Policy Announcement Game: Valence Candidates and Ambiguous Policies

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Abstract

We construct a model to explain the phenomenon that in the course of election campaigns, candidates often use ambiguous language in the early stage of the campaigns while they sometimes make their attitudes clear later. In the model, two candidates obtain opportunities to make their policies unambiguous, which arrive stochastically until the election at a predetermined time. While there is no incentive to keep policies ambiguous if two candidates are perfectly symmetric with respect to valence, there is a strategic incentive to keep policies ambiguous if one candidate is slightly stronger than the other.

Preliminary and Incomplete

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1 Introduction

In elections such as those for the US presidency, candidates' policy announcements are often ambiguous. Aragones and Neeman (1994) quoted Nicholas Biddle, the manager of William Henry Harrison's campaign for the presidency, advising "Let him say not a single word about his principles, or his creed - let him say nothing - promise nothing. Let no Committe, no convention - no town meeting ever exract from him a single word, about what he thinks now, or what he will do hereafter." More recently, at the beginning of US presidential primary election 2008, John Edwards criticized Barack Obama in that he abstained from many votes at Senator House, trying not to clarify his political position. On the process of the election, Barack Obama clarified his policies afterwards.

In this paper, we propose a "policy announcement game," in which candidates strategically use ambiguous language, which is sometimes refined later in equilibrium. In our model, each of two candidates obtains opportunities to announce their policies according to a Poisson process. At each opportunity, candidates can either clarify their policies or remain ambiguous. Once a candidate has made his policy clear, he cannot change it afterwards. We first show that, if two candidates are perfectly symmetric with respect to valence, there are no interesting strategic considerations. Specifically, each candidate makes their policy clear as soon as possible. Next we show that, if one candidate is slightly stronger than the other, there are rich strategic considerations involved in equilibrium. For example, the weak candidate will not make his policy clear in early stages of the election campaign because if he does so then the strong candidate will simply copy that policy afterwards, so that the weak candidate loses for sure. In the above example, at the beginning of the primary, Barack Obama was a "weak" candidate compared to Hillary Clinton, which implies that he would have losen if he had specified his policies at the beginning and let Hillary Clinton optimally react to him.

The mechanism that generates ambiguous policy announcement is starkly different from those obtained in the existing literature. In the literature on ambiguous policies such as Shepsle (1972) and Aragones and Postlewaite (2002), it is assumed that candidates choose their policy positions simultaneously once and for all. Ambiguity obtains because voters are assumed to possess convex utility functions. Since voters' utility is higher with uncertain policies than with certain policies when the utility functions are convex, the ambiguity result is not very surprising in their works. Pn the other hand, we explicitly model the dynamic aspects of the election campaign. It is a strategic concern of the weak candidate about the strong opponent's future play that causes ambiguity. Especially, we do not assume convexity; rather, in one of the variants of our model, we show concavity still obtains ambiguity.

Another closely related literature is on elections with valence candidates. First of all, in the standard simultaneous-move Hotelling model, there exists no pure strategy equilibrium: The strong candidate always wants to copy the weak candidate's policy, while the weak candidate does not want to be copied (This is similar to "matching pennies" game). There are two approaches to address this issue. The first approach is to assume that the strong candidate is the incumbent and the weak candidate is the entrant (Bernhardt and Ingberman (1985), Berger et al. (2000)). In this approach, a typical result is that the strong candidate positions close to the median and the weak candidate positions at a policy slightly away from the strong candidate's policy, where the distance between two policies is determined by the degree of assymmetry between candidate's valences. The second approach is that of Aragones and Palfrey (2002): They argue that the above "[r]esults typically depend on order of moves," and "What is the correct sequential model"? Given this question, they consider a mixed strategy equilibrium in simultaneous-move game. They show that the strong candidate puts high probability around the median while the weak candidate puts small probability on it. But the question of the "correct order" is still open. Our policy announcement game explicitly models the order of policy announcement, so serves as an answer to this question.

To formally model the dynamic policy announcement game, we employ the framework with continuous time, finite horizon, and Poisson revision process, which is extensively explored recently. Kamada and Kandori (2009)'s "revision games" consider this setting, while they consider a little different model than ours: In their model, revisions of actions are not restricted, in the sense that players can freely choose their actions at each of their opportunities to move, as opposed to our assumption that once a candidate makes his intention clear, he cannot change it afterwards. They show that, with some restrictions such as continuous action space, non-static-Nash "collusive" action profiles can be played at the deadline. This result is similar to our result that the action profile that is played at the deadline does not correspond to the equilibrium of the stage game (in our model there is no pure equilibrium but the action profile played at the deadline is pure, so this result trivially holds in our setting).[1] Kamada and Sugaya (2010) propose a new method of equilibrium selection using the revision game. They show that, even if there are multiple equilibria in a stage game, the outcome of the revision game can be unique. The driving force of the uniqueness is also a strategic concern about the opponents' future play. Ambrus and Lu (2009) consider a group bargaining problem in this framework. The difference from our paper is that in their model the game stops when the agreement is reached. They show, in particular, that with conditions such as superadditivity of the characteristic function, a player's expected payoff is proportional to his arrival rate. We expect that this type of result, i.e. the positive effect of increasing arrival rate, also holds in our model, but we have not yet verified this property.

As for the idea of being ambiguous expecting the future events, Gale (1995, 2001)'s model of "monotone games" also considers a similar problem. In his model, in each period, players can only (weakly) increase their actions. Thus, in effect, in each period players commit to a smaller and smaller subset of action space, and they will never be able to "expand" that subset. The main difference from our paper is that they analyze "games with positive spillovers" and show that collusive outcomes can be achieved, while we analyze a constant sum game, thus his results are not applicable to our context at all.

The paper proceeds as follows. In Section 2, we introduce the model of policy announcement game. The key assumption we impose is that one candidate is slightly stronger than the other. Section 3 analyzes the model. In Subsection 3.1, we estblish that if two candidates are perfectly symmetric then both candidates would want to be clear as soon as possible. In Subsection 3.2, we consider the case in which our "key assumption" holds, and demonstrate that there is a rich strategic consideration in equilibrium. In Section 4, we discuss other variants of the model. These models have qualitatavely different result than the model in Section 2. Section 5 concludes.

2 The Model

There are two candidates, S and W, interpreted as a "strong candidate" and a "weak candidate," respectively. The policy space is $X = \{0, 1\}$, and voters are distributed over this policy space.

[1] Kamada and Kandori (2009) analyze "unrestricted" "unambiguous" policy announcement game, in which the policy announcements are not restricted by the previously announced policies, and obtained policy divergence in an equilibrium.

Voters' distribution is unknown, but the distribution of the median voter is known, and follows the probability mass function: $f(0) = p$, $f(1) = 1 - p$, where $p < 1/2$. Notice that this is the minimal environment in which we could potentially have strategic ambiguity. We will consider other (more complicated) specification of the model later.

Time is continuous, $-t \in [-T, 0]$ where T is large. Each candidate obtains an opportunity to announce their "policy sets," which is a subset of X, according to the Poisson proccess with arrival rate λ. Note that we are considering the case of asynchronized announcements. The case of synchronized announcements is discussed in Section 4. Candidate i's policy set at time $-t$, i.e. what i can say at time $-t$, is restricted by his previously announced policy sets: If he has already set $\{0\}$ in the past, then he can only set $\{0\}$; If he has already set $\{1\}$ in the past, then he can only set $\{1\}$; If he has announced only $\{0, 1\}$ in the past, then he can announce either $\{0\}$, $\{1\}$, or $\{0, 1\}$. We let the policy set at time $-T$ be just $\{0, 1\}$.

The voter distribution is unknown during $[-T, 0)$, but is revealed at time 0, at which the election takes place. The candidate who obtains more votes wins, and obtains the payoff of 1. If a candidate loses, he obtains the payoff of 0. Hence we are assuming purely office-motivated candidates. Each candidate tries to maximize his expected payoff, so in essence he tries to maximize the winning probability.

Let us now specify voters' utility function and behavior rules. Let x_i be the policy that candidate i chooses. Let $w(x_S, x_W) \in \{S, W\}$ be the winner of the election, given the policy profile (x_S, x_W). A voter with position y obtains the payoff of

$$u(|x_{w(x_S,x_W)} - y|) + \delta \cdot \mathbb{I}_{w(x_S,x_W)=S},$$

where $u(0) > u(1)$ and $\delta > 0$ is small. In particular, we assume that $\delta < (u(0) - u(1))/2$. Small δ makes it possible to investigate the effect of a very slight asymmetry in candidates' valences. We assume that, given that candidate i is announcing policy set $\{0, 1\}$, voters believe that i will take each of policies 0 and 1 equally likely.[2] Each voter votes for the candidate who generates more payoff, if elected, than the other. For completeness, we assume that in the case of tie (which does not occur in equilibrium), each voter randomizes between two candidates. We will analyze subgame

[2] The model is not a knife-edge case with respect to this assumption. At least for an open set of environments, our results are basically unchanged.

perfect equilibria of this game.

All the proofs that are not provided in the text are provided in the Appendix.

3 Analysis

3.1 A Benchmark Case: Perfectly Symmetric Candidates

In this subsection, we consider the case of $\delta = 0$ as a benchmark case. It turns out that there are no interesting dynamics in this case.

Proposition 1 *Suppose $\delta = 0$. Then, in equilibrium, each candidate announces $\{1\}$ as soon as possible.*

Proof. Announcing $\{1\}$ is a strict dominant strategy at every subgame. ■

This negative result is very general. In particular, the result holds also in the other versions of models that we will present in Section 4. Hence the assumption of $\delta > 0$ is the key to our results that follow.

3.2 The Cases with Valence Candidates

In this section, we demonstrate that if $\delta > 0$, then there are a rich strategic considerations involved in equilibrium.

Before presenting the characterization of the equilibrium, let us try to provide the intuition for our result. For the time being, consider the case with $p = 1/2$, which is actually outside of the model (Remember that we set $p < \frac{1}{2}$). Suppose that at time $-t$, Both S and W have previously announced $\{0, 1\}$. If there is no further revision, W's payoff is 0. So W needs to do something to obtain a positive payoff. Thus W announces $\{0\}$ or $\{1\}$ at some point in time, if he can. Since $\{0\}$ and $\{1\}$ are symmetric, assume without loss of generality that W announces $\{1\}$ when he clarifies his policy.

S clearly does not have an incentive to say anything until W says something ($1/2$ is the lowest possible payoff). But after W's announcement, S tries to copy W's choice as soon as possible, which gives W the payoff of 0.

If W announces $\{1\}$ at high t, then the probability that S enters afterwards is high. So W wants to defer.[3] But waiting too much is not very good for W: If he waits until time $-\epsilon$, then he may have no chance to revise his policy set, which results in the payoff of 0. So there should exist a "cutoff," $-t^*$, after which W announces $\{1\}$ when he gets a revision opportunity.

Remember that we did not have this type of strategic considerations when $\delta = 0$.

Next, consider the case with $p = 0$. In this case, S would want to commit to $\{1\}$ as soon as possible, because then he can obtain the payoff of 1, which is the highest possible payoff.

A question is what happens when $p \in (0, \frac{1}{2})$, and the next proposition characterizes the form of equilibrium for each p.

Proposition 2 *The equilibrium of the game is as follows:*[4]

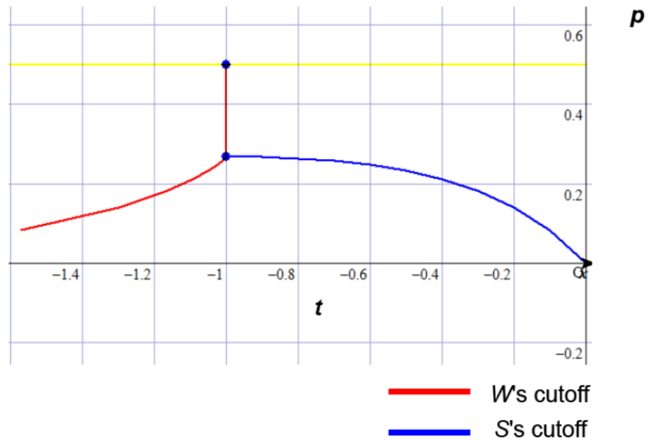
- *Suppose that the previous policy sets are both $\{0,1\}$ at time $-t$. Then, if $t \geq t_S$, S announces $\{1\}$ as soon as possible. If $t < t_S$, he announces $\{0,1\}$.*
- *Suppose that the previous policy sets are both $\{0,1\}$ at time $-t$. Then, if $t > t_W$, S announces $\{0,1\}$; If $t \leq t_W$, he does announces $\{1\}$ as soon as possible.*
- *Given that S has entered, W enters as soon as possible for all t.*
- *Given that W has entered, S enters as soon as possible for all t.*

Here, $\lambda t_S e^{-\lambda t_S} = \frac{p}{1-p}$ and $t_S < \frac{1}{\lambda}$ if such t_S exists, and $t_W = t_S + \frac{1}{2\lambda} \ln\left(\frac{2}{\lambda t_S} - 1\right)$. Otherwise $t_S = \infty$ and $t_W = \frac{1}{\lambda}$.

[3] By "entering" we mean "clarifying the policy," or "announcing the policy $\{0\}$ or $\{1\}$."

[4] This is "essentially unique."

In the following figure, we depict the values of t_S and t_W for various values of p.



As expected, for p close to $\frac{1}{2}$, S does not enter until W enters, and W enters as soon as possible after a finite cutoff. Also, for small p, S enters at high t, but he does not enter at low t, i.e. when the deadline is close. The intuition for this ambiguity near the deadline is as follows: If S obtains an opportunity at t in this range, the probability that W has a chance to announce his policy afterwards is small. So it is likely that W is ambiguous at the deadline. Then, staying ambiguous is good for S, because by doing so S wins for sure with a high probability.

4 Other Variants of the Model

In this section we present three other variants of the model. The first one has a continuous policy space; the second considers three-point voter distribution; and the third one considers the case of synchronized policy announcements.

4.1 Continuous Policy Space

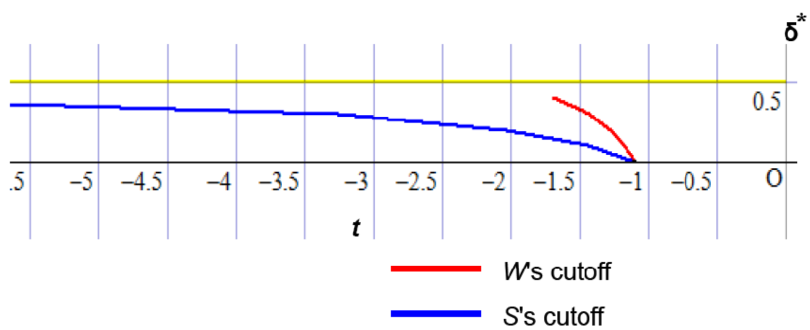
The literature on elections often considers the case in which voters ideal points are distributed uniformly over the interval $[0, 1]$. Thus we investigate this specific context. Ideally, we would assume that candidates can commit to any types of subintervals of $[0, 1]$, which would result in gradual resolution of ambiguity. This analysis is complicated, unfortunately. Instead, we consider the case in which policy set can be either $[0, 1]$ or $\{x\}$ for $0 \leq x \leq 1$. Analogous to the previous model, the policy set at time $-T$ is $[0,1]$. In the next subsection, we will briefly discuss a specification in which gradual resolution of ambiguity is potentially possible.

Again, we assume that the valence term is $\delta > 0$, but is very small. We assume that the voter utility function is the same as before, where u's domain is now $[0,1]$, and u is continuous and decreasing on this domain. We also sasume that u is strictly concave. By assuming concavity, we can purify the effect of dynamic policy annoucement and caniddtates' valence. Let $\delta^* = u(0) - u(\delta)$. Notice that δ^* converges to 0 as δ goes to 0. The next proposition characterizes what happens in equilibrium:

Proposition 3 $t_W = \frac{1}{\lambda} \ln\left(1 + \frac{2}{\sqrt{1-2\delta^*}}\right)$ *and* $t_S = t_W + \frac{2}{\lambda}\left[\frac{1}{1-2\delta^*} - \frac{1}{\sqrt{1-2\delta^*}}\right]$.

Corollary 4 $\lim_{\delta \to 0} t_W = \lim_{\delta \to 0} t_S = \frac{1}{\lambda} \ln(3)$. $\lim_{\delta^* \to \frac{1}{2}} t_W = \lim_{\delta^* \to \frac{1}{2}} t_S = \infty$.

In the following figure, we depicted the values of t_S and t_W for various values of δ^*.



Notice that the result is qualitatively different from the two-point distribution model: In the present model, S prefers waiting in early stages, but chooses to be unambiguous in later stages.

It is worth noting that in the limit that δ converges to 0, t_W and t_S converge to a finite number. Remember that if δ is exactly equal to 0, both of these values should be infinity. Hence, the form of the equilibria are discontinuously different with respect to δ at $\delta = 0$. The reason is rather simple: The payoff matrix, in particular the tie-breaking rule, changes discontinously at $\delta = 0$.

4.2 Synchronized Policy Announcement

So far we have assumed that candidates' policy announcements are asynchronized. But in practice, not all the announcements are asynchronized: For example, in televisioned debates, candidates can state their intentions at the same time. The most realistic would be to assume that there exist both synchronized and asynchronized opportunities, but so far we have only considered the case in which all the opportunities are synchronized.

For sufficiently small valence, the stage game payoff is

$S \setminus W$	$\{0\}$	$\{1\}$	$\{0,1\}$
$\{0\}$	$1,0$	$p,1-p$	$p,1-p$
$\{1\}$	$1-p,p$	$1,0$	$1-p,p$
$\{0,1\}$	$1-p,p$	$p,1-p$	$1,0$

where the first element is S's payoff. Notice that at time 0, if candidates obtain an opportunity, they should use mixed strategies in equilibrium, as suggested by Aragones and Postlewaite (2002). The (unique) mixed strategy equilibrium in the stage game is

$$\left(\frac{p^2}{1-p+p^2}, \frac{(1-p)^2}{1-p+p^2}, \frac{p(1-p)}{1-p+p^2}\right), \left(\frac{(1-p)^2}{1-p+p^2}, \frac{p^2}{1-p+p^2}, \frac{p(1-p)}{1-p+p^2}\right)$$

and the expected value is

$$\left(\frac{2p^2-2p+1}{1-p+p^2}, \frac{p(1-p)}{1-p+p^2}\right)$$

In time interval $[-T, 0]$, opportunities to "specify" their policy arrive at each player with Poisson arrival rate λ. In these opportunities, if a candidate has not yet specified whether her policy is $\{0\}$ or $\{1\}$, that is, if she has taken $\{0,1\}$ up to then, she can choose $\{0\}$, $\{1\}$, and $\{0,1\}$. If she has already specified $\{0\}$ or $\{1\}$, she cannot move at all. We consider the synchronous case, where the opportunities are common to both candidates. Consider the following subgames: the candidates have an opportunity at t, when

- S's current policy choice is $\{0\}$ and W's current policy choice is $\{0,1\}$. In this case, it is optimal for W to choose $\{1\}$ or stay at $\{0,1\}$. In both cases, the payoff is $(p, 1-p)$.
- S's current policy choice is $\{1\}$ and W's current policy choice is $\{0,1\}$. In this case, it is optimal for W to choose $\{0\}$ or stay at $\{0,1\}$. In both cases, the payoff is $(1-p, p)$.
- S's current policy choice is $\{0,1\}$ and W's current policy choice is $\{0\}$. In this case, it is optimal for S to choose $\{0\}$. The payoff is $(1,0)$.

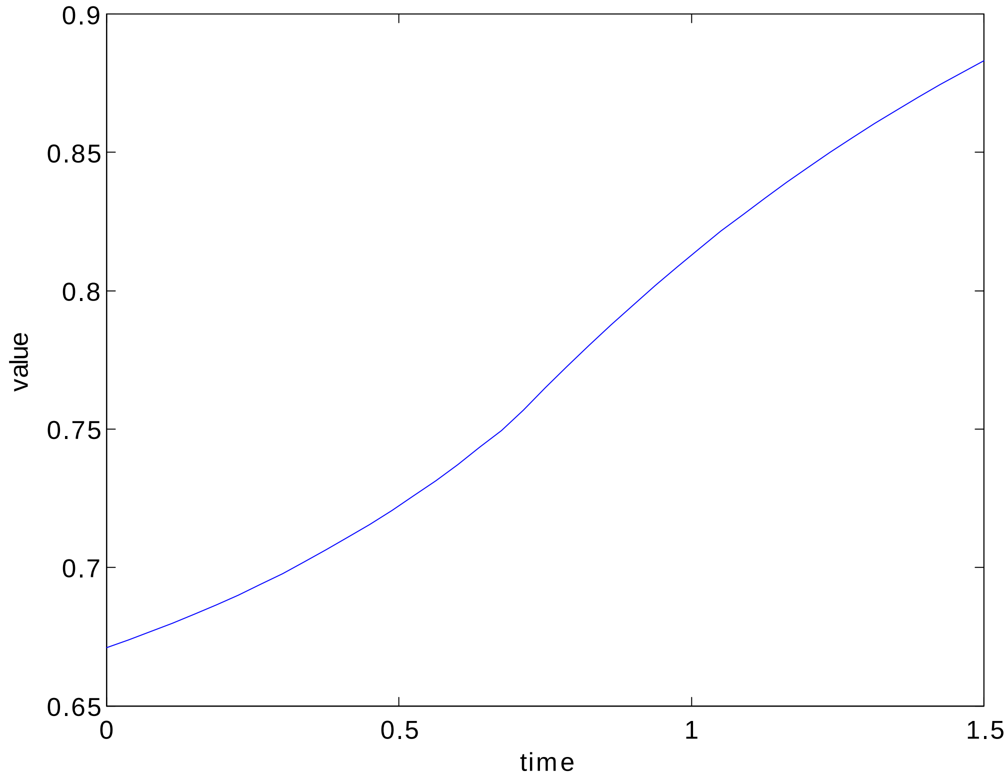
- S's current policy choice is $\{0,1\}$ and W's current policy choice is $\{1\}$. In this case, it is optimal for S to choose $\{1\}$. The payoff is $(1,0)$.
- S's current policy choice is $\{0,1\}$ and W's current policy choice is $\{0,1\}$. In this case, we have the following. For simplicity, we concentrate on the case where each player mixes all the actions.
  - For $t \in [0, \log 2)$,

$$
\begin{aligned}
\begin{pmatrix} p_t^S \\ q_t^S \\ r_t^S \end{pmatrix} &: = \begin{pmatrix} \text{Probability of } S \text{ taking } \{0\} \\ \text{that for } \{1\} \\ \text{that for } \{0,1\}) \end{pmatrix} \\
&= \begin{pmatrix} \frac{p^2 - p\int_0^t e^{-\lambda\tau}\lambda V_{t-\tau}^W dt}{p(1-p)+e^{-\lambda t}(1-2p+2p^2)-\int_0^t e^{-\lambda\tau}\lambda V_{t-\tau}^W dt} \\ \frac{(1-p)^2 e^{-\lambda t}-(1-p)\int_0^t e^{-\lambda\tau}\lambda V_{t-\tau}^W dt}{p(1-p)+e^{-\lambda t}(1-2p+2p^2)-\int_0^t e^{-\lambda\tau}\lambda V_{t-\tau}^W dt} \\ \frac{p(1-p)}{p(1-p)+e^{-\lambda t}(1-2p+2p^2)-\int_0^t e^{-\lambda\tau}\lambda V_{t-\tau}^W dt} \end{pmatrix}
\end{aligned}
$$

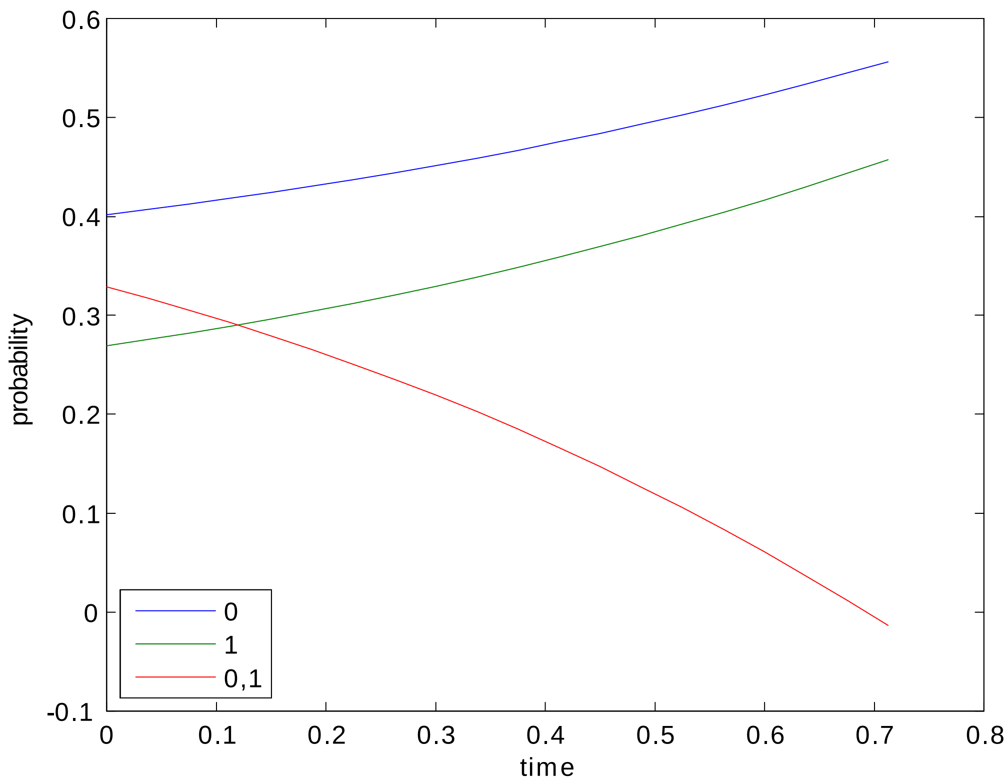
  and

$$
\begin{aligned}
\begin{pmatrix} p_t^W \\ q_t^W \\ r_t^W \end{pmatrix} &: = \begin{pmatrix} \text{Probability of } W \text{ taking } \{0\} \\ \text{that for } \{1\} \\ \text{that for } \{0,1\} \end{pmatrix} \\
&= \begin{pmatrix} \frac{(1-p)\int_0^t e^{-\lambda\tau}\lambda V_{t-\tau}^S dt+(1-p)^2\left(2e^{-\lambda t}-1\right)}{\int_0^t e^{-\lambda\tau}\lambda V_{t-\tau}^S dt+(1-p+p^2)\left(2e^{-\lambda t}-1\right)} \\ \frac{p\int_0^t e^{-\lambda\tau}\lambda V_{t-\tau}^S dt+p^2\left(2e^{-\lambda t}-1\right)}{\int_0^t e^{-\lambda\tau}\lambda V_{t-\tau}^S dt+(1-p+p^2)\left(2e^{-\lambda t}-1\right)} \\ \frac{(1-p)p\left(2e^{-\lambda t}-1\right)}{\int_0^t e^{-\lambda\tau}\lambda V_{t-\tau}^S dt+(1-p+p^2)\left(2e^{-\lambda t}-1\right)} \end{pmatrix}
\end{aligned}
$$

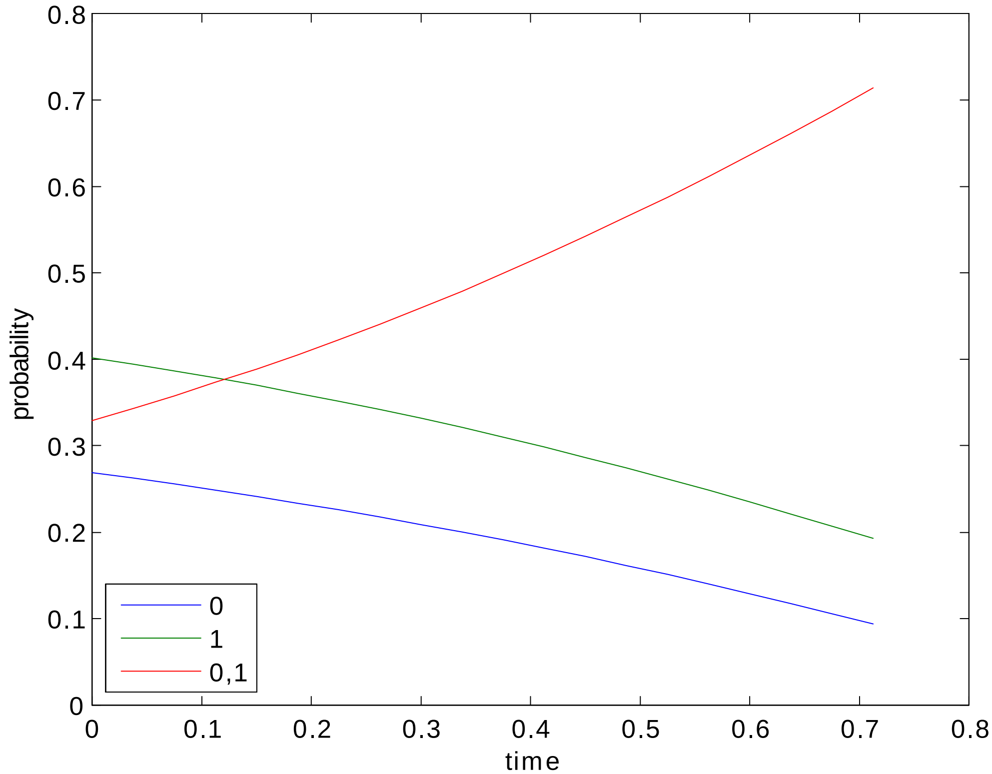
The graph under $p = .45$ is as follows:



Calculating W's strategy $\left(p_t^W, q_t^W, r_t^W\right)$ yields



Calculating S's strategy $\left(p_t^S, q_t^S, r_t^S\right)$ yields



The intuitive explanation of the shape of the equilibrium is as follows. Since S puts high probability on $\{0, 1/2\}$, W is reluctant to take $\{0, 1/2\}$ in the early stage of the game. In turn, since W specifies his intention with high probability in the early stage of the game, S has an incentive to take $\{0, 1/2\}$ with high probability and to wait for the next opportunity in order to mimic W's position afterwards.

5 Conclusion and Future Work

We have proposed the model of "policy announcement game" in which candididates stochastically obtain opportunities to announce their policies. We showed that, if two candidates are perfectly symmetric, each candidate clarifies their policy positions as soon as possible. On the other hand, if one candidate is slightly stronger than the other, then candidates may have incentives to defer a clear announcement of their policies, depending on the opponent's current announcement, and the time left until the election.

Our analysis yields further questions: in future work, we will analyze the case where candidates can commit to some types of subintervals of $[0, 1]$, which would result in gradual resolution of ambiguity. The combination of synchronized and asynchronized opportunities is another possible extension. We will also try to establish the existence and the uniqueness of the equilibrium, in more general setting than in the two-point distribution case. Also, we will analyze the case in which canididates are allowed to make inconsistent announcements, while they must incur "reputational cost" by such announcements. The idea is that if a candidate changes his opinion frequently, voters would infer that it is likely that the candidate would change his policy even after the election.

In addition, we will analyze the distribution of the outcome at the time of election. A question that we will ask is if it corresponds to the distribution in the mixed strategy in the simultaneous-move game. Actually, this type of result may be more general: In general finite games with generic payoffs, we could ask if the distribution of play at the deadline in an "unrestricted" revision game corresponds to some equilibrium of the stage game when T diverges to infinity.

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For John Edward's comment on Barack Obama mentioned in Section 1, see, for example, http://www.youtube.com/watch?v=FsjajB2Gx7U.

6 Appendix

6.1 Proof of Proposition 2

Proof. We consider the case with $\bar{t}_S \geq t_W \geq \underline{t}_S$. To reduce the notational complexity, let $\bar{t}_S = u$, $t_W = w$, and $\underline{t}_S = s$. So we are assuming $u \geq w \geq s$.

It can be shown that s satisfies

$$\lambda s e^{-\lambda s} = \frac{p}{1-p} \quad \text{and} \quad \lambda s < 1.$$

Firstly, we solve the indifference condition at w. W's expected payoff from entering is:

$$e^{-\lambda w}(1-p).$$

W's expected payoff from not entering is:

$$(1-p)\int_0^{w-s} e^{-2\lambda\tau}\lambda e^{-\lambda(w-\tau)}d\tau + p\frac{1-e^{-2\lambda(w-s)}}{2} + (1-p)e^{-2\lambda(w-s)}\int_0^{s} e^{-\lambda\tau}\lambda e^{-\lambda(s-\tau)}d\tau$$

$$= e^{-\lambda w}(1-e^{-\lambda(w-s)}) + p\frac{1-e^{-2\lambda(w-s)}}{2} + (1-p)e^{-2\lambda(w-s)}\lambda s e^{-\lambda s}.$$

Hence, entering is a best response iff:

$$e^{-\lambda w}(1-p) \geq (1-p)e^{-\lambda w}(1-e^{-\lambda(w-s)}) + p\frac{1-e^{-2\lambda(w-s)}}{2} + (1-p)e^{-2\lambda(w-s)}\lambda s e^{-\lambda s}.$$

Notice that the above equation does not hold at $w=\infty$. We need that the above equation holds at $w=s$. So substitute $w=s$ to get:

$$e^{-\lambda s}(1-p) \geq (1-p)\lambda s e^{-\lambda s},$$

which is equivalent to $\lambda s \leq 1$, which is true.

Now, let's try to solve for w.

$$e^{-\lambda w} \geq e^{-\lambda w}(1-e^{-\lambda(w-s)}) + \frac{p}{1-p}\frac{1-e^{-2\lambda(w-s)}}{2} + e^{-2\lambda(w-s)}\lambda s e^{-\lambda s}.$$

$$\iff 0 \geq -e^{-\lambda w}e^{-\lambda(w-s)} + \lambda s e^{-\lambda s}\frac{1-e^{-2\lambda(w-s)}}{2} + e^{-2\lambda(w-s)}\lambda s e^{-\lambda s}.$$

$$\iff 0 \geq -e^{-\lambda w}e^{-\lambda(w-s)} + \lambda s e^{-\lambda s}\frac{1+e^{-2\lambda(w-s)}}{2}.$$

$$\iff 0 \geq -X^2 e^{\lambda s} + \lambda s e^{-\lambda s}\frac{1+X^2 e^{2\lambda s}}{2}, \quad X = e^{-\lambda w}.$$

$$\iff 0 \geq -X^2 e^{2\lambda s} + \lambda s\frac{1+X^2 e^{2\lambda s}}{2}.$$

$$\iff X^2(1-\frac{\lambda s}{2}) \geq \lambda s\frac{e^{-2\lambda s}}{2}.$$

$$\iff X^2 \geq \frac{\lambda s\frac{e^{-2\lambda s}}{2}}{(1-\frac{\lambda s}{2})}.$$

$$e^{-\lambda w} \geq \sqrt{\frac{\lambda s}{2-\lambda s}} e^{-\lambda s}.$$

Notice that $\sqrt{\frac{\lambda s}{2-\lambda s}} < 1$, so as we have shown, $w > s$ holds, which is consistent with our assumption.

We need to show that at w, S prefers entering to not entering. Since W is indifferent between entering (which gives the payoff of $e^{-\lambda w}(1-p)$) and not entering at w, S must prefer $1-p$, the payoff from entering, to $1 - e^{-\lambda w}(1-p)$, the payoff from not entering. So we need that

$$1 - p \geq 1 - e^{-\lambda w}(1-p),$$

which is equivalent to:

$$\sqrt{\frac{\lambda s}{2-\lambda s}} e^{-\lambda s}(1-p) \geq p \iff \sqrt{\frac{\lambda s}{2-\lambda s}} e^{-\lambda s} \geq \lambda s e^{-\lambda s} \iff \sqrt{\frac{\lambda s}{2-\lambda s}} \geq \lambda s$$

$$\iff \frac{\lambda s}{2-\lambda s} \geq (\lambda s)^2 \iff \frac{1}{2-\lambda s} \geq \lambda s \iff 1 \geq \lambda s(2-\lambda s).$$

Since $\lambda s(2-\lambda s)$ is smaller than 1, we have proved that there exists an equilibrium with $u > w > s$.

Next, we consider S's indiffernce codition at u. The payoff from entering is

$$1 - p.$$

The payoff from not entering is:

$$(1 - e^{-\lambda(u-w)})(1-p) + e^{-\lambda(u-w)} \frac{1 - e^{-2\lambda(w-s)}}{2}(1-p)$$

$$+e^{-\lambda(u-w)} \int_0^{w-s} e^{-2\lambda\tau} \lambda(1-(1-p)e^{-\lambda(w-\tau)}) d\tau + e^{-\lambda(u-s)} e^{-\lambda(w-s)} \int_0^{s} e^{-\lambda\tau} \lambda(1-(1-p)e^{-\lambda(s-\tau)}) d\tau$$

$$+e^{-\lambda(u-s)} e^{-\lambda w}.$$

Entering is a best response if:

$$0 \geq -e^{-\lambda(u-w)}(1-p) + e^{-\lambda(u-w)} \frac{1 - e^{-2\lambda(w-s)}}{2}(1-p)$$

$$+e^{-\lambda(u-w)} \int_0^{w-s} e^{-2\lambda\tau} \lambda(1-(1-p)e^{-\lambda(w-\tau)}) d\tau + e^{-\lambda(u-s)} e^{-\lambda(w-s)} \int_0^{s} e^{-\lambda\tau} \lambda(1-(1-p)e^{-\lambda(s-\tau)}) d\tau$$

$$+e^{-\lambda(u-s)}e^{-\lambda w}.$$

$$e^{\lambda w}(1-p)-e^{\lambda w}\frac{1-e^{-2\lambda(w-s)}}{2}(1-p)-e^{\lambda w}\int_0^{w-s}e^{-2\lambda\tau}\lambda(1-(1-p)e^{-\lambda(w-\tau)})d\tau-e^{\lambda s}e^{-\lambda w}$$

$$\geq e^{-\lambda(w-s)}e^{\lambda s}\int_0^{s}e^{-\lambda\tau}\lambda(1-(1-p)e^{-\lambda(s-\tau)})d\tau.$$

Notice that this inequality does not depend on u. Hence, the only way that $u \geq w \geq s$ holds is that the above inequality holds for the already specified w and s.

But we have shown that S's payoff from entering is larger than the payoff from not entering at w. Hence, for other u's such that $u > w$, The above equation must hold. Thus, the equilibrium looks as follows:

- Given that noone has entered, S enters as soon as possible until s.
- Given that noone has entered, W enters as soon as possible after w.
- Given that S has entered, W enters as soon as possible (not entering is also a best response).
- Given that W has entered, S enters as soon as possible.

Now we consider the case with $w > u > s$. We show that there cannot exist an equilibrium (probably unique) with this condition.

First, again, we have

$$\lambda s e^{-\lambda s}=\frac{p}{1-p}.$$

Second, we consider S's indifferennce condition at u. The payoff from entering is $1-p$. The payoff from not entering is:

$$\int_0^{u-s}e^{-2\lambda\tau}\lambda(1-(1-p)e^{-\lambda(u-\tau)})d\tau+\frac{1-e^{-2\lambda(u-s)}}{2}(1-p)$$

$$+e^{-2\lambda(u-s)}\int_0^{s}e^{-\lambda\tau}\lambda(1-(1-p)e^{-\lambda(s-\tau)})d\tau+e^{-\lambda(u-s)}e^{-\lambda u}$$

$$=\frac{1-e^{-2\lambda(u-s)}}{2}-(1-p)e^{-\lambda u}(1-e^{-\lambda(u-s)})+\frac{1-e^{-2\lambda(u-s)}}{2}(1-p)$$

$$+e^{-2\lambda(u-s)}(1-e^{-\lambda s})-(1-p)e^{-2\lambda(u-s)}\lambda s e^{-\lambda s}+e^{-\lambda(u-s)}e^{-\lambda u}.$$

$$= \frac{1-e^{-2\lambda(u-s)}}{2} - (1-p)e^{-\lambda u}(1-e^{-\lambda(u-s)}) + \frac{1-e^{-2\lambda(u-s)}}{2}(1-p)$$
$$+e^{-2\lambda(u-s)} - (1-p)e^{-2\lambda(u-s)}\frac{p}{1-p}.$$
$$= (1-p)\left[\frac{1-e^{-2\lambda(u-s)}}{2(1-p)} - e^{-\lambda u}(1-e^{-\lambda(u-s)}) + \frac{1-e^{-2\lambda(u-s)}}{2} + e^{-2\lambda(u-s)}\right].$$

So entering is a best reponse iff:

$$1 \geq \frac{1-e^{-2\lambda(u-s)}}{2(1-p)} - e^{-\lambda u}(1-e^{-\lambda(u-s)}) + \frac{1+e^{-2\lambda(u-s)}}{2}$$

$$\iff 0 \geq \frac{p}{2(1-p)} - \frac{p}{1-p}\frac{e^{-2\lambda(u-s)}}{2} - e^{-\lambda u}(1-e^{-\lambda(u-s)})$$

$$\iff 0 \geq \lambda s e^{-\lambda s}\frac{1-e^{-2\lambda(u-s)}}{2} - e^{-\lambda u}(1-e^{-\lambda(u-s)})$$

$$\iff 0 \geq \lambda s e^{-\lambda s}(1-X^2e^{2\lambda s}) - 2X(1-Xe^{\lambda s}), \quad \text{with} \quad X = e^{-\lambda u}$$

$$\iff 0 \geq e^{\lambda s}(2-\lambda s)X^2 - 2X + \lambda s e^{-\lambda s}$$

$$\iff 0 \geq (X - e^{-\lambda s})(e^{\lambda s}(2-\lambda s))X - \lambda s)$$

$$\iff \frac{\lambda s}{2-\lambda s}e^{-\lambda s} \leq X \leq e^{-\lambda s}.$$

Notice that we have used the fact that $\lambda s < 1$. Hence it must be the case that

$$e^{-\lambda u} = \frac{\lambda s}{2-\lambda s}e^{-\lambda s},$$

which is consistent with out assumption that $u > s$.

Finally, we will prove that, when W uses cutoff strategy, we must have $w > u$. Notice that S is indifferent between entering and not entering at u. This implies that S's payoff from not entering at u is $1-p$. Hence, W's payoff from not entering at u is p. Thus, all we need to show is that W's payoff from entering at u is higher than p. W's payoff from entering at u is:

$$e^{-\lambda u}(1-p).$$

Hence, it suffices to show that

$$e^{-\lambda u}(1-p) \geq p.$$

This is equivalent to:

$$e^{-\lambda u} \geq \frac{p}{1-p}$$

$$\iff \frac{\lambda s}{2-\lambda s} e^{-\lambda s} > \lambda s e^{-\lambda s} \iff 1 > 2 - \lambda s \iff \lambda s > 1,$$

which contradicts our earlier conclusion that $\lambda s < 1$. ■

6.2 Proof of Proposition 3

Proof. Suppose first that $t_W \geq t_S$. Given that noone has entered, S's payoff from entering at t_S is:

$$1 - (\frac{1}{2} - \delta^*)(1 - e^{-\lambda t_S}).$$

Given that noone has entered, S's payoff from not entering at t_S is:

$$1 - \left[\int_0^{t_S} e^{-2\lambda\tau} \lambda e^{-\lambda(t_S-\tau)} d\tau + (\frac{1}{2} - \delta^*) \int_0^{t_S} e^{-2\lambda\tau} \lambda (1 - e^{-\lambda(t_S-\tau)}) d\tau\right]$$

Since S is indifferent between entering and not entering at t_S, these two should be equal. So we have:

$$(\frac{1}{2} - \delta^*)(1 - e^{-\lambda t_S}) = \int_0^{t_S} e^{-2\lambda\tau} \lambda e^{-\lambda(t_S-\tau)} d\tau + (\frac{1}{2} - \delta^*) \int_0^{t_S} e^{-2\lambda\tau} \lambda (1 - e^{-\lambda(t_S-\tau)}) d\tau$$

$$\iff (\frac{1}{2} - \delta^*) \left[1 - e^{-\lambda t_S} - \frac{1 - e^{-2\lambda t_S}}{2} + e^{-\lambda t_S}(1 - e^{-\lambda t_S})\right] = e^{-\lambda t_S}(1 - e^{-\lambda t_S})$$

$$\iff \frac{1}{2}(\frac{1}{2} - \delta^*) \left[1 - e^{-2\lambda t_S}\right] = e^{-\lambda t_S}(1 - e^{-\lambda t_S})$$

$$\iff \frac{1}{2}(\frac{1}{2} - \delta^*) \left[1 + e^{-\lambda t_S}\right] = e^{-\lambda t_S}$$

$$\iff e^{-\lambda t_S} = \frac{\frac{1}{2}(\frac{1}{2} - \delta^*)}{1 - \frac{1}{2}(\frac{1}{2} - \delta^*)}.$$

For $t_W \geq t_S$ to be true, it must be the case that entering is a best response for W at t_S. W's payoff from entering at t_S is:

$$e^{-\lambda t_S}.$$

W's payoff from not entering at t_S is 1 minus S's payoff from not entering at t_S, which is equal to 1 minus S's payoff from entering at t_S, since S is indifferent at t_S. Hence we need that

$$e^{-\lambda t_S} \geq 1 - (1 - (\frac{1}{2} - \delta^*)(1 - e^{-\lambda t_S})) = (\frac{1}{2} - \delta^*)(1 - e^{-\lambda t_S}).$$

Substituting the value that we have solved, this is equivalent to:

$$\frac{\frac{1}{2}(\frac{1}{2} - \delta^*)}{1 - \frac{1}{2}(\frac{1}{2} - \delta^*)} \geq (\frac{1}{2} - \delta^*)(1 - \frac{\frac{1}{2}(\frac{1}{2} - \delta^*)}{1 - \frac{1}{2}(\frac{1}{2} - \delta^*)}).$$

$$\iff \frac{1}{2}(\frac{1}{2} - \delta^*) \geq (\frac{1}{2} - \delta^*)(1 - (\frac{1}{2} - \delta^*))$$

$$\iff \frac{1}{2} \geq \frac{1}{2} + \delta^*$$

$$\iff \delta^* \leq 0,$$

which contradicts oua assumption that $\delta^* > 0$. Hence we cannot have $t_W \geq t_S$.

Now we consider the case with $t_S > t_W$. Given that noone has entered, W's payoff from entering at t_W is

$$e^{-t_W}.$$

Given that noone has entered, W's payoff from not entering at t_W is

$$\int_0^{t_S} e^{-2\lambda\tau}\lambda e^{-\lambda(t_S-\tau)}d\tau + (\frac{1}{2} - \delta^*)\int_0^{t_S} e^{-2\lambda\tau}\lambda(1 - e^{-\lambda(t_S-\tau)})d\tau.$$

Since W is indifferent between entering and not entering at t_W, these two should be equal. So we have:

$$e^{-t_W} = \int_0^{t_W} e^{-2\lambda\tau}\lambda e^{-\lambda(t_W-\tau)}d\tau + (\frac{1}{2} - \delta^*)\int_0^{t_W} e^{-2\lambda\tau}\lambda(1 - e^{-\lambda(t_W-\tau)})d\tau$$

$$\iff e^{-t_W} = e^{-\lambda t_W}(1 - e^{-\lambda t_W}) + (\frac{1}{2} - \delta^*)\left[\frac{1 - e^{-2\lambda t_W}}{2} - e^{-\lambda t_W}(1 - e^{-\lambda t_W})\right]$$

$$\iff e^{-2t_W} = \frac{1}{2}(\frac{1}{2}-\delta^*)\left[1+e^{-2\lambda t_W}-2e^{-\lambda t_W}\right]$$

$$\iff 1 = \frac{1}{2}(\frac{1}{2}-\delta^*)(\frac{1}{e^{-\lambda t_W}}-1)^2$$

$$\iff \sqrt{\frac{2}{\frac{1}{2}-\delta^*}} = \frac{1}{e^{-\lambda t_W}}-1$$

$$\iff e^{-\lambda t_W} = \frac{1}{1+\sqrt{\frac{4}{1-2\delta^*}}}$$

$$\iff t_W = \frac{1}{\lambda}\ln\left(1+\frac{2}{\sqrt{1-2\delta^*}}\right).$$

Next, consider S's indifference at t_S. S's payoff from entering at t_S is:

$$1-(\frac{1}{2}-\delta^*)(1-e^{-\lambda t_S}).$$

S's payoff from not entering at t_S is:

$$1-\left[(\frac{1}{2}-\delta^*)\int_0^{t_S-t_W} e^{-\lambda\tau}\lambda(1-e^{-\lambda(t_S-\tau)})d\tau + e^{-\lambda(t_S-t_W)}e^{-\lambda t_W}\right]$$

$$= 1-\left[(\frac{1}{2}-\delta^*)\left(1-e^{-\lambda(t_S-t_W)}-\lambda(t_S-t_W)e^{-\lambda t_S}\right)+e^{-\lambda t_S}\right].$$

Here, we used the fact that W's expected payoff at t_W equal to his payoff from entering (since he is indifferent), which is equal to $e^{-\lambda t_W}$, and S's payoff and W's payoff needs to sum up to 1. Hence, we need to have:

$$1-(\frac{1}{2}-\delta^*)(1-e^{-\lambda t_S}) = 1-\left[(\frac{1}{2}-\delta^*)\left(1-e^{-\lambda(t_S-t_W)}-\lambda(t_S-t_W)e^{-\lambda t_S}\right)+e^{-\lambda t_S}\right]$$

$$\iff (\frac{1}{2}-\delta^*)(1-e^{-\lambda t_S}) = (\frac{1}{2}-\delta^*)\left(1-e^{-\lambda(t_S-t_W)}-\lambda(t_S-t_W)e^{-\lambda t_S}\right)+e^{-\lambda t_S}$$

$$\iff -(\frac{1}{2}-\delta^*) = (\frac{1}{2}-\delta^*)\left(-e^{\lambda t_W}-\lambda(t_S-t_W)\right)+1$$

$$\iff \frac{3-2\delta^*}{1-2\delta^*} = e^{\lambda t_W}+\lambda(t_S-t_W)$$

$$\iff t_S = t_W+\frac{1}{\lambda}\left[\frac{3-2\delta^*}{1-2\delta^*}-e^{\lambda t_W}\right]$$

$$\iff t_S = t_W + \frac{1}{\lambda}\left[1 + \frac{3-2\delta^*}{1-2\delta^*} - \frac{2}{\sqrt{1-2\delta^*}}\right]$$

$$\iff t_S = t_W + \frac{2}{\lambda}\left[\frac{1}{1-2\delta^*} - \frac{1}{\sqrt{1-2\delta^*}}\right]$$

■

6.3 Derivation for Section 4.2

Suppose S's current policy choice is $\{0,1\}$ and W's current policy choice is $\{0,1\}$. Given the analysis in Section 4.2, the expected payoff matrix for the last event can be writen as

$S \setminus W$	$\{0\}$	$\{1\}$
$\{0\}$	$1, 0$	$p, 1-p$
$\{1\}$	$1-p, p$	$1, 0$
$\{0,1\}$	$(1-p)e^{-\lambda t} + (1-e^{-\lambda t}), pe^{-\lambda t}$	$pe^{-\lambda t} + (1-e^{-\lambda t}), (1-p)e^{-\lambda t}$

$S \setminus W$	$\{0,1\}$
$\{0\}$	$p, 1-p$
$\{1\}$	$1-p, p$
$\{0,1\}$	$\int_0^t e^{-\lambda\tau}\lambda V^S_{t-\tau} d\tau + e^{-\lambda t}, \int_0^t e^{-\lambda\tau}\lambda V^W_{t-\tau} d\tau$

with $\left(V_0^S, V_0^W\right) = \left(\frac{2p^2-2p+1}{1-p+p^2}, \frac{p(1-p)}{1-p+p^2}\right)$. Here, V_t^i is the value for player i when the opportunity to move arrives at period $-t$. Let us call the game defined by the above payoff matrix "reduced game" as Abreu, Pearce, and Stacchetti. Note that the continuation payoff is also included.

Note that a mixed strategy equilibrium in this reduced game is

$$S \quad : \quad \begin{pmatrix} \frac{p^2 - p\int_0^t e^{-\lambda\tau}\lambda V^W_{t-\tau} d\tau}{p(1-p) + e^{-\lambda t}(1-2p+2p^2) - \int_0^t e^{-\lambda\tau}\lambda V^W_{t-\tau} d\tau} \\ \frac{(1-p)^2 e^{-\lambda t} - (1-p)\int_0^t e^{-\lambda\tau}\lambda V^W_{t-\tau} d\tau}{p(1-p) + e^{-\lambda t}(1-2p+2p^2) - \int_0^t e^{-\lambda\tau}\lambda V^W_{t-\tau} d\tau} \\ \frac{p(1-p)}{p(1-p) + e^{-\lambda t}(1-2p+2p^2) - \int_0^t e^{-\lambda\tau}\lambda V^W_{t-\tau} d\tau} \end{pmatrix}$$

$$W \quad : \quad \begin{pmatrix} \frac{(1-p)\int_0^t e^{-\lambda\tau}\lambda V^S_{t-\tau} d\tau + (1-p)^2\left(2e^{-\lambda t}-1\right)}{\int_0^t e^{-\lambda\tau}\lambda V^S_{t-\tau} d\tau + (1-p+p^2)\left(2e^{-\lambda t}-1\right)} \\ \frac{p\int_0^t e^{-\lambda\tau}\lambda V^S_{t-\tau} d\tau + p^2\left(2e^{-\lambda t}-1\right)}{\int_0^t e^{-\lambda\tau}\lambda V^S_{t-\tau} d\tau + (1-p+p^2)\left(2e^{-\lambda t}-1\right)} \\ \frac{(1-p)p\left(2e^{-\lambda t}-1\right)}{\int_0^t e^{-\lambda\tau}\lambda V^S_{t-\tau} d\tau + (1-p+p^2)\left(2e^{-\lambda t}-1\right)} \end{pmatrix}$$

$$
\begin{aligned}
V_t^S &= \frac{\left(1-p+p^2\right)\int_0^t e^{-\lambda\tau}\lambda V_{t-\tau}^S d\tau + \left(1-2p+2p^2\right)\left(2e^{-\lambda t}-1\right)}{\int_0^t e^{-\lambda\tau}\lambda V_{t-\tau}^S d\tau + \left(1-p+p^2\right)\left(2e^{-\lambda t}-1\right)}, \\
V_t^W &= 1 - V_t^S.
\end{aligned}
$$

Consider V_t^S case:

$$
\begin{aligned}
V_t^S &= \frac{\left(1-p+p^2\right)\int_0^t e^{-\lambda\tau}\lambda V_{t-\tau}^S d\tau + \left(1-2p+2p^2\right)\left(2e^{-\lambda t}-1\right)}{\int_0^t e^{-\lambda\tau}\lambda V_{t-\tau}^S d\tau + \left(1-p+p^2\right)\left(2e^{-\lambda t}-1\right)} \\
&= \left(1-p+p^2\right) - \frac{p^2\left(1-p\right)^2\left(2e^{-\lambda t}-1\right)}{\int_0^t e^{-\lambda\tau}\lambda V_{t-\tau}^S d\tau + \left(1-p+p^2\right)\left(2e^{-t\lambda}-1\right)}
\end{aligned}
$$

$\Leftrightarrow$

$$
\int_0^t e^{-\lambda\tau}\lambda V_{t-\tau}^S d\tau + \left(1-p+p^2\right)\left(2e^{-t\lambda}-1\right) = -\frac{p^2\left(1-p\right)^2\left(2e^{-t\lambda}-1\right)}{V_t^S-\left(1-p+p^2\right)}
$$

Changing variables in the integration by $s := t-\tau$, since $ds = -d\tau$,

$$
\int_t^0 e^{-\lambda(t-s)}\lambda V_s^S\left(-ds\right) + \left(1-p+p^2\right)\left(2e^{-t\lambda}-1\right) = -\frac{p^2\left(1-p\right)^2\left(2e^{-t\lambda}-1\right)}{V_t^S-\left(1-p+p^2\right)},
$$

or

$$
-\int_t^0 e^{-\lambda(t-\tau)}\lambda V_\tau^S d\tau + \left(1-p+p^2\right)\left(2e^{-t\lambda}-1\right) = -\frac{p^2\left(1-p\right)^2\left(2e^{-t\lambda}-1\right)}{V_t^S-\left(1-p+p^2\right)}
$$

$\Leftrightarrow$

$$
\int_0^t e^{\lambda\tau}\lambda V_\tau^S d\tau + \left(1-p+p^2\right)\left(2-e^{t\lambda}\right) = -\frac{p^2\left(1-p\right)^2\left(2-e^{t\lambda}\right)}{V_t^S-\left(1-p+p^2\right)}
$$

Taking derivative with respect to t yields

$$
e^{\lambda t}\lambda V_t^S - \lambda\left(1-p+p^2\right)\lambda e^{t\lambda} = -\frac{-\lambda p^2\left(1-p\right)^2 e^{t\lambda}(V_t^S-\left(1-p+p^2\right)) - V_t^{S\prime}p^2\left(1-p\right)^2\left(2-e^{t\lambda}\right)}{(V_t^S-\left(1-p+p^2\right))^2}
$$

$\Leftrightarrow$

$$
\begin{aligned}
& \lambda e^{\lambda t}V_t^S(V_t^S-\left(1-p+p^2\right))^2 - \lambda\left(1-p+p^2\right)\lambda e^{t\lambda}(V_t^S-\left(1-p+p^2\right))^2 \\
= \; & \lambda p^2\left(1-p\right)^2 e^{t\lambda}(V_t^S-\left(1-p+p^2\right)) + p^2\left(1-p\right)^2\left(2-e^{t\lambda}\right)\frac{dV_t^S}{dt}
\end{aligned}
$$

$\Leftrightarrow$

$$\frac{dV_t^S}{dt} = \frac{\lambda V_t^S (V_t^S - \left(1 - p + p^2\right))^2}{p^2 \left(1 - p\right)^2 \left(2e^{-t\lambda} - 1\right)} - \frac{\left(1 - p + p^2\right) \lambda^2 (V_t^S - \left(1 - p + p^2\right))^2}{p^2 \left(1 - p\right)^2 \left(2e^{-t\lambda} - 1\right)} - \frac{\lambda (V_t^S - \left(1 - p + p^2\right))}{\left(2e^{-t\lambda} - 1\right)}$$

Solving this differential equation numerically yields the result.