Repeated Public Good Provision*

Alexander Wolitzky

MIT

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Abstract

We provide a tractable framework for studying the effects of group size and structure on the maximum level of a public good that can be provided in sequential equilibrium in repeated games with private monitoring. We restrict attention to games with "all-or-nothing" monitoring, in which in every period player i either perfectly observes player j's contribution to the public good or gets no information about player j's contribution; this class of games includes many interesting examples, including random matching, monitoring on networks, and simple kinds of imperfect "quasi-public" monitoring. The first main result is that the maximum level of public good provision can be sustained in grim trigger strategies. In games satisfying a weak form of symmetry, comparative statics on the maximum per capita level of public good provision are shown to depend only on the product of a term capturing the rivalness of the good and a term capturing a simple characteristic of the monitoring technology: its "effective contagiousness." In leading examples, the maximum per capita level of provision of a pure public good is increasing in group size, but the maximum per capita level of provision of a divisible public good is often decreasing in group size. Under broad conditions, making monitoring less uncertain in the second-order stochastic dominance sense increases public good provision. For games played on asymmetric networks, we introduce a new notion of network centrality and show that more central players in social networks make larger contributions, and that every player in better connected networks can contribute more to the public good. We also consider an extension to local public goods.

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1 Introduction

The question of how groups sustain cooperation and the related question of what kinds of groups can sustain cooperation best are fundamental in the social sciences. In economics, existing work on the theory of repeated games provides a framework for answering these questions when individuals can perfectly observe each other's actions (e.g., Abreu, 1988), but has much less to say about the more realistic case where monitoring is imperfect. This weakness is particularly acute when studying large groups, where public signals are very poor signals of each individual's actions, and where high quality—but dispersed—private signals are the basis for cooperation. Consider the construction of a series of infrastructure projects in a small village (wells, schools, roads, etc.). The quality of each project is a poor signal of each villager's contribution to its construction, but each villager may always know whether the other members of her household worked on the project, and may occasionally also observe other villagers working on a project. Similarly, the stock price of a Fortune 500 company is a poor signal of each employee's effort, but each employee may observe her officemates' effort; and price is a poor signal of each firm's output in a large market, but each firm may observe the output of its local competitors. Thus, it is certainly plausible that local, private monitoring plays a larger role than public monitoring in sustaining cooperation in many interesting economic examples, and very little is known about how cooperation is best sustained under this sort of monitoring.

This paper studies the provision of public goods when incentives to contribute are provided through community enforcement under a range of monitoring technologies.[1] We are particularly interested in what strategies sustain the highest possible level of public good provision in sequential equilibrium, how this level varies with group size and structure,[2] and how this depends on characteristics of the public good (e.g., whether it is pure or impure). Thus, our paper is a contribution to the literatures on repeated games with community enforcement (e.g., Kandori, 1992; Ellison, 1994) and repeated public good provision (Bendor and Mookherjee, 1987, 1990; Pecorino, 1999; Haag and Lagunoff, 2007). Unlike most existing work on repeated games with community enforcement, we characterize optimal equilibria at fixed discount factors and emphasize comparative statics with

[1] In this paper, we use the terms "monitoring technology" and "group structure" interchangeably.

[2] How public good provision varies with group size has been a major question dating back at least to Olson (1965). The effect of social structure on public good provision has become more prominent recently, beginning with the work of sociologists such as Coleman (1990) and Putnam (2000) and moving into the economics literature on networks (see, for example, the papers cited in footnote 6).

respect to group size and structure; unlike the existing literature on repeated public good provision, we study games with private monitoring. We also study public good provision in social networks, which makes our paper a contribution to the literature on repeated games on networks.

More precisely, we study N-player repeated games where in every period each player decides how much to contribute to a public good, and contributing nothing is the dominant stage game action. Our analysis is made tractable by the restriction, maintained throughout the paper, to games in which monitoring is "all-or-nothing," in that in every period player i either perfectly observes player j's contribution to the public good or gets no information about player j's contribution. This restriction allows us to cleanly characterize the maximum level of public good provision at any fixed discount factor bounded away from 1, which would be impossible with more general imperfect monitoring; indeed, optimal equilibria under all-or-nothing monitoring have stationary equilibrium path, as discussed below. Examples of all-or-nothing monitoring structures include "uniform monitoring," where each player i observes the contribution of each other player j with probability p independently across i and j; "quasi-public monitoring," where all other players observe player i's contribution with probability p, and none of them observe her contribution with probability $1-p$;[3] "random matching," where players randomly pair off each period and only observe the contributions of their partners; and "monitoring on a network," where players are arranged in fixed positions on an arbitrary graph, and each player observes only the contributions of her neighbors each period. All-or-nothing monitoring also seems like a reasonable approximation to the actual monitoring structures in the motivating applications listed above.

We now outline the remainder of the paper and preview our results. Section 2 relates our paper to the existing economics literatures on community enforcement, dynamic public good provision, repeated games on networks, and repeated games with private monitoring. Section 3 presents our model and our definition of all-or-nothing monitoring, and discusses how it encompasses a wide range of public good environments and monitoring technologies. Section 4 establishes our main theoretical results. We first show that the maximum equilibrium level of public good provision is always sustained in "grim trigger" strategies, under which every individual contributes a fixed amount to the public good each period and stops contributing forever if she ever observes another individual's failure to make her prescribed contribution. The fact that grim trigger strategies provide the strongest possible incentive to contribute to the public good follows from a monotonicity

[3] We refer to this monitoring technology as "quasi"-public, because player i does not know when her opponents observe her actions—this is a technicality that does not affect any of our results.

argument based on a kind of "strategic complementarity" in repeated public good games that to our knowledge has not been previously exploited in the literature. The key observation is that the highest contribution that a player is willing to make from any on-path history onwards is non-decreasing in the contributions that every player makes from each on-path history onwards. This allows us to use a monotonicity argument in the spirit of Milgrom and Roberts (1990) to show that the maximum equilibrium level of public good provision is sustained by grim trigger strategies, rather than by strategies in which players are "rewarded" when they are observed to make their prescribed contributions. We also show that the maximum level of public good provision is sustained in *symmetric* grim trigger strategies—in which each player contributes an amount x^* every period as long as she has never observed a contribution other than x^*, and contributes nothing otherwise—if and (for generic discount factors) only if monitoring also satisfies a weak symmetry property, which requires that all players' actions are "equally observable."

We then use these results to show that, under the equal observability assumption, the maximum per capita level of provision of a public good depends on the monitoring technology *only* through its "effective contagiousness,"[4] defined as

$$\sum_{t=0}^{\infty} \delta^t \mathbb{E}\left[\textit{number of players who learn about a deviation within } \mathrm{t} \textit{ periods}\right].$$

This fact yields simple and intuitive comparative statics results. The key observation driving these comparative statics is that the cost of contributing to the public good does not depend on group size or structure, while the benefit of contributing—the expected discounted value of the future contributions that other individuals make only if the player contributes—depends on group size and structure through both the marginal benefit a player receives from another player's contribution (the "rivalness" of the public good) and the effective contagiousness. In particular, the group size that best supports public good provision is the one that maximizes the product of this marginal benefit and the effective contagiousness.

Section 5 uses the results of Section 4 to derive comparative statics on the maximum per capita level of public good provision with respect to group size (N) for different kinds of public goods and different monitoring technologies that satisfy equal observability. It serves to illustrate the usefulness of the results of Section 4 for applications, and also demonstrates that the maximum

[4]The terminology "effective contagiousness" is due to the fact that this number reflects how quickly information about a defection spreads "contagiously" through the population, measured in terms of payoff consequences for the players.

per capita level of public good provision is increasing in group size in a wide range of important examples when the public good is *pure* (i.e., the marginal benefit a player receives from another player's contribution does not depend on N), but is decreasing in many examples when the public good is *divisible* (i.e., the marginal benefit a player receives from another player's contribution scales by $1/N$).[5]

Section 6 presents a general result comparing monitoring structures under equal observability, holding group size fixed. We provide broad conditions under which making monitoring more uncertain, in the sense of a mean-preserving spread of the distribution of the number of individuals who observe a deviation, reduces the maximum level of public good provision. This suggests that monitoring structures under which a small number of players observe a deviation with high probability are more effective at supporting cooperation than are monitoring structures under which a deviation is publicly observed with low probability.

Section 7 drops the assumption of equal observability and studies public good provision on general, asymmetric networks, using the results of Section 4. We introduce a new notion of network centrality, and show that players who are "more central" make higher contributions in the (effectively unique) equilibrium that sustains the maximum level of public good provision; this follows because a defection by a more central player leads to more defections by other central players (who would otherwise make large contributions), which implies that more central players are less tempted to defect. We also provide simple graph-theoretic tools for determining which players are more central than others and demonstrate their usefulness in an example. Finally, we show that adding a link between any two players strictly increases the contributions of all players in the component of the graph containing those players in the equilibrium that sustains the maximum level of public good provision, which formalizes the idea that individuals in better-connected social groups can contribute more to public goods.

Finally, Section 8 briefly generalizes our model to allow for *local* public goods—like cooperation in a bilateral trading relationship—where players benefit asymmetrically from each other's contributions. Our main theoretical results apply to this more general model, but the effects of group size and structure on public good provision may be different. For example, we find that

[5] A defining property of a public good is that it is non-rival *for a given population.* We distinguish between public goods that remain non-rival as the population grows (e.g., "pure" public goods like national defense and pollution control), and public goods whose benefits become diluted as the population grows (e.g., "divisible" public goods like profits in an organization or fixed prizes from intergroup conflict). Our model also allows for intermediate cases, like infrastructure projects.

more "compact" social structures support a higher level of public good provision when players only benefit from contributions that they observe, which highlights that different social structures are better at supporting the provision of different types of public goods. Section 9 concludes and discusses possible directions for future research, and the appendix contains omitted proofs.

2 Related Literature

As discussed above, this paper lies at the intersection of the literature on repeated games with community enforcement and the literature on repeated public good provision. The study of repeated games with community enforcement was pioneered by Kandori (1992) and Ellison (1994), who introduced the "contagion" strategies that turn out to sustain the maximum level of public good provision in our model. Subsequent important contributions include Greif (1993), Ghosh and Ray (1996), Kranton (1996), Dixit (2003), Takahashi (2008), and Deb (2008). Many of these papers (e.g., Kandori, 1992; Ellison, 1994; Takahashi, 2008; Deb, 2008) focus on sustaining cooperation in the limit as the discount factor goes to 1, rather than on characterizing the maximum level of cooperation that is sustainable in equilibrium for a fixed discount factor. Greif (1993) shows that trade is facilitated by "multilateral" punishments, in which a group of traders stop doing business with an individual if she cheats an individual trader. Ghosh and Ray (1996) study a repeated game in which some players are "uncooperative," and examine how the level of cooperation that can be sustained in a particular class of equilibria varies with the discount factor and the fraction of uncooperative players. Kranton (1996) examines gains from trade under perfect monitoring in a random matching model and shows that gains from trade are larger in larger markets, as there are more frequent future interactions in larger markets. And Dixit (2003) studies the maximal amount of cooperation that can occur in the equilibrium of a two period game with a particular network structure and form of incomplete information.

The maximum level of cooperation that can be sustained for given discount factors has been considered to some degree in the small existing literature on repeated public good provision. Bendor and Mookherjee (1987) study repeated provision of impure public goods with a particular form of imperfect public monitoring, and present numerical evidence suggesting that in this context small groups can provide higher payoffs when only trigger strategies are considered; however, trigger strategies are not optimal in their model, and they do not characterize optimal equilibria. Bendor and Mookherjee (1990) ask when "multilateral" punishments, in which player i may punish j if j

cheats in her relationship with k, can improve on "unilateral" punishments, where this behavior is not present, in a repeated collective action game with perfect monitoring. They find an ambiguous relationship between group size and the maximum level of cooperation.[6] Pecorino (1999) studies repeated public good provision with perfect monitoring. He shows that public good provision is easier in larger groups with perfect monitoring, because the cost of defecting—and thereby inducing everyone else to stop contributing to the public good—is larger in larger groups. Haag and Lagunoff (2007) study repeated collective action games with perfect monitoring when players have heterogeneous discount factors. Restricting attention to stationary equilibria, they show that the maximum level of cooperation is increasing in group size, a result that in their model depends on heterogeneous discounting.[7] Both Pecorino (1999) and Haag and Lagunoff (2007) suggest in their conclusions that imperfect monitoring might lead to worse public good provision in large groups, but do not pursue this possibility in their papers. None of these papers characterize the maximum level of public good provision in games with imperfect monitoring.[8]

Finally, this paper is a contribution to the study of repeated games with private monitoring. The special "all-or-nothing" monitoring structure we consider allows us to characterize efficient equilibria at fixed discount factors under monitoring that is neither almost public nor almost perfect, whereas

[6] Bendor and Mookherjee (1990) can be seen as a precursor to the recent literature on sustaining cooperation on networks—these papers tend to use "all-or-nothing" monitoring, like the current paper, but tend not to focus on comparative statics and often restrict attention to specific classes of strategy profiles. Papers in this literature include Ali and Miller (2008), Ambrus, Möbius, and Szeidl (2008), Bloch, Genicot, and Ray (2008), Fainmesser (2009), Haag and Lagunoff (2006), Karlan et al (2008), Kinateder (2008a, 2008b), Lippert and Spagnolo (2008), Mihm, Toth, and Lang (2009) and Vega-Redondo (2006).

[7] We show that stationary equilibria support the maximum level of public good provision in our model if monitoring is stationary (Theorem 1); thus, we do not restrict attention to stationary equilibria a priori.

[8] Somewhat less closely related to the current paper is the literature on dynamic (non-repeated) public good provision and common resource exploitation (e.g., Bliss and Nalebuff, 1984; Admati and Perry, 1991; Fershtman and Nitzan, 1991; Benhabib and Radner, 1992; Gradstein, 1992; Marx and Matthews, 2000; Lockwood and Thomas, 2002; Compte and Jehiel, 2003, 2004; Bonatti and Hörner, 2009). These papers focus on non-stationary environments in which the current level of contributions to the public good affects the returns of future contributions. Most of these papers do not study the effect of group size and structure on the provision of public goods—for example, Admati and Perry (1991), Benhabib and Radner (1992), Lockwood and Thomas (2002), and Compte and Jehiel (2003, 2004) consider two-player games. Bliss and Nalebuff (1984) provide conditions under which expected delay in public good provision decreases in group size in a war-of-attrition-like model; Fershtman and Nitzan (1991) present a model of dynamic public good provision in which provision is increasing in group size; and Bonatti and Hörner (2009) consider dynamic collaboration on a project of uncertain promise, and show that aggregate effort is independent of group size, though expected delay is increasing in group size.

most of the papers in this literature focus on proving folk theorems or studying robustness to small deviations from public monitoring. Prominent papers in this rapidly expanding literature include Compte (1998), Kandori and Matsushima (1998), Mailath and Morris (2002, 2006), Matsushima (2004), Ely, Hörner and Olszewski (2005), Hörner and Olszewski (2006, 2008), Fong et al (2007), and Yamamoto (2009).

3 Model

There are N players; we also write N for the set of players, abusively. Each period, every player i simultaneously chooses a contribution $x_i \geq 0$. If the players choose contributions $(x_1, \ldots, x_N)$ in period t, player i's period-t payoff is

$$\alpha \sum_{j=1}^{N} x_j - c(x_i),$$

where $\alpha \in (0, 1]$ and $c(\cdot)$ is a strictly increasing, convex, and twice continuously differentiable function, with $c(0) = 0$, $c'(0) \in (\alpha, \alpha N)$, and $\lim_{x \to \infty} c'(x) > \alpha N$.[9] The interpretation is that $c(x)$ is the cost of contributing x to the public good, and α is the common benefit players receive from contributions to the public good. The main role of the parameter α is to examine how the manner in which it changes with N affects comparative statics with respect to N; for example, in the leading cases of *pure public goods*, where $\alpha = 1$ for all N, and *divisible public goods*, where $\alpha = 1/N$. Note that our assumption that $c'(0) \in (\alpha, \alpha N)$ guarantees that the game is a prisoner's dilemma; in particular, setting $x_i = 0$ ("defecting") is a dominant strategy for player i in the one-shot game. Players share a common discount factor, δ. For one of our results, we assume that public randomizations are available, i.e., that the realization of a random variable $Z_t \sim U[0, 1]$ is publicly observed at the end of each period.

We consider "all-or-nothing" private monitoring, in the following sense: For all i and t, there is a set-valued random variable $O(i, t)$ such that, at the end of period t, player j observes $h_{j,t} = \{z_{j,1,t}, \ldots, z_{j,N,t}, z_t\}$, where $z_{j,i,t} = \{x_{i,t}\}$ if $j \in O(i, t)$, $z_{j,i,t} = \emptyset$ if $j \notin O(i, t)$, and $z_t \in [0, 1]$ is the outcome of the public randomizing device. The interpretation is that $O(i, t)$ is the set of players that observe player i's period t action, and that players also observe the outcome of the

[9] Some of our results concern the limit as $N \to \infty$. These results are valid only if $\lim_{x \to \infty} c'(x) = \infty$.

public randomizing device.[10,11,12] Throughout, we let $h_i^t \equiv (h_{i,0}, h_{i,1}, \ldots, h_{i,t-1})$, and denote the null history at the beginning of the game by $h^0 = h_i^0$ for all i. Player i's strategy σ_i specifies a probability distribution over period t actions as a function of h_i^t.

We call a set of outcomes of the $O(i,t)$ a *realization of the monitoring technology*, and denote such a realization by ω. Note that players never observe a set $O(i,t)$; $h_{i,t}$ is all that player i observes at the end of period t. We assume that each $O(i,t)$ is realized at the beginning of play, which implies that the probability distribution over $O(i,t)$ does not depend on the actions any players take during play. We assume throughout the paper that $\left\{\{O(i,t)\}_{i=1}^N\right\}_{t=0}^\infty$ satisfies the following two standard properties:

- *Perfect Recall*: $i \in O(i,t)$ for all i and t.
- *Stationarity*: The vectors $\{O(i,t)\}_{i=1}^N$ are iid across periods.

In Sections 4.2, 5, and 6, we also assume that the monitoring technology has the additional property of *equal observability*. To define this property, we must first introduce an important piece

[10] The public randomizing device plays a role only where explicitly mentioned in the statement of Theorem 1—it can be omitted elsewhere.

[11] Our formulation implies that monitoring is *nonanonymous*, in that player j knows which actions she observes are taken by which of her opponents. One can check, however, that our results that assume equal observability—discussed below—also apply to the case of anonymous monitoring, where player j observes a randomly ordered list of the $\{x_{i,t}\}_{i:j\in O(i,t)}$. The main idea is that, under equal observability, we can restrict attention to symmetric strategy profiles, and in symmetric strategy profiles a player does not need to observe her opponents' identities to know if one of them deviated.

[12] In particular, players do not observe the level of public good provision in period t, $\sum_{i=1}^N x_{i,t}$, even though their payoffs depend on $\sum_{i=1}^N x_{i,t}$. Making $\sum_{i=1}^N x_{i,t}$ observable would introduce a strong form of public monitoring, while our principal motivation is to study different kinds of private monitoring. Our model is equivalent to a model where players' payoffs depend on observable signals of $\sum_{i=1}^N x_{i,t}$ that are so noisy that they cannot practically be used to detect a deviation by a single player (which we view as a reasonable approximation to the monitoring structures in the motivating examples in the introduction). Our model also applies to the case where the level of public good provision is eventually observed perfectly, but not until all public projects are completed (in this case the infinite horizon can be interpreted as an uncertain finite horizon). For example, an individual may be unable to assess the quality of a public school until years later, when she learns whether or not her childrens' skills are valued by the labor market.

of notation: define $D(\tau, t, i)$ recursively by

$$\begin{aligned} D(\tau, t, i) &= \emptyset \text{ if } \tau < t \\ D(t, t, i) &= \{i\} \\ D(\tau+1, t, i) &= D(\tau, t, i) \cup \{j : j \in O(k, \tau) \text{ for some } k \in D(\tau, t, i)\} \text{ if } \tau > t. \end{aligned}$$

That is, $D(\tau, t, i)$ is the set of players in period τ who have observed a player who has observed a player who has observed... player i since time t. The set will be important for our analysis because $j \in D(\tau, t, i)$ is a necessary condition for player j's time τ history to vary with player i's actions starting at time t. For example, if players are using grim trigger strategies and player i defects at time t, then $D(\tau, t, i)$ is the set of players who defect at time τ. Note that stationarity implies that the probability distribution of $D(\tau, t, i)$ is the same as the probability distribution of $D(\tau - t, 0, i)$, for all i, t, and τ. Thus, when we consider the consequences of an initial deviation by player i at some time that is clear from context, we will sometimes simplify notation by writing $D(\tau, i)$ for the set of players who may learn about the deviation within τ periods.

We use this simplified notation to define equal observability:

- *Equal Observability*: $\mathbb{E}[\#D(\tau, j)] = \mathbb{E}[\#D(\tau, k)]$ for all j, k, and τ.

Equal observability requires that the same expected number of players may be influenced by player j's action within τ periods as may be influenced by player k's action within τ periods; this is a weak way of saying that all players are monitored equally well.

Our assumptions on the monitoring technology are satisfied by many important examples. All the examples we consider in this paper satisfy perfect recall and stationarity. Monitoring on an arbitrary, fixed network, where each player observes her neighbors' actions every period and nothing else, is the only case we consider that may not satisfy equal observability; this case is discussed in Section 7. In Section 5, we examine comparative statics with respect to N in four examples that satisfy perfect recall, equal monitoring, and stationarity: uniform monitoring, quasi-public monitoring, random matching, and monitoring on a circle.[13] To fix ideas, we also note an important monitoring technology that does *not* satisfy our assumptions: suppose that players observe only the actions of their neighbors on a random graph that is determined at the beginning of the game and then fixed for the duration of play. This monitoring technology violates stationarity,

[13] See the Introduction for informal definitions and Section 5 for formal ones.

since player i is sure to observe player j's action in period 1 if she observes it in period 0, but observes player j's action in period 0 only if they are linked in the realized graph.

Throughout, we study sequential equilibria (SE) of this model with the property that $\mathbb{E}\left[\sum_{t=0}^{\infty}\delta^t\sigma_i\left(h_i^t\right)\right] < \infty$ for all i. This technical restriction is needed to ensure that payoffs are well-defined. In particular, we will be interested in the highest expected discounted level of public good provision in any SE, which we call the *maximum equilibrium level of public good provision (MELP)*:[14]

Definition 1 *Let Σ_{SE} be the set of SE. The* MELP *is*

$$X^* \equiv \sup_{\sigma\in\Sigma_{SE}} \alpha\left(1-\delta\right)\mathbb{E}\left[\sum_{t=0}^{\infty}\delta^t\sum_{i=1}^{N}\sigma_i\left(h_i^t\right)\right].$$

A strategy profile σ sustains *the MELP if $\sigma\in\Sigma_{SE}$ and $X^*=\alpha\left(1-\delta\right)\mathbb{E}\left[\sum_{t=0}^{\infty}\delta^t\sum_{i=1}^{N}\sigma_i\left(h_i^t\right)\right]$.*

For any strategy profile σ, we refer to $\alpha\left(1-\delta\right)\mathbb{E}\left[\sum_{t=0}^{\infty}\delta^t\sum_{i=1}^{N}\sigma_i\left(h_i^t\right)\right]$ as the corresponding *level of public good provision.*

For future reference, we also define the maximum equilibrium contribution of an individual player:

Definition 2 *Player i's* maximum equilibrium contribution *is*

$$\hat{x}_i \equiv \sup_{\sigma\in\Sigma_{SE}} \left(1-\delta\right)\mathbb{E}\left[\sum_{t=0}^{\infty}\delta^t\sigma_i\left(h_i^t\right)\right].$$

A strategy profile σ sustains *player i's maximum equilibrium contribution if $\sigma\in\Sigma_{SE}$ and $\hat{x}_i = \left(1-\delta\right)\mathbb{E}\left[\sum_{t=0}^{\infty}\delta^t\sigma_i\left(h_i^t\right)\right]$.*

Our main result (Theorem 1) will show that there exists a strategy profile that simultaneously sustains each player's maximum equilibrium contribution (and thus also sustains the MELP). When $\hat{x}_i$ is less than the first-best level of x_i (given by $c'\left(x_i\right)=\alpha N$) for all i, it follows that this strategy profile also maximizes utilitarian social welfare. Therefore, all of our results regarding the MELP

[14] This concept is similar to the "maximal average cooperation" (MAC) studied by Haag and Lagunoff (2007) in games with perfect monitoring and heterogeneous discounting. However, it is unclear how to formulate the MELP with heterogeneous discounting (this issue does not arise in Haag and Lagunoff's paper, as they restrict attention to stationary equilibria). More importantly, our techniques and results are very different from Haag and Lagunoff's, as our focus is on changes in group size and structure under imperfect monitoring, while theirs is on heterogeneous discounting under perfect monitoring.

can also be regarded as results regarding welfare, as long as maximum equilibrium contributions are below the first-best level.[15]

A preliminary observation is that the MELP is finite. This follows because it is impossible to give all players non-negative payoffs when the level of public good provision is too high, by the assumption that $\lim_{x\to\infty} c'(x) > N$.

Lemma 1 *The MELP is finite.*

Proof. Fix a SE σ, and let $X = \alpha(1-\delta)\mathbb{E}\left[\sum_{t=0}^{\infty}\delta^t\sum_{i=1}^{N}\sigma_i\left(h_i^t\right)\right]$. Then the sum of the players' payoffs under σ equals

$$NX - (1-\delta)\sum_{t=0}^{\infty}\delta^t\sum_{i=1}^{N}\mathbb{E}\left[c\left(\sigma_i\left(h_i^t\right)\right)\right].$$

By convexity of $c(\cdot)$ and Jensen's inequality, this is no more than

$$N\left(X - c\left(\frac{X}{\alpha N}\right)\right).$$

Since $\lim_{x\to\infty} c'(x) > \alpha N$, there exists $\bar{X}$ such that $N(X - c(X/\alpha N)) < 0$ for all $X \geq \bar{X}$. Since σ is a SE, each player must receive a non-negative payoff under σ (as each player's minmax payoff is 0), so the sum of the players' payoffs must be non-negative under σ. Therefore, it must be the case that $X \leq \bar{X}$. By definition of the MELP, this implies that $X^* \leq \bar{X}$. ■

4 General Results

4.1 Characterizing the Maximum Equilibrium Level of Public Good Provision

In this section, we show that the MELP can be sustained by the natural generalization of grim trigger strategies. If monitoring satisfies equal observability, then the MELP can be sustained in symmetric grim trigger strategies. Furthermore, all strategy profiles that sustain the MELP have the same equilibrium path of play, so the strategy profile that sustains the MELP is essentially unique.

We begin by defining grim-trigger strategies and symmetric grim-trigger strategies in our environment:

[15] Theorem 1 also shows that the first-best level of public good provision is sustainable in SE even if the MELP is greater than the first-best level. However, in this case it may not be possible to provide the first-best level of public good provision in the least-cost way (i.e., through equal contributions of all players).

Definition 3 *A strategy profile σ is a* grim trigger strategy profile *if there exist contribution levels $\{x_i^*\}_{i=1}^N$ such that $\sigma_i\left(h_i^t\right)=0$ if player i has ever observed a player j contribute $x_j \neq x_j^*$ at h_i^t, and $\sigma_i\left(h_i^t\right)=x_i^*$ otherwise. A grim trigger strategy profile σ is* symmetric *if there exists x^* such that $x_i^*=x^*$ for all i.*

Note that in a grim trigger strategy profile player i's action at an off-path history h_i^t does not depend on the identity of the initial deviator. In particular, since monitoring satisfies perfect recall, player i sets $x_i=0$ in every period following a deviation by player i herself. Note also that in a grim trigger strategy profile a player makes the same contribution when she sees an opponent make her prescribed contribution and when she does not observe the opponent's action; thus, players do not receive "rewards" when they are seen making their prescribed contributions.

We are now ready to present our first result, which says that the MELP can be sustained by grim trigger strategies, that contribution levels in the grim trigger strategy profile that sustains the MELP are given by a simple fixed point condition, and that any level of public good provision below the MELP can be sustained in sequential equilibrium with public randomizations.

Theorem 1 *There exists a unique grim trigger strategy profile σ^* that sustains the MELP, and any strategy profile that sustains the MELP has the same equilibrium path of play as σ^*. σ^* also sustains each player's maximum equilibrium contribution. That is, if player i's on-path contribution under σ^* is x_i^*, then*

$$X^*=\alpha \sum_{i=1}^N x_i^*,$$

and $x_i^*=\hat{x}_i$ for all i. Furthermore, $\{x_i^*\}_{i=1}^N$ is the (component-wise) greatest vector such that

$$c\left(x_i^*\right)=\alpha(1-\delta) \sum_{t=0}^{\infty} \delta^t \sum_{j=1}^N \operatorname{Pr}(j \in D(t, i))\, x_j^* \tag{1}$$

for all i.

Finally, any level of public good provision $X<X^*$ can be sustained in sequential equilibrium, if public randomizations are available.

Theorem 1 is intuitive: one's first thought might be that grim trigger strategies sustain the highest level of public good provision (and the highest equilibrium contribution for each player), because they provide the harshest possible punishment to deviators consistent with deviators understanding how information spreads through the population. However, the theorem is not obvious, and the proof is not trivial. There are two important issues to consider: First, do grim trigger strategies actually provide the strongest incentives to contribute to the public good, or should players receive "rewards" when they seen making high contributions? Second, might grim trigger strategies provide "too strong" incentives to contribute, thus making players unwilling to stop contributing when they observe a deviation? We discuss these issues in turn, and in the process sketch the proof of Theorem 1, the details of which are deferred to the appendix.

We prove Theorem 1 using a novel fixed point approach. The key idea is that a player is willing to contribute more at any on-path history if another player contributes more at any on-path history, because the first player is more likely to benefit from this increased contribution when she conforms than when she deviates. This observation relies on the assumption of all-or-nothing monitoring, since otherwise a deviation by the first player may make some on-path histories more likely. To see how the fixed point approach works, fix a vector of "continuation contributions" $\sum_{\tau=t}^{\infty} \delta^{\tau-t} \mathbb{E}\left[\sigma_j\left(h_j^\tau\right)\middle| h_j^t; \sigma_j\right]$ starting from each (private) on-path history of each player. Let the function ϕ map this vector to the highest continuation contribution that each player is willing to make (at each of her on-path histories) when her opponents make the given continuation contributions. Crucially, the function ϕ is isotone, as per the above observation.[16] As in Lemma 1, there is a level of continuation contribution $\bar{X}$ such that no player's continuation contribution ever exceeds $\bar{X}$ in any sequential equilibrium. Letting $\bar{\mathbf{X}}$ be the vector of contributions $\bar{X}$, $\phi\left(\bar{\mathbf{X}}\right)$ is weakly greater than the highest fixed point of ϕ and is stationary, since monitoring satisfies stationarity. Iterating ϕ on $\bar{\mathbf{X}}$ yields a sequence of vectors of (on-path) contributions that is stationary and weakly greater than the highest fixed point of ϕ at every step, and this sequence converges to the highest fixed point of ϕ. Therefore, the highest fixed point of ϕ is stationary, and it provides an upper bound on the MELP. Since the highest fixed point of ϕ turns out to describe a path of play of a sequential equilibrium (as we discuss in the next paragraph), the path of play of any strategy profile that sustains the MELP must coincide with the (unique) highest fixed point of ϕ.[17] Finally, observe that a player i's incentive to contribute would increase if her opponents'

[16] The reason why we must work with continuation contributions following each history rather than with stage-game contributions is that ϕ would not be isotone if it were defined over the vector of all players' stage-game contributions. This is because a player is not willing to contribute as much today when she expects to contribute more in the future.

[17] The map ϕ is similar to an isotone best response correspondence in a supermodular game, and the approach of iterating ϕ on $\bar{\mathbf{X}}$ to find its highest fixed point is related to the proof of Theorem 5 of Milgrom and Roberts (1990). One important difference is that our model is dynamic and we show that ϕ preserves stationarity. We also note that the iterative procedure described in the text gives a simple and applicable method of computing the grim trigger

contributions were "transferred" from the state in which player i's action is unobserved to the state in which player i is observed to take her prescribed action, but since *every* player is already making her maximum sequentially rational contribution at every on-path history the resulting strategy profile would not satisfy sequential rationality for player i's opponents. This is why strategies that "reward" players when they are seen making their prescribed contributions cannot sustain a higher level of public good provision than can grim trigger strategies.

The discussion of Theorem 1 so far has focused on on-path incentive constraints. As we have suggested, one might be concerned that grim trigger strategies do not satisfy off-path incentive constraints, as a player might want to contribute off-path in order to slow the "contagion" of defecting, as in Kandori (1992) and Ellison (1994). This concern does not apply to strategies that sustain the *maximum* level of contribution (i.e., the highest fixed point of ϕ), however, as under such strategies players must be just indifferent between making their prescribed contributions and not contributing on the equilibrium path, which—by virtually the same argument as in Ellison's paper—implies that they weakly prefer not to contribute off-path. The fact that levels of provision below X^* are also sustainable in sequential equilibrium when public randomizations are available follows from considering two-phase "relenting" strategies as in Ellison's paper.

Finally, once it is established that the MELP can be sustained in a grim trigger strategy profile σ^*, the observation that the vector of on-path contributions $\{x_i^*\}_{i=1}^N$ is the greatest vector that satisfies (1) is intuitive. The left-hand side of (1) is the (per period) cost to player i from conforming to σ^*. The benefit to player i from conforming to σ^* is that, if player i deviated, every player j (including player i herself) would stop contributing to the public good as soon as she found out about the deviation, which occurs as soon as she enters the set $D(t,i)$ (since information about a deviation spreads according to $D(t,i)$ when players use grim trigger strategies). Thus, the right-hand side of (1) is the discounted benefit to player i from conforming to σ^*. The highest vector $\{x_i^*\}_{i=1}^N$ that equalizes costs and benefits for each player sustains the MELP.

We now show that the MELP can be sustained in symmetric grim trigger strategies if monitoring satisfies equal observability, and that the converse holds for generic discount factors (unless the MELP equals 0, in which case the grim trigger strategy profile that sustains the MELP is trivially symmetric). The proof that equal observability is sufficient for symmetry follows easily from the proof of Theorem 1 and is deferred to the appendix, so only the proof that equal observability is necessary for symmetry for generic discount factors is provided in the text.

strategy profile that sustains the MELP.

Theorem 2 *If monitoring satisfies equal observability, then there exists a unique symmetric grim trigger strategy profile σ^* that sustains the MELP, and all strategy profiles that sustain the MELP have the same equilibrium path of play as σ^*. That is, if x^* is the on-path contribution in σ^*, then*

$$X^* = \alpha N x^*,$$

and $x^* = \hat{x}_i$ for all i. If monitoring does not satisfy equal observability, then the set of discount factors in $[0,1]$ for which the MELP is positive and the grim trigger strategy profile σ^* that sustains the MELP is symmetric has Lebesgue measure 0.

Proof. Suppose that monitoring does not satisfy equal observability, i.e., that there exist players i and j and an integer t such that $\mathbb{E}[\#D(t,i)] \neq \mathbb{E}[\#D(t,j)]$. If σ^* is symmetric, (1) becomes

$$c(x^*) = \alpha(1-\delta)\sum_{t=0}^{\infty}\delta^t \sum_{j=1}^{N} \Pr(j \in D(t,i))\, x^*,$$

which can be rewritten as

$$c(x^*) = \alpha(1-\delta)\sum_{t=0}^{\infty}\delta^t \mathbb{E}[\#D(t,i)]\, x^*.$$

For this equality to hold for all i, it must hold for i and j. If the MELP is positive ($x^* > 0$), this implies that

$$\sum_{t=0}^{\infty}\delta^t\left(\mathbb{E}[\#D(t,i)] - \mathbb{E}[\#D(t,j)]\right) = 0. \tag{2}$$

Note that $\mathbb{E}[\#D(0,i)] = \mathbb{E}[\#D(0,j)] = 1$, so $\mathbb{E}[\#D(t,i)] \neq \mathbb{E}[\#D(t,j)]$ for some $t \geq 1$. Therefore, the left-hand side of (2) is a power series in δ with a nonzero coefficient on δ^t for some $t \geq 1$. The set of zeros of such a power series has Lebesgue measure 0, by Sard's Theorem, so (2) can only hold for a set of discount factors of Lebesgue measure 0. Therefore, the set of discount factors for which the MELP is symmetric has Lebesgue measure 0. ■

Before leaving this section, we impose an assumption that guarantees that $\hat{x}_i > 0$. We impose this assumption *for the remainder of the paper.*

Assumption 1 $\alpha(1-\delta)\sum_{t=0}^{\infty}\delta^t\sum_{j=1}^{N}\Pr(j \in D(t,i)) > c'(0)$ for all i.

For any $c'(0) \in (\alpha, \alpha N)$, Assumption 1 holds for high enough δ if $\Pr(j \in D(t,i)) \to 1$ as $t \to \infty$ for all i, j, which says that almost surely player j *eventually* observes a player who has observed a player who has observed... player i. This is a fairly weak condition; for example, it is easy to check that $\Pr(j \in D(t,i)) \to 1$ as $t \to \infty$ for all i, j in all of the examples in Section 5. Assumption 1 is important for our results only because it allows us to make statements about *strict*, rather than weak, comparative statics on the MELP and on players' maximum equilibrium contributions; the issue is that, in the absence of Assumption 1, strict comparisons between two games may not be possible because $\hat{x}_i$ may equal 0 for all i in both games. The weak versions of all of our comparative static results continue to hold in the absence of Assumption 1. We now show why Assumption 1 guarantees that $\hat{x}_i > 0$.

Corollary 1 *$\hat{x}_i > 0$ for all i if Assumption 1 holds.*

Proof. Let $\vec{x} = \{x_i\}_{i=1}^N$, and define the map $\tilde{\phi}(\vec{x}) : \{x_i\}_{i=1}^N \to \{x_i\}_{i=1}^N$ (similar to the map $\phi\left(\vec{X}\right)$ defined in the proof of Theorem 1) by letting $\tilde{\phi}_i(\vec{x})$ be the unique x_i' such that

$$c\left(x_i'\right) = \alpha(1-\delta)\sum_{t=0}^{\infty}\delta^t\sum_{j=1}^{N}\Pr(j \in D(t,i))\,x_j. \quad (3)$$

By the same argument as in the proof of Theorem 1, $\tilde{\phi}$ is isotone, and the highest fixed point of $\tilde{\phi}$ is the unique vector of on-path contributions that sustain the MELP, $\{x_i^*\}_{i=1}^N$.

Suppose that Assumption 1 holds. We claim that there exists $x > 0$ such that $\tilde{\phi}_i(x,\ldots,x) > x$ for all i, where $\tilde{\phi}_i$ is the i^{th} coordinate of $\tilde{\phi}$. To see this, note that $\tilde{\phi}_i(0,\ldots,0) = 0$, and, by (3), the derivative of $\tilde{\phi}_i(x,\ldots,x)$ with respect to x is

$$\frac{d}{dx}c^{-1}\left(\alpha(1-\delta)\sum_{t=0}^{\infty}\delta^t\sum_{j=1}^{N}\Pr(j \in D(t,i))\,x\right) = \frac{\alpha(1-\delta)\sum_{t=0}^{\infty}\delta^t\sum_{j=1}^{N}\Pr(j \in D(t,i))}{c'\left(c^{-1}\left(\alpha(1-\delta)\sum_{t=0}^{\infty}\delta^t\sum_{j=1}^{N}\Pr(j \in D(t,i))\,x\right)\right)}.$$

For small enough x, this derivative is greater than 1, by Assumption 1 and the facts that $c(0) = 0$ and $c(\cdot)$ is increasing and continuously differentiable. Therefore, there exists $x > 0$ such that $\tilde{\phi}_i(x,\ldots,x) > x$ for all i. Since $\tilde{\phi}$ is isotone, this implies that $(x,\ldots,x)$ is strictly lower than the highest fixed point of $\tilde{\phi}$, which in turn implies that $\hat{x}_i = x_i^* > x > 0$ for all i. ■

4.2 Comparative Statics on the MELP Under Equal Observability

Using the characterization of strategies that sustain the MELP developed in the previous subsection, we now characterize comparative statics on the MELP, under the assumption of equal observability. We take up some related issues in our analysis of public good provision on networks in Section 7, where equal observability may not hold.

Formally, let a *game*, Γ, denote a group size $N(\Gamma)$, a benefit from contributions $\alpha(\Gamma)$, and a probability distribution over realizations of the monitoring technology.[18] Our main result in this section describes when the MELP is higher in a game Γ than in another game Γ', when both Γ and Γ' satisfy equal observability. Note that Γ contains all information about both group size (N) and group structure (the distribution over realizations of the monitoring technology).

Assume that a game Γ satisfies equal observability—we impose this condition until Section 7. Let σ^* be the symmetric grim trigger strategy profile sustaining the MELP, which exists and is unique by Theorem 2, and let x^* be the corresponding individual contribution level. Let $D(t,i,\Gamma)$ be the set $D(t,i)$ in game Γ. By stationarity and equal observability, $\mathbb{E}[\#D(t,i,\Gamma)]$ does not depend on the identity of the initial deviator i, so we economize on notation by letting $\mathbb{E}[\#D(t,\Gamma)] \equiv \mathbb{E}[\#D(t,1,\Gamma)]$. With this notation, equation (1) and symmetry imply that the maximum *per capita* level of public good provision, $X^*/(\alpha N)$,[19] (which equals each player's maximum equilibrium contribution) is the highest value of x such that

$$c(x) = \alpha(\Gamma)(1-\delta)\sum_{t=0}^{\infty}\delta^t \mathbb{E}[\#D(t,\Gamma)]\,x. \tag{4}$$

Call the highest solution of (4) $x^*(\Gamma)$. Note that $x^*(\Gamma)$ is the highest zero of $\alpha(\Gamma)(1-\delta)\sum_{t=0}^{\infty}\delta^t\mathbb{E}[\#D(t,\Gamma)]\,x - c(x)$, which is concave in x. Therefore, if $x^* > 0$, then $x^*(\Gamma') > x^*(\Gamma)$ if $\alpha(\Gamma')(1-\delta)\sum_{t=0}^{\infty}\delta^t\mathbb{E}[\#D(t,\Gamma')]\,x - c(x) > \alpha(\Gamma)(1-\delta)\sum_{t=0}^{\infty}\delta^t\mathbb{E}[\#D(t,\Gamma)]\,x - c(x)$ for all x. This observation yields our main comparative statics result:

Theorem 3 *Let Γ' and Γ be two games. Then $x^*(\Gamma') > x^*(\Gamma)$ if and only if*

$$\alpha(\Gamma')\sum_{t=0}^{\infty}\delta^t\mathbb{E}\left[\#D\left(t,\Gamma'\right)\right] > \alpha(\Gamma)\sum_{t=0}^{\infty}\delta^t\mathbb{E}[\#D(t,\Gamma)].$$

The proof of Theorem 3 parallels the above discussion and is deferred to the appendix.

Theorem 3 gives a complete characterization of when $x^*(\Gamma)$ is higher or lower than $x^*(\Gamma')$, for any two games Γ and Γ'. Thus, it shows that all the information needed to determine whether changing Γ increases or decreases the maximum per capita level of public good provision is contained in the product of two terms: the "rivalness" term $\alpha(\Gamma)$ and the "effective contagiousness"

[18] We could also let the discount factor, δ, differ across games, but since none of our applications involve changes in δ we omit this possibility to simplify notation.

[19] Note that the maximum per capita level of public good provision is $X^*/(\alpha N)$, not X^*/N. Thus, the maximum per capita level of public good provision measures contributions, while the MELP measures the benefit players receive from these contributions.

term $\sum_{t=0}^{\infty} \delta^t \mathbb{E}[\#D(t, \Gamma)]$. Information such as group size, higher moments of the distribution of $\#D(t, \Gamma)$, and which players are more likely to observe which other players are all irrelevant, if the rivalness and effective contagiousness terms are held fixed. In particular, the single number $\sum_{t=0}^{\infty} \delta^t \mathbb{E}[\#D(t, \Gamma)]$—the effective contagiousness—completely determines the effectiveness of a given monitoring structure at supporting cooperation in sequential equilibrium.

The finding that comparative statics are determined by the product of the rivalness and effective contagiousness terms yields useful intuitions about the effect of group size on the maximum per capita level of public good provision. In particular, consider indexing a game Γ by its group size, N, and write $\alpha(N)$ for the corresponding benefit from contributions (we will use this simpler notation for the remainder of this section and for Section 5). Normally, one would expect the rivalness term to decrease in N (i.e., a larger population reduces i's benefit from j's contribution to the public good) and the effective contagiousness terms term to increase in N (i.e., a larger population makes it more likely that i's action is observed by more people). The interaction of these terms determines the group size that maximizes per capita public good provision. Consider again the example of constructing a local infrastructure project, like a well. In this case, $\alpha(N)$ is likely to be decreasing, and, over a range, concave: as each individual uses the well only occasionally, there are few externalities among the first few individuals, but once the population reaches a certain size it starts to becomes difficult to find times when the well is available, and water shortages start to become a problem. Similarly, $\sum_{t=0}^{\infty} \delta^t \mathbb{E}[\#D(t, N)]$ is likely to be increasing, and may be concave, if there are "congestion" effects in monitoring. Thus, it seems likely that in many applications $\alpha(N) \sum_{t=0}^{\infty} \delta^t \mathbb{E}[\#D(t, N)]$, and therefore the maximum per capita level of public good provision, is maximized at an intermediate value of N.

Finally, we can use Theorem 3 to easily derive particularly simple comparative statics results for the leading cases of pure public goods ($\alpha(N) = 1$) and divisible public goods ($\alpha(N) = 1/N$), which will be useful for the applications considered in Section 5. Note that in the case of pure public goods the MELP equals $Nx^*(N)$, while in the case of divisible public goods the MELP simply equals $x^*(N)$.

Corollary 2 *If the public good is pure ($\alpha(N) = 1$), then $x^*(N)$ is strictly increasing if $\mathbb{E}[\#D(t, N)]$ is non-decreasing in N for all t and strictly increasing in N for some t.*

For pure public goods, $x^*(N)$ is always increasing unless monitoring degrades so quickly as N increases that the expected *number* of players who find out about a deviation within t periods is *decreasing* in N, for some t. This suggests that both $x^*(N)$ and the MELP are increasing in N in a wide range of applications, which is consistent with the examples in Section 5.

Corollary 3 *If the public good is divisible ($\alpha(N) = 1/N$), then $x^*(N)$ is strictly increasing if $\mathbb{E}[\#D(t, N)]/N$ is non-decreasing in N for all t and strictly increasing in N for some t.*

For divisible public goods, $x^*(N)$ (which equals the MELP) is increasing only if the expected *fraction* of players who observe a deviation within t periods of its occurrence is non-decreasing in N, for all t. This suggests that, for divisible public goods, $x^*(N)$ is decreasing in many applications, which is again consistent with our findings in Section 5.

5 Comparing Group Sizes in Examples with Equal Observability

In this section, we examine the effect of group size on the maximum per capita level of public good provision, $x^*(N)$, for both pure and divisible public goods under four different monitoring technologies satisfying stationarity and equal monitoring: uniform monitoring, quasi-public monitoring, random matching, and monitoring on a circle. We are interested in each of these monitoring technologies in its own right and also in demonstrating the usefulness of the results of Section 4 more generally. With pure public goods, $x^*(N)$ is increasing in N for all of these technologies, and is increasing in N under quasi-public monitoring even if the probability of observing a deviation is declining in N, so long as it is not declining faster than $1/N$. With divisible public goods, $x^*(N)$ may be increasing or decreasing in N under uniform monitoring, and is decreasing in N under quasi-public monitoring, random matching, and monitoring on a circle. We also examine the behavior of $x^*(N)$ in large groups ($N \to \infty$) where possible; recall that for all results about limits as $N \to \infty$, we are assuming that $\lim_{x\to\infty} c'(x) = \infty$. For example, we consider whether the effect of increasing group size on $c(x^*(N))$ vanishes in large groups.[20] To keep the exposition compact, most of the proofs in this section are deferred to the appendix.

5.1 Uniform Monitoring

Definition 4 *Monitoring is* uniform *if there exists $p \in (0, 1]$ such that $j \in O(i, t)$ with probability p, independently across i, j, t.*

[20] This is a more natural question than that of whether the effect of increasing group size on x^* vanishes in large groups, as small increases in x^* may have very large effects on $c(x^*)$ (e.g., if $c(\cdot)$ has a vertical asymptote).

5.1.1 Pure Public Goods

Proposition 1 *With uniform monitoring and pure public goods, $x^*(N)$ is strictly increasing.*

Proof. By Corollary 2, it suffices to show that $\mathbb{E}[\#D(t,N)]$ is non-decreasing in N for all t and strictly increasing in N for $t=1$. Fix $N' > N$. Parametrize the probability distribution over realizations of the monitoring technology as follows: let $\omega = \{\omega_{i,j,t}\}_{i,j,t}$, where $\omega_{i,j,t}$ is a uniform $[0,1]$ random variable, such that $i \in O(j,t)$ if and only if $\omega_{i,j,t} < p$ or $i = j$. Then $D(t,N',\omega) \supseteq D(t,N,\omega)$ for all ω, and $D(1,N',\omega) \supset D(1,N,\omega)$ with positive probability, as there is a positive probability that a player in $\{N+1,\ldots,N'\}$ observes the initial deviation in period 0. ■

In large populations with uniform monitoring, each player contributes for more than two periods after a deviation with vanishing probability, since the number of players who defect in the first period after the initial deviation goes to infinity, and each player observes each of these defections with independent probability $p > 0$. Thus, for large N, increasing N increases $(1-\delta)\sum_{t=0}^{\infty}\delta^t\mathbb{E}[\#D(t,N)]\,x - c(x)$ by a nonvanishing amount, which implies that $c(x^*)$ must increase by a nonvanishing amount. An immediate consequence of this is that $\lim_{N\to\infty} c(x^*(N)) = \infty$.

Proposition 2 *With uniform monitoring and pure public goods, $c(x^*(N+1)) - c(x^*(N))$ does not converge to 0 as N converges to ∞.*

5.1.2 Divisible Public Goods

With uniform monitoring and divisible public goods, larger groups may or may not be able to provide higher levels of public goods. To see this, first suppose that δ is close to 0, so that a player's incentive to contribute to the public good comes almost entirely through her private benefit from contributing, $1/N$.[21] Then increasing N reduces a player's incentive to contribute, and therefore reduces the maximum level of public good provision. On the other hand, suppose that δ is moderate and p is close to 0. When N is small, a deviation is almost never detected quickly enough to have substantial payoff consequences for the deviator, so the incentive to contribute to the public good comes almost entirely from private benefits, which implies that the maximum level of public good provision is very small. When N is extremely large, however, then almost all players find out about an initial deviation at the end of period 1: at the end of period 0, approximately

[21] This is consistent with our assumptions that $c'(0) > 1/N$ and $x^*(N) > 0$ if $c'(0)$ is only slightly greater than $1/N$.

proportion p of the population finds out about the deviation, but, since pN is a very large number if N is large enough, almost all players observe one of these period-1 deviators at the end of period 1. Therefore, incentives to contribute to the public good now come from community enforcement, which can support a larger level of public good provision.

Proposition 3 *With uniform monitoring and divisible public goods, there are examples in which* $x^*(N') > x^*(N)$ *and examples in which* $x^*(N') < x^*(N)$*, with* $N' > N$ *in both cases.*

5.2 Quasi-Public Monitoring

Definition 5 *Monitoring is* quasi-public *if there exists* $p(N) \in (0,1]$ *such that, with probability* $p(N)$*,* $j \in O(i,t)$ *for all* j *and, with probability* $1-p(N)$*,* $j \notin O(i,t)$ *for all* $j \neq i$*, independently across* i,t*.*

Note that we have allowed p to vary with N in the definition of quasi-public monitoring. This allows us to state more general results than if p were fixed.

5.2.1 Pure Public Goods

Due to the simple structure of quasi-public monitoring, we do not need to rely on the sufficient conditions for $x^*(N)$ to be increasing given by Corollary 2. Instead, we use Theorem 3 to establish the following result directly:

Proposition 4 *With quasi-public monitoring and pure public goods,* $x^*(N)$ *is strictly increasing (decreasing) if*

$$p(N+1) - p(N) > (<) - \left(\frac{1-\delta(1-p(N))}{1-\delta}\right)\frac{p(N+1)}{N-1} \tag{5}$$

for all N*.*

Proposition 4 gives a precise characterization of when $x^*(N)$ is increasing, and it can also give some intuition. In particular, $x^*(N)$ is increasing if $p(N)$ is non-decreasing, or if $p(N+1)-p(N)$ is sufficiently close to zero relative to $p(N+1)$. This conveys the important point that $x^*(N)$ is increasing so long as $p(N)$ does not decline too quickly. However, the exact form of (5) is not easy to interpret. This leads us to consider what happens when $p(N) = \beta N^{\zeta}$ for constants $\beta > 0$ and $\zeta \leq 0$, a natural class of functions $p(N)$. In this case, Proposition 5 gives very clean results: $x^*(N)$ is increasing if $\zeta \geq -1$ and is decreasing if N is large and $\zeta < -1$, regardless of β.

Proposition 5 *Suppose that* $p(N) = \beta N^{\zeta}$*, with* $\beta > 0$ *and* $\zeta \leq 0$*. Then* $x^*(N)$ *is strictly increasing if* $\zeta \geq -1$*, and if* $\zeta < -1$*, there exists* $\bar{N}(\zeta) > 0$ *such that* $x^*(N)$ *is strictly decreasing if* $N > \bar{N}(\zeta)$*.*

Now assume that $p(N)$ is fixed at p for all N. Then $\mathbb{E}[\#D(t, N+1)] - \mathbb{E}[\#D(t, N)]$ does not depend on N, for all t. This implies that the effect of increasing N on $c(x^*)$ does not vanish as N converges to ∞, which in turn implies that $\lim_{N\to\infty} c(x^*(N)) = \infty$.

Proposition 6 *With quasi-public monitoring and pure public goods,* $c(x^*(N+1)) - c(x^*(N))$ *does not converge to* 0 *as* N *converges to* ∞*.*

5.2.2 Divisible Public Goods

With quasi-public monitoring and divisible public goods, $x^*(N)$ is strictly decreasing in N as long as $p(N)$ is not *increasing* in N, because the expected fraction of the population that learns about a deviation within t periods does not increase in N if $p(N)$ is non-increasing. Fixing p, $x^*(N)$ may or may not converge to 0 as $N \to \infty$: it converges to 0 if and only if $c'(0) \geq \delta p / (1 - \delta(1-p))$, which holds when δ or p is small.

Proposition 7 *With quasi-public monitoring, divisible public goods, and* $p(N)$ *non-increasing in* N*,* $x^*(N)$ *is strictly decreasing in* N*.*

Proposition 8 *With quasi-public monitoring, divisible public goods, and* $p(N) \equiv p$ *for all* N*,* $x^*(N)$ *converges to* 0 *as* $N \to \infty$ *if* $c'(0) \geq \frac{\delta p}{1-\delta(1-p)}$*, and converges to a positive number otherwise.*

5.3 Random Matching

Definition 6 *Monitoring is* random matching *if in each period each player is randomly paired with another player, and* $j \in O(i,t)$ *if and only if* i *and* j *are paired at* t*.*

5.3.1 Pure Public Goods

With random matching and pure public goods, $x^*(N)$ is increasing in N, even though the probability that player i monitors player j in period t is decreasing in N, for any fixed i and j. This follows because $\mathbb{E}[\#D(t,N)]$ is increasing in N, for $t \geq 2$, as defectors are less likely to match with each other in a larger population.

Proposition 9 *With random matching and pure public goods, $x^* (N)$ is strictly increasing.*

As $N \to \infty$, $c (x^* (N))$ remains bounded if δ is less than 1/2—in contrast to the cases of uniform and quasi-public monitoring—but converges to ∞ if δ is greater than 1/2. This follows because the expected number of defectors t periods after an initial deviation is approximately 2^t when N is large, so whether effective contagiousness converges or diverges as $N \to \infty$ depends on whether δ is greater than or less than 1/2.

Proposition 10 *With random matching and pure public goods, $c (x^* (N))$ converges to a finite number as $N \to \infty$ if $\delta < \frac{1}{2}$, and converges to ∞ as $N \to \infty$ if $\delta \geq \frac{1}{2}$.*

5.3.2 Divisible Public Goods

With random matching and divisible public goods, if $\delta < 1/2$, we can establish that the maximum level of public good provision is decreasing in N (as long as the change in group size under consideration is sufficiently large) and converges to 0 as $N \to \infty$. The proof of Proposition 11 relies directly on Theorem 3 (i.e., on showing that $\sum_{t=0}^{\infty} \delta^t \mathbb{E} [\#D (t, N)] /N$ is decreasing).

Proposition 11 *With random matching and divisible public goods, if $\delta < \frac{1}{2}$ then, for any $\gamma > 0$, there exists $\bar{N}$ such that $x^* (N') < x^* (N)$ if $N' > (1 + \gamma) N \geq \bar{N}$.*

Proposition 12 *With random matching and divisible public goods, $x^* (N) \to 0$ as $N \to \infty$ if $\delta < \frac{1}{2}$.*

The $\delta \geq 1/2$ case presents technical complications and is omitted.

5.4 Monitoring on a Circle

Definition 7 *Monitoring is* on a circle *if the players are arranged in a fixed circle and there exists an integer $k \geq 1$ such that $j \in O (i, t)$ if and only if the distance between i and j is at most k.*

5.4.1 Pure Public Goods

Under monitoring on a circle, that $x^* (N)$ is increasing in N follows immediate from Theorem 3.

Proposition 13 *With monitoring on a circle and pure public goods, $x^* (N)$ is strictly increasing.*

Proof. $\mathbb{E}[\#D(t,N)] = 1+2kt$ if $2kt < N$, and $\mathbb{E}[\#D(t,N)] = N$ if $2kt \geq N$, so $\sum_{t=0}^{\infty}\delta^t\mathbb{E}[\#D(t,N)]$ is strictly increasing in N. The result follows from Theorem 3. ∎

With monitoring on a circle, $c(x^*(N))$ remains bounded as $N \to \infty$. To see why this is true, note that, from the perspective of a potential deviator, increasing N is equivalent to adding players on the opposite side of the circle from the potential defector, as $\#D(t,N+1) = \#D(t,N)$ if $\#D(t,N) < N$. And, when N is large, it takes a long time for a player on the other end of the circle to find out about an initial defection, so increasing N by 1 has a vanishing effect on $\sum_{t=0}^{\infty}\delta^t\mathbb{E}[\#D(t,N)]$.

Proposition 14 *With monitoring on a circle and pure public goods, $x^*(N)$ converges to a finite number as $N \to \infty$.*

5.4.2 Divisible Public Goods

With monitoring on a circle and divisible public goods, the maximum level of public good provision is strictly decreasing in group size and converges to 0 as $N \to \infty$. The intuition for this is simply that the fraction of players who find out about an initial deviation within t periods of its occurrence goes to 0 as $N \to \infty$, for any t, and the results follow from Theorem 3.

Proposition 15 *With monitoring on a circle and divisible public goods, $x^*(N)$ is strictly decreasing.*

Proof. $\mathbb{E}[\#D(t,N)]/N = (1+2kt)/N$ if $2kt < N$, and $\mathbb{E}[\#D(t,N)]/N = 1$ if $2kt \geq N$, so $\sum_{t=0}^{\infty}\delta^t\mathbb{E}[\#D(t,N)]/N$ is strictly decreasing in N. The result follows from Theorem 3. ∎

Proposition 16 *With monitoring on a circle and divisible public goods, $x^*(N) \to 0$ as $N \to \infty$.*

6 Comparing Monitoring Structures with Equal Observability

In this section, we present a general result comparing group structures in terms of the maximum level of public good provision they support, for a fixed group size. We continue to restrict attention to monitoring structures satisfying stationarity and equal observability. By Theorem 3, such a monitoring structure supports a higher level of public good provision if and only if it corresponds to a higher level of $\sum_{t=0}^{\infty}\delta^t\mathbb{E}[\#D(t)]$ (i.e., higher expected contagiousness), where we have omitted $\alpha(N)$ and N because we are now considering comparisons across monitoring structures for fixed

N. Our main motivation is to compare "less reliable, more public monitoring," where it is likely that either a very large or very small fraction about the population finds out about a deviation, with "more reliable, less public monitoring," where it is more certain that an intermediate fraction of the population finds out about it. The latter kind of monitoring is "less uncertain," in that the distribution of the number of individuals who observe player i's action each period is second-order stochastically dominant. We find that, under fairly broad conditions, less uncertain monitoring structures support higher levels of public good provision.

As our focus in this section is on the distribution of the *number* of individuals who observe a deviation, we restrict attention to monitoring structures in which the probability that a given number of individuals observe a defection in period t depends only on the number of defectors in period t:

Assumption 2 There exists a family of functions $\{g_k(\cdot)\}_{k=1}^{N}$ such that, whenever $\#D(t) = k$, $\Pr(\#D(t+1) = k') = g_k(k')$, for all t, k, and k'.

Note that Assumption 2 is satisfied for uniform monitoring, quasi-public monitoring, and random matching, but not for monitoring on a circle, as under monitoring on a circle $\Pr(\#D(t+1) = k')$ depends on the location of the k time-t defectors.

Given a probability mass function $g_k(k')$, denote the corresponding distribution function by $G_k(k') \equiv \sum_{s=0}^{k'} g_k(s)$. Recall that a distribution $\tilde{G}_k$ *strictly second-order stochastically dominates* G_k if $\sum_{s=0}^{N} \eta(s)\,\tilde{g}_k(s) > \sum_{s=0}^{N} \eta(s)\,g_k(s)$ for all non-decreasing and strictly concave $\eta(\cdot)$. Our result is the following:

Theorem 4 *Suppose that $\tilde{G}_k(k')$ and $G_k(k')$ are both non-increasing in k for all k' and strictly convex in k for all $k \leq k'$, and that $\tilde{G}_k$ strictly second-order stochastically dominates G_k for all $k \in \{1, \ldots, N-1\}$. Then the MELP is strictly higher under a monitoring structure corresponding to $\{\tilde{g}_k(\cdot)\}_{k=1}^{N}$ than under a monitoring structure corresponding to $\{g_k(\cdot)\}_{k=1}^{N}$.*

We now discuss the conditions of Theorem 4. The condition that $G_k(k')$ is non-increasing and convex in k means that, as the number of defectors in period t increases, the probability that there are fewer than k' defectors in period $t+1$ decreases at a decreasing rate. The condition that $\tilde{G}_k$ strictly second-order stochastically dominates G_k means that, for any number of defectors k in period t (other than 0 or N), the distribution of the number of defectors in period $t+1$ under $\tilde{G}_k$ strictly second-order stochastically dominates the number of defectors in period $t+1$ under

G_k. If there are 0 (N) defectors in period t, then there are 0 (N) defectors in period $t+1$, so we cannot require strict second-order stochastic dominance if $k \in \{0, N\}$; however, $\tilde{G}_0$ ($\tilde{G}_N$) weakly second-order stochastically dominates G_0 (G_N), trivially.

The intuition for Theorem 4 is fairly simple: If $\tilde{G}_k$ strictly second-order stochastically dominates G_k for all k, then under $\tilde{G}_k$ it is more likely that an intermediate number of players find out about an initial deviation each period. And, since $G_k(k')$ and $\tilde{G}_k(k')$ are non-increasing and convex, the expected number of players who find out about the deviation within t periods increases in t more quickly when it is more likely that an intermediate number of players find out about the deviation each period. This implies that $\sum_{t=0}^{\infty} \delta^t \mathbb{E}[\#D(t)]$ is strictly higher under a monitoring structure corresponding to $\{\tilde{g}_k(\cdot)\}_{k=1}^N$ than under a monitoring structure corresponding to $\{g_k(\cdot)\}_{k=1}^N$, and the result then follows from Theorem 3. The details of the proof are deferred to the appendix.

7 Monitoring on a Network

In this section, we study public good provision under monitoring on a network. That is, there is a fixed network $L = \{l_{i,j}\}_{i,j \in N \times N}$ where $l_{i,j} \in \{0,1\}$, and $i \in O(j,t)$ if and only if $l_{i,j} = 1$, for all t. To simplify the exposition, we restrict attention to undirected networks, which means that we assume that $l_{i,j} = l_{j,i}$ for all i, j, though the extensions of all of the results in this section to directed networks is straightforward. Note that monitoring on a network always satisfies stationarity, but may or may not satisfy equal observability. As in Section 6, we omit the parameter $\alpha(N)$ in this section.

Our main results in this section are that if player i is "more central" than player j in a sense defined below, then player i's maximum equilibrium contribution is greater than player j's; and that adding a link between any two players strictly increases the maximum equilibrium contribution of each player in the connected component containing the players.

7.1 Centrality and Equilibrium Contribution Levels

We begin by introducing our notion of player i's being "more central" than player j. Informally, we say that player i is "more central" than player j if i has more neighbors than j, i's neighbors have more neighbors than j's neighbors, i's neighbors' neighbors have more neighbors than j's neighbors' neighbors, and so on. In the following formal, recursive definition, $d(i,j)$ is the distance (shortest path length) between players i and j, with $d(i,j) \equiv \infty$ if there is no path between i and j:

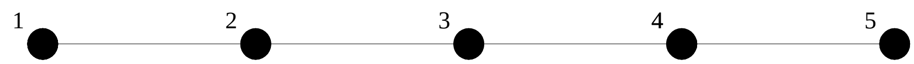

Figure 1: A Five-Player Example

Definition 8 *Player i is* 1-more central *than player j if, for all $t = \{0, 1, \ldots\}$, $\#\{k \in N : d(i,k) \leq t\} \geq \#\{k \in N : d(j,k) \leq t\}$. Player i is* strictly 1-more central *than player j if in addition $\#\{k \in N : d(i,k) \leq t\} > \#\{k \in N : d(j,k) \leq t\}$ for some t.*

Player i is s-more central *than player j if, for all $t = \{0, 1, \ldots\}$, there exists a surjection $\psi : \{k \in N : d(i,k) \leq t\} \to \{k \in N : d(j,k) \leq t\}$ such that, for all k with $d(j,k) \leq t$, there exists $k' \in \psi^{-1}(k)$ such that k' is $s-1$-more central than k. Player i is* strictly s-more central *than player j if in addition k' is strictly $s-1$-more central than k for some t, ψ, k, and k'.*

Player i is more central *than player j if i is s-more central than j for all $s = \{1, 2, \ldots\}$. Player i is* strictly more central *than player j if in addition i is strictly s-more central than j for some s.*

We illustrate this concept with the simple example of five players arranged in a line (see Figure 1)—later in this section, we present a more complicated examples that illustrates the value of tools developed below for determining when one player is more central than another. According to our definition, player 3 is strictly more central than players 2 and 4, who are in turn strictly more central than players 1 and 5; thus, in this example, our definition corresponds to a naive notion of "centrality." To see this, we argue by induction. It is easy to check that player 3 is strictly 1-more central than players 2 and 4, who are in turn each strictly 1-more central than players 1 and 5. For example, player 2 is strictly 1-more central than player 5 because player 2 has 1 "neighbor" within distance 0 (player 2 herself), 3 neighbors within distance 1, 4 neighbors within distance 2, and 5 neighbors within distance 3 or more; while player 5 has 1 neighbor within distance 0, 2 neighbors within distance 1, 3 neighbors within distance 2, 4 neighbors within distance 3, and 5 neighbors within distance 4 or more. Now suppose that player 3 is s-more central than players 2 and 4, and that players 2 and 4 are both s-more central than players 1 and 5. Then it is easy to check that player 3 is also $s+1$-more central than players 2 and 4, who in turn are both $s+1$-more central than players 1 and 5; for example, one surjection $\psi : \{k \in N : d(2,k) \leq 2\} \to \{k \in N : d(5,k) \leq 2\}$ that satisfies the terms of the definition is given by $\psi(1) = \psi(2) = 5$, $\psi(3) = 3$, $\psi(4) = 4$. Thus, by induction on s, player 3 is strictly more central than player 2 and 4, who are in turn strictly more central than players 1 and 5.

We may now state the first result of this section, which says that more central players have greater maximum equilibrium contributions. The proof uses a monotonicity argument similar to that in the proof of Theorem 1, which shows that more central players contribute more at every step of a sequence of contributions that converges to the vector of maximum equilibrium contributions.

Theorem 5 *If player i is more central than player j, then $\hat{x}_i \geq \hat{x}_j$. The inequality is strict whenever player i is strictly more central than player j.*

The proof of the strict inequality in Theorem 5 uses the following lemma, the proof of which is deferred to the appendix:

Lemma 2 *If player i is more central than player j, then for all $t = \{0, 1, \ldots\}$ there exists a surjection $\psi : \{k \in N : d(i,k) \leq t\} \to \{k \in N : d(j,k) \leq t\}$ such that, for all k with $d(j,k) \leq t$, there exists $k' \in \psi^{-1}(k)$ such that k' is more central than k.*

Proof of Theorem 5. Let $\tilde{\phi}(\vec{x})$ be defined as in the proof of Corollary 1, and note that in the present context $\tilde{\phi}_i(\vec{x})$ is the unique x_i' such that

$$c\left(x_i'\right) = (1-\delta) \sum_{t=0}^{\infty} \delta^t \sum_{k:d(i,k)\leq t} x_k. \tag{6}$$

By the same argument as in the proof of Theorem 1, iterating the map $\tilde{\phi}(\cdot)$ on the vector $\bar{\mathbf{x}} = \{\bar{x}\}_{i=1}^N$ yields a sequence of vectors $\{\vec{x}\}_m$converging to the vector of maximum equilibrium contributions, where $\bar{x}$ is a number higher than any contribution that could occur in any SE. Let $x_{i,m} \equiv \tilde{\phi}_i^m(\bar{\mathbf{x}})$, where $\tilde{\phi}^m$ denotes the map given by iterating $\tilde{\phi}(\cdot)$ m times.

To prove the weak inequality, suppose that player i is more central than player j. We use induction on m to show that $x_{i,m} \geq x_{j,m}$ for all m, which implies that $\hat{x}_i = \lim_{m\to\infty} x_{i,m} \geq \lim_{m\to\infty} x_{j,m} = \hat{x}_j$. Since player i is 1-more central than player j, $\#\{k \in N : d(i,k) \leq t\} \geq \#\{k \in N : d(j,k) \leq t\}$ for all t, which implies that $x_{i,1} \geq x_{j,1}$, by (6). Now suppose that $x_{k',m} \geq x_{k,m}$ whenever player k' is more central than player k, for some m. Since player i is $m+1$-more central than player j, for any t there exists a surjection $\psi : \{k \in N : d(i,k) \leq t\} \to \{k \in N : d(j,k) \leq t\}$ such that, for all k with $d(j,k) \leq t$, there exists $k' \in \psi^{-1}(k)$ such that k' is m-more central than k. Since $x_{k',m} \geq x_{k,m}$, this implies that $\sum_{k':d(i,k')\leq t} x_{k',m} \geq \sum_{k:d(j,k)\leq t} x_{k,m}$. This holds for all t, which implies that $x_{i,m+1} \geq x_{j,m+1}$, by (6). By induction, we conclude that $x_{i,m} \geq x_{j,m}$ for all m, completing the proof of the weak inequality.

To prove the strict inequality, suppose that player i is strictly more central than player j, i.e., that player i is more central than player j and is strictly s-more central than player j for some s. We derive a strictly positive lower bound on $x_i^* - x_j^*$ that depends on s, where x_i^* is as in the statement of Theorem 1 (recall that $x_i^* = \hat{x}_i$ for all i). First, rewrite (1) as

$$c\left(x_i^*\right) = (1-\delta) \sum_{t=0}^{\infty} \delta^t \sum_{k:d(i,k) \leq t} x_k^*. \tag{7}$$

By Lemma 2, for any t there exists a surjection $\psi : \{k \in N : d(i,k) \leq t\} \to \{k \in N : d(j,k) \leq t\}$ such that, for all k with $d(j,k) \leq t$, there exists $k' \in \psi^{-1}(k)$ such that k' is more central than k. Suppose that i is more central than j and strictly 1-more central than j, let $\underline{x}^* \equiv \min_i \{x_i^*\}_{i=1}^N$, which is positive by Corollary 1, and let $\bar{d}$ be the diameter of L, i.e., the maximum distance between any two path-connected nodes in L. Then, by Lemma 2 and (7), $c\left(x_i^*\right) \geq c\left(x_j^*\right) + \delta^{\bar{d}-1} \max\{\delta, 1-\delta\} \underline{x}^*$, as player i has at least one more distance-k neighbor than player j for some k.[22] Therefore, there exists $\varepsilon_1 > 0$ such that $x_i^* - x_j^* \geq \varepsilon_1 > 0$ whenever i is more central than j and strictly 1-more central than j. Now suppose that there exists $\varepsilon_s > 0$ such that $x_i^* - x_j^* \geq \varepsilon_s > 0$ whenever i is more central than j and strictly s-more central than j. Suppose that i is more central than j and strictly $s+1$-more central than j. Then $c\left(x_i^*\right) \geq c\left(x_j^*\right) + \delta^{\bar{d}-1} \max\{\delta, 1-\delta\} \varepsilon_s$, by Lemma 2 and (7), which implies that there exists $\varepsilon_{s+1} > 0$ such that $x_i^* - x_j^* \geq \varepsilon_{s+1} > 0$. By induction, we conclude that $x_i^* > x_j^*$ whenever i is strictly more central than j. ■

We remark that Theorem 5 can be used both for determining which players' maximum equilibrium contributions are higher *in a given network*, and for determining how a given player's maximum equilibrium contribution changes *as the network changes*: the latter is accomplished by asking whether the player's position in the new network is more central than her position in the old network. The next subsection contains an application of this idea.

Unfortunately, using Theorem 5 directly for applications may be cumbersome, because checking whether one player is more central than another may be difficult; indeed, this requires checking an infinite number of inequalities, and therefore generally requires a recursive argument like that given in the above five-player example. Sometimes, however, symmetries in the network can be exploited to determine which players are more central than others more easily. Corollaries 4 and 5 and the subsequent seven-player example illustrate how this can be done. Corollary 4 provides the natural idea that, if player i is closer to *all* players $k \neq i, j$ than is player j, then player i is more

[22] The $\max\{\delta, 1-\delta\}$ term thus corresponds to the possibility that player i may have one more distance-$\bar{d}$ neighbor than player j, or may have one more distance-$\bar{d}-1$ neighbor and the same number of distance-$\bar{d}$ neighbors.

central than player j. Corollary 5 shows that if players i and $\rho(i)$ are in "symmetric" positions in the network and player $\rho(i)$ is more central than player j, then player i is more central than player j as well. Combining Corollaries 4 and 5, we see that if player i is in a symmetric position with some player $\rho(i)$ who is closer to all other players than is player j, then player i is more central than player j. This observation is useful both as a way of directly observing when one player is more central than another, and also as a tool for checking the "more central than" relation for pairs of players to whom Corollaries 4 and 5 do not directly apply—for example, knowing that player i is more central than player j may make it easier to check that player i' is more central than player j' if player i is a neighbor of player i' and player j is a neighbor of player j'. The seven-player example following the corollaries gives an example of this kind of argument.

Corollary 4 *If $d(i,k) \leq d(j,k)$ for all $k \neq i,j$, then player i is more central than player j. Player i is strictly more central than player j if in addition the inequality is strict for some $k \neq i,j$.*

Proof. It is immediate that player i is 1-more central than player j, and that player i is strictly 1-more central if at least one of the inequalities is strict. Suppose that player i is s-more central. To see that player i is $s+1$-more central, for any t let $\psi(\cdot)$ be any surjection such that $\psi(k) = k$ if $d(j,k) \leq t$ and $\psi(i) = j$. Then for any k with $d(j,k) \leq t$, there exists $k' \in \psi^{-1}(k)$ such that k' is s-more central than k, since (as is easy to check) any player is s-more central than herself. That player i is more central than player j follows by induction, and player i is therefore strictly more central than player j if at least one of the inequalities is strict. ■

The intuitive statement of Corollary 5 above relied on the notion of two players' being in "symmetric" positions in a network. Mathematically, this is captured by the concept of a *graph automorphism*. Formally, a graph automorphism on L is a permutation ρ on N such that if $l_{i,j} = 1$, then $l_{\rho(i),\rho(j)} = 1$; that is, a graph automorphism is a permutation of vertices that preserves links. Our result is the following:

Corollary 5 *If there exists a graph automorphism $\rho : N \to N$ such that $\rho(i)$ is (strictly) more central than j, then i is (strictly) more central than j.*

Proof. We again proceed by induction. Since $\#\{k : d(i,k) \leq t\} = \#\{k : d(\rho(i),k) \leq t\}$ for all t, it is clear that i is 1-more central than j if $\rho(i)$ is 1-more central than j. Suppose we have shown that i is s-more central than j if $\rho(i)$ is s-more central than j, for all i,j. Suppose that $\psi : \{k \in N : d(\rho(i),k) \leq t\} \to \{k \in N : d(j,k) \leq t\}$ is a surjection such that, for all k with

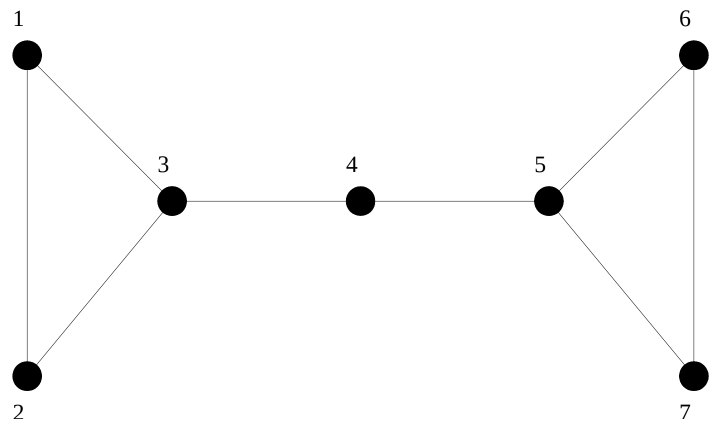

Figure 2: A Seven-Player Example

$d(j,k) \leq t$, there exists $k' \in \psi^{-1}(k)$ such that k' is s-more central than k. We claim that $\psi \circ \rho$ is a surjection from $\{k \in N : d(i,k) \leq t\} \to \{k \in N : d(j,k) \leq t\}$ such that, for all k with $d(j,k) \leq t$, there exists $k'' \in (\psi \circ \rho)^{-1}(k)$ such that k'' is s-more central than k. To see this, note that if $d(\rho(i), k') \leq t$, then $d\left(i, \rho^{-1}(k')\right) \leq t$ (as $\rho(\cdot)$ preserves distances), and if k' is s-more central than k, then $\rho^{-1}(k')$ is s-more central than k as well, by the inductive hypothesis. Therefore, $k'' \equiv \rho^{-1}(k')$ satisfied the required conditions, and repeating this argument for all t shows that i is $s+1$ more central than j if $\rho(i)$ is $s+1$ more central than j. That i is more central than j follows by induction. The argument for i strictly more central than j is similar. ■

The example in Figure 2 illustrates the usefulness of Corollaries 4 and 5.[23] Direct application of Corollary 4 implies that player 3 is more central than players 1 and 2, and that player 5 is more central than players 6 and 7. However, it is not clear how to show that player 3 is more central than players 6 and 7, say, and it is not obvious whether player 4 is more central than the players in $\{1,2,6,7\}$ or than players 3 and 4.

Corollary 5 makes analyzing this network substantially easier. Observe the following map ρ is an automorphism of L: $\rho(1) = 7$, $\rho(2) = 6$, $\rho(3) = 5$, $\rho(4) = 4$, $\rho(5) = 3$, $\rho(6) = 2$, and $\rho(7) = 1$. Since we have already established that player 3 is more central than players 1 and 2 and that player 5 is more central than players 6 and 7, Corollary 5 now implies that each player in $\{3,5\}$ is more

[23] This example is the same as that in Figure 2.13 of Jackson (2008), which Jackson uses to illustrate various network-theoretic concepts of centrality. As we will see, in this example our definition of centrality is similar to the concepts discussed by Jackson in that players 3, 4, and 5 are all more central than players 1, 2, 6, and 7. One impotant difference between our definition of centrality and those discussed by Jackson is that our definition induces an incomplete ordering over nodes, while all of the concepts discussed by Jackson induce complete orderings.

central than each player in $\{1, 2, 6, 7\}$.

Given this observation, it is not hard to show that player 4 is more central than each player in $\{1, 2, 6, 7\}$. Consider player 1. Player 4 is 1-more central than player 1, by inspection. Since player 4 is within distance 2 of every player, showing that player 4 is s-more central than player 1 reduces to finding a bijection $\psi : \{3, 4, 5\} \to \{1, 2, 3\}$ such that k is $s - 1$-more central than $\psi(k)$ for all $k \in \{3, 4, 5\}$. We proceed by induction, supposing that we have already established that player 4 is $s - 1$-more central than player 1. Let $\psi(3) = 3$, $\psi(4) = 1$, and $\psi(5) = 2$. Player 3 is $s - 1$-more central than herself, trivially; player 4 is $s - 1$-more central than player 1, by the inductive hypothesis; and player 5 is $s - 1$-more central than player 2, by the observation in the previous paragraph, which followed from Corollaries 4 and 5. Therefore, player 4 is more central than player 1, and the same argument shows that player 4 is more central than each player in $\{1, 2, 6, 7\}$. The above arguments can easily be replicated for "strictly more central," so Theorem 5 implies that players 3, 4, and 5 have greater maximum equilibrium contributions than do players 1, 2, 6, and 7.

Finally, it is easy to see that players 3 and 4 are not more central than each other, as player 3 has more immediate neighbors while player 4 has more neighbors within distance 2. Therefore, Theorem 5 does not say whether player 3 or player 4 contributes more in an equilibrium that sustains the MELP. This is reassuring, because one can easily construct examples in which player 3 contributes more and others in which player 4 contributes more: for example, if $c(x) = x + x^3$, then $x_1^* \approx 2.167$, $x_3^* \approx 2.215$, and $x_4^* \approx 2.225$ if $\delta = .9$, whereas if $\delta = .4$ then $x_1^* \approx 1.068$, $x_3^* \approx 1.182$, and $x_4^* \approx 1.177$. It is not surprising that player 3 contributes more relative to player 4 when δ is lower, as in this case the fact that player 3 has more immediate neighbors is more important, while player 4's greater number of distance-2 neighbors matters more when δ is higher (since δ^2 is low relative to δ when δ is low).

Before leaving this subsection, we remark that Theorem 5 provides a new perspective on the Olsonian idea of the "exploitation of the great by the small." Olson (1965) noted that small players may free ride on larger players if larger players have greater private incentives to contribute to public goods. Theorem 5 illustrates a reason why larger players might be expected to contribute disproportionately much to public goods even if they do not have greater private incentives to contribute. That is, larger players tend to be observed by more other players (i.e., they tend to be "more central" in the sense of Definition 8), which implies that a defection by a larger player leads more other players to defect. This in turn implies that larger players can contribute more than smaller players in equilibrium, as shown by Theorem 5. This argument contrasts with most previous analyses of "centrality" in networks, which usually suggest that central players receive higher payoffs. In the case of public good provision, it is players who are not central (i.e., who are poorly monitored) who receive higher payoffs, as they are more able to free ride on their opponents' contributions, and therefore cannot be expected to make large contributions in equilibrium.

7.2 The Consequences of Adding or Removing Links

We now show that adding a link between any two players strictly increases the maximum equilibrium contribution of *every* player in the connected component containing these players. This result is a natural formalization of the widespread idea that better-connected societies can provide more public goods. The intuition is that adding a link between players i and j increases the amount that they can contribute in equilibrium, since it decreases their continuation payoffs following a defection, which in turn increases the amount that any player who is path-connected to them can contribute in equilibrium. Theorem 6 could be proved directly using our fixed point characterization of maximum equilibrium contributions (equation (1)), but it is easier to prove by noting that adding a link between two players makes all players in the corresponding connected component strictly more central in the sense of Definition 8 and then applying Theorem 5.

Theorem 6 *Let L' and L be undirected networks such that $l_{k,k'} = l'_{k,k'}$ for all $(k,k') \neq (i,j)$, $l'_{i,j} = 1$, and $l_{i,j} = 0$. Let C be the connected component of L' containing i and j. Then, for any $k \in C$, x_k^* is strictly higher under monitoring on L' than under monitoring on L.*

Proof. Let L'' be the network with two disjoint components, of which one is isomorphic to L' and the other isomorphic to L. For any player $k \in L$, let $\tilde{k}$ be the corresponding player in L'. We claim that if $\tilde{k} \in C$, then $\tilde{k}$ is strictly more central (in L'') than k. Since (1) implies that player k's ($\tilde{k}$'s) maximum equilibrium contribution is the same in L'' as in L (L'), the result then follows from Theorem 5.

Fix $\tilde{k} \in C$. To see that $\tilde{k}$ is more central than k, note that, for any s and t, the surjection $\psi : \left\{k' \in L'' : d\left(\tilde{k}, k'\right) \leq t\right\} \to \psi \{k' \in L'' : d(k,k') \leq t\}$ that maps $\tilde{k}'$ to k' satisfies the conditions of Definition 8. Next, note that $\tilde{\imath}$ is strictly 1-more central than i, since $\tilde{\imath}$ has one more distance-1 neighbor than i does. Therefore, taking the above surjection in Definition 8 with $t \geq d\left(\tilde{k}, \tilde{\imath}\right)$ (which is well-defined because $\tilde{k} \in C$) implies that $\tilde{k}$ is strictly 2-more central than k, because $\tilde{\imath}$ is strictly 1-more central than i. Therefore, $\tilde{k}$ is strictly more central than k. ■

8 An Extension: Local Public Goods

Thus far, we have focused on "global" public goods, in that each player benefits equally from each other player's contribution to the public good. In this section, we consider a generalization of our model where, for example, a player may only benefit from contributions of players she observes, or from contributions of an arbitrary subset of players. This generalization encompasses several models of "local" public goods, which allows our model to cover applications such as cooperation in decentralized trade, exerting effort on a team project within a larger organization, and pricing in a differentiated market where at each date only a subset of firms end up in competition with each other. Versions of our key theoretical results (Theorems 1 through 3) continue to hold in this more general setting, but the effects of group size and structure on cooperation may be different. In particular, public good provision is best supported by monitoring structures in which a given player i is likely to be observed *by those players whose contributions she values.* For example, fixed partnerships (where each player matches with the same partner every period) support a higher level of public good provision than does random matching when players only benefit from contributions of players they observe, but random matching supports a higher level of global public good provision than do fixed partnerships.

Formally, we generalize our model by changing player i's period-t benefit from player j's contribution from αx_j to $\alpha_{i,j}\left(\{O(k,t)\}_{k=1}^{N}\right)x_j$, for an arbitrary function $\alpha_{i,j}:\{O(k,t)\}_{k=1}^{N}\rightarrow\mathbb{R}_+$. Thus, player i's period-t payoff becomes

$$\sum_{j=1}^{N}\alpha_{i,j}\left(\{O(k,t)\}_{k=1}^{N}\right)x_j-c(x_i).$$

Note that $\alpha_{i,j}\left(\{O(k,t)\}_{k=1}^{N}\right)$ is iid across periods, by stationarity. Leading examples include $\alpha_{i,j}\left(\{O(k,t)\}_{k=1}^{N}\right)$ deterministic (i.e., players weight each other's contributions asymmetrically, independently of the realization of the monitoring technology) and $\alpha_{i,j}\left(\{O(k,t)\}_{k=1}^{N}\right)=1$ if $i\in O(j,t)$ and $\alpha_{i,j}\left(\{O(k,t)\}_{k=1}^{N}\right)=0$ otherwise (i.e., a player only benefits from contributions that she observes); these two cases may be thought of as different models of local public goods. Observe that the latter case corresponds to the random matching model of Kandori (1992) and Ellison (1994). We now develop analogs of Theorems 1 through 3 for the generalized model, and then discuss their implications for the effects of group size and structure on cooperation. The proofs of these results are straightforward extensions of the proofs of Theorem 1 through 3 and are therefore omitted.

In the generalized model, there is no single number that measures the overall level of public good provision, as the MELP does in the case of global public goods. Nonetheless, the following direct extension of Theorem 1 holds in the generalized model:

Proposition 17 *There exists a unique grim trigger strategy profile σ^* that sustains each player's maximum equilibrium contribution.*

In the generalized model, players are effectively "equally well monitored" only if they are equally likely to be observed by players whose contributions benefit them. Thus, the appropriate generalization of equal observability is the following:

- *Equal Observability (General Version)*:

$$\sum_{j=1}^{N} \Pr\left(j \in D\left(\tau, t, i\right)\right) \mathbb{E}\left[\alpha_{i,j}\left(\left\{O\left(k,\tau\right)\right\}_{k=1}^{N}\right)\middle| j \in D\left(\tau, t, i\right)\right] =$$
$$\sum_{j=1}^{N} \Pr\left(j \in D\left(\tau, t, i'\right)\right) \mathbb{E}\left[\alpha_{i',j}\left(\left\{O\left(k,\tau\right)\right\}_{k=1}^{N}\right)\middle| j \in D\left(\tau, t, i'\right)\right]$$

  for all i, i', τ, and t.

Note that

$$\Pr\left(j \in D\left(\tau, t, i\right)\right) \mathbb{E}\left[\alpha_{i,j}\left(\left\{O\left(k,\tau\right)\right\}_{k=1}^{N}\right)\middle| j \in D\left(\tau, t, i\right)\right] =$$
$$\Pr\left(j \in D\left(\tau - t, 0, i\right)\right) \mathbb{E}\left[\alpha_{i,j}\left(\left\{O\left(k,\tau - t\right)\right\}_{k=1}^{N}\right)\middle| j \in D\left(\tau - t, 0, i\right)\right],$$

by stationarity. We simplify notation by writing $\Pr\left(j \in D\left(t, i\right)\right) \mathbb{E}\left[\alpha_{i,j}\left(\left\{O\left(k,t\right)\right\}_{k=1}^{N}\right)\middle| j \in D\left(t, i\right)\right]$ for $\Pr\left(j \in D\left(t, 0, i\right)\right) \mathbb{E}\left[\alpha_{i,j}\left(\left\{O\left(k,t\right)\right\}_{k=1}^{N}\right)\middle| j \in D\left(t, 0, i\right)\right]$, paralleling the notation introduced in Section 3.

An analog of Theorem 2 follows immediately:

Proposition 18 *If monitoring satisfies the general version of equal observability, there exists a unique symmetric grim trigger strategy profile σ^* that sustains each player's maximum equilibrium contribution.*

Finally, we have an analog of Theorem 3. In the statement of this result, $\Pr\left(j \in D\left(t, i\right); \Gamma\right)$, $\alpha_{i,j}\left(\left\{O\left(k,t\right)\right\}_{k=1}^{N}; \Gamma\right)$, and $\mathbb{E}\left[\alpha_{i,j}\left(\left\{O\left(k,t\right)\right\}_{k=1}^{N}; \Gamma\right)\middle| j \in D\left(t, i\right); \Gamma\right]$ refer to $\Pr\left(j \in D\left(t, i\right)\right)$, $\alpha_{i,j}\left(\left\{O\left(k,t\right)\right\}_{k=1}^{N}\right)$, and $\mathbb{E}\left[\alpha_{i,j}\left(\left\{O\left(k,t\right)\right\}_{k=1}^{N}; \Gamma\right)\middle| j \in D\left(t, i\right)\right]$ in game Γ, respectively (note that this notation allows both the function $\alpha_{i,j}\left(\cdot\right)$ and the distribution of $\left\{O\left(k,t\right)\right\}_{k=1}^{N}$ to vary with Γ).

Proposition 19 *Let Γ' and Γ be two games that satisfy the general version of equal observability. Then every player's maximum equilibrium contribution is higher in Γ' than in Γ if and only if*

$$\sum_{t=0}^{\infty}\delta^t \sum_{j=1}^{N(\Gamma')} \Pr\left(j \in D(t,i);\Gamma'\right)\mathbb{E}\left[\alpha_{i,j}\left(\{O(k,t)\}_{k=1}^{N};\Gamma'\right)\Big| j \in D(t,i);\Gamma'\right] >$$

$$\sum_{t=0}^{\infty}\delta^t \sum_{j=1}^{N(\Gamma)} \Pr\left(j \in D(t,i);\Gamma\right)\mathbb{E}\left[\alpha_{i,j}\left(\{O(k,t)\}_{k=1}^{N};\Gamma\right)\Big| j \in D(t,i);\Gamma\right]$$

for any i.

Proposition 19 contains the key idea we wish to highlight regarding the difference between supporting cooperation with global and local public goods: the extent to which a monitoring technology supports contributions by player i with general public goods depends on its effective contagiousness *among those players whose contributions benefit player i*. In the case of global public goods, this reduces to Theorem 3, which shows that the extent to which a monitoring technology supports cooperation with global public goods depends on its *overall* effective contagiousness. Proposition 19 is intuitive: with general public goods, player i only benefits from the contributions of some of her opponents, so she is deterred from defecting only when those specific players are likely to find out about a potential defection. Nonetheless, Proposition 3 is useful for analyzing which monitoring structures are more effective at supporting provision of different types of public goods. For example, it is interesting to compare random matching (defined in Section 5) with the following monitoring structure:

Definition 9 *Monitoring is given by* fixed partnerships *if for every player i there exists a player i' such that $O(i,t) = O(i',t) = \{i,i'\}$ for all t.*

Note that fixed partnerships and random matching both satisfy the general version of equal observability. The following result is therefore immediate from Theorem 3 and Proposition 19:

Proposition 20 *Fix $N \geq 4$ and even. With global public goods, every player's maximum equilibrium contribution is greater under random matching than under fixed partnerships. With local public goods, defined as $\alpha_{i,j}\left(\{O(k,t)\}_{k=1}^{N}\right) = 1$ if $i \in O(j,t)$ and $\alpha_{i,j}\left(\{O(k,t)\}_{k=1}^{N}\right) = 0$ otherwise, every player's maximum equilibrium contribution is greater under fixed partnerships than under random matching.*

Proof. Under random matching, $\mathbb{E}[\#D(1)] = 2$ and $\mathbb{E}[\#D(t)] > 2$ for all $t \geq 2$. Under fixed partnerships, $\mathbb{E}[\#D(t)] = 2$ for all $t \geq 1$. Therefore, with global public goods, every player's maximum equilibrium contribution is greater under random matching, by Theorem 3.

Under random matching, $\sum_{j=1}^{N} \Pr(j \in D(1,i)) \mathbb{E}\left[\alpha_{i,j}\left(\{O(k,1)\}_{k=1}^{N}\right)\middle| j \in D(1,i)\right] < 2$ and $\sum_{j=1}^{N} \Pr(j \in D(t,i)) \mathbb{E}\left[\alpha_{i,j}\left(\{O(k,t)\}_{k=1}^{N}\right)\middle| j \in D(t,i)\right] \leq 2$ for all $t \geq 2$. Under fixed partnerships, $\sum_{j=1}^{N} \Pr(j \in D(1,i)) \mathbb{E}\left[\alpha_{i,j}\left(\{O(k,1)\}_{k=1}^{N}\right)\middle| j \in D(1,i)\right] = 2$ for all $t \geq 1$. Therefore, with local public goods, every player's maximum equilibrium contribution is greater under fixed partnerships, by Proposition 19. ■

The intuition for Proposition 20 is straightforward. $\mathbb{E}[\#D(t)]$ is higher under random matching than under fixed partnerships, for all t, because under random matching information about a defection spreads throughout the entire group. This implies that defecting is more costly under random matching when the public good is global. However, under fixed partnerships a player who defects never receives a contribution from her partner again, while under random matching a player can defect and still receive contributions from her future partners. Therefore, defecting is more costly under fixed partnerships when the public good is local (in the sense of Proposition 20).[24]

More generally, Theorem 3 and Proposition 19 show that "compact" social structures (in the sense of high $\sum_{j=1}^{N} \Pr(j \in D(t,i) \cap i \in D(t,j))$) are better at supporting the provision of local (in the sense of Proposition 20) public goods—as in this case it is important that a player observes the same players that observe her—while "diffuse" social structures (in the sense of high $\mathbb{E}[\#D(t)]$) are better at supporting the provision of global public goods—as in this case it is important that a large fraction of the entire population finds out about a defection quickly.

9 Conclusion

This paper introduces a tractable framework for studying repeated public good provision and provides comparative statics on the maximum equilibrium level of public good provision with respect to group size, monitoring structure, and network connectivity. The basis of our results is a characterization of strategies that sustain the maximum level of public good provision: the

[24] A similar result, the proof of which is also straightforward, is that under random matching every player's maximum equilibrium contribution is decreasing in N when the public good is local (in the sense of Proposition 20) and pure, while it is increasing in N when the public good is global and pure, by Proposition 9. This is analogous to the result of Kandori (1992) and Ellison (1994) that a higher discount factor is required to sustain cooperation when the population is larger in the repeated prisoner's dilemma with random matching.

maximum level of public good provision is sustained by grim trigger strategies, and these strategies are also symmetric if monitoring satisfies a weak equal observability condition. The maximum per capita level of public good provision is increasing in group size in the case of pure public goods under quite broad conditions, but is decreasing in group size in the case of divisible public goods in many natural examples; in general, comparative statics on the maximum per capita level of public good provision depend on the product of the rivalness term and the effective contagiousness of the monitoring structure. Less uncertain monitoring, which in some applications may be interpreted as reliable local monitoring rather than unreliable public monitoring, sustains a higher level of public good provision under broad conditions. In social networks, more central individuals can contribute more than less central individuals, and all individuals can contribute more when the network is better connected. Finally, our approach illustrates how provision of global and local public goods are best supported by different social structures.

We conclude by discussing directions for future research. First, our analysis of optimal equilibria with all-or-nothing private monitoring may facilitate further investigations of the relationship between public and private monitoring as means of sustaining cooperation. In the examples in the introduction, it seems likely that only extremely weak incentives can be provided by public monitoring, but this intuition is not captured clearly by existing models of repeated games with imperfect public monitoring. A model in which players learn about each other's play through both all-or-nothing private monitoring and imperfect public monitoring could clarify the extent to which large groups are able to avoid the problems associated with public monitoring by relying on local, private monitoring of the kind studied in this paper.

Second, it would be interesting to see if the concept of all-or-nothing monitoring introduced in this paper has application to settings other than public good provision. In general repeated games with private monitoring, two key challenges are identifying deviations and coordinating punishments. In our model, identifying deviations is made simple by all-or-nothing monitoring, and coordinating punishments is unnecessary due to the specific structure of the public good provision game (in particular, contributing 0 is the strongest possible punishment, and contributing 0 after observing a deviation is optimal in the grim trigger strategy that sustains the MELP). In general, studying models with all-or-nothing monitoring might provide a way to isolate the game-theoretic problem of coordinating punishments from the more statistical problem of identifying deviations.

Finally, our analysis provides several clean predictions about the effects of group size and structure on the level of public good provision, how this differs for pure and divisible public goods and global and local public goods, and what strategies best sustain public good provision. A natural next step would be to study these predictions empirically, either experimentally or using detailed field data like that used in Karlan et al (2008).

Appendix: Omitted Proofs

Proof of Theorem 1. Rather than directly studying the problem of maximizing $\alpha(1-\delta)\mathbb{E}\left[\sum_{t=0}^{\infty}\delta^t\sum_{i=1}^{N}\sigma_i\left(h_i^t\right)\right]$ over all sequential equilibria, we start by considering the relaxed problem of maximizing $\alpha(1-\delta)\mathbb{E}\left[\sum_{t=0}^{\infty}\delta^t\sum_{i=1}^{N}\sigma_i\left(h_i^t\right)\right]$ over all strategy profiles σ such that, for every player i and history h_i^t, player i weakly prefers following σ_i from history h_i^t onwards to setting $x_{i,\tau}=0$ for all $\tau\geq t$ following h_i^t. We first show that a solution σ to the relaxed problem exists that satisfies the following two properties, and that every solution to the relaxed problem has the same path of play:

1. σ is a grim trigger strategy profile.
2. If h_i^t is on the path of play of σ, then player i is indifferent between following σ at h_i^t and setting $x_{i,\tau}=0$ for all $\tau\geq t$ following h_i^t.

We then show that any solution to the relaxed problem that satisfies these properties also solves the full problem.

Let $\mathbb{E}^i\left[\sigma_j\left(h_j^\tau\right)\middle|h_i^t;\sigma_i\right]$ be player i's expectation at history h_i^t of player j's period τ contribution, conditional on player i's following strategy σ_i; and let $\mathbb{E}^i\left[\sigma_j\left(h_j^\tau\right)\middle|h_i^t;0\right]$ be the same expectation conditional on player i's playing $x_i=0$ at history h_i^t and at every subsequent history. The requirement that player i weakly prefers following σ_i at history h_i^t to setting $x_{i,\tau}=0$ for all $\tau\geq t$ following h_i^t may then be written as

$$\left(\sum_{j=1}^{N}\alpha(1-\delta)\sum_{\tau=t}^{\infty}\delta^{\tau-t}\mathbb{E}^i\left[\sigma_j\left(h_j^\tau\right)\middle|h_i^t;\sigma_i\right]\right)-\left((1-\delta)\sum_{\tau=t}^{\infty}\delta^{\tau-t}\mathbb{E}^i\left[c\left(\sigma_i\left(h_i^\tau\right)\right)\middle|h_i^t;\sigma_i\right]\right)$$
$$\geq\sum_{j=1}^{N}\alpha(1-\delta)\sum_{\tau=t}^{\infty}\delta^{\tau-t}\mathbb{E}^i\left[\sigma_j\left(h_j^\tau\right)\middle|h_i^t;0\right],$$

or

$$(1-\delta)\sum_{\tau=t}^{\infty}\delta^{\tau-t}\mathbb{E}^i\left[c\left(\sigma_i\left(h_i^\tau\right)\right)\middle|h_i^t;\sigma_i\right]\leq\sum_{j=1}^{N}\alpha(1-\delta)\sum_{\tau=t}^{\infty}\delta^{\tau-t}\left(\mathbb{E}^i\left[\sigma_j\left(h_j^\tau\right)\middle|h_i^t;\sigma_i\right]-\mathbb{E}^i\left[\sigma_j\left(h_j^\tau\right)\middle|h_i^t;0\right]\right).\tag{8}$$

Observe that

$$\mathbb{E}^i\left[\sigma_j\left(h_j^\tau\right)\middle|\, h_i^t;\sigma_i; j\notin D\left(\tau,t,i\right)\right]=\mathbb{E}^i\left[\sigma_j\left(h_j^\tau\right)\middle|\, h_i^t;0; j\notin D\left(\tau,t,i\right)\right],$$

where $D(\tau,t,i)$ is defined in Section 3; this follows because, conditional on the event $j\notin D(\tau,t,i)$, the probability distribution over histories h_j^τ does note depend on player i's actions following history h_i^t. Therefore, the right-hand side of (8) equals

$$\sum_{j=1}^{N}\alpha\left(1-\delta\right)\sum_{\tau=t}^{\infty}\delta^{\tau-t}\Pr\left(j\in D\left(\tau,t,i\right)\right)\left(\mathbb{E}^i\left[\sigma_j\left(h_j^\tau\right)\middle|\, h_i^t;\sigma_i; j\in D\left(\tau,t,i\right)\right]-\mathbb{E}^i\left[\sigma_j\left(h_j^\tau\right)\middle|\, h_i^t;0; j\in D\left(\tau,t,i\right)\right]\right),$$

which is not more than

$$\sum_{j=1}^{N}\alpha\left(1-\delta\right)\sum_{\tau=t}^{\infty}\delta^{\tau-t}\Pr\left(j\in D\left(\tau,t,i\right)\right)\mathbb{E}^i\left[\sigma_j\left(h_j^\tau\right)\middle|\, h_i^t;\sigma_i; j\in D\left(\tau,t,i\right)\right].$$

Therefore, the following condition is necessary for (8):

$$\left(1-\delta\right)\sum_{\tau=t}^{\infty}\delta^{\tau-t}\mathbb{E}^i\left[c\left(\sigma_i\left(h_i^\tau\right)\right)\middle|\, h_i^t;\sigma_i\right]\leq$$
$$\sum_{j=1}^{N}\alpha\left(1-\delta\right)\sum_{\tau=t}^{\infty}\delta^{\tau-t}\Pr\left(j\in D\left(\tau,t,i\right)\right)\mathbb{E}^i\left[\sigma_j\left(h_j^\tau\right)\middle|\, h_i^t;\sigma_i; j\in D\left(\tau,t,i\right)\right].\quad(9)$$

Let

$$X_i\left(h_i^t,\sigma\right)\equiv\alpha\left(1-\delta\right)\sum_{\tau=t}^{\infty}\delta^{\tau-t}\mathbb{E}\left[\sigma_i\left(h_i^\tau\right)\middle|\, h_i^t;\sigma_i\right].$$

$X_i\left(h_i^t,\sigma\right)$ is the expected value of player i's future contributions starting from history h_i^t. With this notation and a little algebra, the right-hand side of (9) may be rewritten as

$$\sum_{j=1}^{N}\sum_{\tau=t}^{\infty}\delta^{\tau-t}\left(\Pr\left(j\in D\left(\tau,t,i\right)\right)-\Pr\left(j\in D\left(\tau-1,t,i\right)\right)\right)\mathbb{E}^i\left[X_j\left(h_i^\tau,\sigma\right)\middle|\, h_i^t;\sigma_i; j\in D\left(\tau,t,i\right)\backslash D\left(\tau-1,t,i\right)\right];\quad(10)$$

the intuition for this is simply that player j first enters the set $D(T,t,i)$ at $T=\tau$ with probability $\Pr(j\in D(\tau,t,i))-\Pr(j\in D(\tau-1,t,i))$, and when this occurs the expected value of player j's future contributions from player i's perspective is $\mathbb{E}^i\left[X_j\left(h_i^\tau,\sigma\right)\middle|\, h_i^t;\sigma_i; j\in D\left(\tau,t,i\right)\backslash D\left(\tau-1,t,i\right)\right]$. Note also that $\alpha\left(1-\delta\right)\mathbb{E}\left[\sum_{t=0}^{\infty}\delta^t\sum_{i=1}^{N}\sigma_i\left(h_i^t\right)\right]=\sum_{i=1}^{N}X_i\left(h^0,\sigma\right)$, recalling that h^0 is the null history at the beginning of the game. Since (9) is necessary for (8), and (10) equals the right-hand side of (9), the solution to the relaxed problem is bounded from above by the solution to the program

$$\sup_{\sigma\in\Sigma}\sum_{i=1}^{N}X_i\left(h^0,\sigma\right)$$

subject to

$$(1-\delta)\sum_{\tau=t}^{\infty}\delta^{\tau-t}\mathbb{E}^{i}\left[c\left(\sigma_i\left(h_i^{\tau}\right)\right)\middle| h_i^t\right]\leq$$
$$\sum_{j=1}^{N}\sum_{\tau=t}^{\infty}\delta^{\tau-t}\left(\Pr\left(j\in D\left(\tau,t,i\right)\right)-\Pr\left(j\in D\left(\tau-1,t,i\right)\right)\right)\mathbb{E}^{i}\left[X_j\left(h_i^{\tau},\sigma\right)\middle| h_i^t;\sigma_i;j\in D\left(\tau,t,i\right)\backslash D\left(\tau-1,t,i\right)\right]$$

for all h_i^t. Observe that the left-hand side of this inequality is bounded from below by $c\left(X_i\left(h_i^t,\sigma\right)/\alpha\right)$, by convexity of $c(\cdot)$ and Jensen's inequality. Also, observe that playing $x_i=0$ from period t onwards yields a weakly positive payoff to player i, so the same argument as in the proof of Lemma 1 implies that there exists $\bar{X}$ such that $X_i\left(h_i^t,\sigma\right)\leq\bar{X}$ for all i and h_i^t if σ satisfies (8). Let $\bar{\mathbf{X}}$ be an infinite-dimensional vector of $\bar{X}$'s, one for each player i and private history h_i^t. Combining the two preceding observations implies the solution to the relaxed problem is no higher than the solution to the following program:

$$\sup_{\left\{X_i\left(h_i^t,\sigma\right)\right\}\leq\bar{\mathbf{X}}}\sum_{i=1}^{N}X_i\left(h^0,\sigma\right)$$

subject to

$$c\left(X_i\left(h_i^t,\sigma\right)/\alpha\right)\leq$$
$$\sum_{j=1}^{N}\sum_{\tau=t}^{\infty}\delta^{\tau-t}\left(\Pr\left(j\in D\left(\tau,t,i\right)\right)-\Pr\left(j\in D\left(\tau-1,t,i\right)\right)\right)\mathbb{E}^{i}\left[X_j\left(h_i^{\tau},\sigma\right)\middle| h_i^t;\sigma_i;j\in D\left(\tau,t,i\right)\backslash D\left(\tau-1,t,i\right)\right] \tag{11}$$

for all i,h_i^t. Finally, the inequality in (11) may be replaced with equality for all i,h_i^t without loss of generality, as in the case of strict inequality $X_i\left(h_i^t,\sigma\right)$ may be increased without decreasing the objective or violating any of the constraints, since $\Pr\left(i\in D\left(\tau,t,j\right)\right)-\Pr\left(i\in D\left(\tau-1,t,j\right)\right)\geq 0$ for all j, τ, and t (by definition of $D\left(\tau,t,j\right)$).

We now characterize the solution to this program, and then show that the solution can be obtained in a strategy profile that satisfies the constraints of the relaxed problem. Given a vector of the $X_i\left(h_i^t\right)$ for all i and h_i^t, $\vec{X}\equiv\left\{X_i\left(h_i^t\right)\right\}$, define the map

$$\phi\left(\vec{X}\right)=\left\{\phi_{i,h_i^t}\left(\vec{X}\right)\right\}$$

where $\phi_{i,h_i^t}\left(\vec{X}\right)$ is the unique $X_i\left(h_i^t\right)'$ such that

$$c\left(X_i\left(h_i^t\right)'/\alpha\right) =$$
$$\sum_{j=1}^{N}\sum_{\tau=t}^{\infty}\delta^{\tau-t}\left(\Pr j\in D\left(\tau,t,i\right)-\Pr\left(j\in D\left(\tau-1,t,i\right)\right)\right)\mathbb{E}^i\left[X_j\left(h_i^\tau\right)\middle|\,h_i^t; j\in D\left(\tau,t,i\right)\backslash D\left(\tau-1,t,i\right)\right].$$

$\phi\left(\vec{0}\right)=\vec{0}$, so $\phi\left(\cdot\right)$ has at least one fixed point that lies below $\bar{\mathbf{X}}$ (i.e., at least one fixed point with $X_i\left(h_i^t\right)\leq\bar{X}$ for all i,h_i^t). Since $\Pr\left(j\in D\left(\tau,t,i\right)\right)-\Pr\left(j\in D\left(\tau-1,t,i\right)\right)\geq 0$ for all i, τ, and t, $\phi\left(\cdot\right)$ is an isotone function, so by Tarski's theorem it has a highest fixed point below $\bar{\mathbf{X}}$, which we denote by $\vec{X}^*$. Furthermore, the highest fixed point of $\phi\left(\cdot\right)$ below $\bar{\mathbf{X}}$ solves the above program providing an upper bound on the solution to the (original) relaxed problem, because the fact that $\phi\left(\cdot\right)$ is isotone implies that the highest fixed point of $\phi\left(\cdot\right)$ below $\bar{\mathbf{X}}$ involves the highest $X_i\left(h^0\right)\leq\bar{X}$ consistent with (11), for all i. In particular, $\sum_{i=1}^{N}X_i^*\left(h^0\right)$ is an upper bound on the solution to the relaxed problem.

We now claim that $\vec{X}^*$ is stationary, in that, for each i, there exists X_i^* such that $X_i^*\left(h_i^t\right)=X_i^*$ for all h_i^t. Let $\phi^k\left(\vec{X}\right)$ be the map obtained from iterating $\phi\left(\cdot\right)$ k times on $\vec{X}$, and let $\phi^k_{i,h_i^t}\left(\vec{X}\right)$ be the $\left(i,h_i^t\right)^{\text{th}}$ coordinate of $\phi^k\left(\vec{X}\right)$. By definition of $\vec{X}^*$, $\bar{X}\geq X_i^*\left(h_i^t\right)$ for all i,h_i^t. Therefore, by isotonicity of $\phi\left(\cdot\right)$, $\phi^k\left(\bar{\mathbf{X}}\right)\geq\vec{X}^*$ for all k. Furthermore, $\lim_{k\to\infty}\phi^k\left(\bar{\mathbf{X}}\right)=\vec{X}^*$, because the fact that $\phi\left(\phi^K\left(\bar{\mathbf{X}}\right)\right)\leq\phi^K\left(\bar{\mathbf{X}}\right)$ for all K implies that $\phi\left(\lim_{k\to\infty}\phi^k\left(\bar{\mathbf{X}}\right)\right)\leq\lim_{k\to\infty}\phi^k\left(\bar{\mathbf{X}}\right)$ (since limits preserve weak inequalities and $\phi\left(\cdot\right)$ is continuous in the product topology), which implies that $\phi\left(\lim_{k\to\infty}\phi^k\left(\bar{\mathbf{X}}\right)\right)=\lim_{k\to\infty}\phi^k\left(\bar{\mathbf{X}}\right)$. Finally, it is clear from the definition of $\phi\left(\cdot\right)$ and the fact that monitoring satisfies stationarity that $\phi\left(\vec{X}\right)$ is stationary if $\vec{X}$ is. Since $\bar{\mathbf{X}}$ is stationary, it follows that $\phi^k\left(\bar{\mathbf{X}}\right)$ is stationary for all k, which implies that $\lim_{k\to\infty}\phi^k\left(\bar{\mathbf{X}}\right)=\vec{X}^*$ is stationary.

From the definition of $X_i\left(h_i^t,\sigma\right)$, every strategy profile σ with $X_i\left(h_i^t,\sigma\right)=X_i^*$ for all h_i^t on the path of play of σ must satisfy $\sigma_i\left(h_i^t\right)=X_i^*/\alpha$ for all h_i^t on the path of play of σ. We claim that the following grim trigger strategy profile σ^* with this path of play satisfies the constraints of the relaxed problem (and therefore solves the relaxed problem): for all i, let $\sigma_i^*\left(h_i^t\right)=0$ if there exists a $z_{i,j,\tau}\in h_i^t$ such that $z_{i,j,\tau}\notin\left\{X_j^*/\alpha,\emptyset\right\}$, and let $\sigma_i^*\left(h_i^t\right)=X_i^*/\alpha$ otherwise.

Under σ^*, the constraint of the relaxed problem ((8)) becomes

$$c\left(X_i^*/\alpha\right)\leq$$
$$\sum_{j=1}^{N}\alpha\left(1-\delta\right)\sum_{\tau=t}^{\infty}\delta^{\tau-t}\Pr\left(j\in D\left(\tau,t,i\right)\right)\left(\mathbb{E}^i\left[\sigma_j^*\left(h_j^\tau\right)\middle|\,h_i^t;\sigma_i^*;j\in D\left(\tau,t,i\right)\right]-\mathbb{E}^i\left[\sigma_j^*\left(h_j^\tau\right)\middle|\,h_i^t;0;j\in D\left(\tau,t,i\right)\right]\right)$$

for all i. This holds trivially if $X_i^* = 0$, so assume that $X_i^* > 0$. Suppose that player i deviates to playing $x_i = 0$ at history h_i^t, that $\Pr(j \in D(\tau, t, i)) > 0$, and that the realization of the monitoring technology up to time τ, which we denote by ω^τ, is such that $j \in D(\tau, t, i)$ given ω^τ and $\Pr\left(\left\{\{O(i,t)\}_{i=1}^N\right\}_{t=0}^\tau = \omega^\tau\right) > 0$. We claim that $\sigma_j^*\left(h_j^\tau\right) = 0$ given ω^τ. This claim is trivial if $X_j^* = 0$, so assume that $X_j^* > 0$. Proceed by induction on τ: If $\tau = t+1$, then player j observes player i's off-path play at time t given ω^τ, and therefore $\sigma_j^*\left(h_j^\tau\right) = 0$. Suppose that the claim holds for all $\tau \leq T$, and consider the case where $\tau = T+1$. Since $j \in D(T+1, t, i)$, player j observes the action of some player $k \in D(T, t, i)$ at time T given ω^τ, and the fact that $\Pr\left(\left\{\{O(i,t)\}_{i=1}^N\right\}_{t=0}^\tau = \omega^\tau\right) > 0$ implies that $\Pr(j \in D(T+1, T, k)) > 0$. Since $X_j^* > 0$, the fact that $\Pr(j \in D(T+1, T, k)) > 0$ implies that $X_k^* > 0$, by the definition of ϕ. Therefore, by the inductive hypothesis, $\sigma_k^*\left(h_j^T\right) = 0$ given ω^τ, which is off the equilibrium path of σ^*. This implies that $\sigma_j^*\left(h_j^\tau\right) = 0$, completing the proof of the claim.

Therefore, when $X_i^* > 0$, the constraint of the relaxed problem becomes

$$\begin{aligned} c\left(X_i^*/\alpha\right) &\leq \sum_{j=1}^N \alpha(1-\delta) \sum_{\tau=t}^\infty \delta^{\tau-t} \Pr(j \in D(\tau,t,i))\, \mathbb{E}^i\left[\sigma_j^*\left(h_j^\tau\right) \middle| h_i^t; \sigma_i^*; j \in D(\tau,t,i)\right] \\ &= \sum_{j=1}^N \sum_{\tau=t}^\infty \delta^{\tau-t} \left(\Pr(j \in D(\tau,t,i)) - \Pr(j \in D(\tau-1,t,i))\right) X_j^*, \end{aligned}$$

which holds with equality by the definition of $\vec{X}^*$. Therefore, σ^* satisfies the constraints of the relaxed problem, and therefore solves the relaxed problem. Furthermore, every strategy profile that solves the relaxed problem has the same path of play as σ^*, σ^* is a grim-trigger strategy profile, and at every history h_i^t on the path of play of σ^*, player i is indifferent between following σ^* at h_i^t and deviating to $x_i = 0$ forever.

We now claim that σ^* also solves the full problem, i.e., that σ (together with consistent beliefs) is a SE. Clearly, no deviation at an on-path history is more profitable for player i than setting $x_i = 0$, and the constraint of the relaxed problem guarantees that this deviation is not profitable, so we must only check that no player has a profitable deviation at an off-path history (by the one-shot deviation principle). We claim that in any grim trigger strategy in which players are indifferent between conforming and playing $x = 0$ on-path, every player weakly prefers playing $x = 0$ at every off-path history. This follows from the standard argument (originally due to Ellison, 1994) that a player's incentive to play $x_i^* > 0$ rather than 0 in a grim trigger strategy profile is reduced after a deviation by another player. Formally, we establish this fact in our setting, using notation similar to Ellison's: Let $D(t)$ be the set of players such that there exists a $z_{i,j,\tau} \in h_i^t$ such that $z_{i,j,\tau} \notin \left\{ X_j^*/\alpha, \emptyset \right\}$ (i.e., the set of players "in the defection phase" at time t). Recall that ω denotes a realization of the monitoring technology. For $\tau \geq t$, define $C(\tau, D(t), \omega)$ by

$$\begin{aligned} C(t, D(t), \omega) &= N \backslash D(t) \\ C(\tau+1, D(t), \omega) &= \{ i \in C(\tau, D(t), \omega) : i \notin O(j, \tau) \text{ for all } j \notin C(\tau, D(t), \omega) \}. \end{aligned}$$

Define $\tilde{D}(\tau, i^*, \omega)$ by

$$\begin{aligned} \tilde{D}(t, i^*, \omega) &= \{i^*\} \\ \tilde{D}(\tau+1, i^*, \omega) &= \left\{ \tilde{D}(\tau, i^*, \omega) \cup i : i \in O(j, \tau) \text{ for some } j \in \tilde{D}(\tau, i^*, \omega) \right\}. \end{aligned}$$

Given any ω, the benefit to player i^* from playing $x_i^* > 0$ rather than 0 at time t is the expected present value of the contributions made by players in $C(\tau, D(t), \omega) \cap \tilde{D}(\tau, i^*, \omega)$. For all t, τ and ω, the set of such players is smaller when $D(t)$ is larger, as $C(\tau, D(t), \omega) \subseteq C(\tau, D'(t), \omega)$ if $D(t) \supseteq D'(t)$. Therefore, the benefit to player i^* from playing $x_{i^*}^* > 0$ off-path is weakly smaller than her benefit from playing $x_{i^*}^*$ on-path, so the fact that she is indifferent between playing $x_{i^*}^*$ and 0 on-path implies that she weakly prefers to play 0 off-path.

We have shown that there exists a grim trigger strategy profile σ^* that sustains the MELP. Furthermore, any strategy profile that solves the relaxed problem must have the same path of play as σ^*, which implies that any strategy profile that sustains the MELP must also have the same path of play as σ^*. Finally, each player i's maximum equilibrium contribution is bounded from above by the maximum of $X_i\left(h_i^0, \sigma\right)/\alpha$ over $\left\{X_i\left(h_i^t, \sigma\right)\right\} \leq \bar{\mathbf{X}}$ satisfying (11), which equals X_i^*/α by isotonicity of $\phi(\cdot)$. This implies that player i's maximum equilibrium contribution equals X_i^*/α, and that this is sustained by σ^*.

Finally, we sketch a proof of the fact that any level of public good provision below X^* can be sustained in sequential equilibrium if public randomizations are available. Fix a grim trigger strategy profile σ^* that sustains the MELP and $X < X^*$, and let $\sigma_i\left(h_i^t\right) \equiv \frac{X}{X^*}\sigma^*\left(h_i^t\right)$ for all h_i^t. The level of public good provision under σ equals X. We first claim that players do not have a profitable on-path deviation under σ. To see this, note that no deviation at h_i^t is more profitable than setting $x_i = 0$, so it suffices to show that

$$(1-\delta)\sum_{\tau=t}^{\infty}\delta^{\tau-t}\mathbb{E}^i\left[c\left(\sigma_i\left(h_i^\tau\right)\right)\middle| h_i^t;\sigma_i\right] \leq$$
$$\sum_{j=1}^{N}\alpha(1-\delta)\sum_{\tau=t}^{\infty}\delta^{\tau-t}\Pr\left(j\in D(\tau,t,i)\right)\mathbb{E}^i\left[\sigma_j\left(h_j^\tau\right)\middle| h_i^t;\sigma_i; j\in D(\tau,t,i)\right], \quad (12)$$

which is the same as (9). Since σ^* is a SE,

$$(1-\delta)\sum_{\tau=t}^{\infty}\delta^{\tau-t}\mathbb{E}^i\left[c\left(\sigma_i^*\left(h_i^\tau\right)\right)\middle|\, h_i^t;\sigma_i^*\right] \leq$$

$$\sum_{j=1}^{N}\alpha\left(1-\delta\right)\sum_{\tau=t}^{\infty}\delta^{\tau-t}\Pr\left(j\in D\left(\tau,t,i\right)\right)\mathbb{E}^i\left[\sigma_j^*\left(h_j^\tau\right)\middle|\, h_i^t;\sigma_i^*;j\in D\left(\tau,t,i\right)\right], \quad (13)$$

The right-hand side of (12) is X/X^* times the right-hand side of (13). Since $c(0)=0$ and $c(\cdot)$ is convex, the left-hand side of (12) is less than X/X^* times the left-hand side of (13). Therefore, (12) is satisfied.

Since players weakly prefer to conform to σ on-path and public randomizations are available, there exists a two-phase strategy profile of the form described in Ellison (1994, p. 571) such that every player i is indifferent between conforming to σ and playing $x_i=0$ on-path. Since we have shown that a player's benefit from playing $x_i=0$ is weakly higher off-path than on-path, this implies that players weakly prefer playing $x_i=0$ off-path under such a two-phase strategy profile. Therefore, there exists a two-phase strategy profile that is a sequential equilibrium and that has on-path play identical to σ; this strategy profile is a sequential equilibrium that sustains the level of public good provision X. ■

Proof of "Sufficient" Direction of Theorem 2. Consider the map $\phi(\cdot)$ defined in the proof of Theorem 1. We claim that equal observability implies that, if $\vec{X}$ is symmetric, in that $X_i\left(h_i^t\right) = X_1\left(h_1^t\right)$ for all players i and histories h_i^t and h_1^t of the same length, then $\phi\left(\vec{X}\right)$ is symmetric. To see this, suppose that $\vec{X}$ is symmetric, and note that

$$\begin{aligned}
&\sum_{j=1}^{N}\sum_{\tau=t}^{\infty}\delta^{\tau-t}\left(\Pr j\in D\left(\tau,t,i\right)-\Pr\left(j\in D\left(\tau-1,t,i\right)\right)\right)\mathbb{E}^i\left[X_j\left(h_i^\tau\right)\middle|\, h_i^t;j\in D\left(\tau,t,i\right)\backslash D\left(\tau-1,t,i\right)\right]\\
=\;&\sum_{j=1}^{N}\sum_{\tau=t}^{\infty}\delta^{\tau-t}\left(\Pr j\in D\left(\tau,t,i\right)-\Pr\left(j\in D\left(\tau-1,t,i\right)\right)\right)X_1\left(h_1^t\right)\\
=\;&\sum_{\tau=t}^{\infty}\delta^{\tau-t}\left(\left(\mathbb{E}\left[\#D\left(\tau-t,i\right)\right]-1\right)-\left(\mathbb{E}\left[\#D\left(\tau-t-1,i\right)\right]-1\right)\right)X_1\left(h_1^t\right),
\end{aligned}$$

which is the same for all i, by equal observability. The definition of $\phi_{i,h_i^t}\left(\vec{X}\right)$ then implies that $\phi\left(\vec{X}\right)$ is symmetric.

As in the proof of Theorem 1, let $\phi^k\left(\vec{X}\right)$ be the map obtained from iterating $\phi(\cdot)$ k times on $\vec{X}$, and let $\phi_{i,h_i^t}^k\left(\vec{X}\right)$ be the $\left(i,h_i^t\right)^{\text{th}}$ coordinate of $\phi^k\left(\vec{X}\right)$. Note that $\bar{\mathbf{X}}$ is symmetric as well as stationary. Therefore, for all k, $\phi^k\left(\bar{\mathbf{X}}\right)$ is symmetric and stationary, and $\phi_{i,h_i^t}^k\left(\bar{\mathbf{X}}\right) \geq \phi_{i,h_i^t}^{k+1}\left(\bar{\mathbf{X}}\right) \geq$

$X_i^*\left(h_i^t\right)$. Therefore, the sequence $\left\{\phi^k\left(\bar{\mathbf{X}}\right)\right\}_k$ converges monotonically to $\vec{X}^*$, and is symmetric and stationary at every step, which implies that $\vec{X}^*$ is symmetric and stationary.

Finally, the same argument as in the proof of Theorem 1 implies that there exists a SE in grim trigger strategies with $\sigma_i^*\left(h_i^t\right) = X_i^*$ for all i, h_i^t on-path, and we have shown that this yields an upper bound on the MELP, so it must yield the MELP itself; and the same argument as in the proof of Theorem 1 implies that this is the equilibrium path of any strategy profile that sustains the MELP. Since $\vec{X}^*$ is symmetric, $X_i^* = X_1^*$ for all i, so $\sigma_i^*\left(h_i^t\right) = \sigma_1^*\left(h_1^t\right)$ for all i, h_i^t, h_1^t. Thus, σ^* is symmetric. ■

Proof of Theorem 3. $x^*(\Gamma') > x^*(\Gamma)$ means that the highest zero of $\alpha(\Gamma')(1-\delta)\sum_{t=0}^{\infty}\delta^t\mathbb{E}[\#D(t,\Gamma')]x - c(x)$ is greater than the highest zero of $\alpha(\Gamma)(1-\delta)\sum_{t=0}^{\infty}\delta^t\mathbb{E}[\#D(t,\Gamma)]x - c(x)$. So if $x^*(\Gamma') > x^*(\Gamma)$ then, by concavity of $\alpha(\Gamma)(1-\delta)\sum_{t=0}^{\infty}\delta^t\mathbb{E}[\#D(t,\Gamma)]x - c(x)$,

$$\alpha\left(\Gamma'\right)(1-\delta)\sum_{t=0}^{\infty}\delta^t\mathbb{E}\left[\#D\left(t,\Gamma'\right)\right]x^*(\Gamma) - c\left(x^*(\Gamma)\right) > 0$$

and

$$\alpha(\Gamma)(1-\delta)\sum_{t=0}^{\infty}\delta^t\mathbb{E}[\#D(t,\Gamma)]x^*(\Gamma) - c\left(x^*(\Gamma)\right) = 0.$$

Therefore, $\alpha(\Gamma')\sum_{t=0}^{\infty}\delta^t\mathbb{E}[\#D(t,\Gamma')] > \alpha(\Gamma)\sum_{t=0}^{\infty}\delta^t\mathbb{E}[\#D(t,\Gamma)]$.

Similarly, if $\alpha(\Gamma')\sum_{t=0}^{\infty}\delta^t\mathbb{E}[\#D(t,\Gamma')] > \alpha(\Gamma)\sum_{t=0}^{\infty}\delta^t\mathbb{E}[\#D(t,\Gamma)]$, then

$$\alpha\left(\Gamma'\right)(1-\delta)\sum_{t=0}^{\infty}\delta^t\mathbb{E}\left[\#D\left(t,\Gamma'\right)\right]x - c(x) > \alpha(\Gamma)(1-\delta)\sum_{t=0}^{\infty}\delta^t\mathbb{E}[\#D(t,\Gamma)]x - c(x)$$

for all $x \geq 0$, which implies that the highest zero of $\alpha(\Gamma')(1-\delta)\sum_{t=0}^{\infty}\delta^t\mathbb{E}[\#D(t,\Gamma')]x - c(x)$ is greater than the highest zero of $\alpha(\Gamma)(1-\delta)\sum_{t=0}^{\infty}\delta^t\mathbb{E}[\#D(t,\Gamma)]x - c(x)$ (since both of these are positive, by Corollary 1). That is, $x^*(\Gamma') > x^*(\Gamma)$. ■

Proof of Proposition 2. As N converges to ∞, the fraction of players who observe an initial defection in period 0 converges to p almost surely. Therefore, for any $\varepsilon > 0$, there exists $\bar{N}(\varepsilon) > 0$ such that, if $N > \bar{N}(\varepsilon)$, the probability that player $N+1$ is in $D(2, N+1)$ is at least $1 - (1+\varepsilon)(1-p)^{pN}$. Furthermore, the probability that any player $i \in \{1, \ldots, N\}$ is in $D(t, N+1)$ is weakly higher than the probability that she is in $D(t, N)$, for all t. Therefore, if $N > \bar{N}(\varepsilon)$, then

$$\begin{aligned}
& (1-\delta)\sum_{t=0}^{\infty}\delta^t\mathbb{E}[\#D(t,N+1)]x - c(x) - \left((1-\delta)\sum_{t=0}^{\infty}\delta^t\mathbb{E}[\#D(t,N)]x - c(x)\right) \\
= \; & (1-\delta)\sum_{t=0}^{\infty}\delta^t\left(\mathbb{E}[\#D(t,N+1)] - \mathbb{E}[\#D(t,N)]\right)x \\
\geq \; & \delta^2\left(1-(1+\varepsilon)(1-p)^{pN}\right)x.
\end{aligned}$$

This converges to $\delta^2 x$ as N converges to ∞. Since $(1-\delta)\sum_{t=0}^{\infty}\delta^t\left(\mathbb{E}\left[\#D\left(t,N+1\right)\right]-\mathbb{E}\left[\#D\left(t,N\right)\right]\right)x$ is increasing in x, this implies that $c\left(x^*\left(N+1\right)\right)-c\left(x^*\left(N\right)\right)$ is at least $\delta^2 x^*\left(N\right)$, which is bounded away from 0. ■

Proof of Proposition 3. By Theorem 2, $x^*\left(N'\right)>(<)x^*\left(N\right)$ if

$$(1-\delta)\sum_{t=0}^{\infty}\delta^t\mathbb{E}\left[\#D\left(t,N'\right)\right]/N'>(<)(1-\delta)\sum_{t=0}^{\infty}\delta^t\mathbb{E}\left[\#D\left(t,N\right)\right]/N,$$

or, since the initial deviator is always in $D\left(t,N\right)$,

$$\frac{1}{N'}+(1-\delta)\sum_{t=1}^{\infty}\delta^t\mathbb{E}\left[\#D\left(t,N'\right)-1\right]/N'>(<)\frac{1}{N}+(1-\delta)\sum_{t=1}^{\infty}\delta^t\mathbb{E}\left[\#D\left(t,N\right)-1\right]/N. \quad (14)$$

It is immediate that the left-hand side of (14) is strictly less than the right-hand side of (14) for sufficiently small $\delta>0$. Choosing such a δ and then taking $c'(0)$ close enough to $1/N'$ that Assumption 1 is satisfied then yields an example where $x^*\left(N'\right)<x^*\left(N\right)$.

Next, note that as $N\to\infty$, $\#D\left(1,N\right)/N\overset{a.s.}{\to}p$, by the strong law of large numbers, since all players except the initial deviator are in $D\left(1,N\right)$ with independent probability p. Let $\varepsilon\in(0,p)$. Then $\Pr\left(i\in D\left(2,N\right)|\#D\left(1,N\right)/N>p-\varepsilon\right)\to 1$ as $N\to\infty$ for all i, independently across i. Therefore, by the weak law of large numbers, $\mathbb{E}\left[\#D\left(2,N\right)|\#D\left(1,N\right)/N>p-\varepsilon\right]/N\to 1$ as $N\to\infty$. So, since $\#D\left(1,N\right)/N\overset{a.s.}{\to}p$, we have that $\mathbb{E}\left[\#D\left(2,N\right)\right]/N\to 1$ as $N\to\infty$. Therefore, as $N'\to\infty$ the left-hand side of (14) converges to $(1-\delta)\,\delta p+\delta^2$. And if $N=2$, the right-hand side of (14) equals

$$\frac{1}{2}\left(1+(1-\delta)\frac{\delta p}{1-\delta\left(1-p\right)}\right),$$

which is less than $(1-\delta)\,\delta p+\delta^2$ if p is close to 0 and δ is slightly larger than $\sqrt{1/2}$. Therefore, an example exists where $x^*\left(N'\right)>x^*\left(N\right)$. ■

Proof of Proposition 4. By Theorem 3, $x^*\left(N\right)$ is strictly increasing in N if $(1-\delta)\sum_{t=0}^{\infty}\delta^t\mathbb{E}\left[\#D\left(t,N\right)\right]x$ is strictly increasing in N. With quasi-public monitoring, we have

$$\begin{aligned}(1-\delta)\sum_{t=0}^{\infty}\delta^t\mathbb{E}\left[\#D\left(t,N\right)\right]x &= x+\delta\sum_{t=0}^{\infty}\delta^t p\left(N\right)\left(1-p\left(N\right)\right)^t\left(N-1\right)x\\ &= x+\frac{\delta p\left(N\right)}{1-\delta\left(1-p\left(N\right)\right)}\left(N-1\right)x\\ &= \frac{1-\delta+\delta Np\left(N\right)}{1-\delta\left(1-p\left(N\right)\right)}x,\end{aligned}$$

so $x^*\left(N\right)$ is increasing in N if $\frac{1-\delta+\delta Np(N)}{1-\delta(1-p(N))}$ is increasing in N. This is the case if

$$\frac{1-\delta+\delta\left(N+1\right)p\left(N+1\right)}{1-\delta\left(1-p\left(N+1\right)\right)}>\frac{1-\delta+\delta Np\left(N\right)}{1-\delta\left(1-p\left(N\right)\right)}$$

for all N, which, with a little algebra, can be rearranged as (5).

The argument for $x^*(N)$ strictly decreasing is identical. ■

Proof of Proposition 5. If $p(N) = \beta N^{\zeta}$, Proposition 4 implies that a sufficient condition for $x^*(N)$ to be increasing ((5)) becomes

$$\beta (N+1)^{\zeta} - \beta N^{\zeta} > -\left(\frac{1-\delta\left(1-\beta N^{\zeta}\right)}{1-\delta}\right)\frac{\beta (N+1)^{\zeta}}{N-1}.$$

Dividing both sides by $\beta (N+1)^{\zeta}$ and rearranging gives

$$\left(\frac{N}{N+1}\right)^{\zeta} < 1 + \left(\frac{1-\delta\left(1-\beta N^{\zeta}\right)}{1-\delta}\right)\left(\frac{1}{N-1}\right).$$

If $\zeta \geq -1$, the left-hand side of this inequality is not more than $\frac{N+1}{N}$. Therefore, a sufficient condition for $x^*(N)$ to be increasing for all $\zeta \geq -1$ is

$$\frac{N+1}{N} < 1 + \left(\frac{1-\delta\left(1-\beta N^{\zeta}\right)}{1-\delta}\right)\left(\frac{1}{N-1}\right),$$

or

$$1 < \left(\frac{1-\delta\left(1-\beta N^{\zeta}\right)}{1-\delta}\right)\left(\frac{N}{N-1}\right).$$

Since $\frac{1-\delta\left(1-\beta N^{\zeta}\right)}{1-\delta} > 1$ for all N and all $\zeta \leq 0$, and $\frac{N}{N-1} > 1$, this condition holds.

Similarly, rearranging (5) implies that a sufficient condition for $x^*(N)$ to be decreasing is

$$\left(\frac{N}{N+1}\right)^{\zeta} > 1 + \left(\frac{1-\delta\left(1-\beta N^{\zeta}\right)}{1-\delta}\right)\left(\frac{1}{N-1}\right),$$

which can be further rearranged as

$$\zeta < \frac{\ln\left(1 + \left(\frac{1-\delta\left(1-\beta N^{\zeta}\right)}{1-\delta}\right)\left(\frac{1}{N-1}\right)\right)}{\ln\left(\frac{N}{N+1}\right)}.$$

Using L'Hopital's rule, it can be shown that the limit as $N \to \infty$ of the right-hand side of this inequality equals -1 whenever $\zeta < 0$. Therefore, if $\zeta < -1$ there exists $\bar{N}(\zeta) > 0$ such that this inequality is satisfied for all $N > \bar{N}(\zeta)$. ■

Proof of Proposition 6. Using the computation of $\sum_{t=0}^{\infty} \delta^t \mathbb{E}[\#D(t,N)]$ from the proof of Proposition 4,

$$\begin{aligned} & (1-\delta)\sum_{t=0}^{\infty} \delta^t \mathbb{E}[\#D(t,N+1)]\,x - c(x) - \left((1-\delta)\sum_{t=0}^{\infty} \delta^t \mathbb{E}[\#D(t,N)]\,x - c(x)\right) \\ = & \left(\frac{\delta p}{1-\delta(1-p)}\right)x. \end{aligned}$$

Therefore, since $(1-\delta)\sum_{t=0}^{\infty}\delta^t\mathbb{E}[\#D(t,N)]\,x$ is increasing in x, $c(x^*(N+1)) - c(x^*(N)) \geq \left(\frac{\delta p}{1-\delta(1-p)}\right)x^*(N)$ (recalling that $c(x^*(N)) = (1-\delta)\sum_{t=0}^{\infty}\delta^t\mathbb{E}[\#D(t,N)]\,x^*$), which is bounded away from 0, since $x^*(N)$ is increasing. ■

Proof of Proposition 7. Following the proof of Proposition 4 with $\alpha(N) = 1/N$, we see that $x^*(N)$ is strictly decreasing in N if $x - c(x) - \left(\frac{1-\delta}{1-\delta(1-p(N))}\right)\left(\frac{N-1}{N}\right)x$ is strictly decreasing in N. By inspection, this is the case if $p(N)$ is non-increasing. ■

Proof of Proposition 8. By the same argument as in the proof of Proposition 4, $x^*(N)$ is the highest zero of $x - c(x) - \left(\frac{1-\delta}{1-\delta(1-p)}\right)\left(\frac{N-1}{N}\right)x$, which converges to $\left(\frac{\delta p}{1-\delta(1-p)}\right)x - c(x)$ as $N \to \infty$. Therefore, $x^*(N)$ converges to 0 as $N \to \infty$ if $c'(0) \geq \frac{\delta p}{1-\delta(1-p)}$ and converges to a positive number otherwise. ■

Proof of Proposition 9. The result is immediate if N increases from 1 to 2, so we restrict attention to $N \geq 3$. By Corollary 2, it suffices to show that $\mathbb{E}[\#D(t,N)]$ is non-decreasing in N for all t and strictly increasing in N for all $t \geq 2$. $\mathbb{E}[\#D(0,N)] = 1$ and $\mathbb{E}[\#D(1,N)] = 2$, for all N, so it suffices to show that $\mathbb{E}[\#D(t,N)]$ is strictly increasing for all $t \geq 2$. Let $D(l,t,k,N)$ be the probability that there are at least l defectors t periods in the future when there are k defectors today out of N total players. We claim that $D(\cdot,t,k,N)$ is increasing in N in the first-order stochastic dominance sense, for all $t \geq 2$ and $k \geq 1$, which suffices for the result. First, $D(l,t,k,N)$ is increasing in k for all t and N, as the number of deviators in period t is non-decreasing in k for all realizations of the monitoring technology ω, and is strictly increasing in k with positive probability. Second, $D(\cdot,1,k,N)$ is increasing in the first-order stochastic dominance sense in N if $k \geq 2$, as increasing N weakly increases the number of deviators in period 1 for all ω, when ω is interpreted as first randomly assigning matches among players in the larger set and then randomly rematching those players whose partners are not in the smaller population. Now fix $N' > N$ and use induction on t: If $D(\cdot,t-1,k,N')$ first-order stochastically dominates $D(\cdot,t-1,k,N)$, then there exists a measure-preserving bijection between realizations ω and ω' such that $\#D(t-1,k,N',\omega') > \#D(t-1,k,N,\omega)$ for all ω. Then, since $N' > N$, $D(\cdot,1,\#D(t-1,k,N',\omega'),N')$ first-order stochastically dominates $D(\cdot,1,\#D(t-1,k,N,\omega),N)$, which means that the distribution $D(\cdot,t,k,N'|\omega')$, which is conditional on the realization ω' of the monitoring technology up to time $t-1$, first-order stochastically dominates the distribution $D(\cdot,t,k,N|\omega)$, which is conditional on the realization ω of the monitoring technology up to time $t-1$, for any ω. This implies that the unconditional distribution $D(\cdot,t,k,N')$ first-order stochastically dominates $D(\cdot,t,k,N)$, completing the proof. ■

Proof of Proposition 10. Suppose that $\delta < 1/2$. Note that $\mathbb{E}[\#D(t,N)] \leq 2^t$ for all t, N. Therefore,

$$\begin{aligned}(1-\delta)\sum_{t=0}^{\infty}\delta^t\mathbb{E}[\#D(t,N)] &\leq (1-\delta)\sum_{t=0}^{\infty}\delta^t 2^t \\ &= \frac{1-\delta}{1-2\delta},\end{aligned}$$

so $c(x^*(N)) \leq \frac{1-\delta}{1-2\delta}x^*(N)$. Since $\lim_{x\to\infty} c'(x) = \infty$, this implies that $c(x^*(N))$ is bounded (and therefore converges, since it is increasing).

Suppose that $\delta \geq 1/2$. For every T, the probability that any two defectors match with each other within the first T periods after an initial deviation converges to 0 as $N \to \infty$. Therefore, for any $T > 0$ and $\varepsilon > 0$, there exists $\bar{N} > 0$ such that, for all $N \geq \bar{N}$,

$$\begin{aligned}(1-\delta)\sum_{t=0}^{\infty}\delta^t\mathbb{E}[\#D(t,N)] &\geq (1-\delta)(1-\varepsilon)\sum_{t=0}^{T}\delta^t 2^t \\ &\geq (1-\delta)(1-\varepsilon)T.\end{aligned}$$

So, for any T, there exists $\bar{N} > 0$ such that, for all $N \geq \bar{N}$, $c(x^*(N)) \geq (1-\delta)Tx^*(N)$. Therefore, $\lim_{N\to\infty} c(x^*(N)) = \infty$. ■

Proof of Proposition 11. By Theorem 3, to show that $x^*(N') < x^*(N)$ it suffices to show that

$$\sum_{t=0}^{\infty}\delta^t\frac{\mathbb{E}[\#D(t,N')]}{N'} < \sum_{t=0}^{\infty}\delta^t\frac{\mathbb{E}[\#D(t,N)]}{N},$$

or

$$\sum_{t=0}^{\infty}\delta^t\left(N\mathbb{E}\left[\#D\left(t,N'\right)\right] - N'\mathbb{E}[\#D(t,N)]\right) < 0.$$

For any t and any $\varepsilon > 0$, there exists $\bar{N}''$ such that $\mathbb{E}[\#D(t,N')] - \mathbb{E}[\#D(t,N)] < \varepsilon$ for any $N', N \geq \bar{N}''$, as the probability that any two defectors match with each other within the first t periods when there are either N' or N players converges to 0 as $N \to \infty$. Furthermore, $\mathbb{E}[\#D(t,N')] - \mathbb{E}[\#D(t,N)] \leq 2^t$, since $\#D(t,N',\omega) \leq 2^t$ for all ω and $\#D(t,N) \geq 0$. Therefore, for any $\varepsilon' > 0$, there exists $\bar{N}'$ such that $\sum_{t=0}^{\infty}\delta^t\mathbb{E}[\#D(t,N') - \#D(t,N)] < \varepsilon'$ for any $N', N \geq \bar{N}'$, as each of the first T terms in the sum converges to 0 as $\bar{N}' \to \infty$, for any T, and the sum of the remaining terms is less than $\sum_{t=T}^{\infty}\delta^t 2^t = \frac{(2\delta)^T}{1-2\delta}$, which converges to 0 as $T \to \infty$, under the assumption that $\delta < \frac{1}{2}$. Let $\varepsilon' = \gamma$, let $\bar{N}'(\gamma)$ be the corresponding $\bar{N}'$, and let $\bar{N} \equiv (1+\gamma)\bar{N}'(\gamma)$, which guarantees that

$N \geq \bar{N}'(\gamma)$ if $(1+\gamma)N \geq \bar{N}$. Then,

$$\begin{aligned}
& \sum_{t=0}^{\infty} \delta^t \left(N\mathbb{E}\left[\#D\left(t, N'\right)\right] - N'\mathbb{E}\left[\#D\left(t, N\right)\right] \right) \\
= \; & N \sum_{t=0}^{\infty} \delta^t \left(\mathbb{E}\left[\#D\left(t, N'\right)\right] - \mathbb{E}\left[\#D\left(t, N\right)\right] \right) \\
& - \left(N' - N\right) \sum_{t=0}^{\infty} \delta^t \mathbb{E}\left[\#D\left(t, N\right)\right] \\
\leq \; & N\gamma - \left(N' - N\right) \sum_{t=0}^{\infty} \delta^t \mathbb{E}\left[\#D\left(t, N\right)\right] \\
\leq \; & N\gamma - \left(N' - N\right) \\
< \; & 0,
\end{aligned}$$

where the first inequality follows because $N', N \geq \bar{N}'(\gamma)$, the second inequality follows because $\sum_{t=0}^{\infty} \delta^t \mathbb{E}\left[\#D\left(t, N\right)\right] \geq 1$, and the third inequality follows because $N' > (1+\gamma)N$, completing the proof. ■

Proof of Proposition 12. Note that $\mathbb{E}\left[\#D\left(t, N\right)\right] \leq 2^t$ for all t, N. Therefore,

$$\begin{aligned}
(1-\delta) \sum_{t=0}^{\infty} \delta^t \frac{\mathbb{E}\left[\#D\left(t, N\right)\right]}{N} \; & \leq \; \frac{1}{N}(1-\delta) \sum_{t=0}^{\infty} \delta^t 2^t \\
& = \; \frac{1}{N}\left(\frac{1-\delta}{1-2\delta}\right).
\end{aligned}$$

So $c\left(x^*(N)\right) \leq \frac{1}{N}\left(\frac{1-\delta}{1-2\delta}\right) x^*(N)$. Since $x^*(N)$ is bounded from above, by Proposition 11, this implies that $c\left(x^*(N)\right) \to 0$ as $N \to \infty$, which implies that $x^*(N) \to 0$ as $N \to \infty$. ■

Proof of Proposition 14. Note that $\mathbb{E}\left[\#D\left(t, N\right)\right] \leq 1 + 2kt$ for all t, N. Therefore,

$$\begin{aligned}
(1-\delta) \sum_{t=0}^{\infty} \delta^t \mathbb{E}\left[\#D\left(t, N\right)\right] \; & \leq \; 1 + (1-\delta) \sum_{t=0}^{\infty} \delta^t 2kt \\
& = \; 1 + \frac{\delta}{1-\delta} 2k.
\end{aligned}$$

So $c\left(x^*(N)\right) \leq \left(1 + \frac{\delta}{1-\delta} 2k\right) x^*(N)$. Since $\lim_{x \to \infty} c'(x) = \infty$, this implies that $x^*(N)$ is bounded. ■

Proof of Proposition 16. Note that $\mathbb{E}\left[\#D\left(t, N\right)\right] \leq 1 + 2kt$ for all t, N. Therefore, as in the proof of Proposition 14,

$$(1-\delta) \sum_{t=0}^{\infty} \delta^t \frac{\mathbb{E}\left[\#D\left(t, N\right)\right]}{N} \leq \frac{1}{N}\left(1 + \frac{\delta}{1-\delta} 2k\right).$$

So $c(x^*(N)) \leq \frac{1}{N}\left(1+\frac{\delta}{1-\delta}2k\right)x^*(N)$. Since $x^*(N)$ is decreasing, by Proposition 15, this implies that $c(x^*(N)) \to 0$ as $N \to \infty$, which implies that $x^*(N) \to 0$ as $N \to \infty$. ■

Proof of Theorem 4. We first introduce some new notation: let

$g_k^t(k') \equiv \Pr(\#D(t) = k' | \#D(0) = k)$, let $G_k^t(k')$ be the corresponding distribution function, and let $\mathbb{E}_{g_k^t}[k'] \equiv \sum_{k'=0}^{N} k' g_k^t(k')$. That is, $g_k^t(k')$ is the probability that there will be k' defectors in t periods when there are k defectors today, and $G_k^t(k')$ and $\mathbb{E}_{g_k^t}[k']$ are the corresponding cumulative distribution function and expected number of defectors.

We first claim that $\tilde{G}_k^t(k')$ strictly second-order stochastically dominates $G_k^t(k')$ for all $t \geq 1$ and $k \in \{1, \ldots, N-1\}$, which is equivalent to $\sum_{s=0}^{k'} \tilde{G}_k^t(s) < \sum_{s=0}^{k'} G_k^t(s)$ for all $k' \geq k$. We proceed by induction on t. The $t = 1$ case is simply the assumption that $\tilde{G}_k$ strictly second-order stochastically dominates G_k. Assume that $\sum_{s=0}^{k'} \tilde{G}_k^\tau(s) < \sum_{s=0}^{k'} G_k^\tau(s)$ for all $k' \geq k$ and all $\tau \leq t-1$. Then

$$\begin{aligned}
\sum_{s=0}^{k'} \tilde{G}_k^t(s) &= \sum_{s=0}^{k'} \sum_{r=0}^{s} \tilde{g}_k^{t-1}(r)\, \tilde{G}_r(s) \\
&= \sum_{r=0}^{k'} \tilde{g}_k^{t-1}(r) \sum_{s=r}^{k'} \tilde{G}_r(s) \\
&= \sum_{r=0}^{k'} \tilde{g}_k^{t-1}(r) \sum_{s=0}^{k'} \tilde{G}_r(s) \\
&\leq \sum_{r=0}^{k'} \tilde{g}_k^{t-1}(r) \sum_{s=0}^{k'} G_r(s) \\
&< \sum_{r=0}^{k'} g_k^{t-1}(r) \sum_{s=0}^{k'} G_r(s) \\
&= \sum_{s=0}^{k'} G_k^t(s),
\end{aligned}$$

where the first line follows because $\tilde{G}_k^t(s) = \sum_{r=0}^{s} \tilde{g}_k^{t-1}(r)\, \tilde{G}_r(s)$, the second line reverses the order of sums, the third line follows because $\tilde{G}_r(s) = 0$ if $s < r$, the fourth line follows because $\tilde{G}_r(s)$ weakly second-order stochastically dominates $G_r(s)$ for all r, the fifth line follows because $\tilde{G}_k^{t-1}(r)$ strictly second-order stochastically dominates $G_k^{t-1}(r)$ (by the inductive hypothesis) and $\sum_{s=0}^{k'} G_r(s)$ is non-increasing and strictly convex (because $G_r(s)$ is non-increasing and weakly convex for all r, and strictly convex for $r \leq s$, so the sum of such functions is non-increasing and strictly convex for $r \leq k'$), and the sixth line follows from undoing the rearrangement of the first two lines for $G_k^t(s)$ rather than $\tilde{G}_k^t(s)$. This completes the proof of the claim.

By Theorem 3, showing that $\sum_{t=0}^{\infty} \delta^t \mathbb{E}_{\tilde{g}_1^t}[k'] > \sum_{t=0}^{\infty} \delta^t \mathbb{E}_{g_1^t}[k']$ suffices to prove the theorem. Trivially, $\mathbb{E}_{\tilde{g}_1^0}[k'] = \mathbb{E}_{g_1^0}[k'] = 1$, and $\mathbb{E}_{\tilde{g}_1^1}[k'] \geq \mathbb{E}_{g_1^1}[k']$ because $\tilde{G}_k$ second-order stochastically dominates G_k. We claim that $\mathbb{E}_{\tilde{g}_1^t}[k'] > \mathbb{E}_{g_1^t}[k']$ for all $t \geq 2$. This follows because

$$\begin{aligned}
\mathbb{E}_{\tilde{g}_k^t}\left[k'\right] &= \sum_{s=0}^{k'} \tilde{g}_1^{t-1}(s)\, \mathbb{E}_{\tilde{g}_s^1}\left[k'\right] \\
&\geq \sum_{s=0}^{k'} \tilde{g}_1^{t-1}(s)\, \mathbb{E}_{g_s^1}\left[k'\right] \\
&> \sum_{s=0}^{k'} g_1^{t-1}(s)\, \mathbb{E}_{g_s^1}\left[k'\right] \\
&= \mathbb{E}_{g_k^t}\left[k'\right],
\end{aligned}$$

where the first line follows by the law of iterated expectation, the second line follows because $\tilde{G}_s^{t-1}(k')$ second-order stochastically dominates $G_s^{t-1}(k')$ if $t \geq 2$, by Lemma 2, the third line follows because $\mathbb{E}_{g_s^1}[k']$ is non-decreasing and strictly concave in s (since $G_k(k')$ is non-increasing and strictly convex in k, for all $k \leq k'$) and $\tilde{G}_1^{t-1}(s)$ strictly second-order stochastically dominates $G_1^{t-1}(s)$, and the fourth line follows from undoing the rearrangement of the first line. Summing over t completes the proof of the theorem. ■

Proof of Lemma 2. We first claim that if player i is s-more central than player j, then player i is $s-1$-more central than player j. We proceed by induction on s. If i is 2-more central than j, then for all $t = \{0, 1, \ldots\}$ there exists a surjection $\psi : \{k \in N : d(i,k) \leq t\} \rightarrow \{k \in N : d(j,k) \leq t\}$, which implies that $\#\{k \in N : d(i,k) \leq t\} \geq \#\{k \in N : d(j,k) \leq t\}$, so i is 1-more central than j. Suppose now that, if $\tilde{\imath}$ is $s-1$-more central than $\tilde{\jmath}$, then $\tilde{\imath}$ is $s-2$-more central than $\tilde{\jmath}$, for all $\tilde{\imath}$ and $\tilde{\jmath}$, and suppose that i is s-more central than j. Then for all $t = \{0, 1, \ldots\}$ there exists a surjection $\psi : \{k \in N : d(i,k) \leq t\} \rightarrow \{k \in N : d(j,k) \leq t\}$ such that, for all k with $d(j,k) \leq t$, there exists a $k' \in \psi^{-1}(k)$ such that k' is $s-1$-more central than k. By the inductive hypothesis, this implies that k' is $s-2$-more central than k, which, by the definition of $s-1$-more central, implies that i is $s-1$-more central than j. This establishes the claim.

The claim shows that, for any k, the set of players k' such that $d(i,k') \leq t$ and k' is s-more central than k is (weakly) shrinking in s. Since $\{k' \in N : d(i,k') \leq t\}$ and $\{k \in N : d(j,k) \leq t\}$ are finite, this implies that there exists $\bar{s}$ such that, for all k with $d(j,k) \leq t$, the set of players k' such that $d(i,k') \leq t$ and k' is s-more central than k is the same for all $s \geq \bar{s}$. By the above claim, if k' is $\bar{s}$ more-central than k then she is also $\bar{s} - m$-more central than k for all m, so if she is also s-more central than k for all $s > \bar{s}$ then she is more central than k.

If player i is more central than player j, there exists a surjection $\psi : \{k \in N : d(i,k) \leq t\} \rightarrow \{k \in N : d(j,k) \leq t\}$ such that, for all k with $d(j,k) \leq t$, there exists a $k' \in \psi^{-1}(k)$ such that k' is $\bar{s}$-more central than k, and the preceding paragraph shows that such a player k' is more central than k. ■

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