Auctions for Priority Access

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Abstract

This paper analyzes an auction which allocates a perfectly divisible good among competing agents by granting them access in the order of their bids. The highest bidder is granted the first opportunity to purchase as many units as desired; if any remains, the next highest bidder is then given access, and so forth. This auction has immediate application to rent seeking behavior and waiting as a rationing mechanism.

With homogeneous agents, bidding fully dissipates all rent, and expected bid revenue is maximized when exactly one customer is left unable to procure the good. This mimics the behavior of a two-part tariff with the same per-unit price. With heterogeneous agents, similar results occur; in addition, the agent with more to gain from winning first priority is more likely to do so if agents are more distinct.

The auction is also compared to raffles for priority access, which is an extension of the single prize contest. The auction raises more bid revenue than the raffle, provided that agents are not too different. Also, the auction is more likely to award first priority to the agent who gains the most from it.

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1 Introduction

A simple method for allocating a perfectly divisible good among several agents is to conduct an auction for priority access. Each agent simultaneously submits a bid, and then the agent with the highest bid is given the first opportunity to procure some of the good. If any remains, access to the good is then granted to the second highest bidder, and so forth. Any ties are randomly broken.

Of course, if the highest bidder always intends to procure the entire available supply, the auction would be no different from a standard auction of a single indivisible item. However, there are a number of interesting environments in which some supply is left over for second place (and beyond).

For example, consider the grant application process. The organization offering the grants has some pool of money available. The applicant's bid for this grant money can be seen as the effort in drafting a proposal, with the best proposals (those with the highest bid) given first priority on the pool of monies. Even so, the best grant seldom absorbs the entire pool. This is because there is some marginal cost in justifying a larger budget for the proposal.

A similar competition describes consumer's queuing decisions for a good in short supply (such as with gasoline or holiday door-buster sales). If the retailer opens at a particular time, each consumer bids by choosing how early to arrive and stand in line. The earliest arrival will be the first to make purchases, but does not typically purchase the whole supply; perhaps capacity or credit constraints prevent this.

Another appropriate environment is rent seeking. The politician has some supply of money or favors that she can distribute. Rent seekers compete through their campaign contributions, with the largest contributors receiving the first priority. Note that only in this last environment is the bid actually transferred from one group to another. In grant writing and queueing, the bid destroys resources rather than transferring them.

An additional common thread among all three environments is that the competing agents pay their bids regardless of the auction outcome. The unsuccessful grant applicant is not refunded his expended effort, nor can the unsuccessful rent seeker request a refund of campaign donations. In the case of queuing, this could be modeled either way. It is possible that one only learns she arrived too late after waiting for the amount of time she bid; it is equally plausible that she can observe the queue on arrival and infer that she has arrived too late, thereby sparing herself the cost of her bid (referred to as *balking* from a queue). Our model would only apply when balking is not possible.

The bid is thus a sunk cost of procuring the good. For arbitrary preferences, the sunk cost would reduce wealth and potentially affect demand for the good. To avoid any wealth effect, we assume quasi-linear preferences with respect to money. Demand is thus independent of the bid, which can only reduce surplus from consuming the good. We also assume that these preferences are fixed and known by all bidders.

We are able to prove existence of equilibrium as well as some basic characterization of its features; chief among these is that mixed strategies must be used. When all agents have identical utility, the auction fully dissipates all expected rent as long as at least one agent will be left unable to procure some of the good. Moreover, the expected expenditure on bid fees is maximized when there is exactly one agent left unable to procures any of the good, while all others can be full satiated.

Indeed, this result can explain why a monopolist might hold their output below market clearing levels. If the seller is able to collect bid fees, the auction for priority access resembles a two-part tariff. By holding output down, they are able (in expectation) to capture all the consumer surplus from the sale of the item.[1] Indeed, this is typically more profitable than the typical single-price monopoly outcome.

When agents differ in their underlying utility, each may demand a different quantity of the good. Thus, agents must be concerned not only with the probability of being outbid, but also how much the higher bidder will take of the available supply. Here, our results are characterized in terms of the *net gain* in utility when a participant fully satisfies his demand, rather than being left with partial fulfillment behind some other agent.

Due to the use of mixed strategies, the auction for priority access will award first priority randomly, but the agent with higher net gain will always be more likely to win. Even so, in equilibrium, it is possible for this agent to have lower expected utility than one with lower net gain — his aggressive bidding dissipates more of the consumer surplus than the bidding of his opponent. Also, in this environment, a monopolist

[1]This is similar to the outcome of *pay-to-bid auctions*, where bidders pay a fee to increase the current auction price. As modeled by Platt, Price and Tappen (2010), the pay-to-bid auction fully dissipates expected consumer surplus even though the typical closing price is well below retail. Several nascent internet sites have implemented such auctions; data from one such site provides strong evidence confirming the model.

would do well to limit supply: revenues from bid fees are maximized when the lower agent would leave no supply available if he happens to win first priority.

Beginning with Tullock (1980), competition for rents has often been modeled as a contest (or raffle) in which each person's chance of winning the rents is equal to their effort divided by the sum of all participants' effort. For purposes of comparison, we develop the logical extension: a raffle for priority access. The probability of winning first priority access is the same as in the standard contest: the person's bid divided by the sum of all bids. Second priority is similarly determined from the same bids, except that the first winner is excluded. This continues until the supply is exhausted.

In the environment with identical utility functions, the raffle always produces less bid revenue than the auction, but leaves higher expected utility to the participants. When utility differs across agents, the auction produces higher revenue when the agents' net gains are still relatively similar. The agent with lower net gain always receives greater expected utility under the raffle, while the comparison is ambiguous for the higher agent. Finally, the auction format is always more likely to award first priority to the higher agent, which is more efficient than the alternative.

It is worth noting the relationship that the auction for priority access has to other auction systems. The fact that even losers pay their bid would categorize this as a form of *all-pay auction*. These began with the study of awarding a single prize among various participants who may have the same (Hillman and Samet, 1987) or differing (Hillman and Riley, 1989; Baye, Kovenock, and de Vries, 1996) valuations of the prize. Others (Clark and Riis, 1998; Barut and Kovenock, 1998; Siegel, 2009) have extended the all-pay auction to contests with multiple prizes; still, each prize is indivisible and each agent can win no more than one prize. Ours is the first in which agents may choose varying amounts of a perfectly divisible good. The assumption of full information, while not common in first- and second-price auctions, is frequently used in all-pay auctions, including all papers mentioned above, as well as in other models of rent seeking.

Priority auctions are somewhat related. These are queuing models in which a server must determine an order in which to serve randomly arriving customers, who can bump themselves up in the queue through their bids. Typically, all of the customers can eventually be served; the only question is, in what order? This does not precisely map to our environment, where order only matters in how much supply will remain.

The paper proceeds as follows: Section 2 defines the environment and mechanism for the auction for priority access. Section 3 then proves existence of equilibrium and characterizes its basic features. In Section 4, we analyze equilibrium when all agents have the same utility function; Section 5 provides similar analysis for a two person case where utility functions differ. Section 6 defines the raffle for priority access and compares its equilibrium outcomes to those of the auction, and Section 7 concludes.

2 Model

A seller has S units of a perfectly divisible commodity, to be sold among n buyers via an auction for priority access. Each buyer submits a sealed bid b_i which are simultaneously opened by the seller. The buyer with the largest b_i is then allowed to purchase any amount $q_1 \leq S$ units of the good at an exogenously set price $p \geq 0$ per unit. If any supply is available after this purchase, the buyer with the second largest bid is allowed to purchase from the remaining $S - q_1$ units at the same price p. This continues until supply is exhausted. Ties are broken randomly.

All players forfeit their bid, regardless of whether they obtain any supply (an *all-pay* auction). The model is only slightly more complicated if only those who obtain some supply are required to pay (a *pay-as-bid* auction). The bid is constrained to be positive but less than an exogenous maximum M.[2]

Buyers are assumed to have preferences that are quasi-linear with respect to money: $u_i(q_i, b_i) = v_i(q_i) - pq_i - b_i$. Moreover, they have diminishing marginal utility in the auctioned commodity: $v'' < 0$. Thus, we can define $d_i(p) \equiv \arg\max_{q_i} u_i(q_i, b_i)$ as the quantity buyer i will attempt to purchase if he gains access to the available supply.[3] With the assumed utility, if available supply is less than $d_i(p)$, the consumer would optimally choose to purchase all the available supply.

Thus, for a given profile of bids $b = \{b_i\}_{i=1}^n$, if agent i is not tied with another

[2]In making use of this model, one would typically set M sufficiently large to never be binding, such as larger than the maximum value of winning. For existence purposes, the space of actions must be compact.

[3]Note that d_i is independent of b_i, which is precisely the purpose of assuming quasi-linear utility. Eliminating wealth effects on demand greatly simplifies bidding strategies.

agent ($b_i \neq b_j$ for all $j \neq i$), he will obtain:

$$q_i(b) = \min\left\{d_i(p),\ \max\left\{0,\ S - \sum_{j:\, b_j > b_i} d_j(p)\right\}\right\} \tag{1}$$

If he is involved in a tie, the tied bidders are randomly ordered (with equal probability on each permutation) and their demand is filled in order. Thus, if k agents (including agent i) have a bid of b_i, then with probability $\frac{1}{k!}$, the outcome is:

$$q_i(b) = \min\left\{d_i(p),\ \max\left\{0,\ S - \sum_{j:\, b_j > b_i} d_j(p) - \sum_{\substack{j:\, b_j = b_i,\\ \pi(j) < \pi(i)}} d_j(p)\right\}\right\} \tag{2}$$

where $\pi(\cdot)$ is a permutation that creates a strict order of the k tied bidders. Note that the tie could be *inconsequential*, meaning that $q_i(b)$ is the same for any of the permutations. For instance, $q_i(b) = d_i(b)$ for all permutations when there is sufficient remaining supply for all the tied agents. Similarly, if all supply has been exhausted by agents with bids higher than b_i, $q_i(b) = 0$ for all permutations.

Typically, there are no pure strategy equilibria for this game, as will be shown later. Thus, we define each bidder's mixed strategy as a probability measure μ_i on $[0, M]$. By way of notation, let B_i be the support[4] of μ_i. Also let B_{-i} be the cross product of the strategy support of each agent besides i, and $\mu_{-i}(b_{-i})$ be the product of $\mu_j(b_j)$ for all $j \neq i$.

The Nash equilibrium of this game is a strategy profile μ^* such that for all i, any bid in the support B_i^* maximizes i's expected utility, given other players' strategies μ_{-i}^*. That is, for all $b_i^* \in B_i^*$ and all $\hat{b}_i \in [0, M]$,

$$\begin{aligned} EU_i(b_i^*, \mu_{-i}^*) \equiv \int_{B_{-i}^*} &\Big(v_i(q_i(b_{-i}, b_i^*)) \ - \ pq_i(b_{-i}, b_i^*)\Big)\, d\mu_{-i}^*(b_{-i}) \ - \ b_i^* \\ \geq \int_{B_{-i}^*} &\Big(v_i(q_i(b_{-i}, \hat{b}_i)) \ - \ pq_i(b_{-i}, \hat{b}_i)\Big)\, d\mu_{-i}^*(b_{-i}) \ - \ \hat{b}_i \end{aligned}$$

[4]The *support* of a probability measure μ_i is the set B_i such that $\mu_i(B_i) = 1$ and $\mu_i(B') < 1$ for any proper closed subset $B' \subset B_i$.

3 Existence and Characterization of Equilibrium

As is often the case for games with continuous action spaces, this game will not have a Nash equilibrium in pure strategies. This is because payoffs can be discontinuous if multiple agents choose the same bid. By bidding an infinitesimal amount more, an agent can avoid the tie and ensure that he gets first access to the goods — and if supply is constraining, this creates a discrete jump in utility. This is proven formally later in this section. However, we can prove existence in mixed strategies in full generality.

Theorem 1. *For any $p > 0$, a mixed strategy equilibrium exists.*

Beyond existence, we can also significantly narrow the set of potential equilibria. Indeed, the observations made in the following proposition are quite useful in practice as one seeks to calculate such an equilibrium, as is done in the following two sections.

Proposition 1. *If μ^* is an equilibrium strategy in an auction for priority access,*

1. $B_i^* \subset \cup_{j \neq i} B_j^* \cup \{0\}$.

2. *For all $b_i > 0$, there is no i such that $\mu_i^*(\{b_i\}) > 0$.*

3. $\cup_i B_i^*$ *is a connected set, and* $0 \in \cup_i B_i^*$.

4. $EU_i(\mu^*) \geq v_i(0)$.

5. *If $\mu_i^*(0) > 0$ for some i and $S \leq \sum_{j \neq i} d_j(p)$, then $EU_i(\mu^*) = v_i(0)$.*

6. *If $\mu_i^*(0) > 0$ for some i and $\sum_{j \neq i} d_j(p) < S \leq \sum_j d_j(p)$, then*

$$EU_i(\mu^*) = u_i \left(S - \sum_{j \neq i} d_j(p), 0 \right).$$

7. *If $\max B_i^* = \max \cup_j B_j^*$ for some i, then*

$$EU_i(\mu^*) = u_i \left(\min\{d_i(p), S\}, \max B_i^* \right).$$

The first claim establishes that an agent will not include a particular bid in his support unless it also appears in the support of some other player. There is no reason to do so, since in such a case, one could reduce his bid without changing the expected amount of supply available for purchase. The exception, of course, is 0, since it is not possible to reduce the bid any further.

The next observation is that atoms[5] never occur except possibly at zero. If a consequential tie were to occur with positive probability (*i.e.* two agents have an atom on the same bid), at least one of them would strictly prefer to break the tie. If consequential ties almost never occur, two possibilities exist. On the one hand, it could be that if the agent with an atom reduced his bid, he would obtain the same purchases with the same probabilities but lower bid cost. If not, there must be some other agent bidding just below the atom, whose outcome depends on his order relative to the first. But this second agent can do strictly better by raising his bid just above the atom, for then he reduces his chances of purchasing behind the first while incurring an infinitesimal increase in bid cost.

An immediate corollary of the second claim is that no pure strategy equilibrium exists. The only exception is when all agents set $\mu_i(\{0\}) = 1$, and that could only be sustained if supply is weakly greater than aggregate demand.

The third claim states that there can be no gaps in the aggregate support; otherwise, agents bidding just above the gap would do well to reduce their bid inside of the gap. By the same reasoning, the aggregate support always includes 0.

The fourth claim identifies that equilibrium expected utility is bounded below by autarky. This is because agents are always free to bid nothing, $b_i = 0$. The worst that can happen in such a case is that the agent comes in last and obtains nothing. Note that this claim only applies to expected utility; once the randomization on mixed strategies occurs, agents may find themselves having a high bid yet still being beaten by some other agent and left without any opportunity to make a purchase. Yet other ex-post realizations may place the agent first in line in spite of a small bid.

The last three claims are particularly useful in computing equilibria, since they pin down an agent's utility relative to certain aspects of his strategy. Whenever an atom occurs in his mixed strategy, his expected utility is the same as if he bid nothing and won exactly the leftovers after all others are served (or zero if there is insufficient supply to have leftovers). On the other hand, anyone whose support includes the highest bid receives an expected utility as if he always wins the first priority (claiming either his full demand or, if not possible, the entire supply) and

[5] An atom of a measure μ_i is a strategy b_i such that $\mu_i(\{b_i\}) > 0$.

bidding his highest bid for certain.

4 Symmetric Valuations

To provide some initial examples of auctions for priority access, we examine the special case in which all agents have the same underlying utility function, u_i. In this environment, agents have equal demand d for the good; thus, the highest k bidders, where $k \cdot d \leq S < (k+1)d$, will receive their full demand. The $k+1^{\text{th}}$ bidder will receive the leftovers, and all others will receive nothing.

With the assumption of equal utilities, our model can be translated into the environment of Barut and Kovenock (1998). There, one's place after bidding determines which prize is received, and all agents agree on the value of the various prizes. Here, the top k bidders receive a prize worth $\bar{w} = v(d) - pd$. The $k+1^{\text{th}}$ prize is valued at $\bar{x} = v(r) - pr$ where $r = S - kd$, and all others receive a prize of $v(0)$, which we normalize to 0. An application of their results provides the following characterization.

A symmetric equilibrium always exists, in which all agents play the same mixed strategy. If $S < d$, then only the first place bidder receives any amount of the good. The unique equilibrium strategy, expressed as a cumulative distribution function, is $F(b) = \left(\frac{b}{\bar{x}}\right)^{\frac{1}{n-1}}$, which has support $B_i^* = [0, \bar{x}]$.

If a larger supply is available, the symmetric equilibrium strategy $F(b)$ is implicitly expressed as:

$$b = \bar{x}\begin{pmatrix} n-1 \\ k \end{pmatrix} F(b)^{n-k-1}(1-F(b))^k + \bar{w}\sum_{m=0}^{k-1}\begin{pmatrix} n-1 \\ m \end{pmatrix}(1-F(b))^m(F(b))^{n-m-1},$$

where the parenthetical element represents the binomial coefficient of $n-1$ choose k. This will have a support of $B_i^* = [0, \bar{w}]$ (or if $k = n-1$, $B_i^* = [0, \bar{w} - \bar{x}]$), and has no atoms.

By applying Claim 1.7, note that $EU_i(\mu^*) = 0$ if $k < n-1$ and $EU_i(\mu^*) = \bar{x}$ otherwise. The expected expenditures on bids is $R = k\bar{w} + \bar{x}$ if $k < n-1$ and $R = (n-1)\bar{w} - n\bar{x}$ otherwise.

On the topic of allocative efficiency, one might address it on two levels. If we take as given that the quantities are split into k packages of size d and one of size r, then since everyone has the same utility from these packages, any distribution of them is equally efficient. However, if we allow alternative divisions of the S units, this allocation is certainly inefficient because $v(\cdot)$ has diminishing marginal utility; agents who end up with nothing after the auction will value a marginal unit of the good higher than those who obtained d units.

When $1 \leq k < n-1$, there are also a continuum of equilibria with asymmetric mixed strategies, where our identical agents nonetheless employ differing mixed strategies. The defining feature of these strategies is the size of atom placed at 0. Up to $n-k-2$ agents can have an atom at 0, while all others randomize continuously over $[0, \bar{w}]$. Those with atoms will have a gap in their support, from 0 to some bid b_i, where the latter depends on the size of their atom. Where two agents' supports overlap, their CDF will look the same (with the exception of atoms at 0). To solve for the continuous portion of the mixed strategy, one employs functions much like for the symmetric case.

We refer the reader to Barut and Kovenock (1998) for a full characterization of these strategies. Indeed, despite their complexity, they result in the same expected utility and expenditure as the symmetric case, and thus add little to the performance of the auction. When some agents have different utility functions, however, there may not be a symmetric equilibrium, as we shall see in the next section.

We conclude this section by considering these results in the light of applications of the model. First, note that since $EU_i = 0$, rent is full dissipated when $S \leq (n-1)d$; in other words, if at least one agent will be unable to make purchases, they will all expend effort up to the point that ex-ante, they are indifferent between participating in the auction or not.

In political rent seeking, the politician is able to extract these rents in the form of contributions or bribes. If he has discretion as to the size of S, auction revenue is maximized by setting $S = (n-1)d$. Similarly, consider a monopolist producer who limits production to S and holds price p below market clearing. He then allocates the short supply according to whichever clients do the most to ingratiate themselves with the seller. The favors from clients will fully extract their consumer surplus, so long as some clients are left without any supply. Moreover, he extracts the maximum favors (for a given p) by setting $S = (n-1)d$. If allowed to do so, it will be optimal to set p equal to marginal cost to maximize profit.

This gives nearly the same result as the Oi two-part tariff (Oi, 1971) with two exceptions: first, one client is left unfulfilled here, where none would be under the Oi

Tariff (which seeks to full efficiency by excluding no one). Second, while the per-unit price is administered equivalently, here the entry tariff is not a fee explicitly set by the monopolist, but rather the endogenous result of competition among clients.

Of course, in other applications, the bids are not collected by anyone, but are simply destroyed resources, such as productive time wasted by standing in lines or by adding fluff to a grant proposal. In such environment when $S \leq (n-1)d$, participants are no better off for having expended the effort, and yet they destroy $k\bar{w} + \bar{x}$ dollars worth of resources in the process. Here, the efficient outcome occurs with price equal to marginal cost, S approaching nd, and bids falling to zero.

5 Asymmetric Valuations

The auction for priority access becomes even more interesting when bidders differ in their utility from and/or demand for the good. In addition to the fact that mixed strategies will differ across households, this heterogeneity also introduces another complication: agents worry not only about how many people come in ahead of them, but also how much each of them will take.

We provide a complete analysis of two-person auctions, though it is possible (albeit complicated) to solve for any number of bidders.

When there are exactly two agents bidding, the unique mixed strategy equilibrium can be characterized in terms of each agent's value of being first or second. We use the following notation to represent these:

$$\begin{aligned}
\bar{w}_i &\equiv v_i(q_i) - pq_i \quad \text{where} \quad q_i = \min\{d_i(p), S\} \\
\bar{x}_i &\equiv v_i(q_i) - pq_i \quad \text{where} \quad q_i = \min\{d_i(p), \max\{0, S - d_j(p)\}\}
\end{aligned}$$

Thus, $\bar{w}_i$ represents i's utility from realized consumption when he has first priority, and $\bar{x}_i$ when he has second priority. Note that $\bar{w}_i \geq \bar{x}_i$. On the other hand, comparisons between $\bar{w}_i$ and either $\bar{w}_j$ or $\bar{x}_j$ could go either direction. Without loss of generality, we will assume that $\bar{w}_1 - \bar{x}_1 \geq \bar{w}_2 - \bar{x}_2$; if this were not the case, we could just relabel the two agents. The difference $\bar{w}_i - \bar{x}_i$ indicates the net gain from consuming first rather than second.

Proposition 2. *The unique equilibrium is* $F_1(b) = \frac{b}{\bar{w}_2 - \bar{x}_2}$ *and* $F_2(b) = \frac{b + \bar{w}_1 - \bar{x}_1 - \bar{w}_2 + \bar{x}_2}{\bar{w}_1 - \bar{x}_1}$, *resulting in* $EU_1 = \bar{w}_1 - \bar{w}_2 + \bar{x}_2$ *and* $EU_2 = \bar{x}_2$.

Note that both distributions share the same support $[0, \bar{w}_2 - \bar{x}_2]$, and agent 2 has an atom at 0. In effect, the agent who has more to lose by being second (bidder 1) bids more aggressively: his mixed strategy first-order stochastically dominates the other bidders. Indeed, he is more likely to win the first priority: his bid is higher than agent 2's with probability $1 - \frac{(\bar{w}_2 - \bar{x}_2)}{2(\bar{w}_1 - \bar{x}_1)}$, and this becomes more likely the larger the difference in net gains.

This has the flavor of an efficiency result: the total utility generated from consumption is larger when agent 1 consumes first than when agent 2 does($i.e.$ $\bar{w}_1 + \bar{x}_2 \geq \bar{w}_2 + \bar{x}_1$), and that outcome is more likely the bigger the gains. However, the less efficient outcome always occurs with positive probability. Moreover, there is no assurance that marginal utilities will be equated even when 1 does consume first.

It is interesting to note that agent 1 may not fare better than agent two in equilibrium; when $\bar{w}_1 < \bar{w}_2$, agent 2 receives higher expected utility. For instance, this could occur if agent 2 gets a lot of utility from the good, but has minor marginal utility from the additional units obtained by being first priority instead of second. Meanwhile, the reverse must hold for agent 1: his total utility is somewhat small, but the marginal utility from additional units is relatively large.

The expected bid expenditures from the two agents will be $R = \frac{(\bar{w}_2 - \bar{x}_2)}{2} - \frac{(\bar{w}_2 - \bar{x}_2)^2}{2(\bar{w}_1 - \bar{x}_1)}$. This always increases with $\bar{w}_1 - \bar{x}_1$, and is maximized when $\bar{w}_2 - \bar{x}_2 = \frac{1}{2}(\bar{w}_1 - \bar{x}_1)$. Thus, if a politician is awarding rents via an auction for priority access, he would want one agent to have as large a net gain as possible, while the other has half that net gain.

More likely, however, the seller can only indirectly affect the net gain by changing available supply S. In particular, a reduction in S will reduce $\bar{x}_i$ for both agents, and eventually, $\bar{w}_i$ as well; the relative size of the effect will depend on the particular utility from consumption, v_i.

6 Comparisons to Raffles for Priority Access

We now contrast the auction for priority access to the proportional contest, which is frequently used in models of rent seeking. In this environment, each bidder chooses a bid b_i, and wins the contest with probability $\frac{b_i}{\sum_j b_j}$. In this sense, the bid is like a purchase of raffle tickets, where the participant can improve his odds of winning by buying more tickets.

Of course, in our environment, there is not a single prize to win. However, this same mechanism can be used to establish an order of priority. After each agent chooses a bid, the first access is awarded to i with probability $\frac{b_i}{\sum_j b_j}$. If any supply remains, the second priority is awarded to k with probability $\frac{b_k}{\sum_{j \neq i} b_j}$, assuming i won the first round. This continues until either the supply or the bidders are exhausted. As before, bids are sunk; an unlucky participant pays his bid even if he is unable to purchase any of the good.

The key difference is that priority order is no longer a deterministic function of bids, as it is in the auction. In the raffle, a larger bid produces a proportionately larger probability of getting first access, but does not guarantee it. Another difference is that raffles typically result in a unique pure strategy equilibrium. We present this solution for the cases of n-person symmetric valuations and 2-person asymmetric valuations, and compare the results to their auction counterparts.

6.1 Symmetric valuation raffle

Since all individuals are identical, we only investigate the symmetric strategies. Let b_i denote i's bid, while b_j denote the bid used by all other agents. The expected utility of agent i is:

$$
\begin{aligned}
EU_i(b_i, b_j) \quad = \quad & \bar{w}\left(\sum_{s=0}^{k-1} \frac{b_i}{b_i + (n-s-1)b_j} \prod_{t=1}^{s}\left(1 - \frac{b_i}{b_i + (n-t)b_j}\right)\right) \\
& + \bar{x}\frac{b_i}{b_i + (n-k-1)b_j} \prod_{t=1}^{k}\left(1 - \frac{b_i}{b_i + (n-t)b_j}\right) - b_i
\end{aligned}
$$

The first summation represents the k opportunities that agent i has to gain access while there is still enough supply to satisfy his demand. Note that, conditional on having lost on the first $s \leq k$ draws, the probability of winning on the $s + 1^{\text{th}}$ draw, $\frac{b_i}{b_i+(n-s-1)b_j}$, is increasing in s. This is because those who have already been given access are removed from the subsequent draws. The second term indicates the probability of being selected for exactly the $k + 1^{\text{th}}$ priority.

Solving for the symmetric equilibrium, we take the first-order condition, setting the derivative with respect to b_i equal to zero. Then symmetry is imposed, replacing

b_j with b_i. This yields the equilibrium strategy:

$$b_i^* = \frac{\bar{x} - \bar{w} + ((n-k)\bar{w} - \bar{x}) \sum_{t=0}^{k} \frac{1}{n-t}}{n}$$

which produces expected utility

$$EU_i^* = \frac{k\bar{w} + \bar{x}}{n} - b_i^*.$$

Since pure strategies are employed, bid revenue is deterministic, and is equal to nb_i^*.

Comparing the outcomes of a raffle to those of the auction, note that expected utility must be higher in the raffle whenever $k \leq n-1$, since the auction gives expected utility of 0. The raffle, on the other hand, produces strictly positive utility; indeed, this is evidenced by the fact that $b_i = 0$ was available but not selected.

Expected bid revenue, on the other hand, is always higher under the auction format. If we compare the two formats, the auction produces more revenue when

$$\frac{(k+1)\bar{w}}{(n-k)\bar{w} - \bar{x}} > \sum_{t=0}^{k} \frac{1}{n-t}.$$

But note that:

$$\begin{aligned}\frac{(k+1)\bar{w}}{(n-k)\bar{w} - \bar{x}} &> \frac{k+1}{n-k} = \frac{k}{n-k} + \frac{1}{n-k} > \frac{k}{n-k-1} + \frac{1}{n-k} \\ &> \frac{1}{n} + \frac{1}{n-1} + \cdots + \frac{1}{n-k-1} + \frac{1}{n-k} = \sum_{t=0}^{k} \frac{1}{n-t}.\end{aligned}$$

6.2 Asymmetric valuation raffle

Again, we consider a two-person environment using the same notation as in Section 4. As before, let $\bar{w}_1 - \bar{x}_1 > \bar{w}_2 - \bar{x}_2$. Bidder i has expected utility:

$$EU_i(b_i, b_j) = \bar{w}_i \frac{b_i}{b_i + b_j} + \bar{x}_i \frac{b_j}{b_i + b_j} - b_i$$

First-order conditions reveal that the unique Nash equilibrium is

$$b_1^* = \frac{(\bar{w}_1 - \bar{x}_1)^2(\bar{w}_2 - \bar{x}_2)}{(\bar{w}_1 - \bar{x}_1 + \bar{w}_2 - \bar{x}_2)^2} \quad \text{and} \quad b_2^* = \frac{(\bar{w}_1 - \bar{x}_1)(\bar{w}_2 - \bar{x}_2)^2}{(\bar{w}_1 - \bar{x}_1 + \bar{w}_2 - \bar{x}_2)^2},$$

with equilibrium payoffs:

$$EU_1^* = \bar{x}_1 + \frac{(\bar{w}_1 - \bar{x}_1)^3}{(\bar{w}_1 - \bar{x}_1 + \bar{w}_2 - \bar{x}_2)^2} \quad \text{and} \quad EU_2^* = \bar{x}_2 + \frac{(\bar{w}_2 - \bar{x}_2)^3}{(\bar{w}_1 - \bar{x}_1 + \bar{w}_2 - \bar{x}_2)^2}.$$

Regarding efficiency, the probability that agent 1 receives first priority after the raffle is $\frac{\bar{w}_1 - \bar{x}_1}{\bar{w}_1 - \bar{x}_1 + \bar{w}_2 - \bar{x}_2}$; thus, a larger difference in the net gain of the two bidders will create a higher probability of the more efficient ordering. However, an auction for priority access will achieve that outcome with even higher probability:

$$1 - \frac{(\bar{w}_2 - \bar{x}_2)}{2\,(\bar{w}_1 - \bar{x}_1)} > \frac{\bar{w}_1 - \bar{x}_1}{\bar{w}_1 - \bar{x}_1 + \bar{w}_2 - \bar{x}_2} \qquad \Longleftrightarrow \qquad \bar{w}_1 - \bar{x}_1 > \bar{w}_2 - \bar{x}_2,$$

which holds by assumption. This demonstrates that even though mixed strategies are employed in the auction, the atom in agent 2's bid skews the outcome so that agent 1 wins more frequently than in the raffle.

Since bids are chosen in a pure strategy, the expected revenue from bids is simply their sum. This is computes to $R = \frac{(\bar{w}_1 - \bar{x}_1)(\bar{w}_2 - \bar{x}_2)}{\bar{w}_1 - \bar{x}_1 + \bar{w}_2 - \bar{x}_2}$. The auction generates more revenue than the raffle if and only if $w_1 - \bar{x}_1 < \frac{1}{\sqrt{2}-1}(\bar{w}_2 - \bar{x}_2)$. In words, bidder 1's net gain cannot be too large relative to bidder 2's net gain (*i.e.* at most 140% more).

Finally, a comparison of expected utility reveals that bidder 2 fares strictly better under the raffle, rather than the auction. The comparison is ambiguous for bidder 1, who prefers the raffle whenever $(\bar{w}_1 - \bar{x}_1)^3 > (\bar{w}_1 - \bar{w}_2)(\bar{w}_1 - \bar{x}_1 + \bar{w}_2 - \bar{x}_2)^2$. In particular, if $\bar{w}_1 < \bar{w}_2$, this will always hold. Note that this coincides with when agent 2 gets more utility from the auction than agent 1 does, as discussed in Section 4.

7 Conclusion

The auction for priority access provides a method of allocating a good in short supply among potential buyers, allowing customers to make purchases in the order of their bids. Since there is still a marginal cost of procurement even after access is awarded, the first customer may not exhaust supply. We have shown that this auction can be an effective method for the seller to extract consumer surplus from his patrons, similar to a two-part tariff.

From an efficiency standpoint, in one sense the auction for priority access is doomed from the start. Some agents can be left without any of the good, even though their marginal utility from consumption is well above those who did procure a portion. Of course, this is where competitive markets excel, since prices align marginal incentives. These incentives do not arise in our auction because bids are a fixed cost, reflecting total surplus rather than marginal.

Even so, the more agents differ, the more likely our auction is to award first priority in a way that yields highest total surplus. Moreover, the auction performs strictly better on this dimension than the raffle for priority access.

In the context in which bids are collected by the seller, such as political rent seeking or key deposits to circumvent rent control, the auction represents a means for the seller to capture rents. On the other hand, when applied to situations such as grant writing and queuing, the bids represent wasted resources (from a social standpoint). Indeed, applying our same results regarding expected bid revenue, the maximum amount of waste occurs when all but one customer will be able to buy what he wants. Indeed, society may benefit by having *fewer* units available so as to reduce the incentive to queue.

A Proofs

A.1 Proof of Theorem 1

Proof. Existence is proven by applying Corollary 5.2 of Reny (1999). Three conditions are required to apply this result; the first two are clearly satisfied.

- The space of pure strategies $[0, M]^n$ must be compact and Hausdorff.
- The payoff function in pure strategies $v_i(q_i(b_{-i}, b_i)) - pq_i(b_{-i}, b_i) - b_i$ must be bounded and measurable on $[0, M]^n$.
- The mixed strategy game is *better-reply secure.* Formally, let $\bar{\mu}$ be any non-equilibrium mixed strategy profile. Let $\bar{u}$ be a profile of expected payoffs such that for some sequence $\mu^k \to \bar{\mu}$, $u(\mu_k) \to \bar{u}$. The game is better reply secure if there exists some player i and strategy $\hat{\mu}_i$ such that $u_i(\mu_{-i}, \hat{\mu}_i) > \bar{u}_i$ for all μ_{-i} within some open neighborhood of $\bar{\mu}_{-i}$.

The third condition is easily satisfied if $\bar{\mu}$ has no consequential ties occurring with strictly positive probability. If so, there is no discontinuity at $u(\bar{\mu})$. Since $\bar{\mu}$ is not an equilibrium profile, some agent i has a best response to $\bar{\mu}_{-i}$ that strictly increases his utility above $u_i(\bar{\mu})$. By keeping the neighborhood of $\bar{\mu}_{-i}$ sufficiently small, i's utility remains bounded above $u_i(\bar{\mu})$: even if nearby μ_{-i} introduce a consequential tie (and hence a discontinuity), the probability of that tie occurring is limited as small as needed. Hence, any drop in utility at such discontinuities can be kept arbitrarily small, and changes elsewhere are continuous.

Suppose instead that a consequential ties occurs with strictly positive probability under $\bar{\mu}$. In this case, it is not necessarily the case that $\bar{u} = u(\bar{\mu})$. Since $\bar{\mu}$ is not an equilibrium profile, there still exists some i with a best response to $\bar{\mu}_{-i}$ that strictly increases his utility above $u_i(\bar{\mu})$. If he is not involved in any of the consequential ties, the analysis from before still applies.

If the only agents not playing a best reply are involved in a consequential tie, the danger is that $\bar{u}$ may treat some of them as if they always win the tie — and there may not be a $\hat{\mu}_i$ for such a person that strictly provides more utility. However, since this is a consequential tie, at least one of the agents involved has positive probability of not receiving his full demand. That agent can strictly improve on $\bar{\mu}_i$ by shifting the strictly positive atom he placed on the consequential tie(s) to ϵ higher. There can only be a countable number of consequential ties, so he can do this without entering another tie. By doing so, he strictly wins each tie and obtains his full demand, but with an insignificant increase in his expected bid, thus obtaining a strictly higher payoff that $\bar{u}_i$.

With i playing the $\hat{\mu}_i$ so constructed, we can contain μ_{-i} to a small enough neighborhood so that i still receives more than $\bar{u}_i$. Even if they move some positive probability to the atoms of $\hat{\mu}_i$, we can restrict it to be small enough that it only slightly decreases the expected utility of i.

□

A.2 Proof of Proposition 1

Proof of claim 1.1. Assume there is some interval $(a_1, a_2) \subset [0, M]$ such that for all $b \in (a_1, a_2)$, $b \notin B_j^*$ for all $j \neq i$. Suppose that agent i considers choosing some $b_i \in (a_1, a_2)$. If instead he chooses $b_i' = b_i - \epsilon \in (a_1, a_2)$, then for any $b_{-i} \in B_{-i}^*$, $q_i(b_i, b_{-i}) = q_i(b_i', b_{-i})$. In words, whichever bid profile is selected by the other players, both b_i and b_i' will have the same rank compared to the other bids, and thus obtain the same quantity for agent i.

But then $EU_i(b_i', \mu_{-i}^*) = EU_i(b_i, \mu_{-i}^*) + \epsilon$, since both integrate over the same outcomes, but b_i' does so with a smaller bid. Thus, no strategy $b_i \in (a_1, a_2)$ can be utility maximizing; hence $b_i \notin B_i^*$. □

Proof of claim 1.2. Suppose there exists a bid $a \in (0, M]$ such that $\mu_i^*(\{a\}) > 0$ for some agent i. Note that there can only be a finite number of atoms among the n agents; thus, there is some range $(a - \epsilon, a + \epsilon)$ in which there is no other atom, though other agents might have an atom at a.

Let $\tilde{q}_i(b)$ and $\underset{\sim}{q}_i(b)$ denote the most and least agent i could receive under bidding profile b. These only differ when a consequential tie occurs — if so, there is enough remaining supply to satisfy some but not all of those who bid the same as i, and the randomly selected permutation π determines who is served first.

Denote the set of bids where a is consequential to i as:

$$C_i(a) \equiv \left\{ b_{-i} \in B_{-i}^* \ : \ b_i = a \text{ and } \tilde{q}_i(b) > \underset{\sim}{q}_i(b) \right\} \tag{3}$$

Suppose $\mu_{-i}^*(C_i(a)) > 0$. Then agent i can increase his expected utility by shifting his atom $\mu_i^*(a)$ from a to $a+\epsilon$. By doing so, he ensures that he will always obtain $\tilde{q}_i(b)$ for all $b \in C_i(a)$, rather than sometimes obtaining $\underset{\sim}{q}_i(b)$. Moreover, for all $b \notin C_i(a)$, his quantity obtained will weakly increase. Thus, for ϵ sufficiently small, the strict increase in utility from added consumption will outweigh the small increase in bid.

If instead $\mu^*(C_i(a)) = 0$, then when i bids a, two possibilities exist. First, bidding $a - \epsilon$ could give the same outcomes with equal probability. This is to say, for any quantity q,

$$\mu_{-i}^*(\{b_{-i} \in B_{-i}^* \;:\; q_i(a, b_{-i}) = q\}) = \mu_{-i}^*(\{b_{-i} \in B_{-i}^* \;:\; q_i(a-\epsilon, b_{-i}) = q\}).$$

This would happen if no other agents have $(a-\epsilon, a) \subset B_j^*$, or if those who do have no effect on the remaining supply available for i (because the supply is already exhausted, or is so plentiful that both i and j can be satisfied). In such a case, $EU_i(a, \mu_{-i}^*) = EU_i(a-\epsilon, \mu_{-i}^*) - \epsilon$; i can reduce his bid cost while maintaining his average benefit from his opportunities to purchase.

Alternatively, if bidding $a - \delta$ for any $\delta \in (0, \epsilon)$ would have some impact on agent i's outcomes, then there must exist some agent j with $(a-\epsilon, a) \subset B_j^*$. If so, agent k's outcomes are *also* affected by whether agent i is allowed to purchase before him or not. Also, because they are part of an equilibrium best response, any $b_k \in (a-\epsilon, a)$ produces the same expected utility $EU_k(b_k, \mu_{-k}^*)$. However, by bidding $a+\delta$ for some arbitrarily small $\delta > 0$, agent k can strictly increase his utility, because he has reduced by $\mu_i(\{a\})$ the probability of being outbid by agent i, which allows him the opportunity to purchase strictly more of the good, while incurring a slightly higher bid.

Thus, in all cases, we obtain a contradiction if agent i has an atom at a. Note that if an atom were at $a = 0$, the first case regarding consequential bids still apply ; however, if bids are almost always inconsequential bids, the latter arguments cannot be replicated since there is no bid below 0. $\square$

Proof of claim 1.3. Suppose that there is some interval (a_1, a_2) such that for all $b \in (a_1, a_2)$, $b \notin B_j^*$ for all j. Pick an agent i such that $a_2 \in B_i^*$. By claim 2 of Proposition 1, there are no atoms at a_2. Thus, if agent i were to bid $b \in (a_1, a_2)$, he would achieve the same outcomes with the same probability as when bidding $b = a_2$, but with a lower bid. Hence this cannot be an equilibrium.

Thus, the aggregate support $\cup_i B_i^*$ must be connected, having ruled out any gaps.

Moreover, the same logic applies to an interval $(0, a_2)$, hence $0 \in \cup_i B_i^*$. $\square$

Proof of claim 1.4. For any strategy μ_{-i}, $EU_i(0, \mu_{-i}) \geq v_i(0)$; the worst that can happen when $b_i = 0$ is that agent i never wins the opportunity to purchase any amount, but made no expenditure to get there. Thus, since $b_i = 0$ is always a feasible choice, then $EU_i(\mu_i^*, \mu_{-i}^*) \geq EU_i(0, \mu_{-i}^*) \geq\geq v_i(0)$ in equilibrium. $\square$

Proof of claim 1.5. Suppose $\mu_i^*(0) > 0$ for some i, $\sum_{j \neq i} d_j(p) < S \leq \sum_j d_j(p)$, and $EU_i(\mu^*) > v_i(0)$. This also means that $EU_i(0) > v_i(0)$; in other words, with positive probability, agent i is able to buy some positive amount of the good even when bidding $b_i = 0$.

If there were almost never consequential ties at $b = 0$, all agents $j \neq i$ would bid strictly more than 0 with probability 1. If so, $\sum_{j \neq i} d_j(p) < S \leq \sum_j d_j(p)$ implies that agent i would almost always receive 0 of the good, which contradicts.

So consequential ties must occur at $b = 0$ with positive probability; thus, some subset of the bidders also have an atom at 0. But then the same logic applies as in the proof of Claim 1.2: Any one of the agents who ties at 0 would strictly benefit by shifting their atom from 0 to ϵ. This ensures access to a strictly larger amount of the good and only requires an arbitrarily small increase in the bid. Thus μ^* would not be an equilibrium. $\square$

Proof of claim 1.6. Suppose $S < \sum_i d_i(p)$, $\sum_{j \neq i} d_j(p) < S \leq \sum_j d_j(p)$. Again, we can rule out consequential ties at 0, because if they occurred with positive probability, any tied agent would have an incentive to raise his bid slightly.

Instead, suppose that there are almost never consequential ties at $b = 0$. Again, all other agents must bid strictly more than 0 with probability 1. That means when i bids 0, he receives $S - \sum_{j \neq i} d_j(p)$ almost surely. Hence, $EU_i(\mu^*) = u_i\Big(S - \sum_{j \neq i} d_j(p), 0\Big)$. $\square$

Proof of claim 1.7. Let $\hat{b}_i \equiv \max B_i^* = \max \cup_j B_j^*$. Bidding $\hat{b}_i$ will almost always result in i receiving first priority. Recall that there are no atoms if $\hat{b}_i > 0$, and if $\hat{b}_i = 0$ then there are no consequential ties and $S > \sum_j d_j(p)$. Thus $q_i(\hat{b}_i, b_{-i}) = \min\{d_i(p), S\}$ for almost all $b_{-i} \in B_{-i}^*$. Hence, i's expected utility must equal $EU_i(\hat{b}_i, \mu_{-i}^*) = u_i\Big(\min\{d_i(p), S\}, \hat{b}_i\Big)$. $\square$

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