}} C_a(h)$ is s_a^{Σ}. Since $s_a \in Q_a$, we can find a CPS $\mu_a \in \mathcal{C}(S_b)$ so that $s_a \in \rho_a(\mu_a)$ and μ_a strongly believes Q_b.

Let $\mathcal{S}_b^{\Sigma}$ be the set of all $S_b^{\Sigma}(h)$ for $h \in H_a^{\Sigma}$. Define $\nu_a^{\Sigma}(\cdot|\cdot) : \mathcal{B}(S_b^{\Sigma}) \times \mathcal{S}_b^{\Sigma} \to [0,1]$ so that, for each event $E_b \subset S_b$ and each $S_b^{\Sigma}(h) \in \mathcal{S}_b^{\Sigma}$, $\nu_a^{\Sigma}(E_b^{\Sigma}|S_b^{\Sigma}(h)) = \mu_a(E_b|S_b(h))$. It is readily verified that ν_a^{Σ} is indeed a CPS on $(S_b^{\Sigma}, \mathcal{S}_b^{\Sigma})$.

Since s_a allows Σ and s_a is sequentially optimal under μ_a, it follows that s_a^{Σ} is sequentially optimal under ν_a^{Σ}. Fix some $S_b^{\Sigma}(h) \in \mathcal{S}_b^{\Sigma}$. If $Q_b^{\Sigma} \cap S_b^{\Sigma}(h) \neq \emptyset$, then $Q_b \cap S_b(h) \neq \emptyset$. So, in this case, $\nu_a^{\Sigma}(Q_b^{\Sigma}|S_b^{\Sigma}(h)) = \mu_a(Q_b|S_b(h)) = 1$. This establishes that ν_a^{Σ} strongly believes Q_b^{Σ}.

Interchanging a and b establishes the result. ■

We use Lemma C1 to show:

Lemma C2 *Consider the N-repeated Prisoner's Dilemma, as given in Figure 6.2. If $Q_a \times Q_b$ satisfies the best response property for this game, then each strategy profile in $Q_a \times Q_b$ results in the Defect-Defect path.*

Proof. The proof very closely follows the proof of Example 3.2 in Brandenburger-Friedenberg [12, 2010]. It is by induction on N. For $N = 1$, the result is immediate. Assume the result holds for some N and we will show it holds for $N + 1$.

Consider some $Q_a \times Q_b$ of the $N+1$ repeated Prisoner's Dilemma that satisfies the best response property. Suppose, there is a strategy $s_a \in Q_a$ that Cooperates in the first period. Fix a strategy $s_b \in Q_b$. If s_b plays Cooperate (resp. Defect) in the first period, Ann gets c (resp. e) in the first period. By Lemma C1 and the induction hypothesis, Ann gets a payoff of zero, in periods $2, \ldots, N$. So, for each s_b in Q_b, $\pi_a(s_a, s_b) = c$ if s_b plays Cooperate in the first period, and $\pi_a(s_a, s_b) = e$ if s_b plays Defect in the first period.

Now, instead consider the strategy r_a that plays Defect in every period, irrespective of the history. Again, fix a strategy $s_b \in Q_b$. If s_b plays Cooperate in the first period, then $\pi_a(r_a, s_b) \geq d$ and, if $s_b \in Q_b$ plays Defect in the first period, then $\pi_a(r_a, s_b) \geq 0$.

Putting the above together: Under any CPS that strongly believes Q_b, we must have that r_a is a strictly better response than $s_a \in Q_a$, at the first information set. But this contradicts $Q_a \times Q_b$ satisfying the best response property. ■

Corollary C1 *Consider the N-repeated Prisoner's Dilemma, as given in Figure 6.2. If $Q_a \times Q_b$ is an EFBRS, then each strategy profile in $Q_a \times Q_b$ results in the Defect-Defect path.*

Now we turn to Proposition 7.1. We will show the result for a somewhat more general set of games—games where, in a sense, the information structure is determined by the subgames.

Definition C2 *Fix a game Γ. Say a subgame Σ is **sufficient** for an information set $h \in H$ if h is contained in Σ and the set of strategy profiles that allow Σ is exactly $S_a(h) \times S_b(h)$.*

Note, there may be two subgames, viz. Σ and $\bar{\Sigma}$, that are sufficient for h.[14] If so, either Σ is a subgame of $\bar{\Sigma}$ or $\bar{\Sigma}$ is a subgame of Σ. When there are two subgames that are sufficient for h, we will in typically be interested in the **last subgame Σ sufficient for** h—i.e., so that no proper subgame of Σ is sufficient for h.

Also, notice that there may be no subgame that is sufficient for an information set h. Refer to the game in Figure 7.3. There, the only subgame is the entire game. But this subgame is not sufficient for the information set, viz. h, at which Bob moves. To see this, notice that the strategy $s_a = Out$ (trivially) allows the subgame, but does not allow h.

Definition C3 *Say a game Γ is **determined by its subgames** if, for each information set $h \in H$, there is a subgame Σ that is sufficient for h.*

The game in Figure 7.3 is not determined by its subgame; as we have seen, there is no subgame that is sufficient for the information set at which Bob moves. Below, we will characterize Definition C3 in terms of primitives of the game (as opposed to a condition about strategies).

[14] This may happen if there is a node x where no player is active, i.e., $C_a(x)$ and $C_b(x)$ are singletons.

Throughout, we restrict attention to a game Γ determined by its subgames. Fix a pure-strategy SPE, viz. (s_a, s_b), of Γ. Construct maps $f_a : H \to S_a$ and $f_b : H \to S_b$ that depend on this SPE. To do so, fix some $h \in H$, and let Σ be the last subgame sufficient for h. Write x for the root of subgame Σ (which may be Γ itself). If $\Sigma = \Gamma$, set $f_a(h) = s_a$. If Σ is a proper subtree of Γ, then we can write $x = (c^1, ..., c^K)$. In this case, let $f_a(h)$ be the strategy that (i) chooses c_a^1 at $\{\phi\}$, (ii) chooses c_a^k at an information set that contains $(c^1, ..., c^{k-1})$, i.e., an initial segment of $(c^1, ..., c^K)$, and (iii) makes the same choice as s_a at all other information sets. So, if s_a allows h, then $f_a(h) = s_a$. Also, note, $f_a(h)$ is well-defined and allows h precisely because Γ is determined by its subgames. (Again, refer to the game in Figure 7.3, and take h to be the information set at which Bob moves. Consider the SPE $(s_a, s_b) = (Out, Right)$. Then, $f_a(h) = Out$, which precludes h.)

Write $S(h)$ for the set of strategy profiles that allow an information set h. In games determined by their subgames, there is a natural order on sets of the form $S(h)$, for $h \in H$. Specifically, for any pair of information sets h and i (in H), either $S(h) \subseteq S(i)$, $S(i) \subseteq S(h)$, or $S(h) \cap S(i) = \emptyset$.[15] To see this, let Σ_h (resp. Σ_i) be sufficient for h (resp. i). Note, either Σ_h is a subgame of Σ_i, Σ_i is a subgame of Σ_h, or they are disjoint subgames. With this, the order follows from the definition of sufficiency. If $S(h) \subseteq S(i)$, say h **follows** i. Say h and i are **ordered** if either h follows i or i follows h. Say h and i are **unordered** otherwise, i.e., if $S(h) \cap S(i) = \emptyset$.

Let us record a couple of facts, to be used below. The first is immediate.

Lemma C3 *Fix a game Γ that is determined by its subgames, and also fix some SPE (s_a, s_b). Construct (f_a, f_b) as above. If $f_a(h)$ allows i and either h and i are unordered or i follows h, then $f_a(i) = f_a(h)$.*

The next result is immediate from the definition of an SPE.

Lemma C4 *Fix a game Γ that is determined by its subgames and some SPE (s_a, s_b). For each $h \in H_a$,*

$$\pi_a(f_a(h), f_b(h)) \geq \pi_a(r_a, f_b(h)) \quad \text{for all } r_a \in S_a(h).$$

The next result holds quite generally. Again, its proof is immediate.

Lemma C5 *Fix some $\mu_a \in \mathcal{C}(S_b)$. If $s_a \in \rho_a(\mu_a)$, then $[s_a] \subseteq \rho_a(\mu_a)$.*

Proposition C1 *Fix a game Γ that is determined by its subgames, and a pure-strategy SPE, viz. (s_a, s_b).*

(i) There is an EFBRS, viz. $Q_a \times Q_b$, so that $[s_a] \times [s_b] \subseteq Q_a \times Q_b$.

(ii) If Γ satisfies NRT, then $[s_a] \times [s_b]$ is an EFBRS.

[15] Note, in all perfect recall games, whenever $h, i \in H_a$, either $S(h) \subseteq S(i)$, $S(i) \subseteq S(h)$, or $S(h) \cap S(i) = \emptyset$. Here, we have analogous statement, when $h \in H_a$ and $i \in H_b$.

Proof. Fix a pure-strategy SPE, viz. (s_a, s_b). Construct maps $f_a : H \to S_a$ and $f_b : H \to S_b$, as above. We use these maps to construct CPS's $\mu_a \in \mathcal{C}(S_b)$ and $\mu_b \in \mathcal{C}(S_a)$. Specifically, set $\mu_a(f_b(h)|S_b(h)) = 1$ for each $h \in H_a$. And likewise for a and b interchanged.

To see that μ_a is indeed a CPS: Note, it is immediate that μ_a satisfies conditions (i)-(ii) of Definition 3.1. For condition (iii), fix information sets $h, i \in H_a$ so that $S_b(i) \subseteq S_b(h)$. If $f_b(h) \in S_b(i)$, then $f_b(i) = f_b(h)$. (Lemma C3.) So, for each event $E \subseteq S_b(i)$,

$$\mu_a(E|S_b(h)) = \mu_a(E|S_b(i)) \times 1 = \mu_a(E|S_b(i))\,\mu_a(S_b(i)|S_b(h)).$$

If $f_b(h) \notin S_b(i)$, then for each event $E \subseteq S_b(i)$,

$$\mu_a(E|S_b(h)) = 0 = \mu_a(E|S_b(i)) \times 0 = \mu_a(E|S_b(i))\,\mu_a(S_b(h)|S_b(i)),$$

as required. And, likewise, for b.

Now, let $Q_a = \rho_a(\mu_a)$, i.e., the set of all strategies r_a that are sequentially optimal under μ_a. And, likewise, set $Q_b = \rho_b(\mu_b)$. We will show that $Q_a \times Q_b$ is an EFBRS.

Fix some $r_a \in Q_a$. We will show that r_a and μ_a jointly satisfy conditions (i)-(iii) of an EFBRS. In fact, it is immediate that Conditions (i) and (iii) are satisfied. So, we will show condition (ii), i.e., that μ_a strongly believes Q_b.

Fix an information set $h \in H_a$ with $Q_b \cap S_b(h) \neq \emptyset$. We will show that $f_b(h) \in Q_b$, so that $\mu_a(Q_b|S_b(h)) = 1$. To show that $f_b(h) \in Q_b$, it suffices to show that, for each information set $i \in H_b$ allowed by $f_b(h)$,

$$\pi_b(f_a(i), f_b(h)) \geq \pi_b(f_a(i), r_b) \quad \text{for all } r_b \in S_b(i). \tag{C1}$$

Note, if either i follows h or h and i are unordered, then $f_b(h) = f_b(i)$. In either case, we can apply Lemma C4 to the information set i and get the desired result. So, we focus on the case where h follows i.

Take $S(h) \subseteq S(i)$. Since $Q_b \cap S_b(h) \neq \emptyset$, there is a strategy $r_b \in Q_b \cap S_b(h)$. For this strategy r_b, we have that $\pi_b(f_a(i), r_b) \geq \pi_b(f_a(i), f_b(h))$, because r_b is sequentially optimal under μ_b, $\mu_b(f_a(i)|S_a(i)) = 1$, and $f_b(h) \in S_b(h) \subseteq S_b(i)$. We will show that $\pi_b(f_a(i), r_b) = \pi_b(f_a(i), f_b(h))$, establishing Equation C1.

Suppose, contra hypothesis, that $\pi_b(f_a(i), r_b) > \pi_b(f_a(i), f_b(h))$. Consider the information set j, so that the last common predecessor of $(f_a(i), r_b)$ and $(f_a(i), f_b(h))$ is contained in j. Now, use the fact that r_b and $f_b(h)$ both allow h, to get that either j follows h or j and h are unordered. In these cases, we have that $\pi_b(f_a(j), f_b(h)) \geq \pi_b(f_a(j), r_b)$. (This was established in the previous paragraph.) But now note that, since either j follows h or j and h are unordered, we also have that either j follows i or j and i are unordered. In either case, using the fact that $f_a(i)$ allows j, we have $f_a(i) = f_a(j)$. (Lemma C3.) So, putting the above facts together,

$$\begin{aligned} \pi_b(f_a(i), f_b(h)) &= \pi_b(f_a(j), f_b(h)) \\ &\geq \pi_b(f_a(j), r_b) \\ &= \pi_b(f_a(i), r_b) \geq \pi_b(f_a(i), f_b(h)). \end{aligned}$$

But this contradicts the assumption that $\pi_b(f_a(i), r_b) > \pi_b(f_a(i), f_b(h))$.

We have established that $Q_a \times Q_b = \rho_a(\mu_a) \times \rho_b(\mu_b)$ is an EFBRS. By construction, $(s_a, s_b) \in \rho_a(\mu_a) \times \rho_b(\mu_b)$. So, using Lemma C5, $[s_a] \times [s_b] \subseteq Q_a \times Q_b$. Now, suppose the game tree has NRT. We will show that, if $(r_a, r_b) \in Q_a \times Q_b$, then $(r_a, r_b) \in [s_a] \times [s_b]$. To see this, fix some strategy $r_a \notin [s_a]$. Then, there exists some $r_b \in S_b$ with $\zeta(s_a, r_b) \neq \zeta(r_a, r_b)$. Consider the last common predecessor of $\zeta(s_a, r_b)$ and $\zeta(r_a, r_b)$ and let h be the information set that contains this node. Now, we have that $\pi_a(s_a, f_b(h)) \geq \pi_a(r_a, f_b(h))$, by the above analysis. NRT says that, in fact, $\pi_a(s_a, f_b(h)) > \pi_a(r_a, f_b(h))$. This implies that $r_a \notin Q_a$, as required. ■

Lemma C6 *If Γ has observable actions, then Γ is determined by its subgames.*

Proof. Fix an information set h. Since Γ has observable actions, $h = \{x\}$ for some node/history x. Now, consider a node y that follows x. Then, by observable actions, y is contained in the information set $\{y\}$. It follows that there is a subgame whose initial node is x, written Σ. Moreover, the set of strategies that allow Σ is exactly $S_a(h) \times S_b(h)$. So, Γ is determined by its subgames. ■

Proof of Proposition 7.1. Immediate from Proposition C1 and Lemma C6. ■

Finally, we return to characterize the condition that Γ is determined by its subgames, in terms of primitives of the game tree alone (i.e., without reference to strategies). For this, we will need some notation: Given a set of nodes, viz. $\{x^1, \ldots, x^K\}$, write $\mathrm{lcp}(\{x^1, \ldots, x^K\})$ for the last common predecessor of these nodes.

Lemma C7 *A game Γ is determined by its subgames if and only if, for each information set $h \in H$, the following holds:*

(i) the last common predecessor of nodes in h, viz. $\mathrm{lcp}(h)$, is the root of a subgame, and

(ii) the set of terminal nodes allowed by $\mathrm{lcp}(h)$ is exactly the set of terminal nodes allowed by h.

Proof. Fix a game Γ and an information set h. First note that if conditions (i)-(ii) are satisfied for h, then there must be some subgame sufficient for h. To see this claim, take Σ to be the subgame whose root is $\mathrm{lcp}(h)$. (Here we use condition (i).) Fix a strategy profile (s_a, s_b) that allows $\mathrm{lcp}(h)$. Note that the terminal node $\zeta(s_a, s_b)$ is also allowed by h. (Here we use condition (ii).) So, (s_a, s_b) must allow h. This establishes that Σ is sufficient for h.

Now, we suppose that there is some subgame that is sufficient for h, viz. Σ. We will show that conditions (i)-(ii) must be satisfied. For this, we will make use of the fact that Σ must contain $\text{lcp}(h)$.

First, we show condition (i). Suppose, contra hypothesis, $\text{lcp}(h)$ is contained in a non-singleton information set—i.e., there is some $x \neq \text{lcp}(h)$ so that x and $\text{lcp}(h)$ are contained in the same information set. Then, $\text{lcp}(\{\text{lcp}(h), x\})$ is also contained in Σ. Moreover, there is some player who is active at $\text{lcp}(\{\text{lcp}(h), x\})$. This player has a strategy that allows Σ, but not h. This, contradicts the presumption that Σ is sufficient for h.

Next is condition (ii). To see this, note that the set of terminal nodes allowed by h is contained in the set of terminal nodes allowed by $\text{lcp}(h)$. Fix a terminal node, viz. z, allowed by $\text{lcp}(h)$. Then, z is also allowed by Σ (since $\text{lcp}(h)$ is contained in the subtree Σ). So, there is a strategy profile (s_a, s_b) that allows Σ with $\zeta(s_a, s_b) = z$. Note, since Σ is sufficient for h, (s_a, s_b) allows h. It follows that z is allowed by h, as required. ■

Finally, we conclude by pointing out the need for NRT in Proposition C1(ii).

Example C1 *Figure C1 gives a game that fails NRT. Since it is a perfect information game, it is determined by its subgames. Here, $(In, Across)$ is a pure-strategy SPE, but $\{In\} \times \{Across\}$ is not an EFBRS.*

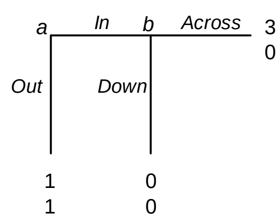

Figure C1

There is an EFBRS, viz. $Q_a \times Q_b$, with $\{In\} \times \{Across\} \subseteq Q_a \times Q_b$, e.g., $\{In\} \times \{Across, Down\}$. (Of course, part (i) of Proposition 7.1 says there must be some such EFBRS.) But every EFBRS, viz. $Q_a \times Q_b$, must have $Q_b = \{Across, Down\}$. (Here we use condition (iii) of an EFBRS.) So, $\{In\} \times \{Across\}$ is not an EFBRS.

Appendix D Examples and Proofs for Section 8

In this appendix, we prove Propositions 8.1-8.2. We also provide examples to better understand the results.

I. No Ties and Proposition 8.1: Part (i) of Proposition 8.1 requires TDI and part (ii) of Proposition 8.1 requires NRT. Example D1 explains why part (i) requires TDI.

Example D1 *Return to Example C1, which fails TDI. There, we saw that* $(In, Down)$ *is contained in an EFBRS. But, it is not outcome equivalent to a pure-strategy Nash equilibrium.*

Observe, when Bob moves, he is indifferent between In and Out. Now turn to a type of Ann that strongly believes Bob is rational. This type has a correct belief about what Bob's payoffs will be if she plays In. But, because the game fails TDI, she may have an incorrect belief about what her own payoff will be if she plays In. As such, a Nash outcome need not obtain.

Example D2 explains why we cannot replace NRT with the (weaker) TDI condition, in part (ii) of Proposition 8.1.

Example D2 *Consider the game in Figure D1, which satisfies TDI, but violates NRT.*

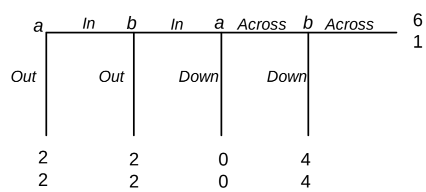

Figure D1

Here, (Out, Out) *is a Nash equilibrium in sequentially justifiable strategies. But, if* $Q_a \times Q_b$ *is a (nonempty) EFBRS, then* $Q_a \times Q_b = \{In\text{-}Across\} \times \{In\text{-}Down\}$*. To see this, let* $Q_a \times Q_b \neq \emptyset$ *be an EFBRS and note that* $Q_a \subseteq \{Out, In\text{-}Across\}$ *and* $Q_b \subseteq \{Out, In\text{-}Down\}$*. (The strategy* In*-*$Down$ *for Ann is dominated at her second information set, and the strategy* In*-*$Across$ *for Bob is dominated at his second information set.) Note, too, that* In*-*$Across$ *is a weakly dominant strategy for Ann. So, condition (iii) of an EFBRS implies that* $In\text{-}Across \in Q_a$*. It follows that, if* μ_b *strongly believes* Q_a*, then* μ_b *must assign probability one to* In*-*$Across$ *conditional on the event "Ann plays* In*." So,* In*-*$Down$ *is Bob's only strategy that is sequentially optimal given a CPS that strongly believes* Q_a*. This implies that* $Q_b = \{In\text{-}Down\}$*, and so* $Q_a = \{In\text{-}Across\}$*.*

In the above example, $\{(Out, Out)\}$ is disjoint from any EFBRS. While it satisfies conditions (i)-(ii) of an EFBRS, it fails condition (iii): If (Out, Out) is played, Ann gets a payoff of 2. But, by going In, she can also assure herself an expected payoff of at least 2. As such, condition (iii) requires that we include In*-*$Across$.

To better understand what is going on, let us recast this at the epistemic level: If (Out, t_a) is rational, so is $(In\text{-}Across, t_a)$. With this, if Bob strongly believes that Ann is rational, then, when his first information set is reached, he must maintain a hypothesis that Ann is playing *In-Across*—that is, he must maintain a hypothesis that Ann is playing a particular strategy that is not in $Q_a = \{Out\}$. As such, *Out* cannot be a best response for Bob.

The key is that the rationality of (Out, t_a) has implications for Ann's rationality at information sets precluded by *Out*. Notice, this happens because Ann is indifferent between the terminal nodes reached by (Out, Out) and $(In\text{-}Across, Out)$. (If Ann's payoffs from $(In\text{-}Across, Out)$ were strictly less than 2, (Out, t_a) can be rational without $(In\text{-}Across, t_a)$ being rational. Similarly, if Ann's payoffs from $(In\text{-}Across, Out)$ were strictly greater than 2, then (Out, Out) would not be a Nash Equilibrium.) This is where the NRT condition comes in—it says that, if Ann is decisive between two terminal nodes (as she is here), then she cannot be indifferent between those nodes.

II. Proof of Proposition 8.1(i): This will follow immediately from the following Lemma.

Lemma D1 *Fix a perfect-information game satisfying TDI. If $Q_a \times Q_b$ satisfies the best response property, then each $(s_a, s_b) \in Q_a \times Q_b$ is outcome equivalent to a Nash Equilibrium.*

The proof of this Lemma closely follows the proof of Proposition 6.1a in Brandenburger-Friedenberg [12, 2004]. It is by induction on the length of the tree. Specifically, fix a game Γ and a subgame Σ. The induction hypothesis states that if a set satisfies the best response property on Σ then it is outcome equivalent to some Nash equilibrium. We saw that, if a set $Q_a \times Q_b$ satisfies the best response property on Γ, it also satisfies the best response property on the subgame Σ. (This was Lemma C1.) So, if we fix a set that satisfies the best response property on the whole tree, then, by the induction hypothesis, it is outcome equivalent to a Nash equilibrium on each reached subgame. The proof uses this fact to construct a pure strategy Nash equilibrium on the whole tree, that is outcome equivalent to each profile in $Q_a \times Q_b$.

Definition D1 *Call $Q_a \times Q_a \subseteq S_a \times S_b$ a **constant set** if, for each (s_a, s_b), $(r_a, r_b) \in Q_a \times Q_b$, $\pi(s_a, s_b) = \pi(r_a, r_b)$.*

Lemma D2 *Fix a perfect-information game satisfying TDI. If $Q_a \times Q_b$ satisfies the best response property, then $Q_a \times Q_b$ is a constant set.*

Proof. The proof is by induction on the length of the tree.

First, fix a tree of length one and suppose Ann moves at the initial node. Then Bob's strategy set is a singleton. So, if $Q_a \times Q_b$ satisfies the best response property, then Ann is indifferent between each (s_a, s_b) and (r_a, s_b) in $Q_a \times Q_b$. By TDI, each profile in $Q_a \times Q_b$ is outcome equivalent.

Assume the result holds for any tree of length l or less. Fix a tree of of length $l + 1$ and a set $Q_a \times Q_b$ satisfying the best response property. Suppose Ann moves at the initial node, and can choose amongst nodes $n_1, \ldots, n_K$. Each n_k can be identified with an information set and each is associated with a subgame $\Sigma = k$.

In particular, fix some subgame k with $Q_a^k \times Q_b^k \neq \emptyset$. Then $Q_a^k \times Q_b^k$ satisfies the best response property for the subgame k. (This is Lemma C1.) So, by the induction hypothesis, $\pi^k\left(s_a^k, s_b^k\right) = \pi^k\left(r_a^k, r_b^k\right)$, for each $\left(s_a^k, s_b^k\right)$ and $\left(r_a^k, r_b^k\right) \in Q_a^k \times Q_b^k$. Now, note that, for each $s_b \in Q_b$, $s_b^k \in Q_b^k$. (Here, we use the fact that Ann moves at the initial node.) Thus, given two strategies $s_a, r_a \in Q_a \cap S_a(\Sigma)$ and $s_b, r_b \in Q_b$, we have that $\pi(s_a, s_b) = \pi(r_a, r_b)$.

Now, fix some $(s_a, s_b), (r_a, r_b) \in Q_a \times Q_b$, where $s_a \in S_a(k)$ and $r_a \in S_a(j)$. We have already established that $\pi(s_a, s_b) = \pi(r_a, r_b)$, for $k = j$. Suppose $k \neq j$. Since $s_a \in Q_a$, s_a is sequentially optimal under some $\mu_a(\cdot|\cdot)$ that strongly believes Q_b. So, in particular, s_a is optimal under $\mu_a(\cdot|S_b)$ with $\mu_a(Q_b|S_b) = 1$. With this,

$$\begin{aligned} \pi_a(s_a, s_b) &= \textstyle\sum_{q_b \in Q_b} \pi_a(s_a, q_b)\, \mu_a(q_b|S_b) \\ &\geq \textstyle\sum_{q_b \in Q_b} \pi_a(r_a, q_b)\, \mu_a(q_b|S_b) \\ &= \pi_a(r_a, r_b)\,. \end{aligned}$$

(The first equality follows from the fact that, for each $q_b \in Q_b$, $\pi_a(s_a, s_b) = \pi_a(s_a, q_b)$. This is a consequence of the last line in the preceding paragraph. Likewise, for the last equality.) By an analogous argument, $\pi_a(r_a, r_b) \geq \pi_a(s_a, s_b)$. So, $\pi_a(r_a, r_b) = \pi_a(s_a, s_b)$. By TDI, $\pi_b(r_a, r_b) = \pi_b(s_a, s_b)$. ■

Proof of Lemma D1. The proof is by induction on the length of the tree.

First, fix a tree of length one and suppose Ann moves at the initial node. Then Bob's strategy set is a singleton. The result follows from the fact that each $s_a \in Q_a$ is sequentially optimal under a CPS.

Now assume the result holds for any tree of length l or less. Suppose Ann moves at the initial node, and can choose among nodes $n^1, \ldots, n^K$. Each n^k can be identified with an information set and each is associated with a subgame $\Sigma = k$.

Fix some $(s_a, s_b) \in Q_a \times Q_b$ and suppose $s_a \in S_a(1)$. Note, $Q_a^1 \times Q_b^1$ satisfies the best response property (Lemma C1). So, by the induction hypothesis, there is a Nash equilibrium of subgame 1, viz. $\left(r_a^1, r_b^1\right)$, so that $\pi\left(s_a^1, s_b^1\right) = \pi\left(r_a^1, r_b^1\right)$. Consider a strategy $r_a \in S_a(1)$ so that the projection of r_a onto $\prod_{h \in H_a^1} C_a(h)$ is r_a^1. We need to show that we can choose $r_b^2, \ldots, r_b^K \in \times_{k=2}^K S_b^k$ so that, for each $q_a \in Q_a$ and associated $q_a^k \in S_a^k$, $\pi_a\left(r_a^1, r_b^1\right) \geq \pi_a\left(q_a^k, r_b^k\right)$. The profile $\left(r_a, \left(r_b^1, r_b^2, \ldots, r_b^K\right)\right)$ will then be a Nash Equilibrium of the game.

Since $s_a \in Q_a$, there exists a CPS and an associated measure $\mu_a(\cdot|S_b)$ so that

$$\sum_{s_b \in S_b} \left[\pi_a(s_a, s_b) - \pi_a(q_a, s_b)\right] \mu_a(s_b|S_b) \geq 0,$$

for all $q_a \in S_a$. Fix k from $2, \ldots, K$. Using Lemma D2,

$$\pi_a\left(r_a^1, r_b^1\right) = \pi_a\left(s_a^1, s_b^1\right) \geq \sum_{s_b^k \in S_k^b} \pi_a\left(q_a^k, s_b^k\right)\left(\operatorname{marg}_{S_b^k} \mu\left(\cdot|S_b\right)\right)\left(s_b^k\right),$$

for any $q_a^k \in S_a^k$. Letting $(\overline{q}_a^k, \overline{q}_b^k) \in \arg\max_{S_a^k} \min_{S_b^k} \pi_a(\cdot, \cdot)$, we have in particular

$$\pi_a\left(r_a^1, r_b^1\right) \geq \sum_{s_b^k \in S_k^b} \pi_a\left(\overline{q}_a^k, s_b^k\right)\left(\operatorname{marg}_{S_k^b} \mu\left(\cdot|S_b\right)\right)\left(s_b^k\right).$$

But $\pi_a(\overline{q}_a^k, q_b^k) \geq \pi_a(\overline{q}_a^k, \overline{q}_b^k)$ for any $q_b^k \in S_b^k$, by definition. So

$$\pi_a\left(r_a^1, r_b^1\right) \geq \sum_{s_b^k \in S_k^b} \pi_a(\overline{q}_a^k, \overline{q}_b^k)(\operatorname{marg}_{S_b^k} \mu\left(\cdot|S_b\right))\left(s_b^k\right) = \pi_a(\overline{q}_a^k, \overline{q}_b^k).$$

Set $(\underline{q}_a^k, \underline{q}_b^k) \in \arg\min_{S_b^k} \max_{S_a^k} \pi_a(\cdot, \cdot)$. By the Minimax Theorem for PI games (see, e.g., Ben Porath [10, 1997]), $\pi_a(\overline{q}_a^k, \overline{q}_b^k) = \pi_a(\underline{q}_a^k, \underline{q}_b^k)$. It follows that $\pi_a(r_a^1, r_b^1) \geq \pi_a\left(\overline{q}_a^k, \overline{q}_b^k\right) = \pi_a(\underline{q}_a^k, \underline{q}_b^k)$. But $\pi_a(\underline{q}_a^k, \underline{q}_b^k) \geq \pi_a(q_a^k, \underline{q}_b^k)$ for any $q_a^k \in S_a^k$, by definition. So $\pi_a(r_a^1, r_b^1) \geq \pi_a(q_a^k, \underline{q}_b^k)$, for each $q_a^k \in S_a^k$. Setting each $r_b^k = \underline{q}_b^k$ gives the desired profile. ■

III. Proof of Proposition 8.1(ii): Let us give the idea of the proof. We will start with a set $Q_a \times Q_b = \{(s_a, s_b)\}$, where (s_a, s_b) is a pure Nash equilibrium in sequentially justifiable strategies. This set will satisfy the best response property. (See Lemma D4 below.) In particular, the set Q_a is associated with a single CPS μ_a, satisfying the conditions of the best response property. We will look at the set P_a of all strategies r_a that are sequentially optimal under μ_a. We use the fact that μ_a strongly believes Q_b (so assigns probability 1 to s_b at the initial information set) to get that Ann is indifferent between all outcomes associated with $P_a \times Q_b$. Indeed, by NRT, these strategy profiles must reach the same terminal node. Likewise, we define P_b and, using standard properties of a PI game tree, we get that all strategies in $P_a \times P_b$ reach the same terminal node.

So, what have we done: We began with a set $Q_a \times Q_b$ and we expanded it to a set $P_a \times P_b$, with (i) $Q_a \times Q_b \subseteq P_a \times P_b$, (ii) all the profiles in $P_a \times P_b$ reach the same terminal node, and (iii) there is a CPS μ_a (resp. μ_b) that strongly believes Q_b (resp. Q_a) and such that P_a (resp. P_b) is the set of strategies that are sequentially optimal under $\mu_a(\cdot|\cdot)$ (resp. $\mu_b(\cdot|\cdot)$). We would have succeeded in constructing an EFBRS if the CPS μ_a (resp. μ_b) strongly believed P_b (resp. P_a) instead of Q_b (resp. Q_a). The key will be that we can similarly expand $P_a \times P_b$ so that the new set satisfies similar properties. Since the game is finite, eventually, the expanded set must coincide with the original set—that is, condition (i) must hold with equality. This gives the desired result.

Now we turn to the proof. First, we give a technical Lemma.

Lemma D3 *Fix some $(\Omega, \mathcal{E})$ where Ω is finite. Let $\mu(\cdot|\cdot)$ be a CPS on $(\Omega, \mathcal{E})$ and let ϖ be a measure on Ω. Construct $\nu(\cdot|\cdot) : \mathcal{B}(\Omega) \times \mathcal{E} \to [0,1]$ as follows: If $F \in \mathcal{E}$ with $\operatorname{Supp} \varpi \cap F \neq \emptyset$ then $\nu(\cdot|F) = \varpi(\cdot|F)$. Otherwise, $\nu(\cdot|F) = \mu(\cdot|F)$. Then $\nu(\cdot|\cdot)$ is a CPS.*

Proof. Let μ, ϖ, and ν be as in the statement of the Lemma. Conditions (i)-(ii) of a CPS are immediate. Turn to condition (iii). For this, fix $E \in \mathcal{B}(\Omega)$ and $F, G \in \mathcal{E}$ with $E \subseteq F \subseteq G$.

First suppose that $\operatorname{Supp} \varpi \cap F \neq \emptyset$. Then

$$\begin{aligned} \nu(E|G) &= \frac{\varpi(E)}{\varpi(G)} \\ &= \frac{\varpi(E)}{\varpi(F)} \frac{\varpi(F)}{\varpi(G)} = \nu(E|F)\,\nu(F|G), \end{aligned}$$

where the first equality makes use of the fact that $E \subseteq G$ and the last makes use of the fact that $E \subseteq F$ and $F \subseteq G$. Next suppose that $\operatorname{Supp} \varpi \cap G = \emptyset$. Then $\operatorname{Supp} \varpi \cap F = \emptyset$, so that

$$\begin{aligned} \nu(E|G) &= \mu(E|G) \\ &= \mu(E|F)\,\mu(F|G) = \nu(E|F)\,\nu(F|G), \end{aligned}$$

as required. Finally, suppose that $\operatorname{Supp} \varpi \cap F = \emptyset$ but $\operatorname{Supp} \varpi \cap G \neq \emptyset$. Then

$$0 \leq \nu(E|G) \leq \nu(F|G) = \varpi(F|G) = 0,$$

where the last equality follows from the fact that $\operatorname{Supp} \varpi \cap F = \emptyset$. Then

$$\begin{aligned} \nu(E|G) &= 0 \\ &= \mu(E|F)\,\varpi(F|G) = \nu(E|F)\,\nu(F|G), \end{aligned}$$

as required. ■

Lemma D4 *Let (s_a, s_b) be a Nash equilibrium in sequentially justifiable strategies. Then $\{(s_a, s_b)\}$ satisfies the best response property.*

Proof. Let (s_a, s_b) be a Nash equilibrium in sequentially justifiable strategies. Then there exists a CPS $\mu_a(\cdot|\cdot)$ so that s_a is sequentially optimal under $\mu_a(\cdot|\cdot)$. Construct a CPS $\nu_b(\cdot|\cdot)$ so that $\nu_b(s_b|S_b(h)) = 1$ if $s_b \in S_b(h)$, and $\nu_b(\cdot|S_b(h)) = \mu_a(\cdot|S_b(h))$ otherwise. By Lemma D3, $\nu_b(\cdot|\cdot)$ is a CPS. It is immediate from the construction that s_a is sequentially optimal under $\nu_b(\cdot|\cdot)$ and $\nu_b(\cdot|\cdot)$ strongly believes $\{s_b\}$. And, similarly, with a and b reversed. ■

Definition D2 *Fix a constant set $Q_a \times Q_a \subseteq S_a \times S_b$. Call $P_a \times P_a \subseteq S_a \times S_b$* ***an expansion of*** *$Q_a \times Q_b$ if there exists a CPS $\mu_a \in \mathcal{C}(S_b)$ so that:*

(i) $Q_a \subseteq P_a = \rho_a(\mu_a)$,

(ii) μ_a strongly believes Q_b, and

(iii) if r_a is optimal under $\mu_a(\cdot|S_b)$ then $\pi_a(r_a, s_b) = \pi_a(s_a, s_b)$ for all $(s_a, s_b) \in Q_a \times Q_b$.

And, likewise, with a and b reversed.

Notice, we only define an expansion of a set $Q_a \times Q_b$, if $Q_a \times Q_b$ is a constant set. Also, note, if $P_a \times P_b$ is an expansion of $Q_a \times Q_b$ then there are CPS's μ_a and μ_b satisfying conditions (i)-(iii) of Definition D2. We will refer to these as **the associated CPS's**.

Lemma D5 *Fix a PI game satisfying NRT. Suppose $P_a \times P_b$ is an expansion of $Q_a \times Q_b$ and fix associated CPS's μ_a and μ_b. Let X_a be the set of strategies that are optimal under $\mu_a(\cdot|S_b)$. And, likewise, define X_b. Then $X_a \times X_b$ is a constant set.*

Proof. Since $P_a \times P_b$ is an expansion of $Q_a \times Q_b$, $Q_a \times Q_b$ is a constant set. (This is by definition.) It follows from condition (iii) of Definition D2 that $X_a \times Q_b$ and $Q_a \times X_b$ are constant sets. Then, using NRT, each profile in $X_a \times Q_b$ reaches the same terminal node. And likewise for $Q_a \times X_b$. In fact, the terminal node reached by $X_a \times Q_b$ and $Q_a \times X_b$ must be the same one, since $(X_a \times Q_b) \cap (Q_a \times X_b) = (Q_a \times Q_b)$. Now fix a profile $(s_a, r_b) \in (X_a \backslash Q_a) \times (X_b \backslash Q_b)$. Note there is a profile $(s_a, s_b) \in (X_a \backslash Q_a) \times Q_b$ and a profile $(r_a, r_b) \in Q_a \times (X_b \backslash Q_b)$. These profiles reach the same terminal node and so (s_a, r_b) must also reach that terminal node. This establishes that $X_a \times X_b$ is a constant set. ■

Corollary D1 *Fix a PI game satisfying NRT. If $P_a \times P_b$ is an expansion of some $Q_a \times Q_b$, then $P_a \times P_b$ is constant.*

The next result is standard, and so the proof is omitted.

Lemma D6 *Fix a measure $\varpi_a \in \mathcal{P}(S_b)$ so that s_a is optimal under ϖ_a given S_a. Then, for any information set h with $s_a \in S_a(h)$ and $\varpi_a(S_b(h)) > 0$, s_a is optimal under $\varpi_a(\cdot|S_b(h))$ given $S_a(h)$.*

Given a measure $\varpi \in \mathcal{P}(\Omega)$, we write $\operatorname{Supp} \varpi$ for the support of the measure.

Lemma D7 *Fix a PI game satisfying NRT. If $P_a \times P_b$ is an expansion of $Q_a \times Q_b$, then there exists some $W_a \times W_b$ that is an expansion of $P_a \times P_b$.*

Proof. Begin with the fact that $P_a \times P_b$ is an expansion of $Q_a \times Q_b$, and choose an associated CPS μ_a (resp. μ_b) satisfying the conditions of Definition D2. Let X_a (resp. X_b) be the set of strategies that are optimal under $\mu_a(\cdot|S_b)$ (resp. $\mu_b(\cdot|S_a)$). By Lemma D5, $X_a \times X_b$ is a constant set.

Construct a measure $\varpi_a \in \mathcal{P}(S_b)$ as follows: Begin with a measure $\overline{\varpi}_a$ with $\operatorname{Supp} \overline{\varpi}_a = P_b$. Construct ϖ_b so that, for each $r_b \in P_b$,

$$\varpi_a(r_b) = (1 - \varepsilon) \mu_a(r_b|S_b) + \varepsilon \overline{\varpi}_a(r_b),$$

where $\varepsilon \in (0, 1)$. Note that μ_a strongly believes $Q_b \subseteq P_b$, $\operatorname{Supp} \mu_a(\cdot|S_b) \subseteq P_b$. With this and the fact that $\operatorname{Supp} \overline{\varpi} = P_b$, we have $\operatorname{Supp} \varpi_a = P_b$. Using the fact that $X_a \times P_b$ is a constant set,

$\pi_a(s_a, \varpi_a) = \pi_a(r_a, \varpi_a)$ for all $s_a, r_a \in X_a$. Moreover, when ε is sufficiently small, $\pi_a(s_a, \varpi_a) > \pi_a(r_a, \varpi_a)$ for all $s_a \in X_a$ and $r_a \in S_a \backslash X_a$. So we can choose ϖ_a so that s_a is optimal under ϖ_a if and only if $s_a \in X_a$.

Now construct a CPS $\nu_a \in \mathcal{C}(S_b)$ as follows: If $P_b \cap S_b(h) \neq \emptyset$, let $\nu_a(\cdot|S_b(h)) = \varpi_a(\cdot|S_b(h))$. (This is well defined since, in this case, $\varpi_a(S_b(h)) > 0$.) If $P_b \cap S_b(h) = \emptyset$, let $\nu_a(\cdot|S_b(h)) = \mu_a(\cdot|S_b(h))$. Lemma D3 establishes that $\nu_a(\cdot|\cdot)$ is a CPS. Construct a measure $\varpi_b \in \mathcal{P}(S_a)$ and a CPS $\nu_b \in \mathcal{C}(S_a)$ analogously.

Take $W_a = \rho_a(\nu_a)$ and $W_b = \rho_b(\nu_b)$. We will show that $W_a \times W_b$ is an expansion of $P_a \times P_b$.

Begin with condition (i). Note, by definition, $W_a = \rho_a(\nu_a)$. So, we only need show that $P_a \subseteq W_a$. Fix some $s_a \in P_a$. By construction, s_a is optimal under ϖ_a. Let $h \in H_a$ with $s_a \in S_a(h)$. If $P_b \cap S_b(h) \neq \emptyset$ then $\varpi_a(\cdot|S_b(h)) = \nu_a(\cdot|S_b(h))$ and s_a is optimal under $\nu_a(\cdot|S_b(h))$ among all strategies in $S_a(h)$. (See Lemma D6.) If $P_b \cap S_b(h) = \emptyset$ then $\nu_a(\cdot|S_b(h)) = \mu_a(\cdot|S_b(h))$. So, again, s_a is optimal under $\nu_a(\cdot|S_b(h))$ given all strategies in $S_a(h)$. With this, $s_a \in \rho_a(\nu_a(\cdot|\cdot))$, as required.

Next, turn to condition (ii). We need to show that ν_a strongly believes P_b. For this notice that if $P_b \cap S_b(h) \neq \emptyset$ then $\nu_a(P_b|S_b(h)) = \varpi_a(P_b|S_b(h)) = 1$.

Finally, we show condition (iii). Suppose r_a is optimal under $\nu_a(\cdot|S_b)$. We will show that $\pi_a(r_a, s_b) = \pi_a(s_a, s_b)$ for all $(s_a, s_b) \in P_a \times P_b$. To see this, recall, $\nu_a(\cdot|S_b) = \varpi_a$. So, if r_a is optimal under $\nu_a(\cdot|S_b)$ then $r_a \in X_a$. The claim now follows from the fact that $X_a \times X_b$ is constant that contains $P_a \times P_b$.

Replacing b with a establishes that $W_a \times W_b$ is an expansion of $P_a \times P_b$. ■

Lemma D8 *Fix a PI game satisfying NRT. Let (s_a, s_b) be a Nash equilibrium in sequentially justifiable strategies. Then there exists an EFBRS, viz. $Q_a \times Q_b$, that contains (s_a, s_b).*

Proof. Fix a Nash equilibrium in sequentially optimal strategies, viz. (s_a, s_b). Let $Q_a^0 \times Q_b^0 = \{s_a\} \times \{s_b\}$. By Lemma D4, $Q_a^0 \times Q_b^0$ satisfies the best response property. So, there is a CPS μ_a (resp. μ_b) that strongly believes $\{s_b\}$ (resp. $\{s_a\}$) and s_a (resp. s_b) is sequentially optimal under μ_a (resp. μ_b). Let $Q_a^1 = \rho_a(\mu_a)$ (resp. $Q_b^1 = \rho_b(\mu_a)$). Note that $Q_a^1 \times Q_b^1$ is an expansion of $Q_a^0 \times Q_b^0$ (associated with the CPS's μ_a and μ_b). Now, repeatedly apply Lemma D7 to get sets $Q_a^0 \times Q_b^0$, $Q_a^1 \times Q_b^1$, $Q_a^2 \times Q_b^2$, $\ldots$, where each $Q_a^{m+1} \times Q_b^{m+1}$ is an expansion of $Q_a^m \times Q_b^m$. Since the game is finite, there is some M with $Q_a^m \times Q_b^m = Q_a^M \times Q_b^M$ for all $m \geq M$. The set $Q_a^M \times Q_b^M$ is an EFBRS. ■

IV. Closing the Gap: In the text, we mentioned that there is a gap between parts (i)-(ii) of Proposition 8.1.

We begin by pointing out that we cannot improve part (ii) to say that, starting from any pure Nash equilibrium, we get an EFBRS. To see this, refer to Figure D2. There is a unique EFBRS, namely $\{In\} \times \{Across\}$. That said, the pair $(Out, Down)$ is a Nash equilibrium—of course, it is not a Nash equilibrium in sequentially justifiable strategies.

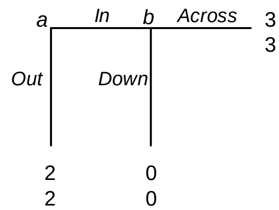

Figure D2

We do not know if part (i) can be improved to read: If $Q_a \times Q_b$ satisfies the best response property, then each $(s_a, s_b) \in Q_a \times Q_b$ is outcome equivalent to a sequentially justifiable Nash Equilibrium. Let us better understand the problem.

Return to Lemma D1 and the proof thereof. Suppose, we strengthened the induction hypothesis, so that we can look at a sequentially justifiable Nash equilibrium of subgame 1, viz. $\left(r_a^1, r_b^1\right)$. Following the proof, we use this, to construct a Nash equilibrium $(r_a, (r_b^1, \underline{q}_b^2, \ldots, \underline{q}_b^K))$, where each $\underline{q}_b^k$ is the minimax strategy on subtree k. But, now we need to show that the constructed equilibrium is sequentially justifiable. Here is where the problem arises—the strategy $\underline{q}_b^k$ (on subtree k) may not be a best response to any strategy on that subtree. Thus, the proof breaks down. Of course, it may very well be that there is another method of proof.

In the text we mentioned a related result (Proposition 8.1) which speaks to the gap. To show this result, it suffices to show the following Lemma.

Lemma D9 *Suppose $Q_a \times Q_b$ is a constant set satisfying the best response property. Then there exists a mixed strategy Nash equilibrium, viz. (σ_a, σ_b), so that:*

(i) $Q_a \times Q_b$ is outcome equivalent to (σ_a, σ_b), and

(ii) each $s_a \in \operatorname{Supp} \sigma_a$ (resp. $s_b \in \operatorname{Supp} \sigma_b$) is sequentially justifiable.

Proof. Pick some $(r_a, r_b) \in Q_a \times Q_b$ and let $\mu_a \in \mathcal{C}(S_b)$ be a CPS so that $r_a \in \rho_a(\mu_a)$ and μ_a strongly believes Q_b. Set $\sigma_b = \mu_a(\cdot|S_b)$. Construct σ_a analogously.

First, notice that (σ_a, σ_b) is a mixed strategy Nash equilibrium: Begin by using the fact that $\mu_b(Q_a|S_a) = 1$ and $\mu_a(Q_b|S_b) = 1$. As such $\operatorname{Supp} \sigma_a \times \operatorname{Supp} \sigma_b \subseteq Q_a \times Q_b$. Since $Q_a \times Q_b$ is a constant set, for each $(s_a, s_b) \in \operatorname{Supp} \sigma_a \times \operatorname{Supp} \sigma_b$, $\pi(s_a, s_b) = \pi(r_a, r_b)$. So, for each $s_a \in \operatorname{Supp} \sigma_a$ and each $q_a \in S_a$,

$$\begin{aligned} \pi_a(s_a, \sigma_b) &= \pi_a(r_a, r_b) \\ &= \pi_a(r_a, \sigma_b) \geq \pi_a(q_a, \sigma_b), \end{aligned}$$

where the inequality holds because $r_a \in \rho_a(\mu_a)$ and $\mu_a(\cdot|S_b) = \sigma_b$. Applying an analogous argument to b, establishes that (σ_a, σ_b) is indeed a Nash equilibrium.

Next, notice that $Q_a \times Q_b$ is outcome equivalent to (σ_a, σ_b): To see this, recall that $\operatorname{Supp}\sigma_a \times \operatorname{Supp}\sigma_b \subseteq Q_a \times Q_b$ and $Q_a \times Q_b$ is a constant set. So, it is immediate that, for each $(s_a, s_b) \in Q_a \times Q_b$, $\pi(s_a, s_b) = \pi(\sigma_a, \sigma_b)$.

Lastly, notice that each $s_a \in \operatorname{Supp}\sigma_a$ is sequentially justifiable, and likewise for b: To see this, recall that $\operatorname{Supp}\sigma_a \times \operatorname{Supp}\sigma_b \subseteq Q_a \times Q_b$. So, if $s_a \in \operatorname{Supp}\sigma_a$, then $s_a \in Q_a$, and so s_a is sequentially justifiable. ■

Proof of Proposition 8.1. Immediate from Lemmata D2-D9. ■

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