When too little is as good as nothing at all:

Rationing a disposable good among satiable people with acceptance thresholds

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Abstract

We study the problem of rationing a divisible good among a group of people. Each person's preferences are characterized by an ideal amount that he would prefer to receive and a minimum quantity that he will accept: he finds any amount less than this threshold to be just as good as receiving nothing at all. Further, any amount beyond his ideal quantity has no effect on his welfare.

The focus of our study is the existence of *Pareto-efficient, strategy-proof,* and *envy-free* rules. While the definitions of these axioms carry through, with minimal changes, from the more commonly studied problem without disposability or acceptance thresholds, we show that these extensions are not compatible in the model that we study. We also adapt the *equal-division lower bound* axiom and propose another fairness axiom called *awardee-envy-freeness*. While these are also incompatible with *strategy-proofness*, we identify the set of all *Pareto-efficient* rules that satisfy these two properties.

We also characterize the class of *conditional sequential priority rules* as the set of all *Pareto-efficient, strategy-proof,* and *non-bossy* rules.

1 Introduction

Imagine a town facing an energy shortage. Suppose that it has access to a fixed number of kilowatt hours of electricity. The town officials must divide these among local business owners. However, these owners have threshold quantities of electricity, below which business is not viable. That is, if the quantity allocated to a particular business owner is lower than his threshold, it is as good as not allocating anything to him at all. Also, an owner is made better off as the quantity that he receives increase beyond his threshold, but only up to a certain level. We propose a model for such situations and study rules for making rationing decisions.

In our model, there is a social endowment to divide among a group of people. Each person in this group has preferences over the quantity that he receives. His preferences are described by a minimum threshold that he finds acceptable and an ideal amount that he would prefer to receive. He is indifferent between receiving any quantity below his threshold and receiving nothing at all. He finds a quantity above his threshold to be better and better as it increases, up to his ideal quantity. He is indifferent between any two quantities that are at least as high as his ideal amount. We do *not* require that the endowment be exhausted. That is, the endowment is disposable. Though it is odd to consider disposing of a good that is being rationed, it may be meaningful when some people have been satiated and the amount left over is unacceptable to those who have not been satiated. Problems where there is no shortage of the good are trivial rationing problems where every person is satiated and the remainder is disposed of.

If we restrict ourselves to rules that never allocate to a person more than his peak, an alternative interpretation of our model is that of a bankruptcy problem (O'Neill 1982), where each claimant has some participation cost: if his award does not exceed his cost, he prefers not to show up and collect it. His peak is interpreted as his claim.

We propose a set of axioms and search for rules that satisfy them. As usual, *Pareto-efficiency* says that no person can be made better off without this hurting another person. *Strategy-proofness*, also defined in the usual way, says that no person can beneficially misrepresent his preferences. Meaningful notions of fairness, on the other hand, are more difficult to define. One familiar notion requires that no person envies another. A weaker version is that people with identical preferences be treated equally. We show that no *Pareto-efficient* rule satisfies even the weaker of these. We propose an axiom that we call *awardee-fairness*, that says no person who receives an amount that he finds acceptable envies any other person. We describe the class of all *Pareto-efficient* rules that are *awardee-fair*. Another notion of fairness, called the *equal-division lower bound*, is that each person finds what he receives to be at least as desirable as an equal share of the social endowment. This requirement is compatible with *Pareto-efficiency* and we provide

a necessary and sufficient condition for it to be met.

Unfortunately, we also show that no rule that is *Pareto-efficient* and satisfies either *awardee-fairness* or the *equal-division lower bound* is also *strategy-proof*. We describe the class of all *Pareto-efficient, strategy-proof,* and *non-bossy* rules. Each of these rules always selects a very inequitable division.

Finally, we consider strengthening the strategic requirement to *group strategy-proofness*, which says that no group of people can misrepresent their preferences in a way that makes at least one member better off without hurting another member. We show that this axiom is incompatible even with very weak notions of efficiency.

In contrast with our model, for the closely related "classical" *division problem with single-peaked preferences*, where the endowment is to be allocated in entirety, a single rule satisfies all of the axioms that we have discussed (Sprumont 1991, Ching 1992, Ching 1994). In fact, this rule uniquely satisfies several other sets of axioms, some of which are discussed in the appendix (Thomson 1994, Sönmez 1994, Thomson 1995, Schummer and Thomson 1997, Thomson 1997, Weymark 1999). To re-iterate, the differences between this classical model and ours are free-disposal of the social endowment and acceptance thresholds.

The incompatibility of *Pareto-efficiency, strategy-proofness,* and *fairness* in our model is a consequence of the introduction of lower bounds on acceptable quantities. In a model without free-disposal, upper bounds on a person's consumption space are also meaningful. When both upper and lower bounds are introduced to classical problems with *single-peaked preferences, strategy-proofness* is incompatible with even a weak notion of *efficiency* (Bergantiños, Massó and Neme 2009). This is different from our model, where *strategy-proofness* and *efficiency* are compatible.

A similarity between our model and one with both upper and lower bounds, without free-disposal, is that for a particular definition of *fairness*, the class of *Pareto-efficient* and *fair* rules (Bergantiños et al. 2009) resembles the class of *Pareto-efficient* and *awardee-fair* rules that we characterize for our model.

The remainder of the paper is organized as follows. In Section 2 we formally define the model and in Section 3 we define our axioms. In Section 4 we provide several classes of rules and discuss the axioms that they satisfy. Section 5 is divided into 5.1 where we discuss incompatibilities between some of the axioms and 5.2 where we characterize two classes of rules from Section 4 by imposing two different sets of axioms. We summarize our results in Section 6. In an appendix, we propose a variable population generalization of the model and study some of the implications of three common variable population axioms.

2 The Model

Let N be a group of people and $M \in \mathbb{R}_+$ be an amount of a perfectly divisible good that is to be divided among them. Each person $i \in N$ has preferences over the quantity that is alloted to him. His preferences, R_i, defined over $[0, \infty)$, are characterized by an acceptance threshold l_i and an ideal amount p_i such that $l_i \leq p_i$. He is indifferent between any two quantities $x, y \in [0, \infty)$ if, either

$$
\begin{aligned}
&i) \quad x, y \leq l_i, \text{ or} \\
&ii) \quad x, y \geq p_i.
\end{aligned}
$$

In all other cases, whenever $x > y$, he prefers x to y. If i finds x to be at least as desirable as y under preference relation R_i, we write $x\, R_i\, y$. If he finds x to be more desirable than y, we write $x\, P_i\, y$. Finally, we write $x\, I_i\, y$ if he is indifferent between x and y.
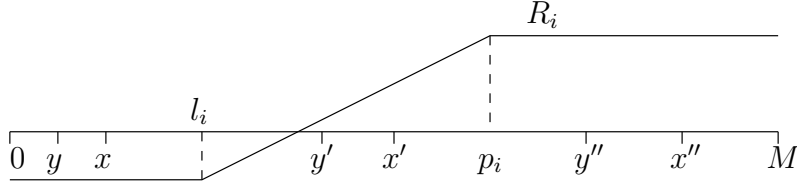
Figure 1: Preferences: Since x and y are both below the threshold level l_i, we have $x\, I_i\, y$. Since x'' and y'' are both greater than p_i, we have $x''\, I_i\, y''$. However, since $p_i \geq x' > y' \geq l_i$, we have $x'\, P_i\, y'$. In fact, we also have $x'\, P_i\, x$ and $x''\, P_i\, x'$.

Notice that a preference relation is completely described by the lower and upper bounds. Though R_i is represented linearly in Figure 1, the only relevant aspects of R_i are l_i and p_i.

Denote the set of all preference relations by $\mathcal{R}$. A **problem involving the people in N** consists of a profile of preferences, $R \in \mathcal{R}^N$, and an amount, $M \in \mathbb{R}_+$, to allocate (not necessarily entirely) among N in such a way that the sum of what is awarded to each person does not exceed M. The presence of lower bounds introduces a certain degree of discreteness to this model which involves only an infinitely divisible good. A **feasible allocation** at M is any vector $x \in \mathbb{R}_+^N$ such that $\sum_{i \in N} x_i \leq M$. Let $F(M)$ denote the set of feasible allocations at M.[1][2] Given an allocation $x \in F(M)$, for each $i \in N$, we denote i's component of x by x_i

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[1]That is, $F(M)$ is an $(|N| - 1)$-simplex.

[2]While our model is reminiscent of "bankruptcy problems with interval claims" (Branzei, Dimitrov, Pickl and Tijs 2002, Dimitrov, Tijs and Branzei 2003, Alparslan Gök and Branzei 2008), the "interval" plays a very different role. In that model, it refers to the possible values

and the list of others' components by x_{-i}. Similarly, let R_i denote i's preference relation and let R_{-i} denote the list of others' preferences. For each $R'_i \in \mathcal{R}^N$, let (R'_i, R_{-i}) be the profile where i has preference R'_i and the list of others' preferences is R_{-i}.

A **rule** for problems involving people in N, $\varphi : \mathcal{R}^N \times \mathbb{R}_+ \to \mathbb{R}^N_+$, associates every problem involving the members of N with a feasible allocation.

3 Axioms

We list some desiderata of rules in this section. Let φ be a rule. The first require-ment is the familiar concept of efficiency. It says that an allocation is chosen only if there is no other allocation that makes at least one person better off without making another worse off.

Pareto-efficiency: For each $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+$, there is no $x \in F(M)$ such that,

$$
\begin{aligned}
&i) \quad \text{for each } i \in N, x_i \ R_i \ \varphi_i(R, M), \text{ and} \\
&ii) \quad \text{there is } i \in N \text{ such that } x_i \ P_i \ \varphi_i(R, M).
\end{aligned}
$$

A weakening of the previous concept of efficiency is that an allocation is chosen only if there is no other allocation that makes *every* person better off.

Weak Pareto-efficiency: For each $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+$, there is no $x \in F(M)$ such that, for each $i \in N, x_i \ P_i \ \varphi_i(R, M)$.

A further weakening says that if the social endowment equals the sum of the ideal amounts, then each person should receive his ideal amount.[3]

Unanimity: For each $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+$, if $\sum_{i \in N} p_i = M$, then $\varphi(R, M) = (p_i)_{i \in N}$.

An implication of *unanimity* is that the range of a the rule includes all possible divisions of the endowment.

The next property is that no person can benefit by misreporting his preferences. In the following definition, and later in the proofs of our results, given $i \in N$ and a pair $R_i, R'_i \in \mathcal{R}$, we place a T above i's true preference relation $(\overset{\text{\tiny T}}{R_i})$, and an F above his false preference relation $(\overset{\text{\tiny F}}{R'_i})$.

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that an uncertain claim could take. It is, thus, the set of possible upper bounds on what a person may be awarded. In our model, the interval is the set of acceptable awards. Further, we consider strategic issues in an environment where these intervals are private information.

[3]A slightly stronger, but normatively more appealing, version of *unanimity* says that if the social endowment is *at least as large as* the sum of each person's ideal amount, then each person should receive his ideal amount.

Strategy-proofness: For each $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+$, and each $i \in N$, there is no $R'_i \in \mathcal{R}$,

$$\varphi_i(\overset{\text{F}}{R'_i}, R_{-i}, M) \overset{\text{T}}{P_i} \varphi_i(\overset{\text{T}}{R_i}, R_{-i}, M).$$

A more demanding property is that no group of people can misreport in a way that makes at least one of its members better off without making any of its members worse off.

Group strategy-proofness: For each $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+$ and each $S \subseteq N$, there is no $R'_S \in \mathcal{R}^S$ such that

$$\text{for each } i \in S, \varphi_i(\overset{\text{F}}{R'_S}, R_{-S}, M) \overset{\text{T}}{R_i} \varphi_i(\overset{\text{T}}{R_S}, R_{-S}, M), \text{ and}$$
$$\text{there is } i \in S \text{ such that } \varphi_i(\overset{\text{F}}{R'_S}, R_{-S}, M) \overset{\text{T}}{P_i} \varphi_i(\overset{\text{T}}{R_S}, R_{-S}, M).$$

We show that *group strategy-proofness* is "too" demanding in the sense that it is not compatible with even the weakest of our notions of efficiency: *unanimity* (Proposition 4).

Next we present two common notions of fairness. First, no person "envies" another.

Envy-freeness: For each $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+$, and each pair $i, j \in N, \varphi_i(R, M) R_i \varphi_j(R, M)$.

The second notion says that two people with identical preferences are treated equally.

Equal treatment of equals: For each $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+$, and each pair $i, j \in N$ such that $R_i = R_j = R_0$, we have $\varphi_i(R, M) I_0 \varphi_j(R, M)$.

Note that *envy-freeness* implies *equal-treatment of equals*. As we will see (Proposition 1), even *equal treatment of equals* is incompatible with *Pareto-efficiency*. We propose another weakening of *envy-freeness* that applies only to those people who receive an acceptable share. This axiom can be interpreted as follows: following a division of the social endowment, only those who find their share acceptable show up to receive it. We require that no person who shows up envies an any other person (whether that person shows up or not).

Awardee-envy-freeness: For each $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+$, and each pair $i, j \in N$, if $\varphi_i(R, M) > l_i$, then $\varphi_i(R, M) R_i \varphi_j(R, M)$.

Our final notion of fairness is that each person finds his component of the allocation to be at least as desirable as an equal share of the social endowment.

Equal-division lower bound: For each $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+$, and each $i \in N$, $\varphi_i(R, M) R_i \frac{M}{|N|}$.

Remark 1. *Among Pareto-efficient rules,* envy-freeness *implies the* equal-division lower bound. If, at $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+$, for each pair $i, j \in N$, $\varphi_i(R, M) \; R_i \; \varphi_j(R, M)$, yet there is $i \in N$ such that $\frac{M}{n} \; P_i \; \varphi_i(R, M)$, then for each $j \in N$, $p_i > \frac{M}{n} > \varphi_i(R, M) \geq \varphi_j(R, M)$. This violates *Pareto-efficiency* of φ. ○

If, at an allocation $x \in F(M)$, there is $i \in N$ who receives an amount that is unacceptable to him, then he is indifferent between receiving x_i and receiving 0. We interpret x as an allocation at which i is given an amount that he finds unacceptable, so he either does not take it, or takes it and disposes of it. The planner may as well dispose of x_i units of the good himself and not give i anything. Similarly, when $x_i > p_i$, the planner may as well dispose of $x_i - p_i$ units of the good and give i only p_i. This is expressed by the next axiom.

For each $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+$, define the **welfare-equivalence class of x at (R, M), $WE(x, R, M)$**, by setting

$$WE(x, R, M) \equiv \{y \in F(M) : \text{ for each } i \in N, x_i \; I_i \; y_i\}.$$

If $WE(x, R, M) = \{x\}$, then **x is welfare-unique at (R, M)**. For each $x \in F(M)$ such that x is not *welfare-unique at* (R, M), there is $x' \in WE(x, R, M)$ such that for each $i \in N$:

$$
\begin{array}{lll}
\text{If} & x_i \leq l_i, & \text{then} \quad x'_i = 0. \\
\text{If} & x_i \in (l_i, p_i), & \text{then} \quad x'_i = x_i. \\
\text{If} & x_i \geq p_i, & \text{then} \quad x'_i = p_i.
\end{array}
$$

Such x' is the **canonical representation of $WE(x, R, M)$ at (R, M)**. If x is either *welfare-unique* or the *canonical representation of $WE(x, R, M)$ at (R, M)*, then x is **canonical at (R, M)**.

Canonicity: For each $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+$, $\varphi(R, M)$ is *canonical at* (R, M).

The next requirement is an application of the "replacement principle" (Moulin 1987, Thomson 1997)[4]. It says that if the preferences of one person change, then all others are affected in the same direction: either each person finds his new share at least as desirable, or each person finds his new share to be no more desirable.

Welfare-domination under preference replacement: For each $(R, M) \in \mathcal{R}^N \times \mathbb{R}$, each $i \in N$ and each $R'_i \in \mathcal{R}$, either:

$$
\begin{array}{ll}
i) & \text{For each } j \in N \setminus \{i\}, \varphi_j(R'_i, R_{-i}, M) \; R_j \; \varphi_j(R_i, R_{-i}, M), \text{ or} \\
ii) & \text{For each } j \in N \setminus \{i\}, \varphi_j(R_i, R_{-i}, M) \; R_j \; \varphi_j(R'_i, R_{-i}, M).
\end{array}
$$

The following requirement is that if a person's preferences change in a way that his own component of the allocation remains the same, then others' components

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[4]For a survey, see (Thomson 1999).

should also remain the same (Satterthwaite and Sonnenschein 1981). While it appears rather technical, one reason this axiom is normatively appealing, in our model, is that it is implied by the combination of the previous requirement, *Pareto-efficiency* and *canonicity*.

Another interpretation of this axiom is that it rules out a particular kind of profitable misreporting by pairs of people. That is, if we impose it alongside *canonicity*, we rule out pairs of people where one member can help the other by misreporting his preferences. Of course, the combination of this axiom with *canonicity* is weaker than *group strategy-proofness*.

Non-bossiness: For each $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+$, each $i \in N$ and each $R'_i \in \mathcal{R}$, if $\varphi_i(R'_i, R_{-i}, M) = \varphi_i(R_i, R_{-i}, M)$ then $\varphi_{-i}(R'_i, R_{-i}, M) = \varphi_{-i}(R_i, R_{-i}, M)$.

Another solidarity requirement is that an increase in the social endowment should not make any person worse off (Chun and Thomson 1988, Roemer 1986, Moulin and Thomson 1988).

Resource monotonicity: For each $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+$, each $M' \in \mathbb{R}_+$ such that $M' > M$, and each $i \in N$, $\varphi_i(R, M') \, R_i \, \varphi(R, M)$.

Our final requirement says that small changes in the data of the problem should cause only small changes in the chosen allocation.

Continuity: If $\{(R^\nu, M^\nu)\}_{\nu=1}^\infty$ is a sequence in $\mathcal{R}^N \times \mathbb{R}_+$,[5] such that $\lim_{\nu \to \infty} (R^\nu, M^\nu) = (R, M) \in \mathcal{R}^N \times \mathbb{R}_+$, then $\lim_{\nu \to \infty} \varphi(R^\nu, M^\nu) = \varphi(R, M)$.

In Section 5.1, we show that no *Pareto-efficient* rule is *continuous*. Intuitively, this is because small changes in lower bounds can sometimes lead to very large changes in the selection of a *Pareto-efficient* rule. We propose a weaker version of *continuity* that is compatible with *Pareto-efficiency* by ignoring sequences where there is a person whose lower bound is below both his peak and the social endowment at each point along the sequence, but is at least as large as one of them in the limit.

Weak continuity: If $\{(R^\nu, M^\nu)\}_{\nu=1}^\infty$ is a sequence in $\mathcal{R}^N \times \mathbb{R}_+$ such that,

$i)$ $\lim_{\nu \to \infty} (R^\nu, M^\nu) = (R, M) \in \mathcal{R}^N \times \mathbb{R}_+$, and

$ii)$ for each $i \in N$ if $l_i \geq \min\{p_i, M\}$, then there is $\nu^* \in \mathbb{N}$ such that for each $\nu \geq \nu^*, l_i^\nu \geq \min\{p_i^\nu, M^\nu\}$,

then $\lim_{\nu \to \infty} \varphi(R^\nu, M^\nu) = \varphi(R, M)$.

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[5]Since each preference relation $R_i \in \mathcal{R}$ is uniquely identified by a pair $(l_i, p_i) \in \mathbb{R}_+$, this is a sequence in $\mathbb{R}^{2n+1}$.

Remark 2. *Allowing strict preferences over $(0, l_i)$:* If, for each $i \in N$, we interpret l_i as the quantity that i requires to recoup a fixed cost, then we can construe any $x_i \in (0, l_i)$ as forcing i to incur a loss. Thus, it makes sense to allow $0 \; I_i \; l_i$, but for each $x_i \in (0, l_i), l_i \; P_i \; x_i$. However, an implication of *Pareto-efficiency* is that $x_i \notin (0, l_i)$. Otherwise, we can find a *Pareto-improvement* by discarding x_i units of the endowment and giving i nothing. ◦

4 Rules

Each member of the first class of rules that we describe is associated with a "sequential priority list." For each possible quantity of the social endowment, a person with highest priority is chosen. He keeps for himself the lowest of his most preferred portions of the endowment. Depending on the social endowment, the identity of the first person, and the quantity that the he keeps for himself, a person with the next highest priority is chosen. Of what is left after the first person is given his ideal quantity, the second person keeps the lowest of his most preferred quantities. Again, depending on the social endowment, the identities of the first two people, and the quantities that they keep for themselves, a person with third highest priority is chosen, and so on.[6]

Conditional sequential priority rules: Let $I \equiv \{i^k\}_{k=1}^{n-1}$ where

$$i^1 : \mathbb{R}_+ \to N,$$
$$i^2 : \mathbb{R}_+ \times N \times \mathbb{R}_+ \to N,$$
$$\vdots$$
$$i^{n-1} : \mathbb{R}_+ \times N^{n-2} \times \mathbb{R}_+^{n-2} \to N$$

be such that for each $M \in \mathbb{R}_+$ and each $x \in \mathbb{R}^{n-2}$, we have a sequence $\{i_k\}_{k=1}^{n-1}$ such that

$$i_1 = i^1(M),$$
$$i_2 = i^2(M, i_1, x_{i_1}) \neq i_1,$$
$$\vdots$$
$$i_{n-1} = i^{n-1}(M, i_{1,\ldots,n-2}, x_{i_{1,\ldots,n-2}}) \notin \{i_1, \ldots, i_{n-2}\}.$$

The **conditional sequential priority rule with respect to I, CSP^I,** is defined as follows for each $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+$.

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[6]Members of this family of rules are similar to those proposed by Pápai (2001) and also studied by Ehlers and Klaus (2003) for the problem of assigning multiple objects.

$$\begin{aligned}
i_1 &\equiv i^1(M), \\
CSP^I_{i_1}(R,M) &\equiv \begin{cases} \min\{M, p_{i_1}\} & \text{if } \min\{M, p_{i_1}\} > l_{i_1}, \text{ and} \\ 0 & \text{otherwise,} \end{cases} \\
i_2 &\equiv i^2(M, i_1, CSP^I_{i_1}(R,M)) \\
CSP^I_{i_2}(R,M) &\equiv \begin{cases} \min\{M - CSP^I_{i_1}(R,M), p_{i_2}\} & \text{if } \min\{M - CSP^I_{i_1}(R,M), p_{i_2}\} > l_{i_2}, \text{ and} \\ 0 & \text{otherwise,} \end{cases} \\
&\;\;\vdots \\
i_{n-1} &\equiv i_1^n(M, i_{1,\ldots,n-2}, CSP^I_{i_{1,\ldots,n-2}}(R,M)) \\
CSP^I_{i_{n-1}}(R,M) &\equiv \begin{cases} \min\left\{M - \sum_{k=1}^{n-2} CSP^I_{i_k}(R,M), p_{i_{n-1}}\right\} & \text{if } \min\left\{M - \sum_{k=1}^{n-2} CSP^I_{i_k}(R,M), p_{i_{n-1}}\right\} > l_{i_{n-1}}, \\ & \quad \text{and} \\ 0 & \quad \text{otherwise,} \end{cases} \\
i_n &\in N \setminus \{i_1, \ldots, i_{n-1}\} \\
CSP^I_{i_n}(R,M) &\equiv \begin{cases} \min\left\{M - \sum_{k=1}^{n-1} CSP^I_{i_k}(R,M), p_{i_n}\right\} & \text{if } \min\left\{M - \sum_{k=1}^{n-1} CSP^I_{i_k}(R,M), p_{i_n}\right\} > l_{i_n} \\ \quad \text{and} \\ 0 & \quad \text{otherwise.} \end{cases}
\end{aligned}$$

Conditional sequential priority rules are *Pareto-efficient, strategy-proof, welfare-dominant under preference replacement, non-bossy,* and *canonical.* However, they are neither *awardee-envy-free* nor do they satisfy the *equal-division lower bound.* They also violate *group strategy-proofness.* While none of these rules are *continuous,* some members are *weakly continuous.*

We now describe a sub-class of *conditional sequential priority rules* for which the priority order is fixed. These rules are "unconditional" *sequential priority* rules.

Sequential priority rules: If I is such that for each $k \in \{1, \ldots, n-1\}$, each pair $M, M' \in \mathbb{R}_+$ and each pair $x, x' \in \mathbb{R}_+^{n-2}$ such that $\sum_{i=1}^{n-2} x_i \leq M$ and $\sum_{i=1}^{n-2} x_i' \leq M'$,

$$\begin{aligned}
i^1(M) &= i_1 = & i^1(M'), \\
i^2(M, i_1, x_{i_1}) &= i_2 = & i^2(M', i_1, x_{i_1}'), \\
&\;\;\vdots \\
i^{n-1}(M, i_{1,\ldots,n-2}, x_{i_{1,\ldots,n-2}}) &= i_{n-1} = & i^{n-1}(M', i_{1,\ldots,n-2}, x_{i_{1,\ldots,n-2}}'),
\end{aligned}$$

then CSP^I is a **sequential priority rule with respect to $\{i_k\}_{k=1}^{n-1}$, SP^I.**

We show that this is exactly the class of *weakly continuous conditional sequential priority rules.*

At the end of the section, Example 1 demonstrates the application of a member of this class and others that follow.

The next rule is analogous to the well-known "uniform rule" for the classical problems with *single-peaked preferences* (Bénassy 1982, Sprumont 1991).

Uniform rule: Define U by setting, for each $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+$, and $i \in N$,

$$U_i(R, M) \equiv \begin{cases} p_i & \text{if } \sum_{i \in N} p_i \leq M, \text{ or} \\ \min\{p_i, \lambda\} & \text{otherwise,} \end{cases}$$

where λ is such that $\sum_{i \in N} U_i(R, M) = M$.

While the *uniform* rule is *weakly Pareto-efficient, strategy-proof, envy-free, welfare-dominant under preference replacement, non-bossy,* and *continuous,* it is not *Pareto-efficient.* In contrast with the classical setting, this rule is not *group strategy-proof.*[7]

Members of the next class of rules are *Pareto-efficienct.* Unfortunately, they are neither *strategy-proof* nor *envy-free.*

Efficient Uniform rules: Before we introduce this class of rules, define for each $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+$, the **efficient uniform coalitions at (R, M), $EUC(R, M)$,** by setting,

$$EUC(R, M) \equiv \left\{ N' \subseteq N : \begin{matrix} \text{there is} \\ \lambda \in \mathbb{R}_+ \\ \text{such that} \end{matrix} \begin{matrix} i) & \text{for each } j \notin N', M - \sum_{i \in N'} \min\{p_i, \lambda\} \leq l_j, \text{ and} \\ ii) & \text{for each } j \in N', \lambda \geq l_j. \end{matrix} \right\}.$$

Before we introduce the next concept, we show that for each problem, there is always at least one *efficient uniform coalition.*

Claim 1. *For each $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+$ such that there is $i \in N$ for whom $l_i < \min\{p_i, M\}$, $EUC(R, M) \neq \varnothing$.*

Proof: Let $\{1, 2, \ldots, k\} \subseteq N$ be such that $l_1 \leq l_2 \leq \cdots \leq l_k < M$. By assumption, $k > 1$. Start with $\{1\}$. If $M - \min\{p_1, M\} \leq l_2$, then $\{1\} \in EUC(R, M)$. Otherwise, $\{1\} \notin EUC(R, M)$.

Suppose $\{1, 2, \ldots, j\} \notin EUC(R, M)$. Then $M - \sum_{i=1}^{j} \min\{p_i, M\} > l_{j+1}$.

If $\{1, \ldots, k\} \notin EUC(R, M)$, then $k \neq n$ and $M - \sum_{i=1}^{k} p_i > l_{k+1} \geq M$, which is a contradiction. $\triangle$

A **selector** is a function, $\sigma : \mathcal{R}^N \times \mathbb{R}_+ \to \mathbb{P}(N)$,[8] such that for each $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+$, $\sigma(R, M) \in EUC(R, M)$.

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[7]To see this, let $R \in \mathcal{R}^N$ and $i \in N$ be such that $0 < \lambda \leq l_i$ and $j \in N \setminus \{i\}$ such that $\lambda \in (l_j, p_j)$. Let $R_i' \in \mathcal{R}$ be such that $p_i' = 0$. Then, $0 = \varphi_i(R_i', R_{-i}) \overset{\text{T}}{I_i} \varphi_i(\overset{\text{T}}{R_i}, R_{-i}) = \lambda$ and since $\varphi_j(\overset{\text{F}}{R_i'}, R_{-i}) > \lambda > l_j$, $\varphi_j(\overset{\text{T}}{R_i'}, R_{-i}) \overset{\text{T}}{P_j} \varphi_j(\overset{\text{T}}{R_i}, R_{-i})$.

[8]Given a set A, let $\mathbb{P}(A)$ be the power set of A.

The **efficient uniform rule with selector σ, EU^{σ},** is defined by setting, for each $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+$ and each $i \in N$,

$$EU_i^{\sigma}(R, M) \equiv \begin{cases} \min\{p_i, \lambda\} & \text{if } i \in \sigma(R, M), \text{ or} \\ 0 & \text{otherwise.} \end{cases}$$

where λ is such that:

$$i) \quad \text{for each } j \notin \sigma(R, M), M - \sum_{i \in \sigma(R, M)} \min\{p_i, \lambda\} \leq l_j,$$

$$ii) \quad \text{for each } j \in \sigma(R, M), \lambda \geq l_j, \text{ and}$$

$$iii) \quad \text{if } \sum_{i \in \sigma(R, M)} \min\{p_i, \lambda\} < M \text{ then } \lambda \geq \max_{i \in \sigma(R, M)} \{p_i\}.$$

Efficient uniform rules are *Pareto-efficient, awardee-envy-free,* and *non-bossy.* They are not, however, *strategy-proof, welfare-dominant under preference replacement,* or *envy-free.* In fact, as we will show, no *Pareto-efficient* rule is *envy-free* (as a consequence of Proposition 1). Some members of this class are *weakly continuous.*

	P	WP	$Unan$	ETE	B_{ed}	$\begin{array}{c}Aw\\E\text{-}F\end{array}$	$St\text{-}P$	$\begin{array}{c}Grp\\\text{-}StP\end{array}$	$\frac{PR}{WD}$	$\begin{array}{c}Non\text{-}\\Boss\end{array}$	$\begin{array}{c}Weak\\Cont\end{array}$	$\begin{array}{c}Res\\Mon\end{array}$	$Canon$
Cond Seq Pri	+	+	+	−	−	−	+	−	−	+	+/−	−	+
Seq Pri	+	+	+	−	−	−	+	−	−	+	+	−	+
Unif	−	+	+	+	+	+	+	−	+	+	+	+	−
Eff Unif	+	+	+	−	+/−	+	−	−	−	+/−	+/−	−	+

Table 1: Rules and axioms: A $+$ indicates that the rule(s) corresponding to the row satisfies the axiom corresponding to the column. A $-$ indicates that it does not. We use $+/-$ if some members of the family satisfy the axiom while others do not.

Example 1. *Application of U, EU^{σ}, and SP^I.*

As shown in Figure 4, let $N = \{1, 2, 3\}$ and $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+$ be such that $M = 10$, $l_1 = l_2 = 0$, $l_3 = 5$, $p_1 = 2, p_2 = 10$, and $p_3 = 7$. Then:

1. $U(R, M) = (2, 4, 4)$, showing that U is not *Pareto-efficient.*
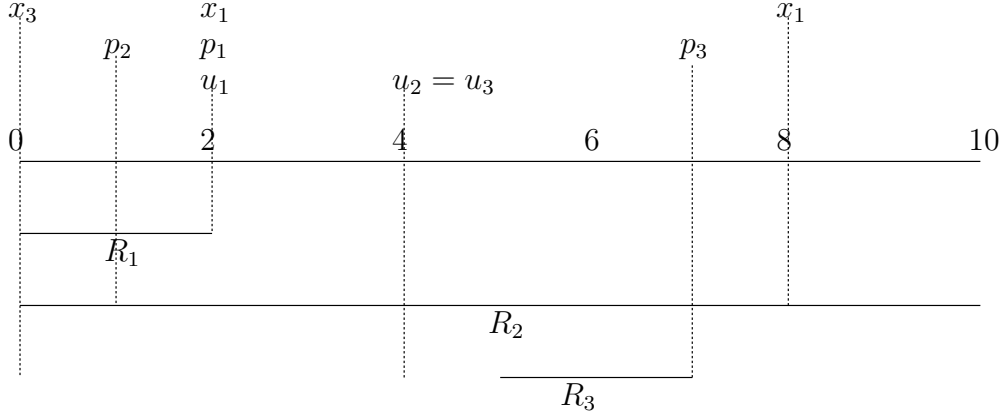
Figure 2: Each person's preference relation is denoted by the segment over which it is strictly increasing. The allocations shown are: $u \equiv U(R, M), x \equiv EU^\sigma(R, M)$, and $p \equiv SP^I(R, M)$.

2. $EUC(R, M) = \{\{1, 2\}, \{1, 3\}, \{2\}\}$.

3. If $\sigma(R, M) = \{1, 2\}$, then $EU^\sigma(R, M) = (2, 8, 0)$.

4. If $I \equiv \{i_1 = 3, i_2 = 1\}$, then $SP^I(R, M) = (2, 1, 7)$. $\diamond$

5 Results

We begin, in 5.1, by showing the incompatibility of some of the axioms presented in Section 3. In 5.2 we provide two characterizations.

5.1 Incompatibilities

The first result is that *Pareto-efficiency* is incompatible with *equal treatment of equals* and therefore with *envy-freeness*.

Proposition 1. *No rule is* Pareto-efficient *and satisfies* equal treatment of equals.

Proof: Let φ be *Pareto-efficient*. Let $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+$ be such that for each $i \in N$, $l_i = \frac{M}{n}$ and $p_i = M$. By *Pareto-efficiency*, there is $i \in N$ such that $\varphi_i(R, M) > l_i = \frac{M}{n}$. This means that there is $j \in N \setminus \{i\}$ such that $\varphi_j(R, M) < \frac{M}{n}$. Thus, φ violates *equal treatment of equals*. $\square$

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Proposition 2. *No rule is* Pareto-efficient *and* continuous.[9]

Proof: Suppose φ is *Pareto-efficient*. Let $R \in \mathcal{R}^N$ be such that for each $i \in N, p_i = M$ and $0 < l_i < M$. By *Pareto-efficiency*, there is $i \in N$ such that $\varphi_i(R, M) > l_i$. Define $r_i : [0, 1] \to \mathcal{R}$ by setting, for each $t \in [0, 1]$,

$$r_i(t) = R_i^t \in \mathcal{R} \text{ associated with } p_i^t = M \text{ and } l_i^t = l_i + t(M - l_i).$$

Then, $r_i(0) = R_i$ and $r_i(1)$ is associated with $l_i^1 = M$.

Define $\gamma : [0, 1] \to [0, M]$ by setting, for each $t \in [0, 1], \gamma(t) \equiv \varphi_i(r_i(t), R_{-i}, M)$. By definition of γ, $\gamma(0) > l_i > 0$, but by *Pareto-efficiency*, $r_i(1)$ is associated with $l_i^t = M$, and there is $j \in N$ such that $l_j < M$ and $p_j = M$, $\gamma(1) = \varphi_i(r_i(1), R_{-i}, M) = 0$. Further, by *Pareto-efficiency* and since there is $j \in N$ such that $l_j < M$ and $p_j = M$, there is no $t \in [0, 1]$ such that $f(t) \in (0, l_i + t(M - l_i)]$. So γ, and thus φ, are *discontinuous*. $\square$

Proposition 3. *No rule is* Pareto-efficient, strategy-proof, *and* awardee-envy-free.[10]

Proof: We show this when $|N| = 2$, but the argument easily generalizes to the remaining cases.

Let φ be a rule that satisfies all of the axioms mentioned in the proposition. Let $N = \{1, 2\}$ and $M = 6$. Let $R \in \mathcal{R}^N$ be such that $l_1 = l_2 = 0$ and $p_1 = p_2 = 6$. By *Pareto-efficiency* and *awardee-envy-freeness*, $\varphi(R, M) \in \{(0, 6), (3, 3), (6, 0)\}$.

If $\varphi(R, M) = (0, 6)$, then let $R_2' \in \mathcal{R}$ be such that $l_2' = 0$, $p_2' = 5$. By *strategy-proofness*, $\varphi_2(R_1, R_2', M) \geq 5$. Otherwise, $6 = \varphi_2(R_1, \overset{F}{R_2}, M) \overset{T}{P_2'} \varphi_2(R_1, \overset{T}{R_2'}, M)$. By *Pareto-efficiency*, $\varphi(R_1, R_2', M) = (1, 5)$. But this violates *awardee-envy-freeness* at (R_1, R_2'). By an analogous argument, $\varphi(R, M) \neq (6, 0)$. Thus, $\varphi(R, M) = (3, 3)$.

Now let $\overline{R}_1 \in \mathcal{R}$ be such that $\bar{l}_1 = 2$ and $\bar{p}_1 = 5$. By *strategy-proofness*, $\varphi_1(\overline{R}_1, R_2, M) = 3$. Let $\overline{R}_2 = \overline{R}_1$. Again, by *strategy-proofness*, $\varphi(\overline{R}, M) = (3, 3)$.

Let $\tilde{R}_1 \in \mathcal{R}$ such that $\tilde{l}_1 = 4$ and $\tilde{p}_1 = 5$. By *strategy-proofness*, $\varphi_1(\tilde{R}_1, \overline{R}_2, M) \leq 3$. By *Pareto-efficiency*, $\varphi_2(\tilde{R}_1, \overline{R}_2, M) \geq 5$. By an analogous argument, letting $\tilde{R}_2 = \tilde{R}_1$, $\varphi_1(\overline{R}_1, \tilde{R}_2, M) \geq 5$.

Finally, by *Pareto-efficiency*, either $\varphi_1(\tilde{R}, M) \geq 5$ or $\varphi_2(\tilde{R}, M) \geq 5$. However, if $\varphi_1(\tilde{R}, M) \geq (5, 0)$, then $\varphi_2(\tilde{R}_1, \overset{F}{\overline{R}_2}, M) \overset{T}{\tilde{P}_2} \varphi_2(\tilde{R}_1, \overset{T}{\tilde{R}_2}, M)$. Thus, $\varphi_2(\tilde{R}, M) \geq 5$. Since $\varphi_1(\overset{F}{\overline{R}_1}, \overset{T}{\tilde{R}_2}, M) \overset{T}{\tilde{P}_1} \varphi_2(\tilde{R}_1, \tilde{R}_2, M)$, this violates *strategy-proofness*. $\square$

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[9]The proof of Proposition 2 amounts to showing that the set of *Pareto-efficient* allocations is not *lower hemicontinuous*, and thus does not permit a continuous selection (Michael 1956).

[10]Note that Proposition 3 is not subsumed by the Theorem 2 since *non-bossiness* is imposed in Theorem 2.

Proposition 4. *No rule is* Pareto-efficient, strategy-proof, *and satisfies the* equal-division lower bound.[11]

Proof: We show this when $|N| = 2$, but the argument easily generalizes to the remaining cases. Let $N = \{1, 2\}$ and let $R \in \mathcal{R}^N$ be such that $R_1 = R_2 = R_0$ such that $(l_0, p_0) = (\frac{M}{2}, M)$. Suppose φ is *Pareto-efficient* and satisfies the *equal-division lower bound*. Then, by *Pareto-efficiency*, $\varphi(R, M) \in \{(M, 0), (0, M)\}$. If $\varphi(R, M) = (0, M)$, then let $R_1' \in \mathcal{R}$ be such that $(l_1', p_1') = (0, M)$. By the *equal-division lower bound*, $\varphi_1(R_1', R_2, M) \geq \frac{M}{2}$ and so $\varphi_2(R_1', R_2, M) \leq \frac{M}{2}$. By *Pareto-efficiency*, $\varphi(R_1', R_2, M) = (M, 0)$. Then, $M = \varphi_1(\overset{F}{R_1'}, R_2, M) \overset{T}{P_i}$ $\varphi_1(\overset{T}{R_1}, R_2, M) = 0$. Thus, φ violates *strategy-proofness*. $\qquad \square$

5.2 Characterizations

We begin with a necessary and sufficient condition for a rule to satisfy the *equal-division lower bound*.

Proposition 5. *A rule, φ, satisfies the* equal-division lower bound, *if and only if, for each $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+$, and for each $i \in N$ such that $l_i < \frac{M}{|N|}$,*

$$\varphi_i(R, M) \geq \min \left\{ p_i, \frac{M}{|N|} \right\}.$$

Proof: Let $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+$ and $i \in N$ be such that $l_i < \frac{M}{|N|}$. By the *equal-division lower bound*, $\varphi_i(R, M) R_i \frac{M}{|N|}$. Thus, if $\varphi_i(R, M) < \frac{M}{|N|}$, then $\varphi_i(R, M) I_i \frac{M}{|N|} I_i p_i$. Thus, $\varphi_i(R, M) \geq p_i$.

Conversely, if for each (R, M) and each $i \in N$ such that $l_i < \frac{M}{|N|}$, $\varphi_i(R, M) \geq \min\{p_i, \frac{M}{|N|}\}$, then $\varphi_i(R, M) R_i \frac{M}{N}$. For each $i \in N$ such that $l_i \geq \frac{M}{|N|}$, $\frac{M}{2} I_i 0$, so $\varphi_i(R, M) R_i 0 I_i \frac{M}{|N|}$. Thus, φ satisfies the *equal-division lower bound*. $\qquad \square$

Each rule that is not *canonical* is equivalent in "welfare terms" to a *canonical* rule. This is formalized in the following lemma.

Lemma 1. *For each rule φ, there is a unique* canonical *rule φ' such that for each $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+$, for each $i \in N$,*

$$\varphi_i(R, M) I_i \varphi_i'(R, M).$$

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[11]Note that Proposition 4 is not subsumed by the Theorem 2 since *non-bossiness* is imposed in Theorem 2.

Proof: Let φ' be such that for each $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+$, for each $i \in N$,

$$\varphi'_i(R, M) = \begin{cases} 0 & \text{if } \varphi_i(R, M) < l_i \\ p_i & \text{if } \varphi_i(R, M) \geq p_i \\ \varphi_i(R, M) & \text{otherwise.} \end{cases}$$

That is, if $\varphi(R, M)$ is *welfare-unique* at (R, M), $\varphi'(R, M)$ coincides with $\varphi(R, M)$. Otherwise, $\varphi'(R, M)$ is the *canonical representation of* $WE(\varphi(R, M), R, M)$ at (R, M). Thus, φ' is *canonical* and for each $i \in N$, $\varphi_i(R, M) \; I_i \; \varphi'_i(R, M)$. $\qquad\square$

We call φ' defined in proof of Lemma 1 the **canonical equivalent of φ**. Since *Pareto-efficiency, envy-freeness, awardee-envy-freness,* and *the equal-division lower bound* deal with a single preference profile and are stated in welfare terms, it is clear that these properties of a rule are inherited by its *canonical equivalent*. However, it is not as obvious that *strategy-proofness* is similarly inherited.

Lemma 2. *The* canonical equivalent *of a* strategy-proof *rule is* strategy-proof.[12]

Proof: Let φ' be the canonical equivalent of φ. Then, for each $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+$ and each $i \in N$,

$$\begin{aligned} \varphi_i(R, M) &= \varphi'_i(R, M) & \text{if } l_i < \varphi_i(R, M) < p_i. \\ \varphi_i(R, M) &\geq \varphi'_i(R, M) & \text{otherwise.} \end{aligned}$$

Suppose φ' is not *strategy-proof*. Then, there is $i \in N$, and $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+$ such that

$$\varphi'_i(\overset{\text{F}}{R'_i}, R_{-i}, M) \; \overset{\text{T}}{P_i} \; \varphi'_i(\overset{\text{T}}{R_i}, R_{-i}, M).$$

Then, $\varphi'_i(R, M) < p_i$. If $\varphi'_i(R, M) > l_i$, then $\varphi_i(R, M) = \varphi'_i(R, M)$ and if $\varphi'_i(R, M) = 0$, then $\varphi_i(R, M) \leq l_i$. Thus,

$$\max\{\varphi'_i(R, M), l_i\} \geq \varphi_i(R, M). \tag{1}$$

Further,

$$\varphi'_i(R'_i, R_{-i}, M) > \max\{\varphi'_i(R, M), l_i\}. \tag{2}$$

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[12]The converse is not true. Consider φ defined by setting, for each $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+$,

$$\varphi_1(R, M) = \begin{cases} M & \text{if } p_1 \text{ and } \leq \frac{M}{2} \\ \frac{M}{2} & \text{otherwise,} \end{cases}$$

and for each $i \in N \setminus \{i\}, \varphi_i(R, M) = 0$. The *canonical equivalent* of φ is defined by setting, for each $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+$,

$$\varphi_1(R, M) = \begin{cases} \max\{p_1, \frac{M}{2}\} & \text{if } l_1 < \frac{M}{2}, \text{ and} \\ 0 & \text{otherwise,} \end{cases}$$

and for each $i \in N \setminus \{i\}, \varphi_i(R, M) = 0$. Clearly, φ' is *strategy-proof* but φ *is not*.

Since $\varphi_i(R'_i, R_{-i}, M) \geq \varphi'_i(R'_i, R_{-i}, M)$, by (1) and (2),

$$\varphi_i(R'_i, R_{-i}, M) \geq \varphi'_i(R'_i, R_{-i}, M) > \max\{\varphi'_i(R, M), l_i\} \geq \varphi_i(R, M).$$

Then, $\varphi_i(\overset{\text{F}}{R'_i}, R_{-i}, M) \overset{\text{T}}{P_i} \varphi_i(\overset{\text{T}}{R_i}, R_{-i}, M)$. So φ is not *strategy-proof*. $\square$

In light of Lemmas 1 and 2, it is without loss of generality to study only *canonical* rules for the remainder of this section.

Theorem 1. *Every* Pareto-efficient *and* awardee-envy-free *rule is an* efficient uniform rule.

Proof: Let φ be a *Pareto-efficient* and *awardee-envy-free* rule. Let σ^φ be a *selection function* defined by setting, for each $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+$,

$$\sigma^\varphi(R, M) \equiv \{i \in N : \varphi_i(R, M) \neq 0\}.$$

Let $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+$, and set $x \equiv \varphi(R, M)$. If $x \neq EU^{\sigma^\varphi}(R, M)$, then we have one of four cases:[13]

Case 1: There is a pair $i, j \in N$ such that $x_i < x_j$ and $x_i < p_i$. In this case, $x_j \, P_i \, x_i$. This violates *awardee-envy-freeness*.

Case 2: There is j such that $x_j = 0$ and $M - \sum_{i \in N} x_i > l_j$. Let $y \in \mathbb{R}_+^N$ be such that $y_j = M - \sum_{i \in N} x_i$ and $y_{-j} = x_{-j}$. For each $i \in N, y_i \, R_i \, x_i$ and $y_j \, P_j \, x_j$. This violates *Pareto-efficiency*.

Case 3: There is $j \in N$ such that $x_j \in (0, l_j]$. This violates *canonicity*.

Case 4: There is $j \in N$ such that $l_j < x_j < p_j$ and $\sum_{i \in N} x_i < M$. Let $y \in \mathbb{R}_+^N$ be such that $y_j = x_j + M - \sum_{i \in N} x_i$ and $y_{-j} = x_{-j}$. For each $i \in N, y_i \, R_i \, x_i$ and $y_j \, P_j \, x_j$. This violates *Pareto-efficiency*.

Thus, $\varphi(R, M) = EU^{\sigma^\varphi}(R, M)$. $\square$

To see that the axioms are independent, note that the *canonical* equivalent of the *uniform* rule is *awardee-envy-free* but not *Pareto-efficient* and *sequential priority* rules are *Pareto-efficient* but not *awardee-envy-free*.

Combining Proposition 5 and Theorem 1, we have the following corollary.

Corollary 1. *A rule φ is* Pareto-efficient, awardee-envy-free *and satisfies the* equal-division lower bound *if and only if there is a selector function σ such that for each $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+$,*

$$\left\{i \in N : l_i < \frac{M}{|N|}\right\} \subseteq \sigma(R, M),$$

and $\varphi = EU^\sigma$.

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[13]These cases correspond to the four parts in the definition of *efficient uniform* rules.

To prove the next characterization, we first prove that each *Pareto-efficient, strategy-proof,* and *non-bossy* rule chooses, for each $M \in \mathbb{R}_+$, a person with highest priority who, regardless of others' preferences, receives the least of his most preferred quantities.

Lemma 3. *Let φ be a* Pareto-efficient, strategy-proof, *and* non-bossy *rule. Then, for each $M \in \mathbb{R}$, there is $i \in N$ such that for each $R \in \mathcal{R}^N$,*

$$\varphi_i(R, M) = \begin{cases} \min\{M, p_i\} & \text{if } \min\{M, p_i\} > l_i, \text{ and,} \\ 0 & \text{otherwise.} \end{cases}$$

Proof: We prove this lemma by induction on $n = |N|$.

We start with $n = 2$. Let $R \in \mathcal{R}^N$ be such that $R_1 = R_2 = R_0$ such that $l_0 = 0$ and $p_0 = M$. Let $x = \varphi(R, M)$. By *Pareto-efficiency* $x_1 + x_2 = M$.

Claim: Either $x_1 = M$ or $x_2 = M$.

Proof: If not, by *Pareto-efficiency*, $x_1, x_2 \in (0, M)$. Let $R' \in \mathcal{R}^N$ be such that $l'_1 = x_1$, $l'_2 = x_2$, and $p'_1 = p'_2 = M$.[14] Let $x^1 \equiv \varphi(R'_1, R_2, M)$. By *strategy-proofness,* $x_1^1 \leq x_1$. Otherwise $x_1^1 = \varphi_1(R'_1, R_2, M) \overset{\text{F}}{P_1} \varphi(\overset{\text{T}}{R}_1, R_2, M)$. By *canonicity* and since $x_1^1 \leq x_1 = l'_1$, we deduce $x_1^1 = 0$ and $x_2^1 = M$. That is, $\varphi(R'_1, R_2, M) = (0, M)$. By an analogous argument, $\varphi(R_1, R'_2, M) = (M, 0)$.

Let $x' \equiv \varphi(R', M)$. Since $x_1 + x_2 = M$, we cannot have $x'_1 \geq l'_1 = x_1$ and $x'_2 \geq l'_2 = x_2$. By *Pareto-efficiency,* either $x' = (0, M)$ or $x' = (M, 0)$. If $x' = (0, M)$, since $M = \varphi_1(\overset{\text{F}}{R}_1, R'_2, M) \overset{\text{T}}{P'_1} \varphi_1(\overset{\text{T}}{R}'_1, R'_2, M) = 0$, this violates *strategy-proofness.* If $x' = (M, 0)$, we reach a similar contradiction. $\circ$

Let $i \in N$ be such that $x_i = M$. For each $R'_i \in \mathcal{R}$ such that $\min M, p'_i > l'_i$, by *strategy-proofness,* $\varphi(R'_i, R_j, M) \geq \min\{M, p'_i\}$. Otherwise, $M = \varphi_i(\overset{\text{F}}{R}_i, R_j, M) \overset{\text{T}}{P'_i} \varphi_i(\overset{\text{T}}{R}'_i, R_j, M)$. By *Pareto-efficiency,* $\varphi_i(R'_i, R_j, M) \leq \min\{M, p'_i\}$. Thus, $\varphi_i(R'_1, R_j, M) = \min\{M, p'_i\}$ and $\varphi_j(R'_i, R_j, M) = M - \min\{M, p'_i\}$. By *strategy-proofness,* for each $R'_j \in \mathcal{R}, \varphi_j(R'_i, R'_j, M) \leq M - \min\{M, p'_i\}$. Otherwise, $\varphi_j(R'_i, \overset{\text{F}}{R}'_j, M) \overset{\text{T}}{P_j} \varphi_j(R_i, \overset{\text{T}}{P}_j, M) = M - \min\{M, p'_i\}$. By *Pareto-efficiency* and *canonicity,* $\varphi_i(R'_i, R'_j, M) = \min\{M, p'_i\}$. If $\min\{M, p'_i\} \leq l'_i$, by *canonicity,* $\varphi_i(R', M) = 0$.

Thus, we conclude that for each $M \in \mathbb{R}$, there is $i \in N$ such that for each $R \in \mathcal{R}^N$,

$$\varphi_i(R, M) = \begin{cases} \min\{M, p_i\} & \text{if } \min\{M, p_i\} > l_i \\ 0 & \text{otherwise.} \end{cases}$$

As an *induction hypothesis,* suppose that if $|N'| = n - 1$ and ψ is a *Pareto-efficient, strategy-proof,* and *non-bossy* rule defined for the problems involving

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[14]Since $x_1, x_2 \in (0, M)$, such a preference profile exists.

people in N', then for each M, there is an $i \in N'$ such that for each $R \in \mathcal{R}^{N'}$,

$$\psi_i(R, M) \;=\; \begin{cases} \min\{M, p_i\} & \text{if } \min M, p_i > l_i, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

We show that for each $M \in \mathbb{R}$, there is $i \in N$ such that for each $R \in \mathcal{R}^N$,

$$\varphi_i(R, M) \;=\; \begin{cases} \min\{M, p_i\} & \text{if } \min\{M, p_i\} > l_i, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Let $R \in \mathcal{R}^N$ be such that for each $i \in N$, $l_i = 0$ and $p_i = M$. Let $x \equiv \varphi(R, M)$. Since φ is *Pareto-efficient*, $\sum_{j \in N} x_j = M$.

Claim: There is $i \in N$ such that $x_i = M$.

Proof: There are two cases to consider:

Case 1: There is $j \in N$ such that $x_j = 0$.

Let $\tilde{R}_j \in \mathcal{R}$ be such that $\tilde{p}_j = 0$. By *canonicity*, for each $\overline{R}_{-j} \in \mathcal{R}^{N \setminus \{j\}}$, and $\overline{M} \in \mathbb{R}_+$, $\varphi_j(\tilde{R}_j, \overline{R}_{-j}, \overline{M}) = 0$. By *non-bossiness*,[15] $\varphi(\tilde{R}_j, R_{-j}, M) = x$. Now, define ψ as a rule for people in $N \setminus \{j\}$ by setting, for each $(\overline{R}_{-j}, \overline{M}) \in \mathcal{R}^{N \setminus \{j\}} \times \mathbb{R}$,

$$\psi(\overline{R}_{-j}, \overline{M}) \equiv \varphi(\tilde{R}_j, \overline{R}_{-j}, \overline{M}).$$

Since φ is *Pareto-efficient, strategy-proof,* and *non-bossy*, ψ inherits these properties. By the *induction hypothesis*, there is $i \in N \setminus \{j\}$ such that $\psi_i(R_{-j}, M) = M$. However, by definition of ψ, we have $\psi(R_{-i}, M) = x_{-j}$. Thus, there is $i \in N$ such that $x_i = M$.

Case 2: For each $j \in N, x_j \in (0, M)$.

Let $R' \in \mathcal{R}^N$ be such that for each $j \in N, x_j = l'_j < p'_j = M$. For each $j \in N$, define $x^j \equiv \varphi(R'_j, R_{-j}, M)$. By *strategy-proofness*, $x^j_j \leq x_j = l'_j$. Otherwise $x^j_j = \varphi_j(R'_j, R_{-j}, M) \overset{\text{F}}{R'_j} \varphi_j(\tilde{R}_j, R_{-j}, M) = x_j$. By *canonicity*, $x^j_j = 0$. By an argument identical to the one used in Case 1, there is $k^j \in N \setminus \{j\}$ such that $x^j_{k^j} = M$.

Let $x' \equiv \varphi(R', M)$. Since $\sum_{j \in N} x_j = M$, there is $j \in N$ such that $x'_j \leq x_j = l'_j$. By *canonicity*, $x'_j = 0$. By an argument identical to the one used in Case 1, there is $i \in N$ such that $x'_i = M$. This implies that for each $j \in N \setminus \{i\}$, $\varphi_j(R', M) = 0$. In particular,

$$\varphi_{k^i}(R'_i, R'_{k^i}, R'_{-\{i, k^i\}}, M) = 0. \tag{3}$$

As shown earlier, $\varphi(R'_i, R_{k^i}, R_{-\{i, k^i\}}, M) = x^i$ and $x^i_{k^i} = M$. Thus, for each $j \in N \setminus \{i, k^i\}$, $\varphi_j(R'_i, R_{k^i}, R_{-\{i, k^i\}}, M) = 0$. Let $\{j_1, \ldots, j_{n-2}\} \equiv N \setminus \{i, k^i\}$. By

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[15]This is the first time we appeal to *non-bossiness*.

strategy-proofness, $\varphi_{j_1}(R'_i, R_{k^i}, R'_{j_1}, R_{-\{i,k^i,j_1\}}, M) = 0$. Otherwise,
$\varphi_{j_1}(R'_i, R_{k^i}, \overset{F}{R'_{j_1}}, R_{-\{i,k^i,j_1\}}, M) \overset{T}{P_{j_1}} \varphi_{j_1}(R'_i, R_{k^i}, \overset{T}{R_{j_1}}, R_{-\{i,k^i,j_1\}}, M) = 0$. By non-bossiness, $\varphi(R'_i, R_{k^i}, \overset{F}{R'_{j_1}}, R_{-\{i,k^i,j_1\}}, M) = x^i$. Repeating this argument $n-3$ more times, $\varphi(R'_i, R_{k^i}, R'_{-\{i,k^i\}}, M) = x^i$. Thus, $\varphi_{k^i}(R'_i, R_{k^i}, R'_{-\{i,k^i\}}, M) = M$. By (3) and the definition of R'_{k_i}, we know that

$$M = \varphi_{k^i}(R'_i, \overset{F}{R_{k^i}}, R'_{-\{i,k^i\}}, M) \overset{T}{P'_{k^j}} \varphi_{k^i}(R'_i, \overset{T}{R'_{k^i}}, R'_{-\{i,k^i\}}, M) = 0.$$

This violates *strategy-proofness*. Thus, there is $i \in N$ such that $x_i = M$. ∘

Let $R' \in \mathcal{R}^N$. To complete the proof of this lemma, we show that

$$\varphi_i(R', M) = \begin{cases} \min\{p'_i, M\} & \text{if } \min\{p'_i, M\} > l'_i, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

By *Pareto-efficiency*, if $\min\{p'_i, M\} \leq l'_i$, then $\varphi_i(R', M) = 0$. If not, we index the members of $N\setminus\{i\}$ as $\{j_1, \ldots, j_{n-1}\}$. By *strategy-proofness*, $\varphi_{j_1}(R'_{j_1}, R_{-j_1}, M) = 0$. Otherwise, $\varphi_{j_1}(\overset{F}{R'_{j_1}}, R_{-j_1}, M) \overset{T}{P_{j_1}} \varphi(\overset{T}{R_{j_1}}, R_{-j_1}, M) = 0$. Then, by *non-bossiness*, $\varphi(R'_{j_1}, R_{-j_1}, M) = x$. Repeating this argument $n-2$ more times, $\varphi(R'_{-i}, R_i, M) = x$. That is, $\varphi_i(R'_{-i}, R_i, M) = M$. By *strategy-proofness*, $\varphi_i(R', M) \geq \min\{M, p'_i\}$. Otherwise, $M = \varphi(\overset{F}{R_i}, R'_{-i}, M) \overset{T}{P_i} \varphi_i(\overset{T}{R_i}, R'_{-i}, M)$. Due to the feasibility constraint, $\varphi_i(R', M) \leq M$ and by *Pareto-efficiency*, $\varphi_i(R', M) \leq p'_i$ and thus, $\varphi_i(R', M) = \min\{M, p'_i\}$. □

Lemma 3 gets us very close to a characterization of the *conditional sequential priority rules* as the only *Pareto-efficient*, *strategy-proof*, and *non-bossy* rules. Intuitively, it says that for each value of the endowment, there is a person with highest priority who is given the least of his most preferred quantities. After this, we are possibly left some of the good to divide among the remaining people and Lemma 3 can be applied again.

Theorem 2. *Every* Pareto-efficient, strategy-proof, *and* non-bossy *rule is a* conditional sequential priority rule.

Proof: We proceed by induction on $n = |N|$. If $n = 2$, the theorem follows directly from Lemma 3. As an *induction hypothesis*, suppose that all *Pareto-efficient*, *strategy-proof*, and *non-bossy* rules defined for the people in N' such that $|N'| = n - 1$ are *conditional sequential priority rules*.

Let φ be a *Pareto-efficient*, *strategy-proof*, and *non-bossy* rule defined for people in N such that $|N| = n$. We will show that there is I_φ such that $\varphi = CSP^{I_\varphi}$.

Step 1: Construction of I_φ.

By Lemma 3, for each $M \in \mathbb{R}$, there is $i_1 \in N$ such that for each $R \in \mathcal{R}^N$,

$$\varphi_{i_1}(R, M) = \begin{cases} \min\{p_{i_1}, M\} & \text{if } \min\{p_{i_1}, M\} > l_{i_1}, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

We begin construction of I_φ by setting $i_\varphi^1(M) \equiv i_1$. Let $x_i \equiv \varphi_i(R, M)$. For each $\tilde{R}_{-i} \in \mathcal{R}^{N \setminus \{i\}}$, $\varphi_i(R, \tilde{R}_{-i}, M) = x_i$.

Let $R_{i_1}^0 \in \mathcal{R}$ be such that $p_{i_1}^0 = 0$. By *Pareto-efficiency*, for each $\tilde{M} \in \mathbb{R}$ and each $\tilde{R}_{-i_1} \in \mathcal{R}^{N \setminus \{i_1\}}$, $\varphi_i(R_{i_1}^0, \tilde{R}_{-i_1}, \tilde{M}) = 0$.

Define the rule $\psi^{(M, i_1, x_{i_1})}$ for people in $N \setminus \{i_1\}$ by setting for each $\overline{M} \in \mathbb{R}$ and each $\overline{R}_{-i_1} \in \mathcal{R}^{N \setminus \{i_1\}}$,

$$\psi^{(M, i_1, x_{i_1})}(\overline{R}_{-i_1}, \overline{M}) = \begin{cases} \varphi_{-i_1}(R_{i_1}, \overline{R}_{-i_1}, \overline{M}) & \text{if } \overline{M} = M - x_{i_1} \\ \varphi_{-i_1}(R_{i_1}^0, \overline{R}_{-i_1}, \overline{M}) & \text{otherwise.} \end{cases}$$

Because $\varphi_i(R_i^0, \overline{R}_{-i}, \overline{M}) = 0$ and φ is *Pareto-efficient, strategy-proof,* and *non-bossy,* $\psi^{(M, i_1, x_{i_1})}$ inherits these properties. Thus, by the *induction hypothesis,* there is $I_{\psi^{(M, i_1, x_{i_1})}}$ such that $\psi^{(M, i_1, x_{i_1})} = CSP^{I_{\psi^{(M, i_1, x_{i_1})}}}$.

Let

$$i_\varphi^2(M, i_1, x_{i_1}) \equiv i^1_{\psi^{(M, i_1, x_{i_1})}}(M - x_{i_1}) = i_2.$$

For each $x_{i_2} \in [0, M - x_{i_1}]$, let

$$i_\varphi^3(M, i_1, i_2, x_{i_1}, x_{i_2}) \equiv i^2_{\psi^{(M, i_1, x_{i_1})}}(M - x_{i_1}, i_2, x_{i_2}) = i_3.$$

$$\vdots$$

For each $x_{i_{n-2}} \in \left[0, M - \sum_{k=1}^{n-3} x_{i_k}\right]$, let

$$i_\varphi^{n-1}(M, i_1, \ldots, i_{n-2}, x_{i_1}, \ldots, x_{i_{n-2}}) \equiv i^{n-2}_{\psi^{(M, i_1, x_{i_1})}}(M - x_{i_1}, i_2, \ldots, i_{n-2}, x_{i_2}, \ldots, x_{i_{n-2}}).$$

Since we can do this for each $M \in \mathbb{R}$ and each $R_{i_\varphi^1(M)} \in \mathcal{R}$, we have a complete description of I_φ.

Step 2: Verify that $\varphi = CSP^{I_\varphi}$.

Let $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+$. By definition of $i_1 \equiv i^1(M)$,

$$x_{i_1} \equiv \varphi_{i_1}(R, M) = \begin{cases} \min\{M, p_{i_1}\} & \text{if } \min\{M, p_{i_1}\} > l_{i_1} \\ 0 & \text{otherwise.} \end{cases} = CSP^{I_\varphi}_{i_1}(R, M).$$

Further, by definition of $\psi^{(M,i_1,x_{i_1})}$,

$$\varphi_{-i_1}(R,M) = \psi^{(M,i_1,x_{i_1})}(R_{-i_1}, M-x_{i_1}) = CSP^{I_{\psi^{(M,i_1,x_{i_1})}}}(R_{-i_1}, M-x_{i_1}) = CSP^{I_\varphi}_{-i_1}(R,M).$$

Thus, $\varphi(R,M) = CSP^{I_\varphi}(R,M)$. Since this holds for each $(R,M) \in \mathcal{R}^N \times \mathbb{R}_+$, we have verified that $\varphi = CSP^{I_\varphi}$. $\qquad\square$

To see that the axioms are independent, note that the *canonical* equivalent of the *uniform* rule violates only *Pareto-efficiency*, some *efficient uniform* rules violate only *strategy-proofness*, and we define the rule φ which violates only *non-bossiness*. Let $I \equiv \{i_k\}_{k=1}^{n-1}$ and $J \equiv \{j_k\}_{k=1}^{n-1}$ be such that $I \neq J$ and $\{i_1, \ldots, i_{n-1}\} = \{j_1, \ldots, j_{n-1}\} = N \setminus \{1\}$. Define φ by setting, for each $(R,M) \in \mathcal{R}^N \times \mathbb{R}_+$,

$$\varphi(R,M) = \begin{cases} SP^I(R,M) & \text{if } l_1 < M \\ SP^J(R,M) & \text{otherwise.} \end{cases}$$

Notice that φ is also *weakly continuous*.

Remark 3. Strategy-proofness *and* Pareto-efficient *are incompatible with any notion of equity.* Every *Pareto-efficient* and *canonical* rule is *non-bossy* whenever $|N| = 2$. Thus, Theorem 2 says that there is always one person who is fully satiated by a *strategy-proof* rule, regardless of the preference profile. This rules out any meaningful notion of equity. $\qquad\circ$

We characterize the *weakly continuous* subset of *conditional sequential priority* below.

Proposition 6. *A* conditional sequential priority rule *is* weakly continuous *if and only if it is a* sequential priority *rule.*

Proof: First we show that each *sequential priority* rule is *weakly continuous*. Let I be associated with the ordering $\{i_1, i_2, \ldots i_{n-1}\}$. Let $\{(R^\nu, m^\nu)\}_{\nu=1}^\infty$ be a sequence in $\mathcal{R}^N \times \mathbb{R}_+$ such that,

    i) $\quad \lim_{\nu \to \infty} (R^\nu, m^\nu) = (R,M) \in \mathcal{R}^N \times \mathbb{R}_+,$ and

    ii) $\quad$ for each $i \in N$ if $l_i \geq \min\{p_i, M\}$, then there is $\nu^* \in \mathbb{N}$ such that for each $\nu \geq \nu^*, l_i^\nu \geq \min\{p_i^\nu, M^\nu\}$.

If $l_{i_1} \geq \min\{M, p_{i_1}\}$, then by *ii)*, there is $\nu^* \in \mathbb{N}$ such that for each $\nu \geq \nu^*, l_{i_1}^\nu \geq \min\{p_{i_1}^\nu, M^\nu\}$ and so $SP_{i_1}^I(R^\nu, M^\nu) = 0$. Thus,

$$\lim_{\nu \to \infty} SP_{i_1}^I(R^\nu, M^\nu) = 0 = SP_{i_1}^I(R,M).$$

If $l_{i_1} < \min\{M, p_{i_1}\}$, then

$$\lim_{\nu \to \infty} SP^I_{i_1}(R^\nu, M^\nu) = \lim_{\nu \to \infty} \min\{p^\nu_{i_1}, M^\nu\} = \min\{p_{i_1}, M\} = SP^I_{i_1}(R, M).$$

This argument can be repeated for $i_2, \ldots, i_{n-1}$, and $i_n \in N \setminus \{i_1, \ldots, i_{n-1}\}$.

Next, we prove that if I is such that CSP^I is *weakly continuous*, then for each pair $M, M' \in \mathbb{R}_+$, and each pair $x, x' \in \mathbb{R}_+$ such that $\sum_{i=1}^{n-2} x_i \leq M$ and $\sum_{i=1}^{n-2} x'_i \leq M$,

$$
\begin{aligned}
i^1(M) \quad &= i_1 = \quad i^1(M'), \\
i^2(M, i_1, x_{i_1}) \quad &= i_2 = \quad i^2(M', i_1, x'_{i_1}), \\
&\vdots \\
i^{n-1}(M, i_{1,\ldots,n-2}, x_{i_{1,\ldots,n-2}}) \quad &= i_{n-1} = \quad i^{n-1}(M', i_{1,\ldots,n-2}, x'_{i_{1,\ldots,n-2}}),
\end{aligned}
$$

Let $R \in \mathcal{R}^N$ be such that for each $i \in N, l_i = 0$ and $p_i = \max\{M, M'\}$. By definition of CSP^I and R, for each $\tilde{M} \leq \max\{M, M'\}$, $CSP^I(R, \tilde{M}) \in \{(\tilde{M}, 0, \ldots, 0), \ldots, (0, \ldots, 0, \tilde{M})\}$. Since for each $i \in N$, $l_i = 0$, *weak continuity* implies that $CSP^I(R, \tilde{M})$ varies continuously in $\tilde{M}$. Thus, $i^1(M) = i^1(M') = i_1$.

For each $\tilde{M} \in \mathbb{R}_+$ and each $\tilde{x}_{i_1} \in \mathbb{R}_+$ such that $\tilde{x}_{i_1} \leq \tilde{M}$, let $R^{\tilde{x}_{i_1}}_{i_1} \in \mathcal{R}$ be such that $l^{\tilde{x}_{i_1}}_{i_1} = 0$ and $p^{\tilde{x}_{i_1}}_{i_1} = \tilde{x}_{i_1}$. Then $CSP^I_{i_1}(R^{\tilde{x}}_{i_1}, R_{-i^1}, \tilde{M}) = \tilde{x}_{i_1}$ and $CSP^I_{-i_1}(R^{\tilde{x}_{i_1}}_{i_1}, R_{-i_1}, \tilde{M}) \in \{(\tilde{M} - \tilde{x}_{i_1}, 0, \ldots, 0), \ldots, (0, \ldots, 0, \tilde{M} - \tilde{x}_{i_1})\}$. Since for each $i \in N \setminus \{i_1\}$, $l_i = 0$, *weak continuity* implies that $CSP^I_{-i_1}(R^{\tilde{x}_{i_1}}_{i_1}, R_{-i_1}, \tilde{M})$ varies continuously with $\tilde{M}$ and $\tilde{x}_{i_1}$. Thus, $i^2(M, i_1, x_{i_1}) = i^2(M', i_1, x'_{i_1}) = i_2$.

Repeating this argument completes the proof. $\qquad \square$

The following is a corollary of Proposition 6 and Theorem 2.

Corollary 2. *Every* Pareto-efficient, strategy-proof, non-bossy *and* weakly continuous *rule is a* sequential priority rule.

The axioms are independent. The *canonical* equivalent of the *uniform* rule violates only *Pareto-efficiency*. There are *efficient uniform* rules that violate only *strategy-proofness*. The rule φ defined to show independence of the axioms in Theorem 2 violates only *non-bossiness*. Finally, each *conditional sequential priority rule* that is not an (unconditional) *sequential priority* rule violates only *weak continuity*.

The next corollary shows that no *Pareto-efficient* and *strategy-proof* rule satisfies *welfare domination under preference replacement*. It follows from the following propositions.

Proposition 7. Pareto-efficiency *and* welfare domination under preference replacement *together imply* non-bossiness.

Proof: Let φ be *Pareto-efficient* and satisfy *welfare domination under preference replacement*. Let $i \in N$ and $R'_i \in \mathcal{R}$ be such that $\varphi_i(R_i, R_{-i}, M) = \varphi_i(R'_i, R_{-i}, M)$. Let $x \equiv \varphi(R_i, R_{-i}, M)$ and $x' \equiv \varphi(R'_i, R_{-i}, M)$. By *canonicity* for each $j \in N \setminus \{i\}$, $l_j < x_j \leq p_j$ and $l_j < x'_j \leq p_j$. By *welfare domination under preference replacement*, without loss of generality, for each $j \in N \setminus \{i\}$, $x_i \leq x_i$. Since $x \neq x'$, there is $j \in N \setminus \{i\}$ such that $x_j < x'_j$. Thus, for each $j \in N$, $x'_j \ R_j \ x_j$ and there is $j \in N$ such that $x'_j \ P_j \ x_j$. So, $\varphi(R, M) = x$ violates *Pareto-efficiency*. $\qquad\square$

Proposition 8. *No* conditional sequential priority *rule satisfies* welfare domination under preference replacement.

Proof: Let $M \in \mathbb{R}_+$ and $i_1 \equiv i^1(M)$. We first show that for each $x_{i_1}, x'_{i_1} \in \mathbb{R}_+$, $i^2(M, i_1, x_{i_1}) = i^2(M, i_1, x_{i_1})$. Let $R_{i_1} \in \mathcal{R}$ be such that $l_{i_1} = 0$ and $p_{i_1} = x_{i_1}$, and $R'_{i_1} \in \mathcal{R}$ be such that $l'_{i_1} = 0$ and $p'_{i_1} = x'_{i_1}$. Let $R_{-i_1} \in \mathcal{R}^{N\setminus\{i_1\}}$ be such that for each $j \in N\setminus\{i_1\}$, $l_j = 0$ and $p_j = M$. Let $i_2 \equiv i^2(M, i_1, x_{i_1})$ and $i'_2 \equiv i^2(M, i_1, x'_{i_1})$. Suppose $i_2 \neq i'_2$. Then, $CSP^I_{i_2}(R_{i_1}, R_{-i_1}, M) = M - x_{i_1} \ P_{i_2} \ 0 = CSP^I_{i_2}(R'_{i_1}, R_{-i_1}, M)$. Similarly, $CSP^I_{i'_2}(R'_{i_1}, R_{-i_1}, M) = M - x'_{i_1} \ P_{i'_2} \ 0 = CSP^I_{i'_2}(R_{i_1}, R_{-i_1}, M)$. This violates *welfare domination under preference replacement*. Thus, $i^2(M, i_1, x_{i_1}) = i^2(M, i_1, x_{i_1}) = i_2$.

Now, we show that CSP^I violates *welfare domination under preference replacement*. Let $R \in \mathcal{R}^N$ be such that $l_{i_1} = 0$, $p_{i_1} = \frac{M}{2}$, $l_{i_2} = \frac{M}{2}$, $p_{i_2} = M$, and for each $j \notin \{i_1, i_2\}$, $l_j = 0$ and $p_j = M$. Let $i_3 \equiv i^3(M, i_1, i_2, \frac{M}{2}, 0)$. By definition of CSP^I, $CSP^I_{i_1}(R, M) = \frac{M}{2}, CSP^I_{i_2}(R, M) = 0$, and $CSP^I_{i_3}(R, M) = \frac{M}{2}$. Let $\alpha \in (0, \frac{M}{2})$, and $R'_{i_1} \in \mathcal{R}$ be such that $l'_{i_1} = 0$ and $p'_{i_1} = \frac{M}{2} - \alpha$. By definition of CSP^I, $CSP^I_{i_1}(R, M) = \frac{M}{2} - \alpha, CSP^I_{i_2}(R, M) = \frac{M}{2} + \alpha$, and $CSP^I_{i_3}(R, M) = 0$. This violates *welfare domination under preference replacement*. $\qquad\square$

Corollary 3. *No rule is* Pareto-efficient, strategy-proof *and satisfies* welfare domination under preference replacement.

The remainder of this section is devoted to showing that *group strategy-proofness* is too demanding a property in that it rules out even the weakest notion of efficiency: *unanimity*.

The following proposition follows from the fact that a *unanimous* rule includes all possible divisions in its range and a *group strategy-proof* rule is *Pareto-efficient* over its range.

Proposition 9. *Every rule that is* group strategy-proof *and* unanimous[16] *is* Pareto-efficient.

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[16]*Unanimity* can be weakened to requiring that the range include all possible divisions. However, *unanimity* is normatively more meaningful.

Proposition 10. *Every* group strategy-proof *rule is* non-bossy.[17]

Proof: If φ is *bossy*, there are $(R, M) \in \mathcal{R}^N \times \mathbb{R}_+, i \in N$, and $R_i' \in \mathcal{R}$ such that $\varphi_i(R_i', R_{-i}, M) = \varphi_i(R, M)$ but $\varphi(R_i', R_{-i}, M) \neq \varphi(R, M)$. Then, there is $j \in N \setminus \{i\}$ such that $\varphi_j(R_i', R_{-i}, M) > \varphi_j(R, M)$. By *canonicity*, $p_j \geq \varphi_j(R_i', R_{-i}, M) > \varphi_j(R, M) > l_j$. But then, setting $S = \{i, j\}$ and $\tilde{R}_S = (R_i', R_j)$, we have $\varphi_i(\tilde{R}_S', R_{-S}, M) \overset{\text{F}}{I_i} \varphi_i(\tilde{R}_S, R_{-S}, M)$ and $\varphi_j(\tilde{R}_S', R_{-S}, M) \overset{\text{F}}{P_j} \varphi_j(\tilde{R}_S, R_{-S}, M)$. This violates *group strategy-proofness*. $\square$

Proposition 11. *No* conditional sequential priority rule *is* group strategy-proof.

Proof: Let $M \in \mathbb{R}_+$. Let $i_1 \equiv i^1(M)$ and let $R_{i_1} \in \mathcal{R}$ be such that $(l_{i_1}, p_{i_1}) = (0, \frac{M}{2})$. Let $\alpha \in (0, \frac{M}{2})$. Let $i_2 \equiv i^2(M, i_1, \frac{M}{2})$ and let $R_{i_2} \in \mathcal{R}$ be such that $(l_{i_2}, p_{i_2}) = (\frac{M}{2} - \alpha, \frac{M}{2})$. For each $j \in N \setminus \{i_1, i_2\}$, let $R_j \in \mathcal{R}$ be such that $(l_j, p_j) = (0, M)$.

Then, $CSP_{i_1}^I(R, M) = \frac{M}{2}$, $CSP_{i_2}^I(R, M) = \frac{M}{2}$, and for each $j \in N \setminus \{i_1, i_2\}$, $CSP_j^I(R, M) = 0$.

Let $R_{i_1}' \in \mathcal{R}$ be such that $(l_{i_1}, p_{i_1}) = (0, \frac{M}{2} + \alpha)$.

Case 1: $i_2 \neq i_2' = i^2(M, i_1, \frac{M}{2} + \alpha)$.

Then, $CSP_{i_2'}^I(R_{i_1}', R_{-i_1}, M) = (\frac{M}{2} - \alpha) \, P_{i_2'} \, 0 = CSP_{i_2'}^I(R, M)$.

Case 2: $i_2 = i^2(M, i_1, \frac{M}{2} + \alpha)$.

Then, $CSP_{i_2}^I(R_{i_1}', R_{-i_1}, M) = 0$. Let $i_3 \equiv i^3(M, i_1, i_2, \frac{M}{2} + \alpha, 0)$. By definition of CSP^I, $CSP_{i_3}^I(R_{i_1}', R_{-i_1}, M) = (\frac{M}{2} - \alpha) \, P_{i_3} \, 0 = CSP_{i_3}^I(R, M)$.

In both cases we see a violation of *group strategy-proofness*. $\square$

As a corollary of Theorem 2, in conjunctions with Propositions 9,10, and 11, we have the following.

Corollary 4. *No rule is* group strategy-proof *and* unanimous.

6 Conclusion

While we have shown that two familiar notions of fairness, *envy-freeness* and *equal treatment of equals*, are incompatible with *Pareto-efficiency*, there are other interesting axioms such as *awardee envy-freeness* and the *equal-division lower bound* that are compatible. However, none of these can be imposed alongside the combination of *Pareto-efficiency* and *strategy-proofness*.

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[17]This implication holds generally. It is easy to prove a version of Proposition 10 that applies to a very general model.

The implication of Theorem 2 is that if we insist on *Pareto-efficiency* and *strategy-proofness*, then we are forced to give up on any notion of fairness. However, if we can forgo *strategy-proofness*, Corollary 1 is a characterization of all *Pareto-efficient* and *awardee-envy-free* rules that satisfy the *equal-division lower bound*.

The results presented here are reminiscent of problems with discrete goods. Particularly the incompatibility, among *Pareto-efficienct* rules, between various axioms of equity on one hand and *strategy-proofness* on the other. Theorem 2 is similar to results from the literature on allocation of objects (Pápai 2001, Ehlers and Klaus 2003). In discrete environments, one way to recover equity is by randomization (Hofstee 1990). We leave the study of randomized rules in our environment for future work.

Appendices

A A variable population extension

We extend our model to variable populations. Let $\mathbb{N}$ be the infinite set of potential agents and let $\mathcal{N}$ be all non-empty subsets of $\mathbb{N}$. For each $N \in \mathcal{N}$, a problem for people in N consists of a profile $R \in \mathcal{R}^N$ and a social endowment $M \in \mathbb{R}_+$. Let $\mathcal{E}^N$ be the set of all problems for people in N. Let $\mathcal{E} \equiv \cup_{N \subseteq \mathcal{N}} \mathcal{E}^N$ be the set of all problems. For a problem involving people in N, let $F(N, M)$ be the feasible allocations. A rule maps each problem to a feasible allocation.

We first present a set of axioms and then extend the definitions of one of our classes of rules: *conditional sequential priority* rules. We then ask which members of this class satisfy our axioms.

A.1 Axioms

The first axiom says that, given a fixed social endowment, the arrival of new people should affect all of those initially present negatively (Thomson 1983a, Thomson 1983b).

Population monotonicity: For each pair $N, N' \in \mathcal{N}$ such that $N' \subseteq N$ and each $(R_N, M) \in \mathcal{E}^N$, for each $i \in N'$, $\varphi_i(R_{N'}, M)\, R_i\, \varphi_i(R_N, M)$.

The next requirement says that the choice made for the replicated problem should be a replication of original choice (Thomson 1997). Given $N \subseteq \mathcal{N}$ and $(R, M) \in \mathcal{E}^N$, $k \in \mathbb{N}$, and $N' \in \mathcal{N}$ such that $|N'| = k|N|$, let $k * (R, M) \in \mathcal{E}^N$ be such that for each $i \in N$, there are k members of N' with the preference relation

R_i and the social endowment is kM. Given $x \in F(N, M)$, let $k * x \in F(N', kM)$ be a similar replication of x.

Replication invariance: For each $N \in \mathcal{N}$, each $(R, M) \in \mathcal{E}^N$, and each $k \in \mathbb{N}$, we have $\varphi(k * (R, M)) = k * \varphi(R, M)$.

The third axiom says that if the rule is re-applied to a smaller group of people to divide only what was originally alloted among them, there should be no change in what each person receives (Davis and Maschler 1965, Peleg 1985, Lensberg 1988, Balinski and Young 1982, Thomson 1988). Let $N \in \mathcal{N}$ and $(R, M) \in \mathcal{E}^N$. For each $x \in F(N, M)$, and $N' \subseteq N$, the **reduction of (R, M) with respect to N' and x**, is $\boldsymbol{r_{N'}^x(R, M)} \equiv (R_{N'}, M - \sum_{i \in N \setminus N'} x_i)$.

Consistency: For each $N \in \mathcal{N}$ and each $(R, M) \in \mathcal{E}^N$, if $x \equiv \varphi(R, M)$, then for each $N' \subseteq N$, $\varphi(r_{N'}^x(R, M)) = x_{N'}$.

A.2   Rules

Conditional sequential priority rules: Let $I \equiv \{i^k\}_{k=1}^\infty$ where

$$i^1 : \mathbb{R}_+ \times \mathcal{N} \to \mathbb{N},$$
$$i^2 : \mathbb{R}_+ \times \mathcal{N} \times \mathbb{N} \times \mathbb{R}_+ \to \mathbb{N},$$
$$\vdots$$

be such that for each $N \in \mathcal{N}$, each $M \in \mathbb{R}_+$ and each $x \in \mathbb{R}^{n-2}$, we have a sequence $\{i_k\}_{k=1}^{n-1}$ such that

$$i_1 = i^1(M, N) \in N,$$
$$i_2 = i^2(M, N, i_1, x_{i_1}) \in N \setminus \{i_1\},$$
$$\vdots$$

The **conditional sequential priority rule with respect to I, CSP^I,** is defined as follows, for each $N \in \mathcal{N}$ and $(R, M) \in \mathcal{E}^N$:

$$
\begin{aligned}
i_1 &\equiv i^1(M,N), \\
CSP^I_{i_1}(R,M) &\equiv \begin{cases} \min\{M, p_{i_1}\} & \text{if } M > l_{i_1}, \text{ and} \\ 0 & \text{otherwise,} \end{cases} \\
i_2 &\equiv i^2(M,N,i_1,CSP^I_{i_1}(R,M)) \\
CSP^I_{i_2}(R,M) &\equiv \begin{cases} \min\{M - CSP^I_{i_1}(R,M), p_{i_2}\} & \text{if } M - CSP^I_{i_1}(R,M) > l_{i_2}, \text{ and} \\ 0 & \text{otherwise,} \end{cases} \\
&\vdots \\
i_{n-1} &\equiv i^n_1(M,N,i_{1,\ldots,n-2}, CSP^I_{i_{1,\ldots,n-2}}(R,M)) \\
CSP^I_{i_{n-1}}(R,M) &\equiv \begin{cases} \min\{M - \sum_{k=1}^{n-2} CSP^I_{i_k}(R,M), p_{i_{n-1}}\} & \text{if } M - \sum_{k=1}^{n-2} CSP^I_{i_k}(R,M) > l_{i_{n-1}}, \\ & \quad \text{and} \\ 0 & \text{otherwise,} \end{cases} \\
i_n &\in N \setminus \{i_1, \ldots, i_{n-1}\} \\
CSP^I_{i_n}(R,M) &\equiv \begin{cases} \min\{M - \sum_{k=1}^{n-1} CSP^I_{i_k}(R,M), p_{i_n}\} & \text{if } M - \sum_{k=1}^{n-1} CSP^I_{i_k}(R,M) > l_{i_n}, \\ & \quad \text{and} \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}
$$

Sequential priority rules: If $I \equiv \{i^k\}_k = 1^{n-1}$ is such that there is a linear ordering $\prec$ of $\mathbb{N}$ such that for each $N \in \mathcal{N}$, each $M \in \mathbb{R}_+$, and each $x \in \mathbb{R}_+^{n-2}$,

$i_1 = i^1(M,N) = \{i \in N : \text{ for each } j \in N \setminus \{i\}, i \prec j\},$
$i_2 = i^2(M,N,i_1,x_{i_1}) = \{i \in N \setminus \{i_1\} : \text{ for each } j \in N \setminus \{i_1, i\}, i \prec j\},$
$\vdots$
$i_{n-1} = i^{n-1}(M, i_{1,\ldots,n-2}, x_{i_{1,\ldots,n-2}})$
$= \{i \in N \setminus \{i_1, \ldots, i_{n-2}\} : \text{ for each } j \in N \setminus \{i_1, \ldots i_{n-2}, i\}, i \prec j\},$

then CSP^I is **sequential priority rule with respect to** $\prec$, $SP^{\prec}$.

A.3   Results

We now study some implications of the above axioms.

Proposition 12. *No* conditional sequential priority *rule is* population monotonic.

Proof: Let $M \in \mathbb{R}_+$ and $N \equiv \{1,2,3\} \in \mathcal{N}$. Without loss of generality, let $1 = i^1(M,N)$ and $2 = i^2(M,N,1,\frac{M}{2})$. Let $R \in \mathcal{R}^N$ be such that $l_1 = l_3 = 0$, $l_2 = p_1 = \frac{M}{2}$, and $p_2 = p_3 = M$. Then, $CSP^I_1(R,M) = \frac{M}{2}, CSP^I_2(R,M) = 0$, and $CSP^I_3(R,M) = \frac{M}{2}$.
   Let $N' = \{2,3\}$.

Case 1: $i^1(M, N') = 2$. Then, $CSP_3^I(R_2, R_3, M) = 0$. Since $CSP_3^I(R, M) = \frac{M}{2}$, this violates *population monotonicity*.

Case 2: $i^1(M, N') = 3$. Let $R' \in \mathcal{R}^{N'}$ be such that $l_2 = l_3 = 0$ and $p_2 = p_3 = M$. Then, $CSP_2^I(R'_2, R'_3, M) = 0$. Since $CSP_2^I(R_1, R'_2, R'_3, M) = \frac{M}{2}$, this violates *population monotonicity*. $\square$

Proposition 13. *No rule is* Pareto-efficient *and* replication invariant.

Proof: Suppose φ is *Pareto-efficient*. Let $N \equiv \{1\} \in \mathcal{N}$ and $(R, M) \in \mathcal{E}^N$ be such that $l_1 = M$ and $p_1 = 2M$. By *Pareto-efficiency*, $\varphi(2 * (R, M)) \in \{(0, 2M), (2M, 0)\}$. In either case, for every value of $\varphi(R, M)$ *replication invariance* is violated. $\square$

Proposition 14. *A* conditional sequential priority *rule is* consistent *if and only if it is a* sequential priority *rule.*

Proof: It is easy to verify that every *sequential priority* rule is *consistent*. We prove that if a $I \equiv \{i^k\}_{k=1}^\infty$ is such that CSP^I is *consistent*, then there is $\prec$ such that $CSP^I = SP^\prec$.

Let $M \in \mathbb{R}_+$. For each pair $i, j \in \mathbb{N}$, let $R \in \mathcal{R}^{\{i,j\}}$ be such that $l_i = l_j = 0$ and $p_i = p_j = M$. If $CSP_i^I(R, M) = 0$, set $j \prec^M i$. Otherwise, set $i \prec^M j$. To verify that $\prec^M$ is a linear order, we note that it is, by definition, complete and antisymmetric. It is only left to verify that it is transitive. Suppose there are $i, j, k \in \mathbb{N}$ such that $i \prec^M j$, $j \prec^M k$ and $k \prec^M i$. Let $R \in \mathcal{R}^{\{i,j,k\}}$ be such that $l_i = l_j = l_k = 0$ and $p_i = p_j = p_k = M$. Without loss of generality, suppose $CSP_i^I(R, M) = M$. Then, by *consistency*, $CSP_i^I(R_i, R_k, M) = M$. But this violates $k \prec^M i$.

Next, we show that for each pair $M, M' \in \mathbb{R}_+$, $\prec^M = \prec^{M'} = \prec$. Suppose $M' < M < 2M'$. Suppose there is a pair $i, j \in \mathbb{N}$ such that $i \prec^{M'} j$ and $j \prec^M i$. Let $k \in \mathbb{N}$ be such that $i \prec^M k$. Let $R \in \mathcal{R}^{\{i,j,k\}}$ be such that $l_i = l_j = M - M'$, $p_i = p_j = M'$, $l_k = 0$ and $p_k = 2M'$. Let $x \equiv CSP^I(R, M)$. By definition of CSP^I, x takes one of the following values:

Case 1: $x_i = 0, x_j = M'$, and $x_k = M - M'$. By *consistency*, $CSP_i^I(R_i, R_j, M') = 0$. This violates $i \prec^{M'} j$.

Case 2: $x_i = M', x_j = 0$, and $x_k = M - M'$. Then $i^1(M, \{i, j, k\}) = i$. Let $R' \in \mathcal{R}^{\{i,j,k\}}$ be such that $l_i = l_j = l_k = 0$ and $p_i = p_j = p_k = M$. By definition, $CSP_i^I(R', M) = M$. By *consistency*, $CSP_i^I(R'_i, R'_j, M) = M$. This violates $j \prec^M i$.

Case 3: $x_i = 0, x_j = 0$, and $x_k = M'$. By *consistency*, $CSP_k^I(R_i, R_k, M) = M$. This violates $i \prec^M k$.

Finally, we verify that $CSP^I = SP^\prec$. If not, then there is $N \in \mathcal{N}$ and $(R, M) \in \mathcal{E}^N$ such that there is a pair $i, j \in N$ for which $i \prec j$, $l_i = l_j = 0$,

$p_i = p_j = M$ and $CSP_i^I(R, M) = 0 < x_j \equiv CSP_j^I(R, M)$. By *consistency*, $CSP_i^I(R_i, R_j, x_j) = 0$. But this violates $i \prec j$. Thus, $\hat{C}SP^I = SP^{\prec}$. $\qquad\square$

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