INVARIANCE TO REPRESENTATION OF INFORMATION

MUHAMET YILDIZ

Abstract. Under weak assumptions on the solution concept, I construct an *invariant* selection across all finite type spaces, in which the types with identical information play the same action. Along the way, I establish an interesting lattice structure for finite type spaces and construct an equilibrium on the space of all finite types.

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1. Introduction

In Game Theory, incomplete information is modeled using a type in a Bayesian game. Unfortunately, the representation is not unique: a given piece of information can be modeled using many distinct types, coming from distinct Bayesian games. It is desirable the solution to be invariant to alternative representations of the same information, in that the types that represent the same information all take the same action according to the solutions to the games they come from. In this note, I explore the implications of such an invariance condition.

To be more precise, consider a researcher. Given any type t_i of any player i in any Bayesian game G, she thinks that the relevant information of t_i is $h_i(t_i, G)$. If there is another type t_i' from a game G' with $h_i(t_i, G) = h_i(t_i', G')$, then she considers (t_i, G) and (t_i', G') as two alternative representations of the same relevant information. Hence, she requires that types t_i and t_i' play the same action according to the solutions of games G and G', respectively. If she selects a solution to each game satisfying her requirement, then she obtains an *invariant selection* from her solution concept. Such a selection is needed to study the solutions that are

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invariant to alternative representations of the same relevant information. Can she find such a selection? Can she ensure that her solutions to the games of her interest are part of an invariant selection without analyzing other hypothetical situations? I answer these questions affirmatively in this note.

There are many candidates for h in the literature, such as the infinite hierarchy of beliefs in Harsanyi (1967), Mertens and Zamir (1985), and Brandenburger and Dekel (1993). I do not take a position on what h should be because the definition of what information is relevant in a given application is best left to the researchers who model the specific application. I only note that under the canonical notions of relevance, the same piece of information can be modeled by an uncountable number of types, coming from Bayesian games with complicated interconnections that are difficult to foresee. Hence, invariance is a strong restriction on selections, requiring that the actions of all these types to be equal. Moreover, given the complicated interconnections between these games, construction of an invariant selection is a difficult task, as it involves equalizing the actions of all these types and doing this for all games at the same time.

I do not take a position on the solution concept, either. I only assume that the solution concept has the following two properties. Given any game G, construct a new game G^h by representing each type t_i in G by its relevant information $h_i(t_i, G)$. The first property is that for every solution σ of G^h, the strategy profile $\sigma \circ h$, in which each t_i plays $\sigma_i(h_i(t_i, G))$, is a solution to G. The second property is that given any finite type space T and any invariant selection for its proper subspaces, there exists a solution on T that extends the selection to T. Both properties are exhibited by canonical solution concepts, such as Bayesian Nash equilibrium and rationalizability. Hence, the results below apply to all such solution concepts.

My main contribution is to construct an invariant selection. I further show that any invariant selection within a class of games can be extended to the set of all games with finite type spaces, maintaining the invariace condition throughout. Conceptually, this shows that the invariance requirement does not impose any extra restrictions on the solutions of individual games or on the selections for subfamilies. Pragmatically, it ensures that if a researcher is only interested in the behavior in a class of games, she can focus on constructing an invariant selection for that class without worrying about the invariance across all games. In particular, if she is interested only in a specific game, she can analyze the game in isolation without analyzing the other possibly hypothetical strategic situations. All she needs to do is to make sure that the types with identical relevant information play the same action in the specific game. In contrast, if she is interested in behavior across a family of games (as in the analysis of comparative statics), then she needs to analyze invariant selections for the family, and invariance imposes many more conditions on such selections than on the solutions to individual games.

There is a one-to-one correspondence between the invariant selections and the equilibria on the subspaces of the universal type space. The above results then lead to equilibrium existence results on that space. First, since there is an invariant equilibrium selection for all games with finite type spaces, there exists an equilibrium on the space of all finite types, which is the subspace of the universal type space that consists of the images of all types from all finite type spaces. This fills an important gap in the literature, in which very little is known on the existence of equilibria in the prominent subspaces of the universal type space. Second, any equilibrium within a subspace can be extended to the space of all finite types. This result is quite useful in equilibrium analysis on such spaces. In such an analysis, one is often interested in the behavior of types within a small class. This result ensures that one can simply focus on the class without worrying about the construction of equilibrium in the entire space, which is often the main difficulty. Third, as a special case of the second result, an equilibrium of a game can be extended to the space of all finite types as long as the types with identical information play the same action in the equilibrium. This result is important for robustness analysis. In such an analysis, one considers an equilibrium in the universal type space and explores its sensitivity to information (see for example Weinstein and Yildiz (2007, 2008)). This result shows that such an analysis is not vacuous and the robustness of any equilibrium as above can be analyzed within this methodology.

In order to construct an invariant selection, I first show that, when the finite type spaces are embedded in a universal type space using the mapping h, they exhibit an interesting and useful lattice structure. With the inclusion of the empty set, the set of all such type spaces is a lattice under the set inclusion, and it is closed under arbitrary number of intersections. In particular, for each type, there is a unique minimal type space that contains the type. (It is the intersection of all type spaces that contain the type.) This structure allows one to rank finite type spaces according to the length of the longest chain of its subspaces under the strict set inclusion. Note that, for a given type, which is in the form of $h_i(t_i, G)$, the minimal type space is the smallest type space in which the information $h_i(t_i, G)$ can be expressed, and its rank is strictly lower than any other type space that contains $h_i(t_i, G)$.

Using the above structure, I construct an invariant selection across all games with finite type spaces as follows. It suffices to find a selection for the type spaces within the universal type space, as one can select the solutions of the other games by using the solutions for their images in the universal type space—thanks to the first property of the solution concept. In the construction within the universal type space, I first consider the type spaces of the first rank. These type spaces do not have any subspace. In particular, they are disjoint because the intersection would be a subspace by the lattice structure. I select a solution from each of these type spaces, which have solutions by the second property of the solution concept. Since they are disjoint, the selection is invariant. Next, I consider the type spaces of the second rank. These type spaces contain only subspaces of the first rank, for which the solution has been selected already. By the second property, each of them has a solution that extends the existing selection for the proper subspaces to the type space itself. I select such a solution from each of these type spaces. Since these type spaces intersect each other, this could have led to a violation of invariance. That is, a type in the intersection could have played different actions according to the solutions of the distinct type spaces. This is not the case. Any such intersection is of the first rank, for which actions have been determined in the previous round. Hence, all of the selected solutions prescribe the same action for any type in the intersection. Iterating this argument, I select a solution for every type space of third rank, fourth rank, and so on. Since each type space has a finite rank, this leads to a selection for every type space.

There are two difficulties in constructing an invariant selection (or constructing an equilibrium on the space of all finite types). First, there are typically uncountable number of type spaces in which the same piece of information is modeled. Hence, in any construction, at some stage, one needs to select the solutions for an uncountable number of such type spaces simultaneously without violating the invariance condition. In the above construction, this is accomplished by fixing the action of any type t_i in any game G at the earliest round at which $h_i(t_i, G)$ is available, and at that round the minimal type space for $h_i(t_i, G)$ is the only type space that contains $h_i(t_i, G)$. When the solution to G is selected at a later round, one may need to select solutions to uncountable number of type spaces that contain $h_i(t_i, G)$, but the solutions will all assign the same fixed action to $h_i(t_i, G)$. The second difficulty is that the space of all finite types (and other interesting type spaces) cannot be partitioned into smaller subspaces. This prevents one from using more straightforward techniques or the existing existence results. For example, if the space could be partitioned into countable subspaces, one could obtain an invariant selection in each subspace, by simply selecting a solution from each of the type spaces one-by-one in the order given by counting the type spaces within the subspace. This would have led to an invariant selection in the entire space. If that were the case, one could also use the existence result of Simon (2003) for finitely generated type spaces.

I study invariance condition of solutions with respect to the alternative representations of incomplete information. More broadly, one wants the solution to be invariant to the alternative representation of the strategic situation. Within this larger context, a number of authors, such as Kohlberg and Mertens (1986) and Govindan and Wilson (2006, 2009a, 2009b), have studied other invariance conditions, such as invariance with respect to the introduction of mixed strategies as pure strategies and the "small worlds" condition, which requires that embedding a game into a larger game with additional players does not affect the solutions induced on the original game. Consistency differs from the above conditions in two ways. First, it focuses on incomplete information. Second, it is a condition on how the solution changes across games rather than being a condition on the solution sets. Indeed, it does not restrict the set of solutions to the individual games with no redundant types at all.

After presenting the basic definitions in the next section, I study the lattice structure of finite type spaces in Section 3, invariant selections in Section 4, invariant equilibrium selections in Section 5, and equilibria of universal type space in Section 6. Section 7 concludes. The omitted proofs are in the Appendix.

2. Model

Fix a set $N = \{1, 2, \ldots, n\}$ of players i, a set $A = A_1 \times \cdots \times A_n$ of action profiles a,[1] and a set Θ^* of payoff parameters θ. For each $i \in N$, fix also a utility function $u_i : \Theta^* \times A \to \mathbb{R}$. A finite type space is a triplet (Θ, T, κ) where $\Theta \subseteq \Theta^*$ is a finite set of parameters, $T = T_1 \times \cdots \times T_n$ is a finite set of type profiles t, and κ_{t_i} is a probability distribution on $\Theta \times T_{-i}$, representing the belief of t_i, for each type $t_i \in T_i$. A Bayesian game is a list $G = (N, A, u, \Theta, T, \kappa)$. The set of all Bayesian games with varying finite type spaces is denoted by $\mathcal{G}$. Throughout $\mathcal{G}$, (N, A, u) is fixed for clarity.

For any game $G = (N, A, u, \Theta, T, \kappa) \in \mathcal{G}$ and any player i, a *strategy of i for G* is a mapping $\sigma_i : T_i \to \Delta(A_i)$, and a *strategy profile for G* is a list $(\sigma_1, \ldots, \sigma_n)$ of strategies. A *solution concept* is any correspondence Σ on $\mathcal{G}$ that picks a set $\Sigma(G)$ of strategy profiles for each game $G \in \mathcal{G}$. Given any solution concept Σ and any $\mathcal{G}' \subseteq \mathcal{G}$, by a *selection from Σ for $\mathcal{G}'$*, I mean a family σ^G, $G \in \mathcal{G}'$, such that $\sigma^G \in \Sigma(G)$ for each $G \in \mathcal{G}'$. A selection for $\mathcal{G}$ is simply called a *selection*.

Relevant Information and Invariance. For every game $G \in \mathcal{G}$ and every type t_i of a player i in G, let $h_i(t_i, G)$ be the relevant information of t_i (according to a researcher). Given that all the relevant information is contained in $h_i(t_i, G)$, if $h_i(t_i, G) = h_i(t'_i, G')$ for some types t_i and t'_i from games G and G', respectively, then (t_i, G) and (t'_i, G') are just alternative representations of the same information. One may then desire the solution to be independent of the representation, requiring that t_i and t'_i play the same action according to the solutions at G and G', respectively. The next definition formalizes this idea.

Definition 1. A selection σ^G, $G \in \mathcal{G}'$, is said to be *h-invariant* iff

$$h_i(t_i, G) = h_i(t'_i, G') \Longrightarrow \sigma_i^G(t_i) = \sigma_i^{G'}(t'_i) \tag{C}$$

for all games $G, G' \in \mathcal{G}'$ and for all types t_i and t'_i in G and G', respectively. Likewise, for any $G \in G$, a strategy profile σ in G is said to be *h-invariant* iff for

[1]**Notation:** For any list $X_1, \ldots, X_n$ of sets, X denotes $X_1 \times \cdots \times X_n$ and X_{-i} denotes $\prod_{j \neq i} X_j$. For any $x_1, \ldots, x_n$, write $x = (x_1, \ldots, x_n) \in X$, $x_{-i} = (x_j)_{j \neq i} \in X_{-i}$, and $(x'_i, x_{-i}) = (x_1, \ldots, x_{i-1}, x'_i, x_{i+1}, \ldots, x_n)$. For any family of functions $f_j : X_j \to Y_j$, write $f(x) = (f_j(x_j))_{j \in N}$ and $f_{-i}(x_{-i}) = (f_j(x_j))_{j \neq i}$. The set of all probability distributions on a set X is denoted by $\Delta(X)$.

all types t_i and t_i' in G,

$$h_i(t_i, G) = h_i(t_i', G) \Longrightarrow \sigma_i(t_i) = \sigma_i(t_i'). \tag{C'}$$

Consistency of selections is a condition on how the solution varies across games, rather than restricting the set of solutions at a given game. On the other hand, invariance of a strategy profile is a condition on the solution of a game, requiring that the types with identical relevant information play the same action. Of course, the latter is the special case of invariance of a selection for $\mathcal{G}' = \{G\}$.

I will now describe three notions of relevant information, h_i, as examples. The first and most canonical notion is the infinite hierarchy of beliefs considered by Harsanyi (1967), Mertens and Zamir (1985) and Brandenburger and Dekel (1993). Given any game G, every type t_i has a belief $h_i^1(t_i, G)$ about θ, a belief $h_i^2(t_i, G)$ about $\left(\theta, h_{-i}^1(\theta, G)\right)$, and so on. According to this notion, one takes the infinite hierarchy $h_i(t_i, G) = h_i^{MZ}(t_i, G) \equiv \left(h_i^1(t_i, G), h_i^2(t_i, G), \ldots\right)$ of beliefs about θ as the relevant information. Ely and Peski (2006) propose a second notion of relevant information: the information used by interim independent rationalizability. According to this notion, one takes $h_i(t_i, G)$ as the infinite hierarchy of beliefs regarding not only θ but also about how the players would have updated their beliefs about θ if they learned the other players' types. A third notion finds everything relevant: $h_i(t_i, G) = (t_i, G)$ for all t_i and G. (This can be taken as an extreme interpretation of Friedenberg and Meier (2008), who propose to use the type space as a way to model the "context" in which the players play the game.)

Universal Type Space. For every notion h of relevant information, there is an abstract universal type space in which each situation is represented by a type that simply describes the relevant information in that situation. Consistent selections are closely related to the strategies in this space. Hence, I embed all games $G \in \mathcal{G}$ to the universal type space as follows.

Definition 2. By *the space of all finite types*, I mean the tuple $\left(\Theta^*, T^{u,h}, \kappa^{u,h}\right)$ where $T^{u,h} = T_1^{u,h} \times \cdots \times T_n^{u,h}$ with

$$T_i^{u,h} = \left\{h_i(t_i, G) \,|\, t_i \in T_i \text{ for some } G = (N, A, \Theta, T, \kappa, u) \in \mathcal{G}\right\} \tag{2.1}$$

for each $i \in N$, and $\kappa^{u,h}$ is defined by

$$\kappa^{u,h}_{h_i(t_i,G)}\left(\theta, h_{-i}\left(t_{-i}, G\right)\right) = \sum_{t'_{-i} \in T_{-i} \cap h^{-1}_{-i}(h_{-i}(t_{-i},G),G)} \kappa_{t_i}\left(\theta, t'_{-i}\right) \tag{2.2}$$

for all i, all $G = (N, A, \Theta, T, \kappa, u) \in \mathcal{G}$, and all $(\theta, t_i, t_{-i}) \in \Theta \times T$.

Here, $\left(\Theta^*, T^{u,h}, \kappa^{u,h}\right)$ is a type space because for each type $\hat{t}_i \in T^{u,h}_i$, which is in the form of $h_i\left(t_i, G\right)$, (2.2) yields a probability distribution $\kappa^{u,h}_{\hat{t}_i}$ on $\Theta^* \times T^{u,h}_{-i}$. It is the probability distribution on $\Theta^* \times T^{u,h}_{-i}$ induced by the probability distribution κ_{t_i} on $\Theta \times T_{-i}$ and the mapping $(\theta, t_{-i}) \mapsto (\theta, h_{-i}\left(t_{-i}, G\right))$ from $\Theta \times T_{-i}$ into $\Theta^* \times T^{u,h}_{-i}$. I will assume that $\kappa^{u,h}$ is well-defined. In general, the type space $\left(\Theta^*, T^{u,h}, \kappa^{u,h}\right)$ is uncountable. Note that (2.2) ensures that embedding yields a belief morphism in the sense of Mertens and Zamir (1985), preserving the beliefs. To simplify the notation, I will suppress the notation h unless it is needed for clarity, e.g., by writing T^u instead of $T^{u,h}$ and κ^u instead of $\kappa^{u,h}$.

Models. A subset $T = T_1 \times \cdots \times T_n \subset T^u$ is said to be a *belief-closed subspace* if (Θ, T, κ^u) is a type space for some $\Theta \subseteq \Theta^*$. That is, for each $t_i \in T_i$, $\kappa^u_{t_i}\left(\Theta \times T_{-i}\right) = 1$. A belief-closed subspace T is said to be *finite* if Θ and T above are finite. Finite belief-closed subspaces are simply called *models*. I will include the empty set to the set of models and write

$$M = \{T \subset T^u | \ T \text{ is a finite belief-closed subspace}\} \cup \{\varnothing\}.$$

Given any two models $T, T' \in M$, define the collage of T and T' as $T \vee T' \equiv (T_1 \cup T'_1) \times \cdots \times (T_n \cup T'_n)$, which is clearly also a model.

Note that, by (2.2), the image $h\left(T, G\right)$ of any game G is a model:

$$h\left(T, G\right) \in M \qquad \left(\forall G = (N, A, u, \Theta, T, \kappa) \in \mathcal{G}\right). \tag{2.3}$$

Indeed, T^u is simply the collage of the images of games with finite type spaces, and it can be written as the collage of all models in T^u.

Properties of h. I make two assumptions on h. First, I assume that h is such that (2.2) leads to a well-defined belief map $\kappa^{u,h}$. That is, for any (t_i, G) and (t'_i, G') with $h_i\left(t_i, G\right) = h_i\left(t'_i, G'\right)$, we have $\kappa^{u,h}_{h_i(t_i,G)} = \kappa^{u,h}_{h_i\left(t'_i,G'\right)}$:

$$\sum_{t'_{-i} \in h^{-1}_{-i}(h_{-i}(t_{-i},G),G)} \kappa_{t_i}\left(\theta, t'_{-i}\right) = \sum_{t'_{-i} \in h^{-1}_{-i}(h_{-i}(t_{-i},G'),G')} \kappa_{t'_i}\left(\theta, t'_{-i}\right).$$

This condition is necessary for embedding all finite type spaces in a universal type space by representing the types with their relevant information, directly. Of course, this assumption holds for the examples above. Second, in order to simplify the exposition, I assume that for any $T \in M$ and any $t_i \in T_i$,

$$h_i\left(t_i, G^T\right) = t_i \tag{2.4}$$

where $G^T \in \mathcal{G}$ is the Bayesian game with type space T. That is, h becomes the identity mapping on $T^{u,h}$. This is a natural restriction on h. It formalizes the idea that h_i represents the relevant information. If $h_i(T_i, G)$ represents the relevant information in the situation described by some (t_i, G), then the relevant information represented by $h_i(T_i, G)$ must also be $h_i(T_i, G)$, as stated by the assumption. This assumption is without loss of generality because even if it does not hold, one can modify h by setting it to the identity on $T^{u,h}$ and on its belief-closed subspaces. Since these spaces are auxiliary constructs, the modification does not affect the actual results; it just complicates the exposition.

Invariant Selections and T^u. There is a one-to-one correspondence between invariant selections and the strategy profiles on belief-closed subspaces of T^u. If a selection σ^G, $G \in \mathcal{G}'$, is h-invariant, then the actions prescribed by the selection yields a well-defined strategy profile σ^* on $T^{\mathcal{G}'} \equiv \bigvee_{G=(N,A,\Theta,T,\kappa,u)\in\mathcal{G}'} h(T,G)$, defined by

$$\sigma_i^*\left(h_i\left(t_i, G\right)\right) = \sigma_i^G\left(t_i\right) \qquad \left(\forall G = (N, A, \Theta, T, \kappa, u) \in \mathcal{G}', i \in N, t_i \in T_i\right). \tag{2.5}$$

Conversely, for any strategy profile σ^* on $T^{\mathcal{G}'}$, (2.5) yields an h-invariant selection σ^G, $G \in \mathcal{G}'$. In order to explore invariant selections, I will first establish a useful and interesting structure for finite belief-closed subspaces of T^u.

3. Lattice Structure of Models

In this section, I show that models form a lattice under the set inclusion, exhibiting many useful properties. In particular, one can rank the models depending on how far they are removed from the empty set.

Proposition 1. *$(M, \supseteq)$ is a lattice with $T \vee T' \in M$ and $T \wedge T' \equiv T \cap T' \in M$ for all $T, T' \in M$. Moreover, $(M, \supseteq)$ is a complete meet-semilattice, i.e., for any $M' \subseteq M$, $\cap_{T \in M'} T \in M$.*

Proof. That $T \vee T' \in M$ is immediate. Fixing any $M' \subseteq M$, I will show that $\bar{T} \equiv \cap_{T \in M'} T \in M$. If $\bar{T} = \varnothing$, $\bar{T} \in M$ by definition. Assume that $\bar{T} \neq \varnothing$. By definition, for each $T \in M'$, there exists a finite set Θ^T, such that $\kappa^u_{t_i}\left(\Theta^T \times T_{-i}\right) = 1$ for each $t_i \in T_i$. Define $\bar{\Theta} = \cap_{T \in M'} \Theta^T$. In order to show that $\bar{T}$ is a finite belief-closed subspace, it suffices to show that $\kappa^u_{t_i}\left(\bar{\Theta} \times \bar{T}_{-i}\right) = 1$ for every $i \in N$ and $t_i \in \bar{T}_i$. To this end, take any $t_i \in \bar{T}_i$ and (θ, t_{-i}) with $\kappa^u_{t_i}(\theta, t_{-i}) > 0$. Then, $(\theta, t_{-i}) \in \Theta^T \times T_{-i}$ for each $T \in M'$, as $t_i \in T_i$. Hence, $(\theta, t_{-i}) \in \cap_{T \in M'}\left(\Theta^T \times T_{-i}\right) = \bar{\Theta} \times \bar{T}_{-i}$. This shows that $\kappa^u_{t_i}\left(\theta', t'_{-i}\right) = 0$ for every $(\theta, t_{-i}) \notin \bar{\Theta} \times \bar{T}_{-i}$. Since $\kappa^u_{t_i}$ has a finite support, this further shows that $\kappa^u_{t_i}\left(\bar{\Theta} \times \bar{T}_{-i}\right) = 1$. □

Proposition 1 shows that M is a lattice under the set inclusion, with join $\vee$ and meet $\wedge$ are defined as above. Moreover, it is a complete meet-semilattice, as it is closed under all intersections. It is not a complete lattice because infinite collages of finite models are not necessarily finite. In particular, $T^u \notin M$.

By (2.3), each type $t_i \in T^u_i$ is in a model $T \in M$, but usually there are many such models. Proposition 1 implies that there is a unique minimal model $T^{t_i} \in M$ that is included in all models that contain t_i. Here,

$$T^{t_i} = \bigcap_{T \in M, t_i \in T_i} T. \tag{3.1}$$

T^{t_i} is the minimal type space in which t_i can be expressed. Note that $T^{t_i} \neq \varnothing$ because $(t_i, t_{-i}) \in T^{t_i}$ for each $(\theta, t_{-i}) \in \mathrm{supp}\kappa^u_{t_i}$.

I will next rank the models according to how far they are removed from the empty set. Define $R_0 = \{\varnothing\}$. Define R_1 as the set of models $T \in M \backslash R_0$ for which there is no model $T' \in M \backslash R_0$ with $T' \subsetneqq T$. That is, T does not have any proper belief-closed subspace other than the empty set. Note that every finite model is either in R_1 or contains a subspace that is in R_1. Proceeding in the same fashion, one can inductively define the sets R_k, $k = 1, 2, \ldots$, by defining R_k as the set of models $T \in M$ such that (i) $T \notin R_{k'}$ for any $k' < k$, and (ii) for any model $T' \subsetneqq T$, $T' \in R_{k'}$ for some $k' < k$. I will say that a model $T \in M$ *has rank* k iff $T \in R_k$. The next lemma establishes some useful facts about the ranks of the models.

Lemma 1. *The following are true.*

(1) *For every $T \in M$, $T \in R_k$ iff k is the largest integer for which there exist models $T^0, \ldots, T^k \in M$ with $\varnothing = T^0 \subsetneqq \cdots \subsetneqq T^k = T$.*

(2) *Every* $T \in M$ *has a rank* $k_T \leq |T|$.
(3) *For any* $T, T' \in M$ *with* $T \subsetneq T'$, $k_T < k_{T'}$.
(4) *For any* $M' \subseteq M$, $\bar{T} \equiv \cap_{T \in M'} T$ *has rank* $k_{\bar{T}}$ *such that* $k_{\bar{T}} \leq k_T$ *for each* $T \in M'$, *with strict inequality whenever* $T \neq \bar{T}$.

The first three properties above are shared by any lattice formed by a family of finite sets under inclusion. The last property relies also on the fact that the meet is the intersection and the lattice is closed under arbitrary intersections. The second and fourth properties are the most crucial properties for this paper. The second property states that every model has a rank, and the fourth property states that intersection of any two models has a rank lower than the rank of either model.

4. Invariant Selections

In this section, I show that there exists an invariant selection from any solution concept that satisfies two basic properties. I further show that any invariant selection in a subfamily can be extended to a larger family. In particular, invariance of a selection has only one implication to the solutions of a given game: the solution is invariant.

In order to state the properties, I introduce two auxiliary games. For any $T \in M$, I write $G^T = \left(N, A, \Theta, T, \kappa^{u,h}, u\right) \in \mathcal{G}$ for the Bayesian game with type space T. For any game $G = (N, A, \Theta, T, \kappa, u) \in \mathcal{G}$, I write $G^h = \left(N, A, \Theta, h(T, G), \kappa^{u,h}, u\right)$ for the image of G in the universal type space under h. The first property is that the solution set $\Sigma(G)$ to G includes all of the solutions to the image G^h of G in universal type space:

Assumption 1. *For all* $G \in \mathcal{G}$, $\Sigma(G) \supseteq \Sigma^h(G) \equiv \left\{\sigma \circ h(\cdot, G) \,|\sigma \in \Sigma\left(G^h\right)\right\}$.

Note that the solutions of the form $\sigma \circ h(\cdot, G)$ do not use any irrelevant information, in the sense that each type t_i plays $\sigma_i(h_i(t_i, G))$. If an invariant selection selects $\sigma \in \Sigma\left(G^h\right)$ at G^h, it must also select $\sigma \circ h(\cdot, G)$ at G. Hence, it is necessary for invariant selection that *some* such solution is available at $\Sigma(G)$, i.e., $\Sigma(G) \cap \Sigma^h(G) \neq \varnothing$. Assumption 1 strengthens this necessary condition by requiring that *all* such solutions are available at $\Sigma(G)$. The necessary condition is not enough because invariance imposes many similar conditions. Assumption 1 ensures that all such conditions are satisfied. Due to (2.2), Assumption 1 holds for canonical solution concepts such as Bayesian Nash equilibrium and interim rationalizability.

The second property is a basic extension (and existence) property.

Assumption 2. *For any $T \in M$ and any invariant selection $\left(\sigma^{T'}\right)$ from Σ for games $G^{T'}$ with $T' \in 2^T \cap M \backslash \{T\}$, there exists $\sigma \in \Sigma\left(G^T\right)$ such that for every $t \in T' \subset T$, $\sigma(t) = \sigma^{T'}(t)$.*

That is, given a finite type space, the invariant solutions for its subspaces can be extended to the type space. Canonical solution concepts, such as Bayesian Nash equilibrium and interim rationalizability, have this property. The crucial restriction in this property is that T is finite (or countable). Extension to uncountable type spaces may not be possible (Friedenberg and Meier (2008)). It is necessary for invariant selection that *some* invariant selections for the subspaces to be extendable to T. Assumption 2 strengthens this necessary condition by requiring that *all* such selections are extendable to T.

Towards constructing an invariant selection, I rank the games in $\mathcal{G}$ as follows. Recall that for any game $G \in \mathcal{G}$ with type space T, $h(T,G) \in M$ by (2.3). Hence, by Lemma 1, $h(T,G)$ has some finite rank k, i.e., $h(T,G) \in R_k$ for some finite $k \geq 1$. A game G is said to be *of rank* k if $h(T,G)$ is of rank k. By (2.4), G and G^h are of the same rank. Write $\mathcal{G}^k$ for the set of all games $G \in \mathcal{G}$ of rank k, and write $\hat{\mathcal{G}}^k = \cup_{l \leq k} \mathcal{G}^l$. We are now ready to state and prove the main result:

Proposition 2. *Under Assumptions 1 and 2, there exists an h-invariant selection from Σ.*

Proof. Using induction on the rank k, the proof constructs an invariant selection σ^G, $G \in \mathcal{G}$, from Σ, rank by rank. Take $k = 1$. For every model $T \in R_1$ (with rank $k = 1$), pick an arbitrary $\sigma^{G^T} \in \Sigma\left(G^T\right)$. Here, $\Sigma\left(G^T\right) \neq \varnothing$ by Assumption 2. For any other $G \in \mathcal{G}^1$, pick $\sigma^G = \sigma^{G^h} \circ h$. Note that, since $\sigma^{G^h} \in \Sigma\left(G^h\right)$, by Assumption 1, $\sigma^G \in \Sigma(G)$. Note also that this constructs a selection σ^G, $G \in \mathcal{G}^1$, from Σ for $\mathcal{G}^1$. The selection is invariant because the games in $\mathcal{G}^1$ have disjoint images under h by the last part of Lemma 1.

Now consider any $k > 1$, and assume that an invariant selection σ^G, $G \in \hat{\mathcal{G}}^{k-1}$, from Σ has been constructed. Consider any model $T \in R_k$ (with rank k). By Part 3 of Lemma 1, each proper subspace T' of T is of rank $k-1$ or lower. Hence, by the inductive hypothesis, an invariant selection $\sigma^{G^{T'}}$, $T' \in 2^T \cap M \backslash \{T, \varnothing\}$, from Σ has been constructed already. Then, by Assumption 2, there exists $\sigma^{G^T} \in \Sigma\left(G^T\right)$ such that $\sigma^{G^T}(t) = \sigma^{G^{T'}}(t)$ for all $t \in T' \in 2^T \cap M \backslash \{T, \varnothing\}$. Pick σ^{G^T} as the solution at G^T, and repeat this for every $T \in R_k$. For all other games $G \in \mathcal{G}^k$ with rank k, pick $\sigma^G = \sigma^{G^h} \circ h$, where $\sigma^G \in \Sigma(G)$ by Assumption 1. (Note that $G^h = G^T$ for some $T \in R_k$.) This constructs a selection from Σ for $\hat{\mathcal{G}}^k$.

In order to complete the inductive construction, check that the selection σ^G, $G \in \hat{\mathcal{G}}^k$, is indeed invariant. To this end, take any distinct (t_i, G) and (t_i', G') with $h_i(t_i, G) = h_i(t_i', G')$ where $G = (N, A, \Theta, T, \kappa, u), G' = (N, A, \Theta, T, \kappa, u) \in \hat{\mathcal{G}}^k$, $t_i \in T_i$ and $t_i' \in T_i'$. Note that, since $G^h = G^{h(T,k)}$ is of rank k, by construction, $\sigma_i^{G^{T''}}(t_i'') = \sigma_i^{G^h}(t_i'')$ for any $t_i'' \in T_i''$ with $T'' \subseteq h(T, G)$. But, by definition, $h_i(t_i, G) \in T^{h_i(t_i,G)}$ and $T^{h_i(t_i,G)} \subseteq h(T, G)$, where $T^{h_i(t_i,G)}$ is the unique minimal model that contains $h_i(t_i, G)$. Therefore,

$$\sigma_i^{G^{T^{h_i(t_i,G)}}}\left(h_i(t_i, G)\right) = \sigma_i^{G^h}\left(h_i(t_i, G)\right). \tag{4.1}$$

Likewise, $\sigma_i^{G^{T^{h_i(t_i',G')}}}\left(h_i(t_i', G')\right) = \sigma_i^{G'^h}\left(h_i(t_i', G')\right)$. Therefore,

$$\begin{aligned}
\sigma_i^G(t_i) &= \sigma_i^{G^h}\left(h_i(t_i, G)\right) \\
&= \sigma_i^{G^{T^{h_i(t_i,G)}}}\left(h_i(t_i, G)\right) \\
&= \sigma_i^{G^{T^{h_i(t_i',G')}}}\left(h_i'(t_i', G')\right) \\
&= \sigma_i^{G'^h}\left(h_i(t_i', G')\right) = \sigma_i^{G'}(t_i'),
\end{aligned}$$

where the first and the last equalities are by construction, the second equality by (4.1), and the third equality holds simply because $h_i(t_i, G) = h_i(t_i', G')$ and $T^{h_i(t_i,G)}$ is unique.

Since each game has a finite rank (i.e., $\mathcal{G} = \cup_k \mathcal{G}^k$), this construction picks a solution $\sigma^G \in \Sigma(G)$ for each $G \in \mathcal{G}$ at some round k. In order to check that σ^G, $G \in \mathcal{G}$, is invariant, note that for any $G, G' \in \mathcal{G}$, since $G, G' \in \hat{\mathcal{G}}^k$ for some k, $\sigma_i^G(t_i) = \sigma_i^{G'}(t_i')$ whenever $h_i(t_i, G) = h_i(t_i', G')$, as established in the previous paragraph. □

Under Assumptions 1 and 2, Proposition 2 establishes that there exists an h-invariant selection from Σ, and its proof explicitly constructs such a selection. As discussed above, Assumptions 1 and 2 are strengthenings of basic necessary conditions for invariant selection. These are weak assumptions in that they hold for canonical solution concepts, such as Bayesian Nash equilibrium and interim rationalizability.

The straightforwardness of the construction in the proof of Proposition 2 may be misleading, as it finesses the following inherent difficulty. The same piece of relevant information can be modeled by types in uncountable number of games. In order to construct an invariant selection, one then needs to select the solutions for uncountable number of such games simultaneously and maintain invariance. In the construction, such a difficult task is made possible by the lattice structure established in the previous section, as follows.

Recall from the previous section that for any type t_i in any game $G \in \mathcal{G}$, there exists a unique minimal type space $T^{h_i(t_i,G)} \in M$ in which the relevant information $h_i(t_i, G)$ of t_i can be modeled. This type space has the lowest rank $k_{T^{h_i(t_i,G)}}$ among the type spaces that can model $h_i(t_i, G)$. The action of all types t_i' from games G' with $h_i(t_i', G') = h_i(t_i, G)$ is selected at round $k_{T^{h_i(t_i,G)}}$, which is the first time it is possible to express $h_i(t_i, G)$, using a solution for the minimal type space $T^{h_i(t_i,G)}$, which is the only model that contains $h_i(t_i, G)$ at that rank. Of course, many of these games have higher ranks than $k_{T^{h_i(t_i,G)}}$, and the solution to these games are selected in later rounds. In the construction, these selections respect the specification of the action for $h_i(t_i, G)$ that has been made at round $k_{T^{h_i(t_i,G)}}$.

I will next explore the restrictions imposed by invariance requirement. I will first establish that any invariant selection in a subfamily can be extended to all games. Conceptually, this establishes that the invariance requirement on a larger family of games does not impose any extra restriction on the subfamilies. Pragmatically, it ensures that if one is only interested in the behavior in a class of games (e.g. in the solution of a particular game), she can focus on constructing an invariant selection for that class without worrying about the invariance across all games.

Proposition 3. *Under Assumptions 1 and 2, for any $\mathcal{G}' \subseteq \mathcal{G}$ and any h-invariant selection σ^G, $G \in \mathcal{G}'$, from Σ for $\mathcal{G}'$, there exists an h-invariant selection $\hat{\sigma}^G$, $G \in \mathcal{G}$, from Σ such that $\hat{\sigma}^G = \sigma^G$ for every $G \in \mathcal{G}'$.*

Proof. I will construct a refinement Σ' of Σ that satisfies Assumptions 1 and 2 and such that $\Sigma'(G) = \{\sigma^G\}$ for all $G \in \mathcal{G}'$. Then, by by Proposition 2, there exists an invariant selection $\hat{\sigma}^G$, $G \in \mathcal{G}$, from Σ' as in the proposition. Since Σ' is a refinement of Σ, $\hat{\sigma}^G$, $G \in \mathcal{G}$, is also a selection from Σ.

To define Σ', note that, since σ^G, $G \in \mathcal{G}'$, is h-invariant, $\sigma^G = \sigma^{G^h} \circ h$ for some solution $\sigma^{G^h} \in \Sigma\left(G^h\right)$ for each $G \in \mathcal{G}'$. Write $\bar{\mathcal{G}}' = \mathcal{G}' \cup \left\{G^h | G \in \mathcal{G}'\right\}$ and set

$$\Sigma'(G) = \begin{cases} \{\sigma^G\} & \text{if } G \in \bar{\mathcal{G}}', \\ \{\sigma^{G^T}|_{T'}\} & \text{if } G = G^{T'},\, T' \subset T,\, G^T \in \bar{\mathcal{G}}', \\ \Sigma(G) & \text{otherwise,} \end{cases}$$

where $\sigma^{G^T}|_{T'}$ is the restriction of σ^{G^T} to T'. Since σ^G, $G \in \mathcal{G}'$, is invariant, Σ' is well-defined.

To check Assumption 1, note that $\Sigma'(G) = \left\{\sigma \circ h(\cdot, G) | \sigma \in \Sigma'\left(G^h\right)\right\}$ for $G \in \mathcal{G}'$ by construction and for $G \in \left\{G^T | T \in M\right\}$ by (2.4). For any $G \notin \mathcal{G}' \cup \left\{G^T | T \in M\right\}$,

$$\Sigma'(G) = \Sigma(G) \supseteq \left\{\sigma \circ h(\cdot, G) | \sigma \in \Sigma\left(G^h\right)\right\} \supseteq \left\{\sigma \circ h(\cdot, G) | \sigma \in \Sigma'\left(G^h\right)\right\},$$

where the first equality is by construction of Σ', the next inclusion is by Assumption 1 for Σ, and the last inclusion is due to the fact that Σ' is a refinement of Σ.

To check Assumption 2, note first that when $G^T \in \bar{\mathcal{G}}'$, Assumption 2 holds for T and Σ' by construction. Now consider any $T \in M$ with $G^T \notin \bar{\mathcal{G}}'$ and any invariant selection $\left(\sigma^{T'}\right)_{T' \in 2^T \cap M \setminus \{T\}}$ from Σ'. This is also an invariant selection from Σ, and $\Sigma\left(G^T\right) = \Sigma'\left(G^T\right)$. Hence, by Assumption 2 for Σ, there exists $\sigma \in \Sigma\left(G^T\right) = \Sigma'\left(G^T\right)$ such that for every $t \in T' \subset T$, $\sigma(t) = \sigma^{T'}(t)$. Therefore, Assumption 2 holds for Σ'. $\square$

I will next characterize the implications of invariance to the solutions of a given game. As discussed before, invariance of a selection trivially implies that the solution is invariant, i.e., the types with identical relevant information play the same action. The next corollary establishes that this is the only implication of invariance of selections to the set of solutions of a given game. In the sequel, given a game G^*, a solution $\sigma \in \Sigma(G^*)$ is said to be *selected by* σ^G, $G \in \mathcal{G}$, iff $\sigma^{G^*} = \sigma$.

Corollary 1. *Under Assumptions 1 and 2, for any $G \in \mathcal{G}$ and $\sigma \in \Sigma(G)$, σ is selected by an h-invariant selection from Σ if and only if σ is h-invariant.*

Proof. The necessity immediately follows from the definition of invariance (applied to the types in G). To prove the sufficiency, in Proposition 3, take $\mathcal{G}' = \{G\}$ and note that σ is an invariant selection for $\{G\}$. $\square$

Under Assumptions 1 and 2, the above results establish that there is always an invariant selection and that invariance does not impose any extra restriction on the solutions for the subfamilies of games. In particular, a solution of a game is selected by an invariant selection if and only if the types with identical relevant information play the same action according to the solution. Hence, if one is interested only in behavior in a game G, then she can analyze G in isolation by focusing on h-invariant solutions for G.

5. Invariant Equilibrium Selection

In this section, under the usual regularity conditions, I show that Bayesian Nash equilibrium satisfies the sufficient conditions for invariant selection, and hence the conclusions of the previous results are true for Bayesian Nash equilibrium: there exists an invariant equilibrium selection and an equilibrium is selected by an invariant equilibrium selection if and only if the equilibrium is invariant.

Given any game $G = (N, A, u, \Theta, T, \kappa)$, by a *Bayesian Nash equilibrium of* G, I mean any strategy profile $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$ such that $\sigma_i^*(t_i) \in BR_{t_i}\left(\sigma_{-i}^*|G\right)$ for each $t_i \in T_i$, where $BR_{t_i}\left(\sigma_{-i}^*|G\right)$ denotes the set of all mixed best replies of type t_i to σ_{-i}^* in game G. I write $BNE(G)$ for the set of Bayesian Nash equilibria of G. By an *equilibrium selection*, I mean a selection from BNE. I will consider the following regularity condition.

Assumption 3. *The set Θ^* of parameters is compact. For each $i \in N$, action set A_i is a compact metric space, and each u_i is continuous in a and measurable in θ.*

Note that Assumption 3 holds in most games considered in Game Theory and its applications, including finite games. It is made to ensure that Nash equilibrium exists when types are considered as players. Under this weak assumption, the next familiar result establishes that Bayesian Nash equilibria satisfies the sufficient conditions for invariant selection. (There are similar results in the literature, including an equilibrium extension result by Friedenberg and Meier (2008).)

Lemma 2. *Under Assumption 3, Assumptions 1 and 2 hold for BNE and h.*

Hence, the conclusions of the previous section apply to equilibrium selection:

Proposition 4. *Under Assumption 3, the following are true.*

(1) *For any* $\mathcal{G}' \subseteq \mathcal{G}$ *and any* h*-invariant equilibrium selection* σ^G, $G \in \mathcal{G}'$, *there exists an* h*-invariant equilibrium selection* $\hat{\sigma}^G$, $G \in \mathcal{G}$, *such that* $\hat{\sigma}^G = \sigma^G$ *for every* $G \in \mathcal{G}'$.
(2) *For any* $G \in \mathcal{G}$ *and any* $\sigma \in BNE(G)$, σ *is selected by an* h*-invariant equilibrium selection if and only if* σ *is* h*-invariant.*
(3) *There exists an* h*-invariant equilibrium selection.*

Proof. Note that Part 1 implies both Part 2 (for $\mathcal{G}' = \{G\}$) and Part 3 (for $\mathcal{G}' = \varnothing$). Part 1 immediately follows from Lemma 2 and Proposition 3. $\square$

Under usual regularity conditions, Proposition 4 establishes that any invariant equilibrium selection for a subset of games can be extended to all games with finite type spaces. In particular, there is an invariant equilibrium selection for all such games. It also implies that, beyond the basic restriction on the actions of types with identical information, invariance does not lead to any equilibrium refinement; it only restricts the way the solutions vary across games.

6. Equilibrium on T^u

There is a one-to-one correspondence between the invariant equilibrium selections and the equilibria on T^u, the universal space of finite types. In this section, using this correspondence and the results of the previous section, I will show that there exists an equilibrium on T^u, and indeed, every equilibrium on its belief-closed subspaces can be extended to T^u.

For any player i, by *strategy*, I mean any function $\sigma_i : T_i^u \to \Delta(A_i)$. By a *Bayesian Nash equilibrium* on T^u, I mean any strategy profile $\sigma^* = (\sigma_1^*, \dots, \sigma_n^*)$ such that $\sigma_i^*(t_i) \in BR_{t_i}\left(\sigma_{-i}^*\right)$ for each $t_i \in T_i^u$ where $BR_{t_i}\left(\sigma_{-i}^*\right)$ is the set of best replies for t_i against σ_{-i}^*. The strategies and equilibria on subspaces are defined similarly. It is crucial here that I do not impose any measurability condition on the strategies. Since strategies are conditioned on players' types already, measurability is not needed for players' knowing their own actions. It is not needed for expectations either because the types' beliefs have finite support, yielding well-defined beliefs on $\Theta^* \times A_{-i}$ for each t_i and each σ_{-i}. Consequently, $BR_{t_i}\left(\sigma_{-i}^*\right)$ is well-defined. The following lemma establishes the one-to-one correspondence between the invariant equilibrium selections and the equilibria on the subspaces of T^u.

Lemma 3. *For any $\mathcal{G}' \subseteq \mathcal{G}$, an equilibrium selection σ^G, $G \in \mathcal{G}'$, is h-invariant if and only if there exists a Bayesian Nash equilibrium σ^* on $T^{\mathcal{G}'} \equiv \bigvee_{G=(N,A,\Theta,T,\kappa,u)\in\mathcal{G}'} h(T,G)$ such that*

$$\sigma_i^*(h_i(t_i,G)) = \sigma_i^G(t_i) \qquad (\forall G = (N,A,\Theta,T,\kappa,u) \in \mathcal{G}', i \in N, t_i \in T_i). \tag{6.1}$$

That is, invariant selections on $\mathcal{G}'$ are precisely the selections obtained by restricting the equilibria on $T^{\mathcal{G}'}$ to its subspaces. Due to this correspondence, the previous results on invariant equilibrium selection immediately yield the following existence result for equilibrium on the space of all finite types. (See the appendix for a detailed proof.)

Proposition 5. *Under Assumption 3, the following are true.*

(1) *For any $\mathcal{G}' \subseteq \mathcal{G}$ and any h-invariant equilibrium selection σ^G, $G \in \mathcal{G}'$, there exists a Bayesian Nash equilibrium σ^* on T^u such that $\sigma_i^*(h_i(t_i,G)) = \sigma_i^G(t_i)$ for every $G \in \mathcal{G}'$ and every type t_i in G.*
(2) *For any $M' \subseteq M$ and any Bayesian Nash equilibrium σ on $T^{M'} \equiv \bigvee_{T\in M'} T$, there exists a Bayesian Nash equilibrium σ^* on T^u such that $\sigma^* = \sigma$ on $T^{M'}$.*
(3) *For any $G \in \mathcal{G}$ and any h-invariant $\sigma \in BNE(G)$, there exists a Bayesian Nash equilibrium σ^* on T^u such that $\sigma_i^*(h_i(t_i,G)) = \sigma_i^G(t_i)$ for all t_i in G.*
(4) *There exists a Bayesian Nash equilibrium on T^u.*

Part 2 states that any equilibrium defined on any subspace can be extended to T^u, the space of all finite types. By Lemma 3, this is equivalent to stating that for any invariant selection for any family of games, there is an equilibrium on T^u that specifies the actions of the types in the family according to the selection—Part 1. This result is very useful in equilibrium analysis on T^u. In such an analysis, one often needs to specify a partial strategy for a given set of types and have an equilibrium on T^u in which the given types play according to the specification. For a suitably selected set of types, it is relatively easy to verify that the specified behavior of the given types is part of an equilibrium of the games the types come from and that these equilibria form an invariant selection. On the other hand, specifying a nontrivial equilibrium on T^u is a prohibitively difficult task because of the complex interconnections between the types in T^u and between their best responses. The result stated in Parts 1 and 2 frees the researcher from the latter daunting task. Thanks to this result, she can focus on specifying the equilibrium behavior on the relevant games without worrying about whether the specified behavior is part of an equilibrium on T^u. (See Weinstein and Yildiz (2008) for such an application.)

As a special case (for $\mathcal{G}' = \{G\}$), this result implies that any invariant equilibrium of any game G can be extended to the space of all finite types, as in Part 3. That is, one can focus on the equilibria on T^u without ruling out any equilibrium of games in which distinct types have distinct relevant information. For example, in robustness analysis for equilibria of these games, it suffices to analyze the sensitivity of equilibria on T^u to perturbations in T^u.

Finally, as another special case (for $\mathcal{G}' = \varnothing$), the result establishes existence of an equilibrium on the space of finite types. This fills an important gap in the literature, in which very little is known on the existence of equilibrium on the universal type space and on its prominent subspaces, such as T^u. Simon (2003) shows existence of equilibrium on spaces that can be partitioned into countable subspaces and in which the types have finite support. Unfortunately, one cannot partition T^u because given any two types in T^u, there is another type in T^u that puts positive probabilities on both of the given types. (This property is exhibited by most prominent subspaces of the universal type space, such as the spaces of all finite types with common prior and all countable types, and one cannot partition them, either.)

The crucial modeling assumption here is that I do not require that the strategies are measurable, which is not necessary here. In a larger type space with types that have uncountable supports, one needs to impose a measurability restriction on the strategies. In that case, the above results may not be true. For example, in a particular class his existence result applies, Simon (2003) shows that all equilibria must be in non-measurable strategies (cf. Part 4). Under the measurability restriction, Friedenberg and Meier (2008) show that some equilibria of a given game may not be extendable to a larger space even if there is an equilibrium in the larger space (cf. Part 3).

Finally, note that if A is convex and u_i is concave in own action, the above results also apply to equilibria in pure strategies.

7. Concluding Remarks

A piece of relevant information can be modeled through multiple types coming from various Bayesian games. In order to avoid the solution to depend on the choice of modeling the information, one may want to require that all these types take the same action according to the solutions to these games. That is, the solutions to the Bayesian games form an invariant selection. In this note, without subscribing to any notion of relevance of information, I construct an invariant selection for the space of all games with finite type space from arbitrary solution concepts that satisfy two basic conditions, which are satisfied by canonical solution concepts, such as Bayesian Nash equilibrium and interim rationalizability. I further show that any invariant selection within a subfamily can be extended to the family of all games with finite type spaces. Constructing such a selection is a difficult task because one needs to equalize the actions of uncountable number of types from various games with complicated interconnections that are difficult to foresee. In order to construct such a selection, I first establish a very interesting and useful lattice structure for the finite type spaces within the universal type space, a structure that is clearly useful beyond the scope of this paper. It is this structure that enables me to construct an invariant selection in a straightforward manner without making any significant assumption.

There is a one-to-one correspondence between the invariant equilibrium selections and the equilibria on the space of all finite types. Using this correspondence, I show that there exists an equilibrium in that space, filling an important gap in the literature, and show that indeed any equilibrium in any type space can be extended to entire space, which is a quite useful result in equilibrium analysis on the space of all finite types.

In this paper, I focus on the finite type spaces. The specific construction I use relies on the finiteness because infinite type spaces may have infinite ranks, and my construction may not specify the actions of all types in such type spaces. It seems, however, that one can extend the analysis here to the space of all countable type spaces using more elaborate techniques.[2] For such type spaces, one does not need the strategies to be measurable, and the underlying results used in the construction hold. In particular, the lattice structure still applies and Bayesian

[2] I thank Jonathan Weinstein for pointing this out.

Nash equilibrium satisfies the relevant sufficient conditions for an invariant selection. The only difficulty is that one needs to use infinite ordinals to rank the type spaces. Hence, in the construction, one needs to use transfinite induction, instead of the usual induction employed here. In order to avoid the technical difficulties, I have not considered that extension.

Appendix A. Omitted Proofs

Proof of Lemma 1. Part 1 immediately follows from the definition of R_k, and Part 3 immediately follows from Parts 1 and 2.

(Part 2) Take any $T \in M$ and integer K such that $T \notin R_k$ for any $k \leq K$. Then, by Part 1, there exist models $T^1, \ldots, T^K \in M \backslash \{\varnothing\}$ with $T^1 \subsetneqq \cdots \subsetneqq T^K \subsetneqq T$. In particular, $|T| > K$. Therefore, for any $T \in M$, $T \in R_{k_T}$ for some $k_T \leq |T|$.

(Part 4) By Proposition 1, $\bar{T} \in M$. Hence, by Part 2, it has a rank $k_{\bar{T}}$. For any $T \in M'$, since $\bar{T} \subseteq T$, by Part 3, $k_{\bar{T}} \leq k_T$, with strict inequality whenever $T \neq \bar{T}$. $\square$

Proof of Lemma 2. In order to show that Assumption 1 holds for BNE, take any $\sigma \in BNE\left(G^h\right)$ for some G. I will show that for each t_i in G, the belief $\pi_{t_i}\left(\cdot|\sigma_{-i} \circ h_{-i}\right) \in \Delta\left(\Theta \times A_{-i}\right)$ of type t_i on $\Theta \times A_{-i}$ is equal to the belief $\pi_{h_i(t_i,G)}\left(\cdot|\sigma_{-i}\right) \in \Delta\left(\Theta \times A_{-i}\right)$ of type $h_i\left(t_i, G\right)$ on $\Theta \times A_{-i}$. Since $\sigma \in BNE\left(G^h\right)$, this implies that $\sigma_i\left(h_i\left(t_i, G\right)\right) \in BR_{h_i(t_i,G)}\left(\sigma_{-i}|G^h\right) = BR_{t_i}\left(\sigma_{-i} \circ h_{-i}|G\right)$ for each t_i in G, showing that $\sigma \circ h \in BNE\left(G\right)$. To show $\pi_{t_i}\left(\cdot|\sigma_{-i} \circ h_{-i}\right) = \pi_{h_i(t_i,G)}\left(\cdot|\sigma_{-i}\right)$, take any $\left(\theta, a_{-i}\right)$. Then,

$$\begin{aligned}
\pi_{h_i(t_i,G)}\left(\theta, a_{-i}|\sigma_{-i}\right) &= \sum_{h_{-i}(t_{-i},G) \in h_{-i}(T_{-i},G)} \kappa^{u,h}_{h_i(t_i,G)}\left(\theta, h_{-i}\left(t_{-i}, G\right)\right) \sigma_{-i}\left(a_{-i}|h_{-i}\left(t_{-i}, G\right)\right) \\
&= \sum_{h_{-i}(t_{-i},G) \in h_{-i}(T_{-i},G)} \sum_{t'_{-i} \in T_{-i} \cap h^{-1}_{-i}(h_{-i}(t_{-i},G),G)} \kappa_{t_i}\left(\theta, t'_{-i}\right) \sigma_{-i}\left(a_{-i}|h_{-i}\left(t'_{-i}, G\right)\right) \\
&= \sum_{t'_{-i} \in T_{-i}} \kappa_{t_i}\left(\theta, t'_{-i}\right) \sigma_{-i}\left(a_{-i}|h_{-i}\left(t'_{-i}, G\right)\right) \\
&= \pi_{t_i}\left(\theta, a_{-i}|\sigma_{-i} \circ h_{-i}\right),
\end{aligned}$$

where the first and last equalities are by definition, the second equality is by (2.2), and the third one is by rearrangement of the sum.

In order to show that Assumption 2 holds for BNE, take any $T \in M$ and any invariant equilibrium selection $\sigma^{G^{T'}}$ for the proper subspaces T' of T. Let $\hat{T} = \bigvee_{T' \in M \cap 2^T \backslash \{T\}} T'$ and note that, by Lemma 3, there exists an equilibrium σ on $\hat{T}$ such that $\sigma^{G^{T'}}_i\left(t_i\right) = \sigma_i\left(t_i\right)$ for each $t_i \in \hat{T}_i$. If $\hat{T} = T$, this already proves the result. Assume $T \backslash \hat{T} \neq \varnothing$, and define the complete-information game with set $\bar{N} \equiv \cup_i \left(T_i \backslash \hat{T}_i\right)$ of players, with action set A_i for each player $t_i \in \bar{N}$, and with utility function U_{t_i} on strategy profiles $s : t_i \mapsto s_i(t_i)$, $t_i \in \bar{N}$, defined as the expected payoff of t_i when each $t_j \in \bar{N}$ plays $s_j(t_j)$ and each $t_j \in \hat{T}_j$ plays $\sigma_j(t_j)$ for all $j \in N$. Since the expectation operator preserves continuity (by the bounded convergence theorem), under Assumption 3, each player $t_i \in \bar{N}$ has a continuous utility function with a compact action space A_i. Since $\bar{N}$ is also finite, by Glicksberg's (1952) Theorem, the constructed game has a Nash equilibrium $\hat{\sigma}$ in possibly mixed strategies. Define the Bayesian Nash equilibrium $\sigma^* : T \to \Delta(A)$ by $\sigma_i^*(t_i) = \sigma_i(t_i)$ if $t_i \in \hat{T}_i$ and $\sigma_i^*(t_i) = \hat{\sigma}_i(t_i)$ otherwise. □

Proof of Lemma 3. As in Section 2, a strategy profile σ^* on $T^{\mathcal{G}'}$ is well-defined by (6.1) iff the family σ^G is h-invariant. Moreover, as in the proof of Lemma 2, for any $G \in \mathcal{G}'$ and t_i in G, $BR_{t_i}\left(\sigma_{-i}^G | G\right) = BR_{h_i(t_i,G)}\left(\sigma_{-i}^*\right)$. Hence, by (6.1), $\sigma_i^G(t_i) \in BR_{t_i}\left(\sigma_{-i}^G\right)$ iff $\sigma_i^*(h_i(t_i,G)) \in BR_{h_i(t_i,G)}\left(\sigma_{-i}^*\right)$. Therefore, σ^G is an equilibrium of G for every $G \in \mathcal{G}'$ iff σ^* is a Bayesian Nash equilibrium on $T^{\mathcal{G}'}$. □

Proof of Proposition 5. (Part 1) By Proposition 4, there exists an h-invariant equilibrium selection $\hat{\sigma}^G$, $G \in \mathcal{G}$, such that $\hat{\sigma}^G = \sigma^G$ for every $G \in \mathcal{G}'$. Then, by Lemma 3, there exists a Bayesian Nash equilibrium σ^* on T^u such that $\sigma_i^*(h_i(t_i,G)) = \hat{\sigma}_i^G(t_i)$ for every $G \in \mathcal{G}$ and every type t_i in G. When $G \in \mathcal{G}'$, $\sigma_i^*(h_i(t_i,G)) = \hat{\sigma}_i^G(t_i) = \sigma_i^G(t_i)$.

(Part 2) Take $\mathcal{G}' = \left\{G^T | T \in M'\right\}$. For each $T \in M'$, set $\sigma^{G^T} = \sigma_{|T} \in BNE\left(G^T\right)$, where $\sigma_{|T}$ is the restriction of σ to T. Then, by Lemma 3, σ^{G^T}, $G^T \in \mathcal{G}'$, is an h-invariant equilibrium selection for $\mathcal{G}'$. Hence, by Part 1, there exists a Bayesian Nash equilibrium σ^* on T^u such that $\sigma_i^*(t_i) = \sigma_i^*\left(h_i\left(t_i, G^T\right)\right) = \sigma_i^{G^T}(t_i) = \sigma_i(t_i)$ for all $t_i \in T_i$ and $T \in M$. Here, the first equality is by (2.4).

Finally, Part 1 implies both Part 3 (for $\mathcal{G}' = \{G\}$) and Part 4 (for $\mathcal{G}' = \varnothing$). □

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