A characterization of farsightedly stable networks [†]

Gilles Grandjean[a], Ana Mauleon[b,c], Vincent Vannetelbosch[c]

[a]CORE, Université catholique de Louvain, 34 voie du Roman Pays, B-1348 Louvain-la-Neuve, Belgium.

[b]FNRS and CEREC, Facultés universitaires Saint-Louis, Boulevard du Jardin Botanique 43, B-1000 Brussels, Belgium.

[c]FNRS and CORE, Université catholique de Louvain, 34 voie du Roman Pays, B-1348 Louvain-la-Neuve, Belgium.

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Abstract

We study the stability of social and economic networks when players are farsighted. We adopt Herings, Mauleon and Vannetelbosch's [*Games and Economic Behavior* 67, 526-541 (2009)] notions of farsightedly stable set and of myopically stable set. We first provide an algorithm that characterizes the unique pairwise and groupwise farsightedly stable set of networks under the componentwise egalitarian allocation rule. We then show that this set coincides with the unique groupwise myopically stable set of networks but not with the unique pairwise myopically stable set of networks. We conclude that, (i) if groupwise deviations are allowed then whether players are farsighted or myopic does not matter; (ii) if players are farsighted then whether players are allowed to deviate in pairs only or in groups does not matter. Finally, we provide some primitive conditions on value functions so that the set of strongly efficient networks belongs to the unique farsightedly stable set.

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1 Introduction

The organization of agents into networks and groups or coalitions plays an important role in the determination of the outcome of many social and economic interactions.[1] For instance, networks of personal contacts are important in obtaining information on goods and services, like product information or information about job opportunities. Many commodities are traded through networks of buyers and sellers. A simple way to analyze the networks that one might expect to emerge in the long run is to examine the requirement that individuals do not benefit from altering the structure of the network. An example of such a condition is the pairwise stability notion defined by Jackson and Wolinsky (1996). A network is pairwise stable if no player benefits from severing one of her links and no two players benefit from adding a link between them, with one benefiting strictly and the other at least weakly. Pairwise stability is a myopic definition. Players are not farsighted in the sense that they do not forecast how others might react to their actions. For instance, the adding or severing of one link might lead to subsequent addition or severing of another link. If individuals have very good information about how others might react to changes in the network, then these are things one wants to allow for in the definition of the stability concept. For instance, a network could be stable because players might not add a link that appears valuable to them given the current network, as that might in turn lead to the formation of other links and ultimately lower the payoffs of the original players.

In this paper we address the question which networks one might expect to emerge in the long run when players are either farsighted or myopic.

Herings, Mauleon and Vannetelbosch (2009) have first extended the Jackson and Wolinsky pairwise stability notion to a new set-valued solution concept, called the pairwise myopically stable set. A set of networks G is pairwise myopically stable (i) if all possible myopic pairwise deviations from any network $g \in G$ to a network outside the set are deterred by the threat of ending worse off or equally well off, (ii) if there exists a *myopic improving path* from any network outside the set leading to some network in the set, and (iii) if there is no proper subset of G satisfying Conditions (i) and (ii). The myopically pairwise stable set is non-empty, unique and contains all pairwise stable networks. They have then introduced the pairwise

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[1]See Jackson (2008) or Goyal (2007) for a comprehensive introduction to the theory of social and economic networks.

farsightedly stable set, to predict which networks may be formed among farsighted players. The definition corresponds to the one of a pairwise myopically stable set with myopic deviations and myopic improving paths replaced by farsighted deviations and farsighted improving paths. A farsighted improving path is a sequence of networks that can emerge when players form or sever links based on the improvement the end network offers relative to the current network. Each network in the sequence differs by one link from the previous one. If a link is added, then the two players involved must both prefer the end network to the current network. If a link is deleted, then it must be that at least one of the two players involved in the link prefers the end network. Similarly, it is straightforward to define the notions of groupwise myopically stable sets and of groupwise farsightedly stable sets.

We first provide an algorithm that characterizes the unique pairwise and groupwise farsightedly stable set of networks under the componentwise egalitarian allocation rule. We then show that this set coincides with the unique groupwise myopically stable set of networks but not with the unique pairwise myopically stable set of networks. We conclude that, (i) if groupwise deviations are allowed then whether players are farsighted or myopic does not matter; (ii) if players are farsighted then whether players are allowed to deviate in pairs only or in groups does not matter. In addition, we show that alternatives notions of farsighted stability also single out G^v as the unique farsighted stable set. We then analyze the possibility of having small transfers among deviating players. Finally, we provide some primitive conditions on value functions so that the set of strongly efficient networks belongs to the unique pairwise farsightedly stable set.

The paper is organized as follows. In Section 2 we introduce some notations and basic properties and definitions for networks. In Section 3 we define the notions of pairwise (groupwise) myopically stable sets and of pairwise (groupwise) farsightedly stable sets. In Section 4 we characterize the unique farsightedly stable set of networks under the componentwise egalitarian allocation rule. In Section 5 we study the relationship between farsighted stability and other concepts of farsighted stability such as the largest consistent set, the von Neumann-Morgenstern farsightedly stable set and the path dominance core. In Section 6 we look at the relationship between farsighted stability and efficiency of networks. In Section 7 we conclude.

2 Networks

Let $N = \{1, \ldots, n\}$ be the finite set of players who are connected in some network relationship. The network relationships are reciprocal and the network is thus modeled as a non-directed graph. Individuals are the nodes in the graph and links indicate bilateral relationships between individuals. Thus, a network g is simply a list of which pairs of individuals are linked to each other. We write $ij \in g$ to indicate that i and j are linked under the network g. Let g^S be the set of all subsets of $S \subseteq N$ of size 2.[2] So, g^N is the complete network. The set of all possible networks or graphs on N is denoted by $\mathbb{G}$ and consists of all subsets of g^N. The network obtained by adding link ij to an existing network g is denoted $g + ij$ and the network that results from deleting link ij from an existing network g is denoted $g - ij$. Let

$$g|_S = \{ij \mid ij \in g \text{ and } i \in S, \, j \in S\}.$$

Thus, $g|_S$ is the network found deleting all links except those that are between players in S. For any network g, let $N(g) = \{i \mid \exists j \text{ such that } ij \in g\}$ be the set of players who have at least one link in the network g. A path in a network $g \in \mathbb{G}$ between i and j is a sequence of players $i_1, \ldots, i_K$ such that $i_k i_{k+1} \in g$ for each $k \in \{1, \ldots, K-1\}$ with $i_1 = i$ and $i_K = j$. A non-empty network $h \subseteq g$ is a component of g, if for all $i \in N(h)$ and $j \in N(h) \setminus \{i\}$, there exists a path in h connecting i and j, and for any $i \in N(h)$ and $j \in N(g)$, $ij \in g$ implies $ij \in h$. The set of components of g is denoted by $C(g)$. Knowing the components of a network, we can partition the players into groups within which players are connected. Let $\Pi(g)$ denote the partition of N induced by the network g.

A value function is a function $v : \mathbb{G} \to \mathbb{R}$ that keeps track of how the total societal value varies across different networks. The set of all possible value functions is denoted by $\mathcal{V}$. An allocation rule is a function $Y : \mathbb{G} \times \mathcal{V} \to \mathbb{R}^N$ that keeps track of how the value is allocated among the players forming a network. It satisfies $\sum_{i \in N} Y_i(g, v) = v(g)$ for all v and g.

Jackson and Wolinsky (1996) have proposed a number of basic properties of value functions and allocation rules. A value function is *component additive* if $v(g) = \sum_{h \in C(g)} v(h)$ for all $g \in \mathbb{G}$. Component additive value functions are the ones for which the value of a network is the sum of the value of its components. An allocation

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[2]Throughout the paper we use the notation $\subseteq$ for weak inclusion and $\subsetneq$ for strict inclusion. Finally, $\#$ will refer to the notion of cardinality.

rule Y is *component balanced* if for any component additive $v \in \mathcal{V}$, $g \in \mathbb{G}$, and $h \in C(g)$, we have $\sum_{i \in N(h)} Y_i(h, v) = v(h)$. Component balancedness only puts conditions on Y for v's that are component additive, so Y can be arbitrary otherwise. Given a permutation of players π and any $g \in \mathbb{G}$, let $g^\pi = \{\pi(i)\pi(j) \mid ij \in g\}$. Thus, g^π is a network that is identical to g up to a permutation of the players. A value function is *anonymous* if for any permutation π and any $g \in \mathbb{G}$, $v(g^\pi) = v(g)$. Given a permutation π, let v^π be defined by $v^\pi(g) = v(g^{\pi^{-1}})$ for each $g \in \mathbb{G}$. An allocation rule Y is *anonymous* if for any $v \in \mathcal{V}$, $g \in \mathbb{G}$, and permutation π, we have $Y_{\pi(i)}(g^\pi, v^\pi) = Y_i(g, v)$.

An allocation rule that is component balanced and anonymous is the *componentwise egalitarian allocation rule*. For a component additive v and network g, the componentwise egalitarian allocation rule Y^{ce} is such that for any $h \in C(g)$ and each $i \in N(h)$, $Y_i^{ce}(g, v) = v(h)/\#N(h)$. For a v that is not component additive, $Y^{ce}(g, v) = v(g)/n$ for all g; thus, Y^{ce} splits the value $v(g)$ equally among all players if v is not component additive.

In evaluating societal welfare, we may take various perspectives. A network g is *Pareto efficient* relative to v and Y if there does not exist any $g' \in \mathbb{G}$ such that $Y_i(g', v) \geq Y_i(g, v)$ for all i with at least one strict inequality. A network $g \in \mathbb{G}$ is *strongly efficient* relative to v if $v(g) \geq v(g')$ for all $g' \in \mathbb{G}$. This is a strong notion of efficiency as it takes the perspective that value is fully transferable.

Which networks are likely to emerge in the long run? The game-theoretic approach to network formation uses two different notions of a deviation by a coalition. *Pairwise deviations* (Jackson and Wolinsky, 1996) are deviations involving a single link at a time. That is, link addition is bilateral (two players that would be involved in the link must agree to adding the link), link deletion is unilateral (at least one player involved in the link must agree to deleting the link), and network changes take place one link at a time. *Groupwise deviations* (Jackson and van den Nouweland, 2005) are deviations involving several links within some group of players at a time. Link addition is bilateral, link deletion is unilateral, and multiple link changes can take place at a time. Whether a pairwise deviation or a groupwise deviation makes more sense will depend on the setting within which network formation takes place. The definitions of stability we consider allow for a deviation by a coalition to be valid only if all members of the coalition are *strictly* better off. It is customary to require that a coalition deviates only if all members are made better off since

changing the status-quo is *costly*, and players have to be compensated for doing so.[3]

3 Definitions of stable sets of networks

3.1 Myopic definitions

We first introduce the notion of pairwise myopically stable sets of networks due to Herings, Mauleon and Vannetelbosch (2009) which is a generalization of Jackson and Wolinsky (1996) pairwise stability notion.[4] Pairwise stable networks do not always exist. A pairwise myopically stable set of networks is a set such that from any network outside this set, there is a myopic improving path leading to some network in the set, and each deviation outside the set is deterred because the deviating players do not prefer the resulting network. The notion of a myopic improving path was first introduced in Jackson and Watts (2002). A myopic improving path is a sequence of networks that can emerge when players form or sever links based on the improvement the resulting network offers relative to the current network. Each network in the sequence differs by one link from the previous one. If a link is added, then the two players involved must both prefer the resulting network to the current network. If a link is deleted, then it must be that at least one of the two players involved in the link prefers the resulting network.

Formally, a *pairwise myopic improving path* from a network g to a network $g' \neq g$ is a finite sequence of networks $g_1, \ldots, g_K$ with $g_1 = g$ and $g_K = g'$ such that for any $k \in \{1, \ldots, K-1\}$ either: (i) $g_{k+1} = g_k - ij$ for some ij such that $Y_i(g_{k+1}, v) > Y_i(g_k, v)$ or $Y_j(g_{k+1}, v) > Y_j(g_k, v)$, or (ii) $g_{k+1} = g_k + ij$ for some ij such that $Y_i(g_{k+1}, v) > Y_i(g_k, v)$ and $Y_j(g_{k+1}, v) > Y_j(g_k, v)$. For a given network g, let $m(g)$ be the set of networks that can be reached by a pairwise myopic improving path from g.

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[3]But sometimes some players may be indifferent between the network they face and an alternative network, while others are better off at this network structure. Then, it should not be too difficult for the players who are better off to convince the indifferent players to join them to move towards this network structure when very small transfers among the deviating group of players are allowed.

[4]A network $g \in \mathbb{G}$ is pairwise stable with respect v and Y if no player benefits from severing one of their links and no two players benefit from adding a link between them. The original definition of Jackson and Wolinsky (1996) allows for a pairwise deviation to be valid if one deviating player is better off and the other one is at least as well off.

Definition 1. A set of networks $G \subseteq \mathbb{G}$ is pairwise myopically stable with respect v and Y if

(i) $\forall \, g \in G$,

    (ia) $\forall \, ij \notin g$ such that $g + ij \notin G$, $Y_i(g + ij, v) \leq Y_i(g, v)$ or $Y_j(g + ij, v) \leq Y_j(g, v)$,

    (ib) $\forall \, ij \in g$ such that $g - ij \notin G$, $Y_i(g - ij, v) \leq Y_i(g, v)$ and $Y_j(g - ij, v) \leq Y_j(g, v)$,

(ii) $\forall g' \in \mathbb{G} \setminus G$, $m(g') \cap G \neq \emptyset$,

(iii) $\nexists \, G' \subsetneq G$ such that G' satisfies Conditions (ia), (ib), and (ii).

Conditions (ia) and (ib) in Definition 1 capture deterrence of external deviations. In Condition (ia) the addition of a link ij to a network $g \in G$ that leads to a network outside G is deterred because the two players involved do not prefer the resulting network to network g. Condition (ib) is a similar requirement, but then for the case where a link is severed. Condition (ii) requires external stability. External stability asks for the existence of a pairwise myopic improving path from any network outside G leading to some network in G. Condition (ii) implies that if a set of networks is pairwise myopically stable, it is non-empty. Notice that the set $\mathbb{G}$ (trivially) satisfies Conditions (ia), (ib), and (ii) in Definition 1. This motivates Condition (iii), the minimality condition.

Jackson and Watts (2002) have defined the notion of a closed cycle. A set of networks C is a *cycle* if for any $g \in C$ and $g' \in C \setminus \{g\}$, there exists a pairwise myopic improving path connecting g to g'. A cycle C is a maximal cycle if it is not a proper subset of a cycle. A cycle C is a *closed cycle* if no network in C lies on a pairwise myopic improving path leading to a network that is not in C. A closed cycle is necessarily a maximal cycle. Herings, Mauleon and Vannetelbosch (2009) have shown that the set of networks consisting of all networks that belong to a closed cycle is the unique pairwise myopically stable set.

The notion of pairwise myopically stable set only considers deviations by at most a pair of players at a time. It might be that some group of players could all be made better off by some complicated reorganization of their links, which is not accounted for under pairwise myopic stability. A network $g' \in G$ is *obtainable* from $g \in G$ via

deviations by group $S \subseteq N$ if (i) $ij \in g'$ and $ij \notin g$ implies $\{i, j\} \subseteq S$, and (ii) $ij \in g$ and $ij \notin g'$ implies $\{i, j\} \cap S \neq \varnothing$.

A *groupwise myopic improving path* from a network g to a network $g' \neq g$ is a finite sequence of networks $g_1, \ldots, g_K$ with $g_1 = g$ and $g_K = g'$ such that for any $k \in \{1, \ldots, K - 1\}$: g_{k+1} is obtainable from g_k via deviations by $S_k \subseteq N$ and $Y_i(g_{k+1}, v) > Y_i(g_k, v)$ for all $i \in S_k$. For a given network g, let $M(g)$ be the set of networks that can be reached by a groupwise myopic improving path from g.

Definition 2. A set of networks $G \subseteq \mathbb{G}$ is groupwise myopically stable with respect v and Y if

(i) $\forall\, g \in G$, $S \subseteq N$, $g' \notin G$ that is obtainable from g via deviations by S, there exists $i \in S$ such that $Y_i(g', v) \leq Y_i(g, v)$,

(ii) $\forall g' \in \mathbb{G} \setminus G$, $M(g') \cap G \neq \emptyset$,

(iii) $\nexists\, G' \subsetneq G$ such that G' satisfies Conditions (ia), (ib), and (ii).

Replacing the notion of pairwise improving path by the notion of groupwise improving path in the definition of a closed cycle, we have that the set of networks consisting of all networks that belong to a closed cycle is the unique groupwise myopically stable set. The notion of groupwise myopically stable set is a generalization of Dutta and Mutuswami (1997) strong stability notion.[5] In Figure 1 we have depicted an example where the unique pairwise myopically stable set is $\{g_0, g_7\}$ while the unique groupwise myopically stable set is $\{g_7\}$. The networks g_0 and g_7 are pairwise stable but only g_7 is strongly stable, and there are no cycles of networks when players can modify their links either in pairs or in groups. There is no network such that there is a pairwise myopic improving path from any other network leading to it: $m(g_0) = \emptyset$, $m(g_1) = \{g_0, g_4, g_6, g_7\}$, $m(g_2) = \{g_0, g_4, g_5, g_7\}$, $m(g_3) = \{g_0, g_5, g_6, g_7\}$, $m(g_4) = \{g_7\}$, $m(g_5) = \{g_7\}$, $m(g_6) = \{g_7\}$, and $m(g_7) = \emptyset$. Hence, a set formed by the empty network g_0 and the complete network g_7 is a pairwise myopically stable set. However, the groupwise myopically stable set consists only of the complete network

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[5] A set g is strongly stable stable with respect v and Y if $\forall\, S \subseteq N$, g' that is obtainable from g via deviations by S, there exists $i \in S$ such that $Y_i(g', v) \leq Y_i(g, v)$. Jackson and van den Nouweland (2005) have introduced a slightly stronger definition where a deviation is valid if some members are better off and others are at least as well off. For many value functions and allocation rules these definitions coincide.

since $g_7 \in M(g)$ $\forall g \neq g_7$ and $M(g_7) = \varnothing$. Indeed, we have $M(g_0) = \{g_4, g_5, g_6, g_7\}$, $M(g_1) = \{g_0, g_4, g_5, g_6, g_7\}$, $M(g_2) = \{g_0, g_4, g_5, g_6, g_7\}$, $M(g_3) = \{g_0, g_4, g_5, g_6, g_7\}$, $M(g_4) = \{g_7\}$, $M(g_5) = \{g_7\}$, $M(g_6) = \{g_7\}$, and $M(g_7) = \emptyset$.
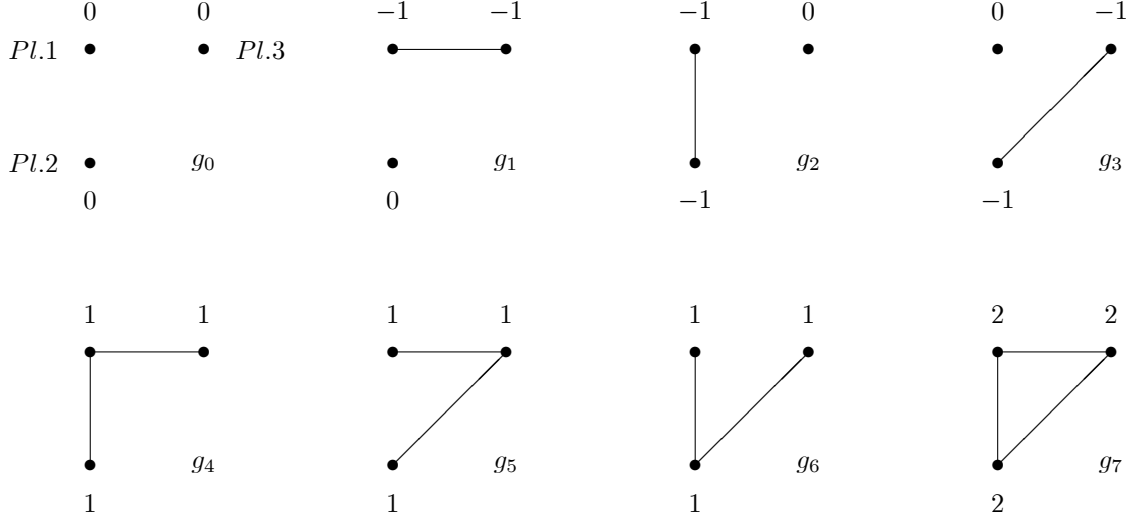

Figure 1: An example without cycles.

In Figure 2 we have depicted Jackson and Wolisnky co-author model with three players. It is easy verified that the complete network g_7 is the unique pairwise stable network. Moreover, it is easy to demonstrate that the pairwise myopically stable set is $\{g_7\}$. However, there is no strongly stable network. The groupwise myopically stable set is $\{g_1, g_2, g_3, g_4, g_5, g_6, g_7\}$ and consists only of cycles. Indeed, we have $M(g_0) = \{g_1, g_2, g_3, g_4, g_5, g_6, g_7\}$, $M(g_1) = \{g_1, g_2, g_3, g_4, g_5, g_6, g_7\}$, $M(g_2) = \{g_1, g_2, g_3, g_4, g_5, g_6, g_7\}$, $M(g_3) = \{g_1, g_2, g_3, g_4, g_5, g_6, g_7\}$, $M(g_4) = \{g_1, g_2, g_3, g_4, g_5, g_6, g_7\}$, $M(g_5) = \{g_1, g_2, g_3, g_4, g_5, g_6, g_7\}$, $M(g_6) = \{g_1, g_2, g_3, g_4, g_5, g_6, g_7\}$, and $M(g_7) = \{g_1, g_2, g_3, g_4, g_5, g_6, g_7\}$.

3.2 Farsighted definitions

A *pairwise farsighted improving path* is a sequence of networks that can emerge when players form or sever links based on the improvement the end network offers relative to the current network. Each network in the sequence differs by one link from the previous one. If a link is added, then the two players involved must both strictly prefer the end network to the current network. If a link is deleted, then it must be that at least one of the two players involved in the link prefers the end
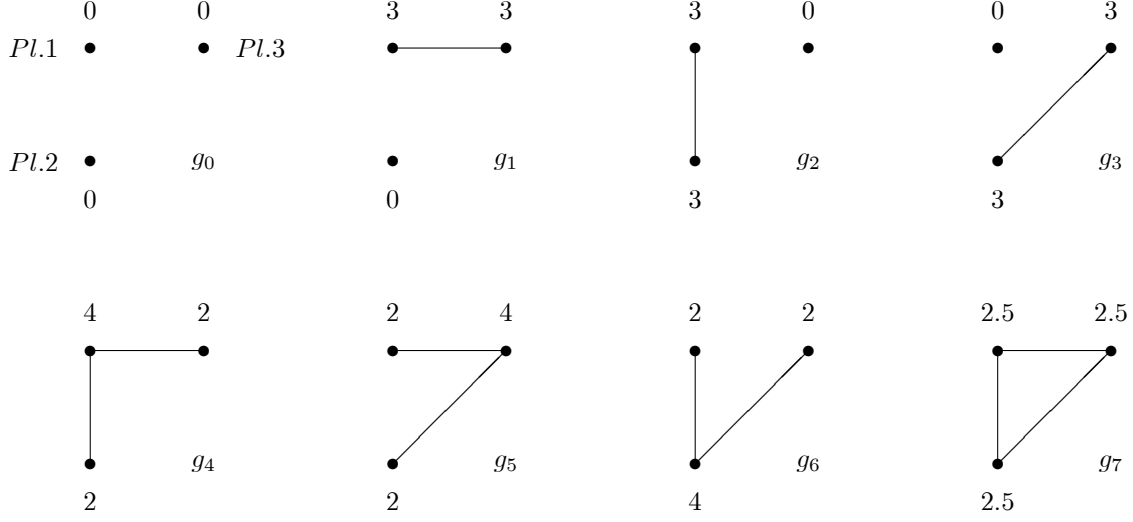
$Pl.1$ 0   0 $Pl.3$ 3 —— 3   3   0   0   3

$Pl.2$ • g_0 • g_1 g_2 g_3

0   0   3   3

4   2   2   4   2   2   2.5   2.5

g_4 g_5 g_6 g_7

2   2   4   2.5

Figure 2: The co-author model with three players.

network. Formally, a pairwise farsighted improving path from a network g to a network $g' \neq g$ is a finite sequence of networks $g_1, \ldots, g_K$ with $g_1 = g$ and $g_K = g'$ such that for any $k \in \{1, \ldots, K-1\}$ either: (i) $g_{k+1} = g_k - ij$ for some ij such that $Y_i(g_K, v) > Y_i(g_k, v)$ or $Y_j(g_K, v) > Y_j(g_k, v)$, or (ii) $g_{k+1} = g_k + ij$ for some ij such that $Y_i(g_K, v) > Y_i(g_k, v)$ and $Y_j(g_K, v) > Y_j(g_k, v)$. For a given network g, let $f(g)$ be the set of networks that can be reached by a pairwise farsighted improving path from g.

We now give the definition of a pairwise farsightedly stable set due to Herings, Mauleon and Vannetelbosch (2009).

Definition 3. A set of networks $G \subseteq \mathbb{G}$ is a pairwise farsightedly stable set with respect v and Y if

(i) $\forall\, g \in G$,

    (ia) $\forall\, ij \notin g$ such that $g + ij \notin G$, $\exists\, g' \in f(g + ij) \cap G$ such that $Y_i(g', v) \leq Y_i(g, v)$ or $Y_j(g', v) \leq Y_j(g, v)$,

    (ib) $\forall\, ij \in g$ such that $g - ij \notin G$, $\exists\, g', g'' \in f(g - ij) \cap G$ such that $Y_i(g', v) \leq Y_i(g, v)$ and $Y_j(g'', v) \leq Y_j(g, v)$,

(ii) $\forall g' \in \mathbb{G} \setminus G$, $f(g') \cap G \neq \emptyset$.

9

(iii) $\nexists\, G' \subsetneq G$ such that G' satisfies Conditions (ia), (ib), and (ii).

Condition (i) in Definition 3 requires the deterrence of external deviations. Condition (ia) captures that adding a link ij to a network $g \in G$ that leads to a network outside of G, is deterred by the threat of ending in g'. Here g' is such that there is a pairwise farsighted improving path from $g + ij$ to g'. Moreover, g' belongs to G, which makes g' a credible threat. Condition (ib) is a similar requirement, but then for the case where a link is severed. Condition (ii) in Definition 3 requires external stability and implies that the networks within the set are robust to perturbations. From any network outside of G there is a farsighted improving path leading to some network in G. Condition (ii) implies that if a set of networks is pairwise farsightedly stable, it is non-empty. Notice that the set $\mathbb{G}$ (trivially) satisfies Conditions (ia), (ib), and (ii) in Definition 3. This motivates the requirement of a minimality condition, namely Condition (iii). Herings, Mauleon and Vannetelbosch (2009) have shown that a pairwise farsightedly stable set of networks always exists.[6]

A network g strictly Pareto dominates all other networks if g is such that for all $g' \in \mathbb{G} \setminus \{g\}$ it holds that, for all i, $Y_i(g,v) > Y_i(g',v)$. Although the network that strictly Pareto dominates all others is pairwise stable, there might be many more pairwise stable networks. Herings, Mauleon and Vannetelbosch (2009) have shown that, if there is a network g that strictly Pareto dominates all other networks, then $\{g\}$ is the unique pairwise farsightedly stable set. Thus, pairwise farsighted stability singles out the Pareto dominating network as the unique pairwise farsightedly stable set.

A *groupwise farsighted improving path* from a network g to a network $g' \neq g$ is a finite sequence of networks $g_1, \ldots, g_K$ with $g_1 = g$ and $g_K = g'$ such that for any $k \in \{1, \ldots, K-1\}$: g_{k+1} is obtainable from g_k via deviations by $S_k \subseteq N$ and $Y_i(g_K,v) > Y_i(g_k,v)$ for all $i \in S_k$. For a given network g, let $F(g)$ be the set of networks that can be reached by a groupwise farsighted improving path from g.

Definition 4. A set of networks $G \subseteq \mathbb{G}$ is groupwise farsightedly stable with respect v and Y if

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[6]Other approaches to farsightedness in network formation are suggested by the work of Chwe (1994), Xue (1998), Herings, Mauleon and Vannetelbosch (2004), Mauleon and Vannetelbosch (2004), Dutta, Ghosal and Ray (2005), Page, Wooders and Kamat (2005), and Page and Wooders (2009).

(i) $\forall\ g \in G,\ S \subseteq N,\ g' \notin G$ that is obtainable from g via deviations by S, there exists $g'' \in F(g') \cap G$ such that $Y_i(g'', v) \leq Y_i(g, v)$ for some $i \in S$,

(ii) $\forall g' \in \mathbb{G} \setminus G,\ F(g') \cap G \neq \emptyset$,

(iii) $\nexists\ G' \subsetneq G$ such that G' satisfies Conditions (ia), (ib), and (ii).

Let us reconsider the co-author model with three players depicted in Figure 2. No singleton set is pairwise farsightedly stable. Indeed, there is no network such that there is a farsighted improving path from any other network leading to it. More precisely, $f(g_0) = \{g_1, g_2, g_3, g_4, g_5, g_6\}$, $f(g_1) = \{g_4, g_5\}$, $f(g_2) = \{g_4, g_6\}$, $f(g_3) = \{g_5, g_6\}$, $f(g_4) = \{g_7\}$, $f(g_5) = \{g_7\}$, $f(g_6) = \{g_7\}$, and $f(g_7) = \emptyset$. However, a set formed by the complete and two star networks is a pairwise farsightedly stable set of networks. The pairwise farsightedly stable sets are $\{g_4, g_5, g_7\}$, $\{g_4, g_6, g_7\}$, $\{g_5, g_6, g_7\}$, and $\{g_1, g_2, g_3, g_7\}$ in the co-author model with three players. Suppose that we allow now for groupwise deviations. Then, we have $F(g_0) = \{g_1, g_2, g_3, g_4, g_5, g_6\}$, $F(g_1) = \{g_4, g_5\}$, $F(g_2) = \{g_4, g_6\}$, $F(g_3) = \{g_5, g_6\}$, $F(g_4) = \{g_3, g_7\}$, $F(g_5) = \{g_2, g_7\}$, $F(g_6) = \{g_1, g_7\}$, and $F(g_7) = \{g_1, g_2, g_3\}$. Hence, $\{g_1, g_2, g_3\}$ becomes a groupwise farsightedly stable set. But, this is not the unique groupwise farsightedly stable set. The others are $\{g_2, g_3, g_5, g_6\}$, $\{g_2, g_3, g_4, g_6\}$, $\{g_1, g_3, g_4, g_5\}$, $\{g_1, g_3, g_5, g_6\}$, $\{g_1, g_2, g_4, g_5\}$, $\{g_1, g_2, g_4, g_6\}$, $\{g_4, g_5, g_7\}$, $\{g_4, g_6, g_7\}$, $\{g_5, g_6, g_7\}$.

4 Stable sets of networks under the componentwise egalitarian allocation rule

We now investigate whether the pairwise or groupwise farsighted stability coincide or not with the pairwise or groupwise myopically stability under the componentwise egalitarian allocation. Let

$$g(v, S) = \left\{ g \subseteq g^S \ \middle|\ \frac{v(g)}{\#N(g)} \geq \frac{v(g')}{\#N(g')} \forall g' \subseteq g^S \right\}$$

be the set of networks with the highest per capita value out of those that can be formed by players in $S \subseteq N$. Given a component additive value function v, find a network g^v through the following algorithm due to Banerjee (1999). Pick some $h_1 \in g(v, N)$. Next, pick some $h_2 \in g(v, N \setminus N(h_1))$. At stage k pick some

$h_k \in g(v, N \setminus \cup_{i \leq k-1} N(h_i))$. Since N is finite this process stops after a finite number K of stages. The union of the components picked in this way defines a network g^v. We denote by G^v the set of all networks that can be found through this algorithm. More than one network may be picked up through this algorithm since players may be permuted or even be indifferent between components of different sizes.

Lemma 1. *Consider any anonymous and component additive value function v. For all $g \in G^v$ we have $f(g) = \varnothing$ and $F(g) = \varnothing$ under the componentwise egalitarian allocation rule Y^{ce}.*

Proof. Take any $g \in G^v$ where $g = \bigcup_{k=1}^{K} h_k$ with $h_k \in g(v, N \setminus \cup_{i \leq k-1} N(h_i))$. Players belonging to $N(h_1)$ in g who are looking forward will never engage in a move since they can never be strictly better off than in g given the componentwise egalitarian allocation rule Y^{ce}. Players belonging to $N(h_2)$ in g who are forward looking will never engage in a move since the only possibility to obtain a strictly higher payoff is to end up in h_1 (if h_1 gives a strictly higher payoff than h_2) but players belonging to $N(h_1)$ will never engage a move. So, players belonging to $N(h_2)$ can never end up strictly better off than in g given the componentwise egalitarian allocation rule Y^{ce}. Players belonging to $N(h_k)$ in g who are forward looking will never engage in a move since the only possibility to obtain a strictly higher payoff is to end up in h_1 or h_2 ... or h_{k-1} but players belonging to $\cup_{i \leq k-1} N(h_i)$ will never engage a move. So, players belonging $N(h_k)$ can never end up strictly better off than in g given the componentwise egalitarian allocation rule Y^{ce}; and so on. Thus, $f(g) = \varnothing$ and $F(g) = \varnothing$. $\qquad \square$

Corollary 1. *Consider any anonymous and component additive value function v. For all $g \in G^v$ we have $m(g) = \varnothing$ and $M(g) = \varnothing$ under the componentwise egalitarian allocation rule Y^{ce}.*

Lemma 2. *Consider any anonymous and component additive value function v. For all $g' \notin G^v$ there exists $g \in G^v$ such that $g \in f(g')$ under the componentwise egalitarian allocation rule Y^{ce}.*

Proof. We show in a constructive way that for all $g' \notin G^v$ there exists $g \in G^v$ such that $g \in f(g')$ under the componentwise egalitarian allocation rule Y^{ce}. Take any $g' \notin G^v$.

Step 1: If there exists some $h_1 \in g(v, N)$ such that $h_1 \in C(g')$ then go to Step 2 with $g_1 = g'$. Otherwise, two cases have to be considered. (A) There exists $h \in C(g')$ such that $h_1 \subsetneq h$ with $h_1 \in g(v, N)$. In g' all players are strictly worse off than the players belonging to $N(h_1)$ under the componentwise egalitarian allocation rule Y^{ce}. From g', let the players who belong to $N(h_1)$ and who look forward to $g \in G^v$ delete successively all their links to reach $g_1 = g' - \{ij \mid i \in N(h_1) \text{ and } ij \notin h_1\}$. Along the sequence from g' to g_1 all players who are moving always prefer the end network g to the current network. (B) There does not exist $h \in C(g')$ such that $h_1 \subsetneq h$ with $h_1 \in g(v, N)$. Pick some $h_1 \in g(v, N)$. In g' all players are strictly worse off than the players belonging to $N(h_1)$ under the componentwise egalitarian allocation rule Y^{ce}. From g', let the players who belong to $N(h_1)$ and who are looking forward to $g \in G^v$ such that $h_1 \in C(g)$ first deleting successively all their links and then building successively the links in h_1 leading to $g_1 = g' - \{ij \mid i \in N(h_1) \text{ and } ij \notin h_1\} + \{ij \mid i \in N(h_1), ij \in h_1 \text{ and } ij \notin g'\}$. Along the sequence from g' to g_1 all players who are moving always prefer the end network g to the current network. Once g_1 and h_1 are formed, we move to Step 2.

Step 2: If there exists some $h_2 \in g(v, N \setminus N(h_1))$ such that $h_2 \in C(g_1)$ then go to Step 3 with $g_2 = g_1$. Otherwise, two cases have to be considered. (A) There exists $h \in C(g')$ such that $h_2 \subsetneq h$ with $h_2 \in g(v, N \setminus N(h_1))$. In g_1 all the remaining players who belong to $N \setminus N(h_1)$ are strictly worse off than the players belonging to $N(h_2)$ under the componentwise egalitarian allocation rule Y^{ce}. From g_1 let the players who belong to $N(h_2)$ and who look forward to $g \in G^v$ such that $h_1 \in C(g)$ and $h_2 \in C(g)$ delete successively all their links to reach $g_2 = g_1 - \{ij \mid i \in N(h_2) \text{ and } ij \notin h_2\} + \{ij \mid i \in N(h_2), ij \in h_2 \text{ and } ij \notin g_1\}$. Along the sequence from g_1 to g_2 all players who are moving always prefer the end network g to the current network. (B) There does not exist $h \in C(g')$ such that $h_2 \subsetneq h$ with $h_2 \in g(v, N \setminus N(h_1))$. Pick some $h_2 \in g(v, N \setminus N(h_1))$. In g_1 all the remaining players who are belonging to $N \setminus N(h_1)$ are strictly worse off than the players belonging to $N(h_2)$ under the componentwise egalitarian allocation rule Y^{ce}. From g_1 let the players who belong to $N(h_2)$ and who are looking forward to $g \in G^v$ such that $h_1 \in C(g)$ and $h_2 \in C(g)$ first deleting successively all their links and then building successively the links in h_2 leading to $g_2 = g_1 - \{ij \mid i \in N(h_2)$

and $ij \notin h_2\} + \{ij \mid i \in N(h_2), ij \in h_2 \text{ and } ij \notin g_1\}$. Along the sequence from g_1 to g_2 all players who are moving always prefer the end network g to the current network. Once g_2 and h_2 are formed, we move to Step 3.

Step k: If there exists some $h_k \in g(v, N \setminus \{N(h_1) \cup ... \cup N(k-1)\})$ such that $h_k \in C(g_{k-1})$ then go to Step $k+1$ with $g_k = g_{k-1}$. Otherwise, two cases have to be considered. (A) There exists $h \in C(g_{k-1})$ such that $h_k \subsetneq h$ with $h_k \in g(v, N \setminus \{N(h_1) \cup ... \cup N(k-1)\})$. In g_{k-1} all the remaining players who are belonging to $N \setminus \{N(h_1) \cup ... \cup N(k-1)\}$ are strictly worse off than the players belonging to $N(h_k)$ under the componentwise egalitarian allocation rule Y^{ce}. From g_{k-1} let the players who belong to $N(h_k)$ and who look forward to $g \in G^v$ such that $h_1 \in C(g)$, $h_2 \in C(g)$, ... $h_k \in C(g)$ delete successively all their links to reach $g_k = g_{k-1} - \{ij \mid i \in N(h_k) \text{ and } ij \notin h_k\} + \{ij \mid i \in N(h_k), ij \in h_k \text{ and } ij \notin g_{k-1}\}$. Along the sequence from g_{k-1} to g_k all players who are moving always prefer the end network g to the current network. (B) There does not exist $h \in C(g_{k-1})$ such that $h_k \subsetneq h$ with $h_k \in g(v, N \setminus \{N(h_1) \cup ... \cup N(k-1)\})$. Pick some $h_k \in g(v, N \setminus \{N(h_1) \cup ... \cup N(k-1)\})$. In g_{k-1} all the remaining players who are belonging to $N \setminus \{N(h_1) \cup ... \cup N(k-1)\}$ are strictly worse off than the players belonging to $N(h_k)$ under the componentwise egalitarian allocation rule Y^{ce}. From g_{k-1} let the players who belong to $N(h_k)$ and who are looking forward to $g \in G^v$ such that $h_1 \in C(g)$, $h_2 \in C(g)$, ... $h_k \in C(g)$ first deleting successively all their links and then building successively the links in h_k leading to $g_k = g_{k-1} - \{ij \mid i \in N(h_k) \text{ and } ij \notin h_k\} + \{ij \mid i \in N(h_k), ij \in h_k \text{ and } ij \notin g_{k-1}\}$. Along the sequence from g_{k-1} to g_k all players who are moving always prefer the end network g to the current network. Once g_k and h_k are formed, we move to Step $k+1$; and so on until we reach the network $g = \bigcup_{k=1}^{K} h_k$ with $h_k \in g(v, N \setminus \cup_{i \le k-1} N(h_i))$. Thus, we have build a pairwise farsightedly improving path from g' to g; $g \in f(g')$. Since $f(g') \subseteq F(g')$, we also have that for all $g' \notin G^v$ there exists $g \in G^v$ such that $g \in F(g')$ under the componentwise egalitarian allocation rule Y^{ce}. $\qquad \square$

The next proposition tells us that once players are farsighted it does not matter whether groupwise or only pairwise deviations are feasible. Both pairwise farsighted stability and groupwise farsighted stability single out the same unique set.

Proposition 1. *Consider any anonymous and component additive value function*

v. The set G^v is both the unique pairwise farsightedly stable set and the unique groupwise farsightedly stable set under the componentwise egalitarian allocation rule Y^{ce}.

Proof. Consider any anonymous and component additive value function v. From Lemma 1 we know that $f(g) = \varnothing$ and $F(g) = \varnothing$ for all $g \in G^v$ under the componentwise egalitarian allocation rule Y^{ce}. From Lemma 2 we have that for all $g' \notin G^v$ there exists $g \in G^v$ such that $g \in f(g')$ under the componentwise egalitarian allocation rule Y^{ce}. Using Theorem 5 in Herings, Mauleon and Vannetelbosch (2009) which says that G is the unique pairwise farsightedly stable set if and only if $G = \{g \in \mathbb{G} \mid f(g) = \varnothing\}$ and for every $g' \in \mathbb{G} \setminus G$, $f(g') \cap G \neq \varnothing$, we have that G^v is the unique pairwise farsightedly stable set. In case of groupwise deviations, Theorem 5 says that G is the unique groupwise farsightedly stable set if and only if $G = \{g \in \mathbb{G} \mid F(g) = \varnothing\}$ and for every $g' \in \mathbb{G} \setminus G$, $F(g') \cap G \neq \varnothing$. Since $f(g) \subseteq F(g)$ $\forall g \in \mathbb{G}$, we have that G^v is also the unique groupwise farsightedly stable set. $\qquad \square$

Lemma 3. *Consider any anonymous and component additive value function v. For all $g' \notin G^v$ there exists $g \in G^v$ such that $g \in M(g')$ under the componentwise egalitarian allocation rule Y^{ce}.*

Proof. We show in a constructive way that for all $g' \notin G^v$ there exists $g \in G^v$ such that $g \in M(g')$ under the componentwise egalitarian allocation rule Y^{ce}. Take any $g' \notin G^v$.

Step 1: If there exists some $h_1 \in g(v, N)$ such that $h_1 \in C(g')$ then go to Step 2 with $g_1 = g'$. Otherwise, pick some $h_1 \in g(v, N)$. In g' all players are strictly worse off than the players belonging to $N(h_1)$ under the componentwise egalitarian allocation rule Y^{ce}. Then, we have that all members of $N(h_1)$ have incentives to deviate from g' to $g_1 = g'|_{N \setminus N(h_1)} \cup h_1$. Indeed, g_1 is obtainable from g' via deviations by $N(h_1) \subseteq N$ and $Y_i(g_1, v) > Y_i(g', v)$ for all $i \in N(h_1)$. In words, players who belong to $N(h_1)$ delete their links in g' with players not in $N(h_1)$ and build the missing links of h_1. Once g_1 and h_1 are formed, we move to Step 2.

Step 2: If there exists some $h_2 \in g(v, N \setminus N(h_1))$ such that $h_2 \in C(g_1)$ then go to Step 3 with $g_2 = g_1$. Otherwise, pick some $h_2 \in g(v, N \setminus N(h_1))$. In g_1 all the remaining players who are belonging to $N \setminus N(h_1)$ are strictly worse

15

off than the players belonging to $N(h_2)$ under the componentwise egalitarian allocation rule Y^{ce}. Then, we have that all members of $N(h_2)$ have incentives to deviate from g_1 to $g_2 = g_1|_{N \setminus N(h_2)} \cup h_2$. Indeed, g_2 is obtainable from g_1 via deviations by $N(h_2) \subseteq N$ and $Y_i(g_2, v) > Y_i(g_1, v)$ for all $i \in N(h_2)$. Once g_2 and h_2 are formed, we move to Step 3.

Step k**:** If there exists some $h_k \in g(v, N \setminus \{N(h_1) \cup ... \cup N(k-1)\})$ such that $h_k \in C(g_{k-1})$ then go to Step $k + 1$ with $g_k = g_{k-1}$. Otherwise, pick some $h_k \in g(v, N \setminus \{N(h_1) \cup ... \cup N(k-1)\})$. In g_{k-1} all the remaining players who are belonging to $N \setminus \{N(h_1) \cup ... \cup N(k-1)\}$ are strictly worse off than the players belonging to $N(h_k)$ under the componentwise egalitarian allocation rule Y^{ce}. Then, we have that all members of $N(h_k)$ have incentives to deviate from g_{k-1} to $g_k = g_{k-1}|_{N \setminus N(h_k)} \cup h_k$. Indeed, g_k is obtainable from g_{k-1} via deviations by $N(h_k) \subseteq N$ and $Y_i(g_k, v) > Y_i(g_{k-1}, v)$ for all $i \in N(h_k)$. Once g_k and h_k are formed, we move to Step $k + 1$; and so on until we reach the network $g = \bigcup_{k=1}^{K} h_k$ with $h_k \in g(v, N \setminus \cup_{i \leq k-1} N(h_i))$. Thus, we have build a groupwise myopically improving path from g' to g; $g \in M(g')$. □

The next proposition tells us that groupwise myopic stability singles out the same unique set as pairwise and groupwise farsighted stability do.

Proposition 2. *Consider any anonymous and component additive value function v. The set G^v is the unique groupwise myopically stable set under the componentwise egalitarian allocation rule Y^{ce}.*

Proof. Since the set of networks consisting of all networks that belong to a closed cycle is the unique groupwise myopically stable set, we have to show that the set of all networks that belong to a closed cycle is G^v. From Lemma 3 we know that for all $g' \notin G^v$ there exists $g \in G^v$ such that $g \in M(g')$ under the componentwise egalitarian allocation rule Y^{ce}. By Corollary 1 we have that $M(g) = \varnothing$ for all $g \in G^v$. Thus, it follows that each $g \in M(g)$ is a closed cycle, all closed cycles belong to G^v, and G^v is the unique groupwise myopically stable set. □

Notice that all networks belonging to G^v are pairwise stable networks in a strict sense. However, the pairwise myopically stable set may include networks that do not belong to G^v. Thus, if players are myopic it matters whether groupwise or only pairwise deviations are feasible. So, pairwise farsighted stability, groupwise

farsighted stability and groupwise myopic stability refines the notion of pairwise stability under Y^{ce} when deviations are valid only if all deviating players are strictly better off.

5   Other notions of farsighted stability

In this section we study the relationship between alternative notions of farsighted stability and pairwise farsighted stable sets. The largest consistent set is a concept that has been defined in Chwe (1994) for general social environments. By considering a network as a social environment, we obtain the definition of the largest consistent set.

The largest consistent set

Definition 5. G is a consistent set if $\forall\, g \in G$, $S \subseteq N$, $g' \in \mathbb{G}$ that is obtainable from g via deviations by S, there exists g'', where $g'' = g'$ or $g'' \in F(g') \cap G$ such that $Y_i(g'', v) \leq Y_i(g, v)$ for some $i \in S$. The largest consistent set is the consistent set that contains any consistent set.

Proposition 3. *Consider any anonymous and component additive value function* v. *The set* G^v *is the largest consistent set under the componentwise egalitarian allocation rule* Y^{ce}.

Proof. First, we show in a constructive way that any $g' \notin G^v$ cannot belong to a consistent because there always exists a deviation which is not deterred. Take any $g' \notin G^v$.

Suppose $\nexists h_1 \in g(v, N)$ such that $h_1 \in C(g')$. Then, in g' all players are strictly worse off than the players belonging to $N(h_1)$ under the componentwise egalitarian allocation rule Y^{ce}. We have that the deviation by all members of $N(h_1)$ from g' to $g'' = g'|_{N \setminus N(h_1)} \cup h_1$ cannot be deterred. Indeed, g'' is obtainable from g' via deviations by $N(h_1) \subseteq N$ and $Y_i(g'', v) > Y_i(g', v)$ for all $i \in N(h_1)$. In words, players who belong to $N(h_1)$ delete their links in g' with players not in $N(h_1)$ and build the missing links of h_1. In addition, for any $g^* \neq g''$, we have that $Y_i(g'', v) \geq Y_i(g^*, v)$ for all $i \in N(h_1)$. So, for any $g''' \in F(g'')$ we have $Y_i(g', v) < Y_i(g'', v) = Y_i(g''', v)$ for all $i \in N(h_1)$. Thus, g' cannot belong to a consistent set.

Suppose that $\exists h_1 \in g(v, N)$ such that $h_1 \in C(g')$ but $\nexists h_2 \in g(v, N \setminus N(h_1))$ such that $h_2 \in C(g')$. Then, in g' all players who belong to $N \setminus N(h_1)$ are strictly

worse off than the players belonging to $N(h_2)$ under the componentwise egalitarian allocation rule Y^{ce}. Then, we have that the deviation by all members of $N(h_2)$ from g' to $g'' = g'|_{N \setminus N(h_2)} \cup h_2$ cannot be deterred. Indeed, g'' is obtainable from g' via deviations by $N(h_2) \subseteq N$ and $Y_i(g'', v) > Y_i(g', v)$ for all $i \in N(h_2)$. In addition, for any $g^* \neq g''$, $g^* \subseteq g^{N \setminus N(h_1)}$, we have that $Y_i(g'', v) \geq Y_i(g^*, v)$ for all $i \in N(h_2)$. So, for any $g''' \in F(g'')$ we have $Y_i(g', v) < Y_i(g'', v) = Y_i(g''', v)$ for all $i \in N(h_2)$. Thus, g' cannot belong to a consistent set.

Suppose that $\exists h_1, h_2, h_3, ..., h_{k-1}$ with $h_l \in g(v, N \setminus \{N(h_1) \cup ... \cup N(h_{l-1})\})$, $l = 2, ..., k-1$, and $h_l \in C(g')$ but $\nexists h_k \in C(g')$ such that $h_k \in g(v, N \setminus \{N(h_1) \cup ... \cup N(h_{k-1})\})$. Then, in g' all players who are belonging to $N \setminus \{N(h_1) \cup ... \cup N(h_{k-1})\}$ are strictly worse off than the players belonging to $N(h_k)$ under the componentwise egalitarian allocation rule Y^{ce}. Then, we have that the devaition by all members of $N(h_k)$ from g' to $g'' = g'|_{N \setminus N(h_k)} \cup h_2$ cannot be deterred. Indeed, g'' is obtainable from g' via deviations by $N(h_k) \subseteq N$ and $Y_i(g'', v) > Y_i(g', v)$ for all $i \in N(h_k)$. In addition, for any $g^* \neq g''$, $g^* \subseteq g^{N \setminus \{N(h_1) \cup ... \cup N(h_{k-1})\}}$, we have that $Y_i(g'', v) \geq Y_i(g^*, v)$ for all $i \in N(h_k)$. So, for any $g''' \in F(g')$ we have $Y_i(g', v) < Y_i(g'', v) = Y_i(g''', v)$ for all $i \in N(h_k)$. Thus, g' cannot belong to a consistent set. And so forth.

Second, we have from Lemma 1 that $F(g) = \varnothing$ $\forall$ $g \in G^v$. Hence, each $\{g\}$ with $g \in G^v$ is a consistent set. Thus, G^v is the largest consistent set under the componentwise egalitarian allocation rule Y^{ce}. $\qquad\qquad\square$

von Neumann-Morgenstern farsighted stability

The von Neumann-Morgenstern stable set (von Neumann and Morgenstern, 1953) imposes internal and external stability. Incorporating the notion of farsighted improving paths into the original definition of the von Neumann-Morgenstern stable set, we obtain the von Neumann-Morgenstern farsightedly stable set. von Neumann-Morgenstern farsightedly stable sets do not always exist.

Definition 6. The set G is a von Neumann-Morgenstern pairwise farsightedly stable set if (i) $\forall g \in G$, $f(g) \cap G = \emptyset$ and (ii) $\forall g' \in \mathbb{G} \setminus G$, $f(g') \cap G \neq \emptyset$.

Definition 7. The set G is a von Neumann-Morgenstern groupwise farsightedly stable set if (i) $\forall g \in G$, $F(g) \cap G = \emptyset$ and (ii) $\forall g' \in \mathbb{G} \setminus G$, $F(g') \cap G \neq \emptyset$.

Corollary 5 in Herings, Mauleon and Vannetelbosch (2009) tells us that if G is the unique pairwise (groupwise) farsightedly stable set, then G is the unique von Neumann-Morgenstern pairwise (groupwise) farsightedly stable set. Hence, the set

G^v is both the unique von Neumann-Morgenstern pairwise farsightedly stable set and the unique von Neumann-Morgenstern groupwise farsightedly stable set under the componentwise egalitarian allocation rule Y^{ce}.

Path dominance core

The concept of path dominance core is due to Page and Wooders (2009). We give two versions of their concept, one based on pairwise deviations, another based on groupwise deviations. A network $g' \in \mathbb{G}$ pairwise path dominates network $g \in \mathbb{G}$, if $g' = g$ or if there exists a finite sequence of networks $\{g_k\}_{k=0}^{K}$ in $\mathbb{G}$ with $g_K = g'$ and $g_0 = g$ such that for $k = 1, 2, ..., K$, $g_k \in f(g_{k-1})$. Similarly, a network $g' \in \mathbb{G}$ groupwise path dominates network $g \in \mathbb{G}$, if $g' = g$ or if there exists a finite sequence of networks $\{g_k\}_{k=0}^{K}$ in $\mathbb{G}$ with $g_K = g'$ and $g_0 = g$ such that for $k = 1, 2, ..., K$, $g_k \in F(g_{k-1})$.[7]

Definition 8. A network $g \in \mathbb{G}$ is contained in the pairwise (groupwise) path dominance core $\mathbb{C} \subseteq \mathbb{G}$ with respect v and Y if and only if there does not exist a network $g' \in \mathbb{G}$, $g' \neq g$, such that g' pairwise (groupwise) path dominates g.

The set G^v is both the pairwise and groupwise path dominance core under the componentwise egalitarian allocation rule Y^{ce}. Indeed, a network $g \in \mathbb{G}$ is contained in the pairwise path dominance core $\mathbb{C}_p \subseteq \mathbb{G}$ with respect v and Y if and only if $f(g) = \varnothing$ and a network $g \in \mathbb{G}$ is contained in the groupwise path dominance core $\mathbb{C}_p \subseteq \mathbb{G}$ with respect v and Y if and only if $F(g) = \varnothing$. From Lemma 1 we have that all $g \in G^v$ belong to the (pairwise or groupwise) path dominance core. From Lemma 2 we have that for all $g' \notin G^v$ there exists $g \in G^v$ such that $g \in f(g')$ under the componentwise egalitarian allocation rule Y^{ce}. Hence, for all $g' \notin G^v$ we have $\varnothing \neq f(g') \subseteq F(g')$. Thus, all $g' \notin G^v$ do not belong to the (pairwise or groupwise) path dominance core.

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[7]In general, the pairwise (groupwise) path dominace core is contained in each pairwise (groupwise) farsightedly stable set of networks. However, a path dominance core may fail to exist while a pairwise (groupwise) farsightedly stable set always exists.

6 Efficiency and stability

6.1 Strict or weak deviations

The definition of pairwise (or groupwise) farsighted stability allows for a deviation by a pair (or a coalition) to be valid only if all deviating players are *strictly* better off. However, in some situations the player who is better off at the end network may be able to convince the indifferent player to join her to move towards this end network. For instance, when small transfers between the deviating pair are allowed. The notion of *weak* pairwise farsighted stability captures this idea. Formally, weak farsighted stability is defined as follows. A weak pairwise farsighted improving path from a network g to a network $g' \neq g$ is a finite sequence of graphs $g_1, \ldots, g_K$ with $g_1 = g$ and $g_K = g'$ such that for any $k \in \{1, \ldots, K-1\}$ either: (i) $g_{k+1} = g_k - ij$ for some ij such that $Y_i(g_K, v) > Y_i(g_k, v)$ or $Y_j(g_K, v) > Y_j(g_k, v)$, or (ii) $g_{k+1} = g_k + ij$ for some ij such that $Y_i(g_K, v) > Y_i(g_k, v)$ and $Y_j(g_K, v) \geq Y_j(g_k, v)$. For a given network g, let $f^w(g)$ be the set of networks that can be reached by a weak pairwise farsighted improving path from g. We have that $f(g) \subseteq f^w(g)$.

Definition 9. A set of networks $G \subseteq \mathbb{G}$ is a weak pairwise farsightedly stable set with respect v and Y if

(i) $\forall \, g \in G$,

    (ia) $\forall \, ij \notin g$ such that $g + ij \notin G$, $\exists g' \in f^w(g + ij) \cap G$ such that $(Y_i(g', v), Y_j(g', v)) = (Y_i(g, v), Y_j(g, v))$ or $Y_i(g', v) < Y_i(g, v)$ or $Y_j(g', v) < Y_j(g, v)$,

    (ib) $\forall \, ij \in g$ such that $g - ij \notin G$, $\exists g', g'' \in f^w(g - ij) \cap G$ such that $Y_i(g', v) \leq Y_i(g, v)$ and $Y_j(g'', v) \leq Y_j(g, v)$,

(ii) $\forall g' \in \mathbb{G} \setminus G$, $f^w(g') \cap G \neq \emptyset$.

(iii) $\nexists \, G' \subsetneq G$ such that G' satisfies Conditions (ia), (ib), and (ii).

It is straightforward that if $\{g\}$ is a pairwise farsightedly stable set then $\{g\}$ is a weak pairwise farsightedly stable set. The reverse is not true. Notice that if G is a weak pairwise farsightedly stable set then (i) $\nexists \, G' \supseteq G$ such that G' is a pairwise farsightedly stable set, (ii) $\nexists \, G' \subseteq G$ such that G' is a pairwise farsightedly

stable set as the following example shows. Consider a situation with three players where the payoffs are given in Figure 3. It can be verified that $f^w(g_0) = \{g_1, g_3, g_7\}$, $f^w(g_1) = \{g_0\}$, $f^w(g_2) = \{g_0, g_1, g_7\}$, $f^w(g_3) = \{g_1, g_6, g_7\}$, $f^w(g_4) = \{g_0, g_1, g_7\}$, $f^w(g_5) = \{g_1, g_3, g_6, g_7\}$, $f^w(g_6) = \{g_1, g_7\}$, and $f^w(g_7) = \{g_6\}$. Hence, the weak pairwise farsightedly stable sets are $\{g_0, g_7\}$, $\{g_0, g_3, g_6\}$, $\{g_1, g_6\}$, $\{g_1, g_7\}$. It can also be verified that $f(g_0) = \varnothing$, $f(g_1) = \{g_0\}$, $f(g_2) = \{g_0, g_1\}$, $f(g_3) = \varnothing$, $f(g_4) = \{g_0, g_1\}$, $f(g_5) = \{g_1, g_3\}$, $f(g_6) = \varnothing$, and $f(g_7) = \{g_6\}$. Hence, the unique pairwise farsightedly stable sets is $\{g_0, g_3, g_6\}$, and pairwise farsighted stability refines weak pairwise farsighted stability.
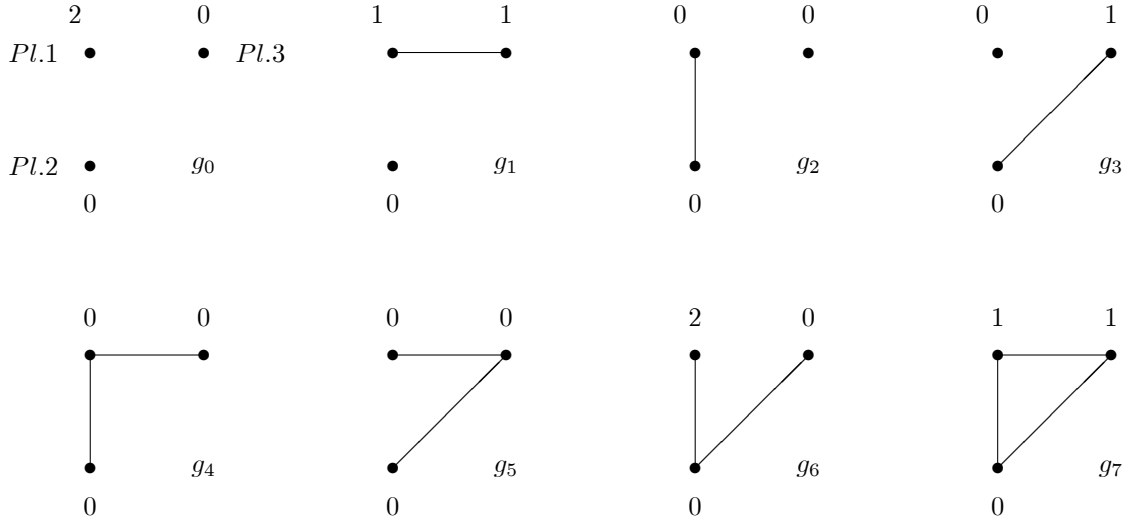

Figure 3: Strict versus weak pairwise farsighted stability: an example.

Consider another situation with three players where the payoffs are given in Figure 4. It can be verified that $f^w(g_0) = \{g_1, g_3, g_7\}$, $f^w(g_1) = \{g_0\}$, $f^w(g_2) = \{g_0, g_1, g_7\}$, $f^w(g_3) = \{g_1, g_7\}$, $f^w(g_4) = \{g_0, g_1, g_7\}$, $f^w(g_5) = \{g_1, g_3, g_4, g_7\}$, $f^w(g_6) = \{g_1, g_7\}$, and $f^w(g_7) = \{g_4\}$. The weak pairwise farsightedly stable sets are $\{g_0, g_7\}$, $\{g_0, g_3, g_4, g_6\}$, $\{g_1, g_4\}$, $\{g_1, g_7\}$. It can also be verified that $f(g_0) = \varnothing$, $f(g_1) = \{g_0\}$, $f(g_2) = \{g_0, g_1\}$, $f(g_3) = \varnothing$, $f(g_4) = \{g_0, g_1\}$, $f(g_5) = \{g_1, g_3\}$, $f(g_6) = \{g_1, g_7\}$, and $f(g_7) = \{g_4\}$. Hence, the pairwise farsightedly stable sets are $\{g_0, g_3, g_7\}$, $\{g_0, g_3, g_4, g_6\}$, $\{g_0, g_1, g_3, g_4\}$. Thus, in general, there are no relationships between pairwise farsighted stability and weak pairwise farsighted stability. However, a pairwise farsightedly stable set always contains a weak pairwise farsightedly stable set
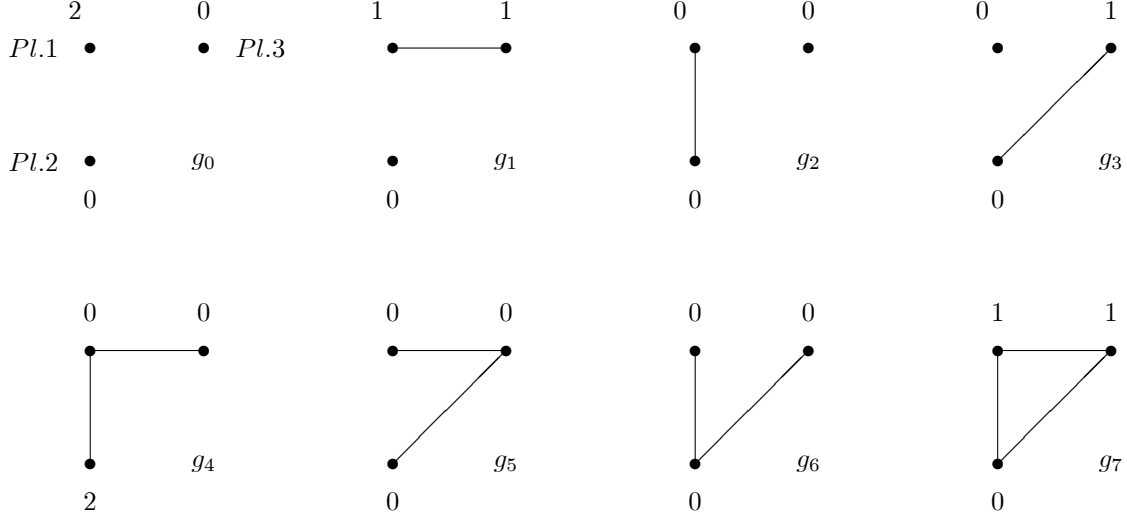
Figure 4: Strict versus weak pairwise farsighted stability: another example.

(but some weak pairwise farsightedly stable set may not be contained in a pairwise farsightedly stable set as illustrated by the example of Figure 3).

Proposition 4. *A pairwise farsightedly stable set of networks always contains a weak pairwise farsightedly stable set of networks.*

Proof. Take $G \subseteq \mathbb{G}$ such that G is a pairwise farsightedly stable set. We have to show that there exists $G' \subseteq G$ such that G' is weak pairwise farsightedly stable set. Notice that $f(g) \subseteq f^w(g)$ for all $g \in \mathbb{G}$. Then, since G satisfies condition (ii) of a pairwise farsightedly stable set, it also satisfies condition (ii) of a weak pairwise farsightedly stable set. In addition, since the set G is immune to external deviations, then it is immune to external weak deviations. Indeed, for all $g \in G$, (a) for all $ij \in g$ such that $g + ij \notin G$, $\exists g'' \in F^s(g + ij) \cap G$ such that $Y_i(g'', v) < Y_i(g, v)$ or $Y_j(g'', v) < Y_j(g, v)$ or $Y_i(g'', v) = Y_i(g, v)$ and $Y_j(g'', v) = Y_j(g, v)$. Since $f(g + ij) \subseteq f^w(g + ij)$, the same network g'' can be reached by a weak farsighted improving path from $g + ij$. (b) idem for link deletion leading to a network outside the set G. We have shown so far that since G satisfies conditions (i) and (ii) of a pairwise farsightedly stable set, then it satisfies conditions (i) and (ii) of a weak pairwise farsightedly stable set. It then follows that either G satisfies condition (iii) of a weak pairwise farsightedly stable set (minimality), in which case G is not only a pairwise farsightedly stable set

but also a weak pairwise farsightedly stable set; or it does not satisfy condition (iii) of a weak pairwise farsightedly stable set, which means that there exists a subset G' of G satisfying conditions (i), (ii) and (iii) of a weak pairwise farsightedly stable set. $\qquad\square$

Corollary 2. *Consider any anonymous and component additive value function v. There exists a subset $G \subseteq G^v$ such that G is a weak pairwise farsighted stable set under the componentwise egalitarian allocation rule Y^{ce}.*

6.2 Primitive conditions on value functions

Grandjean, Mauleon and Vannetelbosch (2009) have shown that the set of strongly efficient networks $E(v)$ is the unique *weak* pairwise farsightedly stable set under Y^{ce} if and only if the value function v is top convex. Remember that weak pairwise farsighted stability is the counterpart of the version of pairwise farsighted stability we use here when deviations are valid if both deviating players are at least as well off and one of them is strictly better off. A value function v is *top convex* if some strongly efficient network also maximizes the per capita value among players. Let $\rho(v, S) = \max_{g \subseteq g^S} v(g)/\#S$. The value function v is *top convex* if $\rho(v, N) \geq \rho(v, S)$ for all $S \subseteq N$.

Top convexity implies that all components of a strongly efficient network must lead to the same per-capita value (if some component led to a lower per capita value than the average, then another component would have to lead to a higher per capita value than the average which would contradict top convexity). It follows that under the componentwise egalitarian allocation rule any $g \in E(v)$ Pareto dominates all $g' \notin E(v)$ Then, it is immediate that $g \in f(g')$ for all $g' \in \mathbb{G} \setminus E(v)$ and that $f(g) = \varnothing$. Using Theorem 5 in Herings, Mauleon and Vannetelbosch (2009) which says that G is the unique pairwise farsightedly stable set if and only if $G = \{g \in \mathbb{G} \mid f(g) = \varnothing\}$ and for every $g' \in \mathbb{G} \setminus G$, $f(g') \cap G \neq \varnothing$, we have that $E(v)$ is the unique pairwise farsightedly stable set. Thus, if v is top convex then the set of strongly efficient networks $E(v)$ is the unique pairwise farsightedly stable set under Y^{ce}.

The following example reveals that under the notion of pairwise farsightedly stable set, top convexity is not necessary to sustain the set of strongly efficient networks as the unique pairwise farsightedly stable set. Let $\#N = 5$. Consider a component additive value function where the value to a component is 30 if it is a

line of 3 players, is 10 if it is a line of 2 players, and is 0 otherwise. The set of strongly efficient networks $E(v)$ is the set of networks composed of two lines, one of 3 players and another of 2 players. Suppose that the value is allocated to the agents according to the componentwise egalitarian allocation rule Y^{ce}. The value function of this game is not top convex. As such, $E(v)$ is not the unique weak pairwise farsightedly stable set of networks. However, the pairwise farsightedly stable set of networks coincide with $E(v)$.

We now show that under the componentwise egalitarian allocation rule Y^{ce}, the set of strongly efficient networks $E(v)$ is contained in the unique pairwise farsightedly stable set of networks G^v if and only if v is weakly top convex. Take an efficient network $g \in E(v)$. A value function is *weakly top convex* if for each network $g' \neq g$ and each component h' of g', the per capita value of the component h' is not greater than the one generated by some component h that belongs to the set of components of g for which at least one player belongs to h and to h'. That is, a value function v is weakly top convex if for all $g' \neq g$ and for all $h' \in C(g')$ such that $N(h') \cap N(g) \neq \emptyset$, $v(h')/\#N(h') \leq v(h)/\#N(h)$ for some $h \in C(g)$ such that $N(h) \cap N(h') \neq \emptyset$. Weak top convexity is a weaker condition than top convexity. The example above with five players shows that a value function may satisfy weak top convexity but not top convexity.

Proposition 5. *Consider any anonymous and component additive value function v. If v is top convex then v is weakly top convex.*

Proof. Let v be an anonymous and component additive value function. Let v also satisfy top convexity. Let $g \in E(v)$. Top convexity implies that all components of a strongly efficient network must lead to the same per-capita value. Thus, for all $h, h' \in C(g)$, $v(h)/\#N(h) = v(h')/\#N(h')$. It follows that each component of a strongly efficient network generates at least the same per capita value than any component of any other network; otherwise, top convexity would be violated. Hence, for all $h \in C(g)$ and $h' \in C(g')$ where $g' \neq g$, we have that $v(h)/\#N(h) \geq v(h')/\#N(h')$. It implies that the value function v is weakly top convex. $\square$

Proposition 6. *Consider any anonymous and component additive value function v. The set of strongly efficient networks $E(v)$ is contained in the unique pairwise farsightedly stable set G^v under the componentwise egalitarian allocation rule Y^{ce} if and only if v is weakly top convex.*

Proof. Consider any anonymous and component additive value function v. Notice that G^v is the unique pairwise farsightedly stable set of networks under the componentwise egalitarian allocation rule Y^{ce}. ($\Leftarrow$) From Theorem 5 in Herings, Mauleon and Vannetebosch (2009), we have that $G^v = \{g \in \mathbb{G} \mid f(g) = \emptyset\}$. Thus, we only have to show that $f(g) = \emptyset$ if $g \in E(v)$ when v is weakly top convex. Without loss of generality, let $g = \cup_{i=1}^k h_i$ be such that $v(h_l)/\#N(h_l) \geq v(h_m)/\#N(h_m)$ if $l < m$. Notice that under the componentwise egalitarian allocation rule, the payoff of player i in component h'' of network $g'' \in \mathbb{G}$ is given by $Y_i^{ce}(g'', v) = v(h'')/\#N(h'')$. Players from $N(h_1)$ do not take part in any pairwise farsighted improving path emanating from the network g since in every other network g', low convexity implies that $Y_i^{ce}(g', v) \leq Y_i^{ce}(g, v)$ for all $i \in N(h_1)$. The rest of the proof proceeds by induction. Suppose players from $N(h_1)$ to $N(h_l)$ do not participate in a pairwise farsighted improving path emanating from the network g. We have to show that players from $N(h_{l+1})$ do not take part in a pairwise farsighted improving path from g. Let $S = N(h_1) \cup ... \cup N(h_l)$. Every network $g' = g|_S \cup \widetilde{g}$ where $\widetilde{g} \subseteq g^{N \setminus S}$ is such that $Y_i^{ce}(g', v) \leq Y_i^{ce}(g, v)$ for all $i \in N(h_{l+1})$ by low convexity. Thus, if agents from S do not take part in a pairwise farsighted improving path, then agents from $N(h_{l+1})$ do not take part in such move either. We have shown so far that any pairwise farsighted improving path emanating from g does not involve players that are connected under g. If every player is connected under g, $f(g) = \emptyset$. If one player is not connected under g, he does not have the power to change the network without the consent of another player, but we have just established that each other player does not take part in a move from g, thus $f(g) = \emptyset$. Finally, if more than one agent is not connected under g, then by strong efficiency of g and by component additivity, $v(\widehat{g}) \leq v(g^{\emptyset})$ for any $\widehat{g} \subseteq g^{N \setminus N(g)}$, implying that there are no pairwise farsighted improving path involving players from $N \setminus N(g)$ only. Thus, $f(g) = \emptyset$.

($\Rightarrow$) Suppose that $E(v) \subseteq G^v$ but weak top convexity is not satisfied. Then, there exists a pair of networks $g \in E(v)$ and $g' \neq g$ such that $v(h')/\#N(h') > v(h)/\#N(h)$ for some $h' \in C(g')$ such that $N(h') \cap N(g) \neq \emptyset$, for all $h \in C(g)$ such that $N(h) \cap N(h') \neq \emptyset$. Without loss of generality, let $g = \cup_{i=1}^k h_i$ be such that $v(h_l)/\#N(h_l) \geq v(h_m)/\#N(h_m)$ if $l < m$. Since $g \in G^v$ we have that $h_1 \in g(v, N)$. Thus, h' is such that $N(h') \cap N(h_1) = \emptyset$. The rest of the proof proceeds by induction. Suppose that h' is such that $N(h') \cap N(h_j) = \emptyset$ for all $h_j \leq h_l$. Then, we have to show that h' is such that $N(h') \cap N(h_{l+1}) = \emptyset$. Let $S = N(h_1) \cup N(h_2) \cup ... \cup N(h_l)$. Since

$g \in G^v$ we have that $h_{l+1} \in v(v, N \setminus S)$. Then, $v(h_{l+1})/\#N(h_{l+1}) \geq v(h')/\#N(h')$ since $N(h') \cap S = \emptyset$. This establishes that $N(h') \cap N(h_{l+1}) = \emptyset$. Thus, $N(h') \cap N(g) = \emptyset$, a contradiction. □

We now provide a sufficient condition on the value function so that pairwise farsighted stability singles out the set of strongly efficient networks, $E(v)$. A network g is connected if for all $i \in N(g)$ and $j \in N(g)\setminus\{i\}$, there exists a path in g connecting i and j. A value function v is *convex* with respect to connected networks if for any connected networks $g \subsetneq g' \subsetneq g'' \subsetneq g'''$ such that $N(g''') = N(g'')\cup\{i\}$ with $i \notin N(g'')$ and $N(g') = N(g) \cup \{j\}$ with $j \notin N(g)$, we have $v(g''') - v(g'') > v(g') - v(g)$. A value function v is *truncated convex* with respect to connected networks if (i) for any connected networks $g \subsetneq g' \subsetneq g'' \subsetneq g'''$ such that $N(g''') = N(g'') \cup \{i\}$ with $i \notin N(g'')$, $N(g') = N(g) \cup \{j\}$ with $j \notin N(g)$, and $\#N(g''') \leq s$ (with $s \in \mathbb{N}_0$), we have $v(g''') - v(g'') > v(g') - v(g)$, and (ii) for any connected networks $g \subsetneq g'$ such that $\#N(g) \geq s$ (with $s \in \mathbb{N}_0$), we have $v(g') \leq v(g)$. Notice that, truncated convexity reverts to convexity if and only if $s \geq n$.

Proposition 7. *Consider any anonymous and component additive value function v. If v is truncated convex then $E(v) = G^v$ under the componentwise egalitarian allocation rule Y^{ce}.*

Proof. □

Corollary 3. *Consider any anonymous and component additive value function v. If v is truncated convex then $E(v)$ is the unique pairwise farsightedly stable set, groupwise farsightedly stable set and groupwise myopically stable set under the componentwise egalitarian allocation rule Y^{ce}.*

A value function v may be truncated convex but not top convex. Let $\#N = 5$. Consider a component additive value function where the value to a component is 10 if it involves 2 players, is 30 if it involves 3 players, is 28 if it involves 4 players, is 25 if it involves 5 players, and is 0 otherwise. This value function is not top convex but is truncated convex (with $s = 3$). Conversely, a value function v may be top convex but not truncated convex. Let $\#N = 4$. Consider a component additive value function where the value to a component is 4 if it involves 2 players, is 3 if

it involves 3 players, is 16 if it involves 4 players, and is 0 otherwise. This value function is not truncated convex but it is top convex.[8]

7 Conclusion

We have studied the stability of social and economic networks when players are farsighted. We have provided an algorithm that characterizes the unique pairwise and groupwise farsightedly stable set of networks under the componentwise egalitarian allocation rule. We have then shown that this set coincides with the unique groupwise myopically stable set of networks but not with the pairwise myopically stable set. Thus, we can conclude that (i) if group deviations are allowed then whether players are farsighted or myopic does not matter, (ii) if players are farsighted then whether players are allowed to deviate in pairs only or in groups does not matter. Finally, we have provided some primitive conditions on value functions so that the set of strongly efficient networks belongs to the unique farsightedly stable set or coincides with the unique farsightedly stable set.

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[8]If v is convex with respect to connected networks, then v is top convex.

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