Market games and the Bargaining Set

Nicholas Ziros

Department of Economics, University of Cyprus, P.O. Box 20537, 1678 Nicosia, Cyprus

Abstract

We present the bargaining set of an economy, where trades among groups of individuals are conducted via the Shapley-Shubik mechanism. Then we prove that in atomless economies the allocations resulting from this equilibrium notion are competitive.

Keywords: Strategic market games; bargaining set; competition

JEL classification: D50; C71; C72

1 Introduction

The idea that in a mass economy individuals act as *price takers,* found some formal proof in two theories of competition the 'cooperative' and the 'noncooperative' that emerged from the works of Edgeworth and Cournot respectively. The cooperative approach with the various equilibrium notions, i.e. the core and the bargaining set, as well as the noncooperative approach with the theory of strategic market games[1], have helped us to formalize terms and shape our understanding as to what it takes for a market to exhibit perfectly competitive characteristics. Despite the great differences of the two approaches, one does not preclude the other. Moreover there is a substantial overlap between the set of conditions, which the two approaches identify as important for the prevalence of perfect competition. One could try to bring together the strategic market games with the coalitional bargaining ideas. This idea was the starting point in Koutsougeras and Ziros (2008), where a synthesis of the two theories was presented by defining the core of an economy where trades are conducted via the Shapley-Shubik mechanism. However, an issue was left unexplored in that paper; whether other cooperative equilibrium notions can be studied in the same framework.

In the current paper, we proceed in that direction by defining in the context of strategic market games the bargaining set, a notion that is complementary to the core. Briefly, we examine the possibility of individuals to form coalitions in order to object (or counterobject) proposed distributions of commodities as in the standard bargaining set theory, but we consider only allocations that are attainable through the norms of strategic market games. In this context we define a kind of *constrained* equilibrium notion, namely the *Shapley-Shubik bargaining set*, which has a more descriptive nature about the rules of trade. In the main part of the paper we address the properties of the resulting allocations. It turns out

[0] Tel: 00357-22892437, Fax: 00357-22892432, E-mail: n.ziros@ucy.ac.cy

I would like to thank the participants of the XVII European Workshop on General Equilibrium Theory in Paestum, Italy (2008) for their comments. I am responsible for any existing errors.

[1] The class of games introduced in Shubik (1973) and in Shapley and Shubik (1977).

that in atomless economies the allocations resulting from this *hybrid* equilibrium notion are competitive. In other words, our results show that in large economies the allocations, which cannot be blocked when arbitrary redistribution of endowments is allowed, are identical to those which cannot be blocked via trades within the rules of a strategic market game.

However the most important contribution of this paper is at the conceptual level. The standard theory of the bargaining set there is characterized by a lack of description of the trading process. For instance, in Mas-Colell (1989) the author states that: *"Given a p (coalition) S is formed by precisely those agents who would rather trade at the price vector p than get the consumption bundle assigned to them by x"*, however in that paper there is no description of the process that leads to these prices. By contrast, the model we adopt in this paper provides an explicit description of the formation of market outcomes, i.e., how individual activities are aggregated to produce the price vector p that a coalition would prefer to trade at.

In the section that follows we develop the model and several equilibrium notions. Next we introduce the new *hybrid* equilibrium notion. Finally we proceed to prove some equivalence results.

2 The economy

Let $(A, \mathcal{A}, \mu)$ be a measure space of agents, where μ is a Borel regular measure on A. In the economy there are L commodity types and the consumption set of each agent is identified with $\Re^L_+$. An individual is characterized by a preference relation, which is represented by a utility function $u_a : \Re^L_+ \to \Re$, and an initial endowment $e(a) \in \Re^L_+$. In order to be able to use some standard results we employ the following assumptions throughout the rest of the paper:

Assumption 1: $e_h \gg 0$ ae.

Assumption 2: Preferences are continuous, strictly monotonic, complete and transitive and indifference surfaces passing through the endowment do not intersect the axis.

Let $\mathcal{P}$ denote the set of utility functions satisfying the above assumption endowed with the appropriate topology (see Mas-Colell, 1985). An economy is a measurable mapping $\mathcal{E} : A \to \mathcal{P} \times \Re^L_+$. The definition of a competitive equilibrium for such an economy is as follows:

Definition 1 *A competitive equilibrium is a price system $p \in \Re^L_+$ and a measurable assignment $x : A \to \Re^L_+$ such that:*

(i) $\int_A x(a) \leq \int_A e(a)$

(ii) $x(a) \in argmax \{u_a(y) : p \cdot y \leq p \cdot e(a)\}$ *ae in A.*

The notion of the bargaining set was introduced in Aumann and Maschler (1964) in order to take into account not only the objections to a given allocation but also the possible counterobjections. Several variants of this notion have evolved and here we employ the definition proposed in Mas-Colell (1989).

Definition 2 *The pair (T, y) where $T \in \mathcal{A}$, with $\mu(T) > 0$, and $y : A \to \Re^L_+$ is an objection to allocation x if:*

(i) $\int_T y(a) \leq \int_T e(a)$ *and*

(ii) $u_a(y(a)) \geq u_a(x(a))$ *ae in* T *and* $u_a(y(a)) > u_a(x(a))$ *for some* $a \in T$.

Definition 3 *Let (T, y) be an objection to allocation x. The pair (V, z) where $V \in \mathcal{A}$, with $\mu(V) > 0$, and $z : A \to \Re^L_+$ is a counterobjection to (T, y) if:*

(i) $\int_V z(a) \leq \int_V e(a)$ *and*

(ii) $u_a(z(a)) > u_a(y(a))$ *ae in* $V \cap T$ *and* $u_a(z(a)) > u_a(x(a))$ *ae in* $V \backslash T$.

Definition 4 *An objection (T, y) is said to be justified if there is no counterobjection to it. The bargaining set is the set of measurable assignments against which there is no justified objection*

Let $\mathcal{W}(A)$ and $\mathcal{B}(A)$ denote respectively the set of competitive equilibria and the bargaining set for this economy. Mas-Colell (1989) proved the equivalence between $\mathcal{W}(A)$ and $\mathcal{B}(A)$. Since the conditions for that result are also satisfied in the model presented here we will employ it for our proofs.

We now turn to describe how trade takes place. The results are developed for the strategic market game studied in Peck et al. (1992) and Postlewaite and Schmeidler (1978).

2.1 The strategic market game

Trade is organized via systems of trading posts where individuals offer commodities for sale and place orders for purchases of commodities. Bids are placed in terms of a unit of account. The action sets of agents are described by a measurable correspondence $S: A \to 2^{\Re^L_+ \times \Re^L_+}$, where

$$S(a) = \{(b, q) \in \Re^L_+ \times \Re^L_+ : q^i \leq e^i(a),\ i = 1, 2, ..., L\}.$$

A strategy profile is a pair of measurable mappings $b : A \to \Re^L_+$ and $q : A \to \Re^L_+$ such that $(b(a), q(a)) \in S(a)$ *ae* in A, i.e., a strategy profile is a measurable selection from the graph of the correspondence S, which we denote by $Gr(S)$. It is easily seen that $S : A \to 2^{\Re^{2L}_+}$ has a measurable graph so such measurable mappings exist by Aumann's measurable selection theorem.

For a given strategy profile $(b, q) \in Gr(S)$, let $B^i = \int_{a \in A} b^i(a)$, where it is understood that $B^i = \infty$ whenever the integral is not defined, and $Q^i = \int_{a \in A} q^i(a)$.

Consumption assignments for $i = 1, 2, ..., L$, are determined as follows:

$$x^i_a(b(a), q(a), B, Q) = \begin{cases} e^i(a) - q^i(a) + b^i(a)\dfrac{Q^i}{B^i} & \text{if } \sum_{i=1}^L \dfrac{B^i}{Q^i} q^i(a) \geq \sum_{i=1}^L b^i(a), \\ e^i(a) - q^i(a) & \text{otherwise,} \end{cases}$$

where it is agreed that divisions over zero are taken to equal zero. The interpretation of this allocation rule is that commodities are distributed to non bankrupt individuals in proportion to their bids, while the purchases of bankrupt individuals are confiscated. Notice that when $B^iQ^i \neq 0$, the vector defined as $\pi(b,q) = (\frac{B^i}{Q^i})_{i=1}^L$ has the natural interpretation as a 'price vector'.

Given a profile $(b,q) \in Gr(S)$ consumers are viewed as solving the following problem:

$$\max_{(\hat{b},\hat{q}) \in S(a)} u_a(x_a(\hat{b},\hat{q},B,Q)) \tag{1}$$

In this way we have a game in normal form that describes trade in this economy. We define below the standard pure strategy Nash equilibrium notion for this game.

Definition 5 *A strategy profile* $(b,q) \in Gr(S)$ *is a Nash equilibrium of the market game, iff:* $u_a(x_a(b(a),q(a),B,Q) \geq u_a(x_a(\hat{b},\hat{q},B,Q)), \forall (\hat{b},\hat{q}) \in S(a)$ *ae in* A.

Due to the bankruptcy rule above, at a Nash equilibrium with positive bids and offers individuals can be viewed as solving the following problem:

$$\max_{(\hat{b},\hat{q}) \in S(a)} u_a(x_a(\hat{b},\hat{q},B,Q)) \ s.t. \ \pi(b,q)\cdot\hat{q} \geq \sum_{i=1}^{L} \hat{b}^i \tag{2}$$

Notice that there is exactly one trading post available in this game. This fact forces together all individuals who wish to trade a commodity. In the next section we add an Edgeworthian flavor in the Cournotian setup of the market game by allowing coalitions to use the market game mechanism exclusively for their members.

2.2 Cooperation in strategic market games

At this point, before defining cooperative notions in the strategic market games framework, one should decide whether coalitions should be allowed to form sub-contained economies. In other words, one should decide whether the members of a coalition have the ability or not, to apply the mechanism described above to exchange among themselves and exclude non members from trading. In markets with exclusion, when a coalition deviates, prices and allocations are calculated by considering the strategies of the members of the deviating coalition only. Therefore, the strategies of non coalition members are insignificant in the determination of prices and allocations within the coalition. Hence, cooperative notions such as the core or the bargaining set can only be defined in the framework of market exclusion. By contrast, in markets without exclusion, prices and allocations are calculated by taking into account the possible reactions of the non coalition members, and such a setup would necessitate the use of notions such the α-core or the β-core.

The bargaining set of an economy where trades take place via the strategic market game mechanism is defined as follows:

Definition 6 *Let the strategy profile* $(b,q) \in Gr(S)$ *and and* $x : A \to \Re^L_+$ *be the corresponding commodity assignment* $x(a) = (x^i_a(b,q))^L_{i=1}$ *ae in* A. *The coalition* $T \in \mathcal{A}$, *with*

$\mu(T) > 0$, *and the strategy profile* $(\hat{b}, \hat{q}) \in Gr(S)$, *where* $(\hat{b}(a), \hat{q}(a)) = (0, 0)$ *ae in* $A \backslash T$, *is an objection to* $(b, q) \in Gr(S)$ *if:*

$u_a(x_a(\hat{b}(a), \hat{q}(a), \hat{B}, \hat{Q})) \geq u_a(x_a(b(a), q(a), B, Q))$ *ae in* T *and*

$u_a(x_a(\hat{b}(a), \hat{q}(a), \hat{B}, \hat{Q})) > u_a(x_a(b(a), q(a), B, Q))$ *for some* $a \in T$.

Definition 7 *Let* $(T, (\hat{b}, \hat{q}))$ *be an objection to the strategy profile* $(b, q) \in Gr(S)$. *Then* $(V, (\bar{b}, \bar{q}))$ *where* $V \in \mathcal{A}$, *with* $\mu(V) > 0$, *and* $(\bar{b}, \bar{q}) \in Gr(S)$, *where* $(\bar{b}(a), \bar{q}(a)) = (0, 0)$ *ae in* $A \backslash V$, *is a counterobjection to* $(T, (\hat{b}, \hat{q}))$ *if:*

$u_a(x_a(\bar{b}(a), \bar{q}(a), \bar{B}, \bar{Q})) > u_a(x_a(\hat{b}(a), \hat{q}(a), \hat{B}, \hat{Q}))$ *ae in* $V \cap T$ *and*

$u_a(x_a(\bar{b}(a), \bar{q}(a), \bar{B}, \bar{Q})) > u_a(x_a(b(a), q(a), B, Q))$ *ae in* $V \backslash T$.

Definition 8 *An objection* $(T, (\hat{b}, \hat{q}))$ *is said to be justified if there is no counterobjection to it. The Shapley-Shubik bargaining set (SSBS) is the set of strategy profiles against which there is no justified objection.*

Conceptually the difference from the standard definitions is that deviating coalitions here can only attain payoffs achievable via the rules of the market game and not just any set of payoffs which results from some arbitrary redistribution of initial endowments.

Let $\mathbf{B}_e^{ss}(A)$ denote the set of *SSBS* strategies and $\mathcal{B}_e^{ss}(A)$ the set of allocations which correspond to the elements of $\mathbf{B}_e^{ss}(A)$.

3 Results

For our results we will employ a well known property of strategic market games, that individual strategies can be altered in a way so that prices, budgets and allocations remain the same. The following fact records this property.

Fact *Given any* $(b, q) \in Gr(S)$, *all strategy profiles* $(\tilde{b}, \tilde{q}) \in Gr(S)$, *which satisfy* $\tilde{b}(a) = (b^i(a) + \pi^i(b, q)(\tilde{q}^i(a) - q^i(a)))_{i=1}^L$ *ae in* A, *give rise to the same prices, budgets and allocations for each* $a \in A$.

Therefore, one we fix the offers at the endowment level and describe a *SSBS* strategy profile in terms of bids.

Next we prove the main results of the paper that relate the *SSBS* with the bargaining set.

Proposition 1 $\mathcal{B}(A) \subset \mathcal{B}_e^{ss}(A)$.

Proof. Let $x \in \mathcal{B}(A)$. Then we must also have that $x \in \mathcal{W}(A)$, so there is $p \in \Re_{++}^L$ so that $p \cdot x(a) \leq p \cdot e(a)$ *ae* in A. Define the strategy profile $(b, q) : A \to \Re_+^{2L}$ as follows: $(b(a), q(a)) = ((p^i x^i(a), e^i(a))_{i=1}^L)$. Clearly, b and q are measurable and by construction $(b(a), q(a)) \in S(a)$, *ae* in A so $(b, q) \in Gr(S)$. Notice that $\pi(b, q) = p$.

For this strategy profile we have that ae in A:

$$\pi(b,q)\cdot q(a) = p\cdot e(a) = p\cdot x(a) = \sum_{i=1}^{L} b^i(a).$$

From the allocation rule we deduce that:

$$x_a(b(a),q(a),B,Q) = (\frac{b^i(a)}{p^i})_{i=1}^{L} = x(a) \quad \text{ae in } A.$$

In other words, the strategy profile (b,q) implements the bargaining set assignment x.

We claim that $x \in \mathcal{B}_e^{ss}(A)$. Indeed, suppose that for some $T \in A$, $\mu(T) > 0$, there is $(\hat{b},\hat{q}) \in Gr(S)$, where $(\hat{b},\hat{q}) = (0,0)$ for $A\backslash T$, so that the corresponding assignment is such that: $u_a((x_a^i(\hat{b}(a),\hat{q}(a),\hat{B},\hat{Q})) \geq u_a(x(a))$ ae in T, $u_a((x_a^i(\hat{b},\hat{q}))_{i=1}^L) > u_a((x_a^i(b,q))_{i=1}^L)$ for some $a \in T$ and there does not exist $V \in \mathcal{A}$, $\mu(V) > 0$ and $(\bar{b},\bar{q}) \in Gr(S)$, where $(\bar{b}(a),\bar{q}(a)) = (0,0)$ ae in $A\backslash V$, such that

$u_a(x_a(\bar{b}(a),\bar{q}(a),\bar{B},\bar{Q})) > u_a(x_a(\hat{b}(a),\hat{q}(a),\hat{B},\hat{Q}))$ ae in $V \cap T$ and

$u_a(x_a(\bar{b}(a),\bar{q}(a),\bar{B},\bar{Q})) > u_a(x_a(b(a),q(a),B,Q))$ ae in $V\backslash T$.

From the definition of the allocation rule it follows that:

$$\int_T x_a((\hat{b},\hat{q},B,Q)) = (\int_T e^i(a) + \frac{Q_T^i}{B_T^i}\int_T b^i(a) - \int_T q^i(a))_{i=1}^L = \int_T e(a)$$

But then there is a feasible objection to allocation x, which is a contradiction to our initial statement. Thus, we conclude that $x \in \mathcal{B}_e^{ss}(A)$. ■

The result that follows establishes that the converse is also true.

Proposition 2 $\mathcal{B}_e^{ss}(A) \subset \mathcal{B}(A)$.

Proof. Let $x \in \mathcal{B}_e^{ss}(A)$ and $(b,q) \in Gr(S)$ the strategy profile that implements x. Suppose that $x \notin \mathcal{B}(A)$, i.e., there is $T \in \mathcal{A}$, where $\mu(T) > 0$, and a measurable assignment $\hat{x} : T \to \Re_+^L$ such that $\int_T \hat{x}(a) = \int_T e(a)$, $u_a(\hat{x}(a)) \geq u_a(x(a))$ *ae* in T, $u_a(\hat{x}(a)) > u_a(x(a))$ for some $a \in T$ and $\nexists\ V \in \mathcal{A}$ where $\mu(V) > 0$ and $\bar{x} : V \to \Re_+^L$ such that $\int_V \bar{x}(a) = \int_V e(a)$, $u_a(\bar{x}(a)) > u_a(\hat{x}(a))$ ae in $V \cap T$ and $u_a(\bar{x}(a)) > u_a(x(a))$ ae in $V\backslash T$.

By proposition 3 in Mas-Colell (1989) $\hat{x}$ must be supportable by a price vector $\hat{p} \in \Re_{++}^L$.

As in the previous proposition we define the following profile of strategies for the members of coalition T: $(\hat{b}(a),\hat{q}(a)) = ((\hat{p}^i\hat{x}^i(a), e^i(a))_{i=1}^L)$. Clearly, $\hat{b}$ and $\hat{q}$ are measurable and by construction $(\hat{b}(a),\hat{q}(a)) \in S_T(a)$. Notice also that $\pi(\hat{b},\hat{q}) = \hat{p}$.

For this strategy profile we have that:

$$\pi(\hat{b},\hat{q})\cdot\hat{q}(a) = \hat{p}\cdot e(a) = \hat{p}\cdot x(a) = \sum_{i=1}^{L}\hat{b}^i(a).$$

From the allocation rule we deduce that:

$$x_a(\hat{b}(a),\hat{q}(a),\hat{B},\hat{Q}) = (\frac{\hat{b}^i(a)}{\hat{p}^i})_{i=1}^{L} = \hat{x}(a) \quad \textit{ae} \text{ in } T.$$

In other words, the strategy profile $(\hat{b},\hat{q}) \in Gr(S)$ implements $(\hat{p},\hat{x})$. Thus, we have found $T \in \mathcal{A}$ with $\mu(T) > 0$, and $(\hat{b},\hat{q}) \in Gr(S)$, where $(\hat{b},\hat{q}) = (0,0)$ for all $A\backslash T$, so that: $u_a((x_a^i(\hat{b}(a),\hat{q}(a),\hat{B},\hat{Q})) \geq u_a(x(a))$ *ae* in T, $u_a((x_a^i(\hat{b},\hat{q}))_{i=1}^L) > u_a((x_a^i(b,q))_{i=1}^L)$ for some $a \in T$ and there does not exist a $V \in \mathcal{A}$, $\mu(V) > 0$ and $(\bar{b},\bar{q}) \in Gr(S)$, where $(\bar{b}(a),\bar{q}(a)) = (0,0)$ ae in $A\backslash V$, such that

$u_a(x_a(\bar{b}(a), \bar{q}(a), \bar{B}, \bar{Q})) > u_a(x_a(\hat{b}(a), \hat{q}(a), \hat{B}, \hat{Q}))$ ae in $V \cap T$ and

$u_a(x_a(\bar{b}(a), \bar{q}(a), \bar{B}, \bar{Q})) > u_a(x_a(b(a), q(a), B, Q))$ ae in $V \backslash T$.

which contradicts the fact that $x \subset \mathcal{B}_e^{ss}(A)$. ■

The following result is a consequence of the preceding two propositions and the coincidence between $\mathcal{W}(A)$ and $\mathcal{B}(A)$.

Theorem 1 $\mathcal{B}_e^{ss}(A) = \mathcal{B}(A) = \mathcal{W}(A)$

References

[1] Aumann, R.J. and M. Maschler,1964, The bargaining set for cooperative games, in: M. Dresher, L.S. Shapley L.S. and A. W. Tucker, eds., Advances in Game Theory. (Princeton University Press, Princeton, N.J.) 443-447.

[2] Koutsougeras, L.C. and N. Ziros, 2008, A three way equivalence, Journal of Economic Theory 139, 380-391.

[3] Mas-Colell, A., 1989, An equivalence theorem for a bargaining set, Journal of Mathematical Economics 18, 129-139.

[4] Mas-Colell, A.,1985, The Theory of General Equilibrium: A Differentiable Approach. (Cambridge University Press, Cambridge).

[5] Peck, J., K. Shell and S. Spear, 1992, The market game: existence and structure of equilibrium, Journal of Mathematical Economics 21, 271-299.

[6] Postlewaite, A. and D. Schmeidler, 1978, Approximate efficiency of non-Walrasian Nash equilibria, Econometrica 46, 127-135.

[7] Shapley, L.S. and M. Shubik, 1977, Trade using one commodity as a means of payment, Journal of Political Economy 85, 937-968.

[8] Shubik, M., 1973, Commodity money, oligopoly, credit and bankruptcy in a general equilibrium model, Western Economic Journal 10, 24-38.