Selling Storable Goods to a Dynamic Population of Buyers: A Mechanism Design Approach

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Abstract

We study the problem of selling K units of identical storable goods to buyers who arrive stochastically over a finite time horizon. Buyers have single unit demand for the product; they are risk neutral and patient, and keep their birthdays and valuations as private information. We characterize the expected surplus maximizing allocation and implement it by a direct mechanism that is periodic ex-post incentive compatible and individually rational. We also propose a sequential simultaneous ascending auction as an outcome equivalent indirect mechanism and compare it with the standard uniform price auction to highlight issues created by market dynamics.

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1 Introduction

In this paper, we study the problem of selling K identical units of storable goods over a finite time horizon. Buyers arrive stochastically over time and have single unit demand for the product. They are risk neutral and patient, and keep their birthdays and valuations (jointly called the types) as private information. We characterize the allocation rule that maximizes expected total surplus and implement it by a direct mechanism that is periodic ex-post incentive compatible and individually rational. We also devise a sequential simultaneous ascending auction as an outcome equivalent indirect mechanism and compare it with the standard uniform price auction to highlight issues created by market dynamics.

Variation in buyers' population arises naturally in many settings: computer stores in school district reach to the freshmen every September to expand their customer pools; owner of a vintage watch will never dump it in garage sale if he expects it to be the target of professional collectors in ten years; more recently, sellers on online trading platforms become experienced at dealing with buyers who come across the product websites at different times.

However, allowing buyers to arrive over time raises several challenges. First of all, note that the seller's allocation decision at any point in time starts to depend on the number of units remain, because for any given inflow of future buyers the seller becomes less willing to sell now as less units become available and raises the hurdle for existing buyers in response. In other words, the seller faces an increasing virtual cost curve even if the products incur the same physical production cost. Instead of charging a single price in the standard setup, she has to design a of payment scheme that is contingent on the units of residual supplies.

Market dynamics also introduces interdependence to our private value environment. When the seller decides whether or not to satisfy buyer A's demand, he has the option to withhold the product from A and reserve it for future sales. This option value depends on the valuations of losing buyers [1] in the current sales, and must be integrated into A's decision if the allocation and transfer scheme is meant to achieve efficiency. In this way, we create interdependence between A and her competitors. In spirit of Dasgupta and Maskin [4], the allocation and transfer scheme required to achieve efficiency in this setting tend to incorporate rich dimensions of buyers' private information.[2] In particular, if an indirect mechanism is used to achieve efficient outcome, it must generate sufficient information disclosure in an endogenous and timely fashion.

Finally, when buyer's birthday becomes private information as well, the seller faces the challenge to illicit truthful reports on multi-dimensional private information. In general, this problem could be exceedingly hard to solve.

Main results in this paper are three-folded. First, assuming that buyers' types become public information upon their arrivals, we characterize the allocation rule that maximizes ex-

[1] Including A and those whose valuations are below A's.

[2] Dasgupta and Maskin [4] propose an efficient auction for buyers with interdependent but single dimensional types. In their auction the bid submitted by individual agent is a function of other bidders' types.

pected total surplus and reduce the seller's decision to a simple algorithm: each period, she gives the first unit to current buyer with the highest valuation if this valuation is above the first cutoff, the second unit to the buyer with the second highest valuation if it is above the second cutoff, etc., and raises cutoff as more and more units are sold out.

To implement this outcome in a private information environment, we propose a direct mechanism that is periodic ex-post incentive compatible and individually rational. In particular, we give each living agent one chance to report his type and charge the expected total externality he incurs when he leaves the market. If all buyers are equally patient, then any entrant who could get assigned now has no incentive to hide his presence, because tomorrow he faces either a higher cutoff if the forgone opportunity is taken advantage by someone else, or an increasing number of competitors if his position remains untaken. Thus, we essentially reduce a multi-dimensional screening problem to a single-dimensional one.

At any time t, the expected total externality incurred by a pivotal buyer (call him A), i.e., buyer who gets assigned at t, has several interesting features. First, it differs systematically according to the number of remaining units when A gets assigned. Since A's position is always uncertain due to the continuous influx of new customers, the seller has to prepare for him a menu of transfer schemes at each interim stage before he exits the market. Second, due to the interdependence created by market dynamics, the expected total externality function incorporates rich dimensions of information announced by non-pivotal agents.

These two features manifest themselves in the sequential simultaneous ascending auction we devise as an outcome equivalent indirect mechanism to implement the efficient outcome. Each period, buyers are required to bid in a simultaneous ascending auction and submit separate bids for different units of the remaining product. For each unit of remaining product, a robot is used to bid against the buyers and to screen out those whose valuations are below the cutoff for that particular unit. In the end, price is determined by the $(k+1)$th bid for the kth unit, where k is the last unit such that at least k buyers outbid the robot. Interestingly, we require buyers to submit a bidding portfolio instead of a single bid even if the products convey the same consumption value, and interpret bids for different units as the buyer's willingness to pay under different demand pressures. Meanwhile, we choose an open auction format to generate full information disclosure and to keep the buyers from information arbitrage. In particular, the price that screens out buyers below the cutoff depends on the valuations of losing bidders and we interpret it as a "generalized reserve price" in the current setting with interdependent values.

Though we focus on efficient outcome and assume fixed supply of homogenous product, we can easily extend the framework to allow for goods with different but commonly ranked qualities, introduce dynamic inflow of products, allow multiple sellers to compete and discuss revenue management issues. Nevertheless, our results hinge on a major assumption that the two parties cannot write rental contract on the product. Though this assumption becomes relatively innocuous when consumption is not time-separable (e.g. collectibles), or potential damages are non-contractible (e.g. service animal, electronic devices), it is not always non-biting. When rental contract is available, the seller could simply ask existing buyers to bid for rental contracts first and auction off all the products in the last period.

Literature Review The idea of selling goods to a dynamic population of buyers is extensively explored in the yield management literature. (See Gershkov and Moldovanu [6], and the book by Talluri and Van Ryzin [11] for reference). Motivated by observations in travel and seasonal goods industry, this literature focuses almost exclusively on transient demand and uses posted price as the primary instrument for revenue management. Allowing buyers to wait, we introduce another dimension of strategic concern and extend the analysis to new forms of businesses, e.g., online auction sites. Moreover, we no longer restrict ourselves to use posted prices. Indeed, several other papers, including Lavi and Nisan [7] and Vulcano et al. [12], are motivated by the same concern and experiment with mechanisms other than posted prices, but they focus primarily on transient demand. For example, [7] derive their insights from bandwidth allocation on communication links where decision about each bid should be made when it is received.

Two recent papers study similar topics as ours. Said [10] considers the problem of selling an exogenous flow of non-storable goods to buyers who arrive over time and points out the need for open auction format to account for the interdependence created by buyers' population dynamics. With storable goods, we endogenize the seller's allocation decision and propose an incentive menu to account for the distinct inter-temporal tradeoffs underlying the seemingly homogenous products. Meanwhile, Board [3] assumes infinite supply of storable goods and solves for the price path that precludes delay in participation, and it is interesting to see how our results behave when both size of the market grow infinitely large.

Our work is closely related to recent advancement in dynamic mechanism design, including Bergemann and Valimaki [2], Athey and Segal [1] and Pavan et.al. [9]. Indeed, the transfer scheme we design is equivalent to the "marginal flow contribution" proposed by Bergemann and Valimaki in [2], as each agent in our model realizes his full contribution once and for all at the time he leaves the market.

2 The Model

Preliminaries Time is discrete. The market opens at $t = 0$ and closes at $t = T$. A single seller is endowed with K units of identical products and wants to sell them between $t = 0$ and $t = T$. Assume that rental contract for the product is unavailable.

At the beginning of $t \in \{0, 1, ..., T\}$, n_t risk neutral buyers with single unit demand arrive at the market. Their values are i.i.d. drawn from a distribution $F_t(.) : \Theta \to [0, 1]$. Let $\Theta = [0, 1]$, $F_t(\theta)$ be atomless on $(0, 1]$, and $F_t(.), F_{t'}(.)$ be independent for all $t \neq t^{'}$. Buyers and the seller share the discount factor $\delta \in (0,1)$.

Information A buyer's value θ_i and birthday τ_i, or his type, is private information, whereas $\{n_t, F_t(.)\}_{t=0}^T \cup \{\delta, T, K\}$ are common knowledge.

Payoffs If a buyer of type (θ_i, τ_i) gets the product at time t, then his payoff is $\delta^{t-\tau_i}\theta_i$ plus the present value of any monetary transfer. The seller is benevolent and wants to maximize expected sum of the buyers' payoffs.

Allocation At time t, let I_t and E_t be the set of incumbent buyers and entrants, respectively, and let $A_t = I_t \cup E_t$ be the set of all active buyers. An time t allocation maps each active buyer in A_t to either one unit of the product or nothing: $x_t : A_t \to \{0,1\}^{|A_t|}$, and the overall allocation policy is a collection of state-contingent allocations $\{x_t(.)\}_{t=0}^T$. Finally, denote the set of buyers who get assigned in period t as $B_t = \{i \in A_t : x_t^i = 1\}$.

2.1 Efficient Allocation Policy

Assume for the time being that the seller observes each buyer's type upon his arrival and wants to design an allocation policy to maximize expected total surplus. First, observe that at any point in time the seller could treat each active buyer as newly born and her allocation decision should be based on the valuation of active buyers only.

Lemma 1. *$\forall t \in \{0,1,...,T\}$, x_t is independent of τ_i for all $i \in A_t$.*

At the beginning of period t, rank incumbents and entrants by their values as $v_t^1 > v_t^2 > ... > v_t^{|I_t|}$ and $\theta_t^1 > \theta_t^2 > ... > \theta_t^{|E_t|}$, respectively, and combine these two rankings to get the ranking for all active buyers $y_t^1 > y_t^2 > ... > y_t^{|I_t|+|E_t|}$. We occasionally drop the time subscript as long as it causes no confusion. At the beginning of period t before entry occurs, denote the number of unsold products as m_t and let $W_t^{m_t}(I_t)$ be the maximum expected surplus to be generated from period t onward. In the last period $t = T$,

$$W_T^{m_T}(I_T) = \max_{x_T : A_T \to \{0,1\}^{|A_T|}} \mathbb{E}_T[\sum_{y_T^i \in A_T} y_T^i x_T^i] \quad s.t. \quad |B_T| \le m_T$$

At $t \le T-1$, the seller balances immediate gain with future surplus:

$$W_t^{m_t}(I_t) = \max_{x_t : A_t \to \{0,1\}^{|A_t|}} \mathbb{E}_t[\sum_{y_t^i \in A_t} y_t^i x_t^i + \delta W_{t+1}^{m_t - |B_t|}(A_t - B_t)] \quad s.t. \quad |B_t| \le m_t \tag{2.1}$$

On top of Lemma 1, we show that at any point in time the allocation is monotone in buyers' values, i.e., high value buyers always get assigned first.

Lemma 2. *$\forall t \in \{0,1,...,T\}$, $x_t^i \ge x_t^j$ for all $y_i, y_j \in A_t$ and $y_i > y_j$.*

Proof. All proofs in this Section are relegated to the Appendix. □

Unfortunately, Lemma 2 provides no clue how many units should be sold in each period. The next Lemma addresses this issue and reduces the seller's allocation decision to a series of inter-temporal tradeoffs.

Lemma 3. *In period t starting with m units of supply, the seller gives out j units iff:*

$$\begin{aligned} [MC_j] &: \quad y_j \geq \delta[W_{t+1}^{m-j+1}(y_j, y_{j+1}, ..., y_m) - W_{t+1}^{m-j}(y_{j+1}, ..., y_m)], \quad \textit{and} \\ \neg[MC_{j+1}] &: \quad y_{j+1} < \delta[W_{t+1}^{m-j}(y_{j+1}, y_{j+2}..., y_m) - W_{t+1}^{m-j-1}(y_{j+2}, ..., y_m)] \end{aligned}$$

Moreover, if $[MC_j]$ is true, then $[MC_i]$ is true for all $1 \leq i < j$.

$[MC_j]$ can be regarded as a cost-benefit analysis regarding the jth unit product: the left hand side is the immediate gain from current sales while the right hand side captures the opportunity cost from future sales. With finite supply, the allocation decision with respect to each unit of product becomes distinct and is independent of products that alreay have been sold.

In the next Lemma, we characterize the opportunity cost of withholding a particular unit of product from the current sales. Define $\Delta W_t^m(v_1, v_2, ..., v_m) = W_t^m(v_1, v_2, ..., v_m) - W_t^{m-1}(v_2, ..., v_m)$ as the maximum expected surplus that could be generated from the first out of the m remaining units from period t onward, and we characterize $\Delta W_t^m(.)$ in the next Lemma:

Lemma 4. *In period t starting with m products:*

(i) $\Delta W_t^m(0, 0, ..., 0) > 0$, *and* $\Delta W_t^m(1, v_2, ..., v_m) = 1$.

(ii) $\Delta W_t^m(v_1, v_2, ..., v_m)$ *is continuous in* $v_j, \forall 1 \leq j \leq m$.

(iii) For any realization of $v_{\geq 2}$, $0 < \Delta W_t^m(v_1', v_2, ..., v_m) - \Delta W_t^m(v_1, v_2, ..., v_m) < v_1' - v_1$

(iv) For any realization of $v_{\geq 2}$, v *single crosses* $\delta \Delta W_t^{q_1,...,q_m}(v, v_2, ..., v_m)$ *from below for only once at an interior point of* $[0, 1]$.

By Lemma 4 Part *(iv)*, $[MC_j]$ in Lemma 3 is true if and only if y_j exceeds the unique solution to the following equation:

$$v = \delta \Delta W_t^{m-j+1}(v, y_{j+1}, ..., y_m)$$

Denote this solution as $r_{t,m}^j(y_{\geq j+1})$. It turns out that we can interpret it as the marginal cost of the jth unit product at time t, as we verify in the next Lemma that the "marginal cost" is indeed independent of buyers' valuationes.

Lemma 5. *The maximum expected surplus function and the associated cutoff satisfy the following independence property:*

(i) $\Delta W_t^m(v_1, ..., v_m)$ *is independent of* $v_{\geq 2}$.

(ii) $r_{t,m}^1(v_{\geq 2})$ *is independent of* $v_{\geq 2}$.

As a Corollary, note that the time t cutoff for the jth last unit is independent of the inventory size the seller starts with at the beginning of period t:

Corollary 1. $\forall\ m' < m$, $r_{t,m}^{m-j} = r_{t,m'}^{m'-j}$ *for all* $0 \leq j \leq m'$.

Proof. By definition, $r_{t,m}^{m-j} = \delta \Delta W_{t+1}^{j+1}(r_{t,m}^{m-j}, y_{\geq m-j+1}) = \delta \Delta W_{t+1}^{j+1}(r_{t,m}^{m-j}, y_{\geq m'-j+1})$. Since $r_{t,m'}^{m'-j}$ is the unique solution to $v = \delta \Delta W_{t+1}^{j+1}(r_{t,m}^{m-j}, y_{\geq m'-j+1})$, we must have $r_{t,m}^{m-j} = r_{t,m'}^{m'-j}$. □

Lemma 1 to 4 reduce the seller's problem to the following algorithm: in period t starting with m products and active buyers $A_t = \{y_1, ..., y_{|A_t|}\}$, he first decides whether or not to give the first unit to the buyer with the current highest value: if $y_1 < r_{t,m}^1$, he reserves it for futures sales and exits immediately; otherwise he gives it to y_1 and proceeds to the decision respect to the second unit, so on and so forth. Observe that at time t the cutoff for the first unit of product should be weakly lower than that for the second unit, which is in turn weakly lower than that for the third unit, etc., because otherwise Lemma 2 will be violated. Indeed, we show that time t cutoffs should increase strictly as the inventory depletes.

Lemma 6. *At time* t *let* $m_t = m$, *then* $r_{t,m}^j < r_{t,m}^k$, $\forall 1 \leq j < k \leq m$.

With Lemma 1-6, we fully characterize the allocation policy that maximizes expected total surplus.

Theorem 1. *The expected surplus maximizing allocation policy takes the following form: in period* $t \leq T - 1$ *starting with* m *units of supply, the seller gives the* j*th unit to* y_t^j *if and only if* $y_t^j \geq r_{t,m}^j$, *where* $r_{t,m}^j \in (0, 1)$ *is independent of buyers' valuations and is strictly increasing in* j. *In the last period* $t = T$, *the seller simple assigns the* m *products to buyers with top* m *values.*

2.2 Direct Mechanism

In this section we propose a direct mechanism Γ that is periodic ex-post incentive compatible and individually rational to implement the expected surplus maximizing allocation.

Message Space Buyers already in the market have one chance to send a message to the seller, whereas those haven't been born are not able to participate. Let the message space at time t be $M_t = \Theta \times \{0, 1, ..., t\}$, i.e. existing buyers who have not sent the message could report his value and birthday to the seller.

Allocation and Transfer At time t, let m_t be the new reports, x_t be the allocation rule and ψ_t be the transfer scheme, respectively. The seller ranks active buyers according to their reported values and assigns the remaining products according to the expected surplus maximizing policy. Suppose j out of m units are sold out, then buyers who fail to get assigned are not charged, whereas those who win the object pay $\max\{b^{j,1}_{t,m}(y_{j+1})(y_{\geq j+2}), b^{j,2}_{t,m}(y_{j+1})(y_{\geq j+2})\}$, where:

$$\begin{aligned} b^{j,1}_{t,m}(y_{j+1})(y_{\geq j+2}) &= \delta[W^{m-j+1}_{t+1}(y_{j+1}, y_{\geq j+2}) - W^{m-j}_{t+1}(y_{j+1}, y_{\geq j+2})] \\ b^{j,2}_{t,m}(y_{j+1})(y_{\geq j+2}) &= y_{j+1} + \delta[W^{m-j}_{t+1}(y_{\geq j+2}) - W^{m-j}_{t+1}(y_{j+1}, y_{\geq j+2})] \end{aligned}$$

Observe that $\max\{b^1_{t,j}(y_{j+1})(y_{\geq j+2}), b^2_{t,j}(y_{j+1})(y_{\geq j+2})\}$ is the expected total externality incurred by each pivotal buyer $y_{i,i<j}$: if $y_{j+1} \geq r^j_{t,m}$, then he will get assigned instead if y_i is removed from the market. As a result, the expected total contribution by $y_{i,i<j}$ is:

$$\begin{aligned} & y_i + \delta W^{m-j}_{t+1}(y_{j+1}, y_{\geq j+2}) - [y_{j+1} + \delta W^{m-j}_{t+1}(y_{\geq j+2})] \\ = \; & y_i - [y_{j+1} + \delta(W^{m-j}_{t+1}(y_{\geq j+2}) - W^{m-j}_{t+1}(y_{j+1}, y_{\geq j+2}))] \\ = \; & y_i - b^{j,2}_{t,m}(y_{j+1})(y_{\geq j+2}) \end{aligned}$$

and $b^{j,2}_{t,m}(y_{j+1})(y_{\geq j+2}) = y_{j+1} + \delta(W^{m-j}_{t+1}(y_{\geq j+2}) - W^{m-j}_{t+1}(y_{\geq j+1}))$ is the expected total externality y_i incurs. On the other hand, if $y_{j+1} < r^j_{t,m}$, then he remains excluded from the period t sales even if y_i is removed from the market. The total contribution by $y_{i,i<j}$ is:

$$\begin{aligned} & y_i + \delta W^{m-j}_{t+1}(y_{j+1}, y_{\geq j+2}) - \delta W^{m-j+1}_{t+1}(y_{j+1}, y_{\geq j+2}) \\ = \; & y_i - \delta[W^{m-j+1}_{t+1}(y_{j+1}, y_{\geq j+2}) - W^{m-j}_{t+1}(y_{j+1}, y_{\geq j+2})] \\ = \; & y_i - b^1_{t,j}(y_{j+1})(y_{\geq j+2}) \end{aligned}$$

and $b^{j,1}_{t,m}(y_{j+1})(y_{\geq j+2}) = \delta[W^{m-j+1}_{t+1}(y_{j+1}, y_{\geq j+2}) - W^{m-j}_{t+1}(y_{j+1}, y_{\geq j+2})]$ is the expected total externality y_i incurs. Note that $b^{j,2}_{t,m}(y_{j+1})(y_{\geq j+2}) \geq b^{j,1}_{t,m}(y_{j+1})(y_{\geq j+2})$ iff $y_{j+1} \geq r^j_{t,m}$.

History and Information Disclosure Let $m^t = \{m_0, ..., m_t\}$, $x^t = \{x_0, ..., x_t\}$. A history is a collection of messages and allocations: $H_0 = \{\emptyset\}$,...,$H_t = (m^t, x^t)$. We are interested in the performance of the mechanism under *full information disclosure*, so let each buyer in the market observe the full history.

Reporting Strategy Buyer i who is born in period τ has the following reporting strategy:

$$\{m_{i,t} : H_{t-1} \times M_t \to M_t \times \emptyset\}^T_{t=\tau}$$

Definition 1. *A mechanism is periodic ex-post incentive compatible if at any point in time, regardeless of the reports of other agents who have arrived at the market, and given the expectation of truthfully reporting by future entrants, truth-telling is a best response for all agents upon their arrival.*

Theorem 2. Γ *is periodic ex-post incentive compatible and individually rational, as the fol-*

lowing strategy and belief profile constitutes a Perfect Bayesian Equilibrium of the game form of Γ: on the equilibrium path, each entrant reports his type truthfully upon his arrival; off the equilibrium path, each buyer in the market who has not yet reported should report his type truthfully and immediately, i.e. $\forall\ t \geq \tau_i$, $\forall\ H_{t-1}$,

$$m_{i,t}(\theta_i, \tau_i, H_{t-1}) = (\theta_i, \tau_i)$$

Furthermore, each buyer believes that all the other buyers use the strategy described above.

Proof. On the equilibrium path, a period t entrant has three types of deviations: he could report immediately but non-truthfully (Type 1 Deviation), delay but report truthfully (Type 2 Deviation), or delay and report non-truthfully (Type 3 Deviation).

Type 1 Deviation First, consider a period t entrant's incentive to under-report his value upon arrival if he happens to be pivotal in period t sales, i.e. he could get assigned if he reports truthfully and timely. Without losing generality, at time t let him be the highest value buyer y_1 and assume that j out of m units are sold out if he reports truthfully and timely. Then:

$$\begin{aligned} y_1 > y_2 > ... > y_j \geq r^j_{t,m} \\ y_{j+1} < r^{j+1}_{t,m} \end{aligned}$$

Furthermore, assume that $y_{j+1} \geq r^j_{t,m}$ so that he will win one unit of the product if any member of $\{y_1, ..., y_j\}$ is absent at time t. By Lemma 4,

$$\begin{aligned} y_{j+1} &< \delta[W^{m-j}_{t+1}(y_{\geq j+1}) - W^{m-j-1}_{t+1}(y_{\geq j+2})] \\ y_{j+1} &+ \delta[W^{m-j}_{t+1}(y_{\geq j+2}) - W^{m-j}_{t+1}(y_{\geq j+1})] \geq \delta[W^{m-j+1}_{t+1}(y_{\geq j+1}) - W^{m-j}_{t+1}(y_{\geq j+1})] \end{aligned}$$

Therefore, y_1's payoff if he reports truthfully and timely is:

$$\begin{aligned} \Pi(y_1) &= y_1 - [y_{j+1} - \delta[W^{m-j}_{t+1}(y_{\geq j+1}) - W^{m-j}_{t+1}(y_{\geq j+2})]] \\ &> y_1 - \delta[W^{m-j}_{t+1}(y_{\geq j+1}) - W^{m-j-1}_{t+1}(y_{\geq j+2}) - [W^{m-j}_{t+1}(y_{\geq j+1}) - W^{m-j}_{t+1}(y_{\geq j+2})]] \\ &= y_1 - \delta[W^{m-j}_{t+1}(y_{\geq j+2}) - W^{m-j-1}_{t+1}(y_{\geq j+2})] \end{aligned}$$

Assume further that $y_1 \geq r^{j+1}_{t,m}$ and consider his incentive to under-report on his birthday and lose the time t sales. Since $y_1 \geq r^{j+1}_{t,m}$, by Lemma 5 we have:

$$\begin{aligned} \Pi(y_1) &> y_1 - \delta[W^{m-j}_{t+1}(y_{\geq j+2}) - W^{m-j-1}_{t+1}(y_{\geq j+2})] \\ &\geq \delta[W^{m-j}_{t+1}(y_1, y_{\geq j+2}) - W^{m-j-1}_{t+1}(y_{\geq j+2})] - \delta[W^{m-j}_{t+1}(y_{\geq j+2}) - W^{m-j-1}_{t+1}(y_{\geq j+2})] \\ &= \delta[\underbrace{W^{m-j}_{t+1}(y_1, y_{\geq j+2})}_{(1)} - \underbrace{W^{m-j}_{t+1}(y_{\geq j+2})}_{(2)}] \end{aligned} \tag{2.2}$$

We claim that (2.2) is y_1's expected payoff if he has the option to re-submit a truthful report at $t+1$. Observe that (1) is the maximum expected surplus evaluated at the beginning of $t+1$ if y_1 re-submits a truthful report to the seller, and let s be a future state in which he wins the object. Furthermore, let k_s be the number of remaining units at the beginning of state s, q_s out of which are sold to the top q_s buyers, including y_1. Then,

$$\begin{aligned} y_s^{q_s} &> r_{s,k_s}^{q_s} \\ y_s^{q_s+1} &< r_{s,k_s}^{q_s+1} \end{aligned}$$

If $y_s^{q_s+1} \geq r_{s,k_s}^{q_s}$, then q_s units are sold in the absence of y_1, and

$$\begin{aligned} (1)-(2)|s &= \delta^{s-t-1}[y_1 + \delta W_{s+1}^{k_s-q_s}(y_s^{\geq q_s+1}) - y_s^{q_s+1} - \delta W_{s+1}^{k_s-q_s}(y_s^{\geq q_s+2})] \\ &= \delta^{s-t-1}\left[y_1 - b_{s,k_s}^{q_s,2}(y_s^{q_s+1})(y_s^{\geq q_s+2})\right] \end{aligned}$$

Similarly, if $y_s^{q_s+1} < r_{s,k_s}^{q_s}$, $q_s - 1$ units are sold in the absence of y_1, and

$$\begin{aligned} (1)-(2)|s &= \delta^{s-t-1}[y_1 + \delta W_{s+1}^{k_s-q_s}(y_s^{\geq q_s+1}) - \delta W_{s+1}^{k_s-q_s+1}(y_s^{\geq q_s+1})] \\ &= \delta^{s-t-1}\left[y_1 - b_{s,k_s}^{q_s,1}(y_s^{q_s+1})(y_s^{\geq q_s+2})\right] \end{aligned}$$

Therefore, $(2.2) = \mathbb{E}_t[\sum_s \delta^{s-t}(y_1 - b_{s,k_s}^{q_s}(y_s^{q_s+1})(y_s^{\geq q_s+2}))]$, which is exactly the expected payoff y_1 would get if he could re-submit a truthful value report at $t+1$. Unfortunately, (2.2) is an upperbound on y_1's payoff if he deviates and under-reports now, because he cannot revise his report and thus may not be able to win in all future states in which he could have won with a corrected type report. Thus, he has no incentive to under-report on his birthday.

The argument above applies to any realization of y_1 and y_{j+1} and any pivotal agent at time t. Moreover, if an entrant happens to be non-pivotal in period t, he could only decrease his expected payoff from future sales if he under-reports now. As a result, no entrant has incentive to under-report on his birthday.

Now consider a non-pivotal entrant's incentive to over-report on his birthday. Without losing generality, at time t assume that j out of m units are sold out if he reports timely and truthfully, and let him be the $(j+1)$th highest value buyer. According to the analysis above, if y_{j+1} sticks to the equilibrium strategy, his expected payoff from future sales is:

$$\Pi(y_{j+1}) = \delta[W_{t+1}^{m-j}(y_{j+1}, y_{\geq j+2}) - W_{t+1}^{m-j}(y_{\geq j+2})]$$

Furthermore, assume that $y_j < r_{t,m}^{j+1}$ so that only $\{y_1, ..., y_{j-1}, y_{j+1}\}$ get assigned if y_{j+1} over-reports. Since $y_j \geq r^j_{t,m}$, y_{j+1}'s current payoff from over-reporting becomes:

$$\begin{aligned} y_{j+1} - b^{j,2}_{t,m}(y_j)(y_{\geq j+2}) &= y_{j+1} - [y_j + \delta W^{m-j}_{t+1}(y_{\geq j+2}) - \delta W^{m-j}_{t+1}(y_j, y_{\geq j+2})] \\ &= -(y_j - y_{j+1}) + \delta[W^{m-j}_{t+1}(y_j, y_{\geq j+2}) - W^{m-j}_{t+1}(y_{j+1}, y_{\geq j+2})] + \Pi(y_{j+1}) \\ &< \Pi(y_{j+1}) \end{aligned}$$

where the last inequality follows Lemma 4 (*iii*). Thus y_{j+1} has no incentive to over-report in order to get assigned on his birthday. Moreover, this conclusion is robust to the realization of y_j and applies to any non-pivotal entrant in period t.

Type 2 Deviation If any entrant delays the message but reports truthfully, he could weakly increase his payoff by reporting truthfully on his birthday.

Type 3 Deviation According to the argument above, it is clear that no buyer has incentive to delay and misreport because it is dominated by a strategy that delays but report truthfully, which is in turn dominated by the equilibrium strategy.

Off Equilibrium Path Repeat the argument above for any off-equilibrium path, assuming that each buyer believes that all the other buyers report truthfully on their birthdays. □

Remark 1. We allow each agent to receive his expected total contribution once and for all at the time when he leaves the market. This transfer scheme turns out to be equivalent to the "marginal flow contribution" (See Bergemann and Valimaki [2]) in our setup.

Remark 2. The key assumption we rely upon to get immediate participation from the entrants is that all buyers are equally patient. If this assumption is violated, then we no longer get immediate participation for free. To see why, imagine that the market lasts for three periods and there is one product for sale. At $t = 0$ a buyer with valuation y_2 enters the market and falls short of the cutoff; at $t = 1$, another buyer with $y_1 > y_2$ enters and $y_1 = r + \varepsilon > r > r - \varepsilon = y_2$, where r is the cutoff at $t = 1$. Now if y_2 always reports his private information truthfully and it becomes common knowledge at $t = 1$ that y_2 has to leave the market at the end of $t = 1$, no matter he wins the object or not. Then y_1 may benefit from waiting if entrants at $t = 2$ have weak demand. Assuming buyers to be equally patient may or may not be appropriate, depending on the situation we try to approximate.

2.3 Multi-Round Simultaneous Ascending Auction

In this section we introduce a sequential simultaneous ascending auction as an outcome equivalent indirect mechanism to achieve the efficient outcome. Observe that the seller basically wants to allocate K homogenous products to buyers who want them the most. This raises the question as to whether efficiency can be achieved by an indirect mechanism we are familiar with. A natural candidate is uniform price auction with reserve prices, as intuition suggests that we can use reserve price to screen out ineligible bidders below the cutoffs and rely on the competitive force in auctions to achieve efficient outcome. Nevertheless, analysis below suggests that our intuition is flawed.

First of all, though products in our model convey the same consumption value to the buyers, they are treated as distinct by the seller for reasons we discussed above: as the inventory depletes, the seller become less willing to sell and raises the cutoff for in response. This suggests the use of simultaneous ascending auction at each stage to provide a menu of incentive schemes in order to align the buyers' incentives with the seller's.

Second, a standard reserve price may not suffice to screen out ineligible buyers. Due to the interdependence created by buyers' population dynamics, the transfer scheme in the direct mechanism incorporates rich dimensions of non-pivotal agents' information that can hardly be summarized into a single reserve price. As an example, consider a market that lasts for two periods. Two buyers $y_1 > y_2$ arrive at $t = 0$, and one more buyer θ arrives at $t = 1$. The seller is endowed with one product, and let $r > 0$ be the cutoff in period 0. Now suppose she holds a second price sealed bid auction with reserve price R at $t = 0$ followed by a standard second price auction at $t = 1$. At $t = 0$, the reserve price should effectively separate buyers above and below the cutoff so that only those above the cutoff participate, and all participants bid up to their own valuations. In particular, the reserve price should make $y_1 = r$ indifferent between participating or not:

$$R = \delta \mathbb{E}_{\tilde{y}_2,\theta}\big[\max\{\tilde{y}_2, \theta\}\mathbb{1}_{\theta<r} \mid \tilde{y}_2 < r\big] \tag{2.3}$$

Moreover, a buyer is willing to participate if and only if her valuation is above the cutoff r, i.e., the following condition holds iff $y_1 \geq r$:

$$\mathbb{P}(y_1 > \tilde{y}_2)(y_1) - \mathbb{P}(\tilde{y}_2 < r)R - \mathbb{E}_{\tilde{y}_2}\big[\tilde{y}_2\mathbb{1}_{y_1>\tilde{y}_2\geq r}\big] \geq \delta\mathbb{E}\big[(y_1 - \max\{\tilde{y}_2, \theta\})\mathbb{1}_{\theta<y_1}\mathbb{1}_{\tilde{y}_2<r}\big] \tag{2.4}$$

From the previous section, we know that $\forall\ \tilde{y}_2 < r$, the next condition hold iff $y_1 \geq r$:

$$y_1 - \delta W(\tilde{y}_2) \geq \delta\mathbb{E}\big[(y_1 - \max\{\tilde{y}_2, \theta\})\mathbb{1}_{\theta<y_1}\big] \tag{2.5}$$

Integrate (2.5) over $\tilde{y}_2 < r$ we have:

$$\mathbb{E}\big[(y_1 - \delta W(\tilde{y}_2))\mathbb{1}_{\tilde{y}_2<r}\big] \geq \delta\mathbb{E}\big[(y_1 - \max\{\tilde{y}_2, \theta\})\mathbb{1}_{\tilde{y}_2<r}\mathbb{1}_{\theta<y_1}\big] \quad iff \quad y_1 \geq r$$

Adding $\mathbb{E}\big[(y_1 - \tilde{y}_2)\mathbb{1}_{y_1>\tilde{y}_2>r}\big]$ to the (LHS), we get the following condition which holds iff $y_1 \geq r$ and observe that it is different from (2.4):

$$\mathbb{P}(\tilde{y}_2 < r)y_1 - \underbrace{\delta\mathbb{E}_{\tilde{y}_2,\theta}\big[\max\{\tilde{y}_2, \theta\}\mathbb{1}_{\theta<\tilde{y}_2}\mathbb{1}_{\tilde{y}_2<r}\big]}_{\neq\, \mathbb{P}(\tilde{y}_2<r)R} > \delta\mathbb{E}\big[(y_1 - \max\{\tilde{y}_2, \theta\})\mathbb{1}_{\theta<y_1}\mathbb{1}_{\tilde{y}_2<r}\big] \tag{2.6}$$

Hence (2.4) may not hold in general.

The argument above suggests us to use an open format auction that induces full information disclosure with a "generalized reserve price" which depends non-pivotal agents' values. In this spirit, we propose a simultaneous simultaneous ascending auction as follows:

Setup In period $t \leq T-1$ starting with an inventory of size m, the seller simultaneously runs m separate ascending button auctions[3] for each individual unit of product. In the auction for the jth unit (or the jth auction), he sets the starting bid to zero and uses a robot of type $r^j_{t,m}$ to bid against the buyers. In the last period $t = T$, he simply auctions off the remaining m units in a standard uniform price auction.

Buyers and Robots In period t, buyers observe full bidding history from the past, H_{t-1}, as well as the drop out history in each on-going auction. They are required to bid in all m auctions.

In the jth auction, the robot bids according to $b^{j,1}_{t,m}(.)(.)$ under the assumption that every other bidder in the jth auction does the same in the following sense: starting from an empty bidding history in the current period, a bidder of value y sets his drop out price to $b^{j,1}_{t,m}(y)(0,0,...,0)$ and continues bidding until this price is reached or someone else exits before him. In the second case he assumes that the drop out bidder uses the same strategy as he does and infers his value as $y^{'}$. Then he sets the next drop out price to $b^{j,1}_{t,m}(y)(y^{'},0,...,0)$. If someone else exits before him again, he infers the second drop out bidder's value as $y^{''}$ and sets the next drop out price to $b^{j,1}_{t,m}(y)(y^{''},y^{'},0,...,0)$, so on and so forth. Thus, we can regard the robot as a buyer who values the product at $r^j_{t,m}$.

Outcome Let q_j be the number of buyers who outbid the robot in the jth auction. Define $k = \max\{j : q_j \geq j, j = 1,2,...,m\}$, and let $k = 0$ if $q_j < j$ for all $1 \leq j \leq m$. If $k > 0$, then the top k bidders in the kth auction win and pay the $(k+1)$th bid in the kth auction. Otherwise the seller gives out nothing and proceeds to the next period.

Theorem 3. *The following strategy and belief profile constitutes a Perfect Bayesian Equilibrium of the multi-round simultaneous ascending auction game. At time t starting with m units of supply and regardless of the bidding history from the past H_t: In the jth auction starting from zero bid, a bidder of value y sets his drop out price to $b^1_{t,m}(v)(0,0,...,0)$ and continues bidding until this price is reached or someone else exits before him. In the second case he infers the drop out bidder's value as $y^{'}$ under the assumption that every other bidder uses the same strategy as he does. Then he sets the next drop out price to $b^1_{t,m}(y^{'},0,...,0)$. If some other bidder exits before him again, he infers the second drop out bidder's value as $y^{'}$ and sets the next drop out price to $b^1_{t,m}(v)(y^{''},y^{'},...,0)$, so on and so forth. This process continues as long as the robot remains active.*

[3] Button auction as described in Milgrom and Weber [8].

When the robot drops out from the jth auction, a remaining bidder of value y sets the next drop out price to $b^{j,2}_{t,m}(y)(y'_{<y})$ [4] *and continues bidding until this price is reached or someone else exits before him. In the second case he infers the drop out bidder's value as y''' under the assumption that all remaining bidders use the same strategy as he does when the robot becomes inactive. He sets the next drop out price to $b^{j,2}_{t,m}(y''', y'_{<y})$ and proceeds.*

As the first step, we verify that $b^{j,1}_{t,m}(.)(.), b^{j,2}_{t,m}(.)(.)$ are indeed valid bidding functions.

Lemma 7. *$\forall t, m, j$:*

(i) Both $b^{j,1}_{t,m}(v)(v'_{<v})$ and $b^{j,2}_{t,m}(v)(v'_{<v})$ are strictly increasing in v and each element of $v'_{<v}$.

(ii) For any $v'_{<v}$, $b^{2}_{t,m}(v)(v'_{<v}) \geq b^{1}_{t,m}(v)(v'_{<v})$ iff $v \geq r^{j}_{t,m}$, and the inequality is strict iff $v > r^{j}_{t,m}$.

Proof. See Appendix. □

Detailed proof for Theorem 3 is relegated to the Appendix. However, we want to highlight key ingredients of the sequential simultaneous ascending auction and compare it with the standard uniform price auction to pin point the challenge raised by population dynamics.

Simultaneous Ascending Auction To get a more intuitive interpretation of the simultaneous ascending auction, consider a market that lasts for two periods. Three buyers $y_1 > y_2 > y_3$ arrive at $t = 0$ and one more buyer arrives at $t = 1$. The seller is endowed with two identical products, and let r_1, r_2 be the cutoff for the first and second unit at $t = 0$, respectively. In period 0, if $y_1 > r_2 > y_2$, then y_1 wins the first auction and pays y_2's bid for the first unit; but if we fix y_1 and raises y_2 to $y'_2 > r_2$, then both y_1, y'_2 win the second auction and pay y_3's bid for the second unit. Though y_1's value for the product is fixed, he wins in the first case because he expresses a sufficiently high willingness to pay when demand by other agents is low, and wins in the second case as he reveals an even more intense interest when demand from his competitors is high. Thus, bids for different units can be interpreted as *one's willingness to pay under different demand pressures.* Since the seller essentially faces an increasing "marginal cost curve", this is the exact instrument he needs to eliminate agents below the marginal cost curve and induce truth-telling from those who remain pivotal.

In the absence of population dynamics or production cost, products that deliver the same consumption value to the buyers are identical from the seller's perspective. Thus, there is no need for more than one price and the uniform price auction suffice for achieving efficiency.

Open Auction Format and Endogenous Information Revelation To account for the interdependence created by population dynamics, we make use of an open auction format to generate full information disclosure within each period and neutralize the information gap between incumbents and entrants (See Said [10] for a detailed discussion). Unfortunately, we cannot

[4] $y'_{<y}$ are the valuations of drop out bidders. "$_{<y}$" means that these bidders exits before y does.

achieve this objective with closed form auctions unless we bring the mechanism designer back (or use some coordination devise). On the contrary, auction format choice is irrelevant in standard uniform price auction.

Generalized Reserve Price $b^{j,1}_{t,m}(r^j_{t,m})(.)$ eliminates buyers below the cutoff for the jth unit product and can be regarded as a "generalized reserve price" in our setting with interdependent values.

Corollary 2. *The sequential simultaneous ascending auction is outcome equivalent to* Γ*, in the sense that buyers receive the same allocations and monetary transfers in the two Perfect Bayesian Equilibria discussed above.*

Remark 3. The sequential simultaneous ascending auction is designed not to promote the use of indirect mechanism but to highlight issues created by market dynamics.

3 Discussion and Conclusion

We study the problem of selling identical storable goods to patient buyers who arrive stochastically to a marketplace. Instead of repeating what we have done, we want to discuss several unaddressed questions and point out avenues for future research.

First, when evaluating the performance of certain trading platforms, e.g. vehicle auction, we may want to allow buyers to have private, yet heterogeneous discounting factors, because in reality they face different borrowing cost. Since the IC constraint for immediate participation is genetically binding in this scenario, we are curious to see if efficiency/maximal revenue can still be achieved by simple and intuitive mechanisms.

Second, we are interested to see what will happen if buyers could trade for speculative purpose, as they often do in reality. To bridge this gap we have to introduce a secondary market to our framework, and we do not have a neat solution to this problem yet.

References

[1] Athey, Susan and Ilya Segal (2007), *"An Efficient Dynamic Mechanism"*, Working paper, Stanford University.

[2] Bergemann, Dirk and Juuso Valimaki (2008), *"The Dynamic Pivot Mechanism"*, Working paper, Yale University.

[3] Board, Simon (2008), *"Durable-Goods Monopoly with Varying Demand"*, Review of Economic Studies, 75(2), pp. 391-413.

[4] Dasgupta, Partha and Eric Maskin (2000), *"Efficient Auctions"*, The Quarterly Journal of Economics, 115(2), pp. 341-388.

[5] Fuchs, William and Andrzej Skrypacz (2008), *"Bargaining with Arrival of New Traders"*, forthcoming, American Economic Review.

[6] Gershkov, Alex and Benny Moldovanu (2009), *"Dynamic Revenue Maximization with Heterogeneous Objects: A Mechanism Design Approach"*, American Economic Journal: Microeconomics, 1(2), pp. 168-198(31).

[7] Lavi, Ron and Noam Nisan (2004), *"Competitive analysis of incentive compatible on-line auctions"*, Theoretical Computer Science, 310(1-3), pp. 159-180.

[8] Milgrom, Paul and Robert J. Weber (1982), *"A Theory of Auction and Competitive Bidding"*, Econometrica, 50(5), pp. 1089-1122.

[9] Pavan, Alessandro, Ilya Segal and Juuso Toikka (2008), *"Dynamic Mechanism Design: Revenue Equivalence, Profit Maximization, and Information Disclosure,"*, Working Paper, Stanford University.

[10] Said, Maher (2009), *"Auctions with Dynamic Populations: Efficiency and Revenue"*, Working paper, Yale University.

[11] Talluri, K.T. and Van Ryzin, G. (2004), *"The Theory and Practice of Revenue Management"*, Springer, New York.

[12] Gustavo Vulcano, Garrett van Ryzin and Costis Maglaras (2002), *"Optimal Dynamic Auctions for Revenue Management"*, Management Science, 48(11), pp. 1388-1407.

Appendix

Proof for Lemma 2

Proof by induction. Apparently this condition holds in the last period T. Now assume it is true in period $\tau \in \{t+1, ..., T\}$ and consider the problem in period t. Let $B_t = \{y_{i_1}, y_{i_2}, ..., y_{i_k}\}$ and suppose $y_1 > y_{i_1}$. Define S as the collection of future states in which y_1 gets the product $S = \{s : x_s(y_1) = 1\}$, and consider a new allocation rule $x^{'}$ that switches the roles between y_1 and y_{i_1}, i.e. in period t, $x^{'}_t(y_1) = 1, x^{'}_t(y_{i_1}) = 0$; $\forall\ s \in S$, $x^{'}_s(y_{i_1}) = 1$ and $x^{'}_s(y_1) = 0$.

Denote the maximum expected total surplus evaluated at time t after entry occurs as $\tilde{W}_t^{m_t}(A_t)$, which is a function of the number of unsold products and the values of active buyers, and let the expected total surplus generated by the new allocation rule as $\tilde{V}_t^{m_t}(A_t)$:

$$\begin{aligned}\tilde{V}_t^{m_t}(A_t) - \tilde{W}_t^{m_t}(A_t) &= (y_1 - y_{i_1}) + \mathbb{E}_t[\sum_{s\in S} \mathbb{1}_s \delta^{s-t}(y_{i_1} - y_1)] \\ &= (y_1 - y_{i_1})(1 - \mathbb{E}_t[\sum_{s\in S} \mathbb{1}_s \delta^{s-t}]) > 0\end{aligned}$$

Thus the new allocation rule generates a strictly higher expected total surplus than the surplus maximizing policy, contradiction, so we must have $y_1 = y_{i_1}$. Similarly, walk through this argument for $\{y_{i_2}, ..., y_{i_k}\}$ and conclude that $B_t = \{y_1, ..., y_k\}$.

Proof for Lemma 3

Since the seller prefers to give out j units than $j-1$ units, it must be that:

$$\sum_{k=1}^{j} y_k + \delta W_{t+1}^{m-j}(y_{j+1}, ..., y_m) \geq \sum_{k=1}^{j-1} y_k + \delta W_{t+1}^{m-j+1}(y_j, y_{j+1}, ..., y_m)$$

In other words, the immediate gain from the jth unit must exceed the maximum expected surplus it could genenerate from future sales, i.e.

$$y_j \geq \delta[W_{t+1}^{m-j+1}(y_j, y_{j+1}, ..., y_m) - W_{t+1}^{m-j}(y_{j+1}, ..., y_m)]$$

On the other hand, the seller withholds the $(j+1)$th unit because he expects to extract a strictly higher surplus from this particular unit in future sales:

$$y_{j+1} < \delta[W_{t+1}^{m-j}(y_{j+1}, y_{j+2} ..., y_m) - W_{t+1}^{m-j-1}(y_{j+2}, ..., y_m)]$$

Finally, since the seller is willing to give out the first $(j-1)$ units before he proceeds to the decision with respect to the jth unit, $[MC_i]$ must be true for all $1 \leq i < j$. Thus, $[MC_j]$ implies $[MC_i]$, $\forall 1 \leq i < j$.

Proof for Lemma 4

(i) is easy to show: observe that even if $v_1 = 0$, the seller is still able to generate a strictly positive surplus from the first unit because he can always give out the 2nd, ..., the mth unit first and reserve the first unit for whomever left in the last period. On the other hand, if $v_1 = 1$, the seller should immediate awards the first unit to v_1. Thus, $\Delta W_t^m(0, 0, ..., 0) > 0, \Delta W_t^m(1, v_2, ..., v_m) = 1$.

(ii) can be easily verified. In order to show (iii), define S as the collection of states in which the highest value incumbent v_1 gets assigned. When v_1 is replaced by $v_1^{'} > v_1$, we can continue to use the old allocation rule and pretend that the replacement did not occur, and achieves something weakly less than the maximum expected surplus, i.e.

$$\begin{aligned} \Delta W_t^m(v_1^{'}, v_2, ..., v_m) - \Delta W_t^m(v_1, v_2, ..., v_m) &= W_t^m(v_1^{'}, v_2, ..., v_m) - W_t^m(v_1, v_2, ..., v_m) \\ &\geq \mathbb{E}_t[\sum_{s \in S_l} \mathbb{1}_s \delta^{s-t}(v_1^{'} - v_1)] > 0 \end{aligned}$$

Switch the roles between v_1 and $v_1^{'}$ and apply the argument above symmetrically, we have:

$$W_t^m(v_1^{'}, v_2, ..., v_m) - W_t^m(v_1, v_2, ..., v_m) \leq \mathbb{E}_t[\sum_{s \in S} \mathbb{1}_s \delta^{s-t}(v_1^{'} - v_1)] < v_1^{'} - v_1$$

By $(i) - (iii)$, v single crosses $\delta \Delta W_t^m(v, v_{\geq 2})$ from below for only once at some interior point of $[0, 1]$.

Proof for Lemma 5

Prove by induction. It's easy to verify that $(i) - (ii)$ hold in the last period $t = T$, and (ii) is true at $t = T - 1$. Now suppose (i) holds in period $t + 1, ..., T$ and (ii) holds in period $t, t + 1, ..., T$. We want to show that (i) is true in period t. Denote the seller's problem at the beginning of period t with m products and incumbents $I_t = \{v_1, v_{\geq 2}\}$ as $\Psi_t^1(m)(v_1, ..., v_m)$, and the problem starting with $m - 1$ products and $I_t = \{v_{\geq 2}\}$ as $\Psi_t^2(m - 1)(v_2, ..., v_m)$. In Ψ_t^1, define S_t^j as the collection of period t states in which v_1 wins the jth unit. At any $s \in S_t^j$, it must be that:

$$\begin{aligned} y_s^1 > y_s^2 \quad &\geq \quad r_{t,m}^2 = r_{t,m-1}^1 \quad (Corollary \quad 1) \\ &... \\ y_s^{j-1} > v_1 \quad &= \quad y_s^j \geq r_{t,m}^j = r_{t,m-1}^{j-1} \\ &... \end{aligned}$$

Since $y_s^1 > ... > y_s^{j-1}$ are all entrants, in problem Ψ_t^2 the seller must give one unit to each member of $\{y_s^1, ..., y_s^{j-1}\}$, and after that the continuation problem becomes identical to the one in Ψ_t^1. Therefore, at any $s \in S_t^j$, the difference in total surplus between the two problems is independent of $v_{\geq 2}$. Combining these states and we get:

$$\begin{aligned} \mathbb{E}_t[\sum_{j=1}^{m} \sum_{s \in S_t^j} \mathbb{1}_s[\tilde{W}_s^m(y_s^1, ..., y_s^{j-1}, v_1, y_s^{j+1}, ..., v_2, ...) \quad &- \quad \tilde{W}_s^{m-1}(y_s^1, ...y_s^{j-1}, y_s^{j+1}, ..., v_2, ...)]] \\ &= \quad v_1 \mathbb{E}_t[\sum_{j=1}^{m} \sum_{s \in S_t^j} \mathbb{1}_s] \end{aligned}$$

Observe that in period t the seller starts with one more unit of product and one more high value incumbent v_1 in problem Ψ_t^1. Call this situation C_1.

Now consider a period t state s in which v_1 fails to get assigned in problem Ψ_t^1. If the seller gives out non-negative units at s, all of them must go to the entrants. Suppose in Ψ_t^1,

$x_s(y_1, y_2, ..., y_j, y_{j+1}, ...) = (1, 1, ..., 1, 0, ...)$, then it must be that:

$$\begin{aligned} y_1 > y_2 &\geq r^2_{t,m} = r^1_{t,m-1} \\ &... \\ y_{j-1} > y_j &\geq r^j_{t,m} = r^{j-1}_{t,m-1} \\ y_{j+1} < r^{j+1}_{t,m} &= r^j_{t,m-1} \leq r^{j+1}_{t,m-1} \quad (Lemma \quad 6) \end{aligned}$$

Therefore, in problem Ψ^2_t, each member of $\{y_1, ..., y_{j-1}\}$ should get one unit and $\{y_i : i \geq j+1\}$ nothing. y_j may or may not get assigned, and depending on his status at the end of period t there are two relevant situations:

1. If y_j wins one unit at s in problem Ψ^2_t , then the seller is left with $m - j - 1$ products and buyers $y_{j+1} > ... > v_2 > ...$ at the end of period t. Define this continuation problem as $\Psi^2_{t+1}(m-j-1)(y_{j+1}, ..., v_2, ...)$ and compare it with $\Psi^1_{t+1}(m-j)(y_{j+1}, ..., v_1, ...v_2, ...)$: since the seller starts with one more unit of product and the same set of incumbents except for v_1 in Ψ^1_{t+1}, we are back to C_1 and can continue to solve the problem using the procedure described above: if v_1 gets assigned in period $t + 1$ we are done; otherwise we end up in C_1 or C_2 (to be defined immediately).

2. If y_j loses at s in Ψ^2_t, then in period $t + 1$ the seller continues with $m - j$ products and incumbents $y_j > y_{j+1} > ... > v_2 > ....$ Define this continuation problem as $\Psi^2_{t+1}(m - j)(y_j, y_{j+1}, ..., v_2, ...)$. Compared with $\Psi^1_{t+1}(m - j)(y_{j+1}, ..., v_1, ..., v_2, ...)$, the seller start with with the same number of products, one more high value incumbent y_j but no v_1 in problem Ψ^2_{t+1}. Define this situation as C_2.

   Since the seller starts with the same set of products in period $t+1$, he must use the same cutoff rule in both problems. If v_1 gets assigned in period $t+1$, Ψ^1_{t+1}, then we can conclude by the argument at the very beginning of this proof; if he doesn't, without losing generality rank the buyers who get assigned in problem Ψ^2_{t+1} as: $\theta_1 > \theta_2 > ... > \theta_k = y_j > ... > \theta_n$.[5] Then we must have:

$$\begin{aligned} \theta_1 > ... > \theta_{k-1} > \theta_k = y_j \geq r^k_{t,m-j} \\ \theta_{k+1} \geq r^{k+1}_{t,m-j} \geq r^k_{t,m-j} \\ ... \\ \theta_n \geq r^n_{t,m-j} \geq r^{n-1}_{t,m-j} \\ \theta_{n+2} < \theta_{n+1} < r^{n+1}_{t,m-j} \end{aligned}$$

[5]If y_j loses in period $t+1$, problem Ψ^2_{t+1}, then the set of buyers who gets assigned in period $t+1$ are identical in both problems, so we remain in C_2.

Furthermore, since the seller uses the same cutoff rules in both problems, in Ψ^1_{t+1}, each member of $\{\theta_1, ..., \theta_{k-1}, \theta_{k+1}, ..., \theta_n\}$ must get assigned, and $\theta_{\geq n+2}$ get nothing. Depending on the value of θ_{n+1}, he may or may not get assigned. If he does, then the first problem becomes $\Psi^1_{t+2}(m-j-n)(\theta_{n+2}, ..., v_1, ..., v_2)$, whereas the second problem becomes $\Psi^2_{t+2}(m-j-n)(\theta_{n+1}, ..., v_2, ...)$, so we remain in C_2; if he doesn't, we are back to C_1.

This algorithm has to stop in finite times, and whenver it stops the difference between the two surplus functions is independent of $v_{\geq 2}$. Therefore, (i) is true in period t, and (ii) is true in period $t-1$.

Proof for Lemma 6

Compare $r^{m-1}_{t,m}$ and $r^m_{t,m}$. Observe that if the following inequality holds, then $r^{m-1}_{t,m} < r^m_{t,m}$:

$$W^2_{t+1}(v, 0) < W^1_{t+1}(v) + W^1_{t+1}(0) \tag{3.1}$$

The reason is straightforward: by Lemma 4 and 5, $r^{m-1}_{t,m}$ and $r^m_{t,m}$ are the unique solutions to the following two equations, respectively:

$$\begin{aligned} v &= \delta[W^2_{t+1}(v, 0) - W^1_{t+1}(0)] \\ v &= \delta W^1_{t+1}(v) \end{aligned}$$

If (3.1) holds, then $r^{m-1}_{t,m} = \delta[W^2_{t+1}(r^{m-1}_{t,m}, 0) - W^1_{t+1}(0)] < \delta W^1_{t+1}(r^{m-1}_{t,m})$, so we must have $r^{m-1}_{t,m} < r^m_{t,m}$ by Lemma 4.

As in the proof of Lemma 5, denote the dynamic assignment problem on LHS of (3.1) as $\Psi^1_{t+1}(2)(v)$, the problems on RHS of (3.1) as $\Psi^2_{t+1}(1)(v)$ and $\Psi^3_{t+1}(1)(0)$. There are two relevant situations:

1. Define S as the collection of states in which v gets the first unit in the first problem, $\Psi^1_{t+1}(2)(v)$, and note that v must be the highest value buyer at any $s \in S$. Denote the second highest buyer at s as y^2_s, we have:

$$\tilde{W}^2_s(v, y^2_s) = v + \tilde{W}^1_s(y^2_s)$$

Observe that in the first problem Ψ^1_{t+1}, v, together with other active buyers, are excluded from the sales before state s occurs. Since $r^{m-1}_{t,m} \leq r^m_{t,m}$, in both Ψ^2_{t+1} and Ψ^3_{t+1} no sales occurs before state s, and depending on the realization of v, he may or may not get assigned at state s in problem Ψ^2_{t+1}. Consequently,

$$\mathbb{E}_{t+1}[\delta^{s-t}(\tilde{W}^1_s(v) + \tilde{W}^1_s(y^2_s))|s, y^2_s] \geq \delta^{s-t}[v + \tilde{W}^1_s(y^2_s)] = \mathbb{E}_{t+1}[\delta^{s-t}\tilde{W}^2_s(v, y^2_s)|s, y^2_s]$$

2. Now suppose that in Ψ^1_{t+1}, $y_1 > v$ wins the first unit at some s_1 and $y_2 \geq v$ wins the second unit at some s_2 (potentially after s_1). Depending on the value of y_1, he may or may not get the first unit at s_1 in $\Psi^2_{t+1}, \Psi^3_{t+1}$, i.e.

$$\mathbb{E}_{t+1}[\delta^{s_1-t}\tilde{W}^1_{s_1}(y_1)|s_1, y_1] \quad \geq \quad \delta^{s_1-t}y_1 > \delta^{s_2-t}y_2$$

As a result, the maximum expected surplus from Ψ^1_{t+1} is strictly less than the sum of the maximum expected surplus from Ψ^2_{t+1} and Ψ^3_{t+1}:

$$\begin{aligned}\mathbb{E}_{t+1}[\tilde{W}^2_{t+1}|s_1, y_1, s_2, y_2] &= \delta^{s_1-t}y_1 + \delta^{s_2-t}y_2 \\ &< \mathbb{E}_{t+1}[\delta^{s_1-t}(\tilde{W}^1_{s_1}(y_1) + \tilde{W}^1_{s_1}(y_1))|s_1, y_1] \\ &= \mathbb{E}_{t+1}[\tilde{W}^1_{t+1} + \tilde{W}^1_{t+1}|s_1, y_1]\end{aligned}$$

Since both situations discussed above occur with strictly positive probability, combine them and we get (3.1).

Proof for Lemma 7

Part (ii) is implicitly stated in Lemma 4. For Part (i), we only verify that for any fixed vector $v'_{<v}$, $b^{j,1}_{t,m}(v)(v'_{<v})$ is strictly increasing in v. Using exactly the same argument we can show that $b^{j,1(2)}_{t,m}(v)(v'_{<v})$ is strictly increasing in each element of $v'_{<v}$.

At the beginning of $t+1$, define the dynamic programming problem starting with $m-j+1$ products and incumbents $I_t = \{v, v'_{<v}\}$ as $\Psi^1_{t+1}(m-j+1)(v, v'_{<v})$, and the problem starting with $m-j$ products and incumbents $I_t = \{v, v'_{<v}\}$ as $\Psi^2_{t+1}(m-j)(v, v'_{<v})$. Note that the seller starts with the same set of incumbents but one more unit in problem Ψ^1_{t+1}. Call this situation C_1. First of all, consider a period $t+1$ state s (if exists) in which v gets assigned in Ψ^1_{t+1}. According to Corollary 1 and Lemm 6, v may or may not get assigned at state s in problem Ψ^2_{t+1}. If he does, then the surplus difference between the two problems is independent of v; but if he does not, he must be the last agent who gets assigned in Ψ^1_{t+1}. Without losing generality, let k units be sold out at state s and rank the winning agents in Ψ^1_{t+1} as $y_1 > ... > y_k = v$. Then it must be that:

$$\begin{aligned}y_1 > ... > y_{k-1} > y_k &= v \geq r^k_{t,m-j+1} = r^{k-1}_{t,m-j} \\ y_{k+1} &< v < r^{k+1}_{t,m-j+1} = y^k_{t,m-j}\end{aligned}$$

Thus in Ψ^2_{t+1}, $\{y_1, ..., y_{k-1}\}$ get assigned whereas $\{v, y_{k+1}, ...\}$ do not, and the realized surplus difference becomes:

$$\mathbb{E}_{t+1}[\tilde{W}^{m-j+1}_{t+1} - \tilde{W}^{m-j}_{t+1}|s] \quad = \quad \mathbb{E}_{t+1}[v - \delta W^{m-j-k+1}_{t+2}(v, y_{k+1}, ...) + \delta W^{m-j-k+1}_{t+2}(y_{k+1}, ...)|s]$$

which is strictly increasing in v by Lemma 4 (iii).

Now consider a period $t+1$ state s in which v fails to get assigned in Ψ^1_{t+1}. Without losing generality, let k out of $m-j+1$ units be assigned in Ψ^1_{t+1} to $y_1 > ... > y_k$, then it must be that:

$$\begin{aligned} y_1 > ... > y_{k-1} > y_k \geq r^k_{t,m-j+1} = r^{k-1}_{t,m-j} \\ y_{k+1} < r^{k+1}_{t,m-j+1} = r^k_{t,m-j} \end{aligned}$$

Thus in Ψ^2_{t+1}, $\{y_1, ..., y_{k-1}\}$ get assigned whereas $\{y_{k+1}, y_{k+2}, ...\}$ do not. y_k may or may not get assigned, and depending on his status at the end of $t+1$ there are two relevant situations:

1. If y_k gets assigned in Ψ^2_{t+1}, then the continuation problem becomes $\Psi^2_{t+2}(m-j-k)(y_{k+1}, ..., v, ...)$. Compared with $\Psi^1_{t+2}(m-j-k+1)(y_{k+1}, ..., y, ...)$, it starts with one unit of product less but the same set of incumbent in period $t+2$. In other words, we are back to C_1.

2. If y_k is excluded from time $t+1$ sales in Ψ^2_{t+1}, then the continuation problem becomes $\Psi^2_{t+2}(m-j-k+1)(y_k, y_{k+1}, ..., v, ...)$. Compared with $\Psi^1_{t+2}(m-j-k+1)(y_{k+1}, ..., v, ...)$, it starts with the same number of units but one more high value incumbent y_k. Call this situation C_2.

   Since the seller starts with the same inventory size in period $t+2$, he should use the same cutoff rule in both problems. If v gets assigned in period $t+2$, problem Ψ^1_{t+2}, then he may or may not get assigned in period $t+2$ in Ψ^2_{t+2} because the second problem involves one more high value agent y_{k+1}, but in any event we can conclude using the argument at the very beginning of this proof. On the other hand, if v fails to get assigned in period $t+2$, problem Ψ^1_{t+2}, then he must be excluded from period $t+2$ sales in Ψ^2_{t+2} as well. Without losing generality rank the agents who get assigned in Ψ^2_{t+2} as $\theta_1 > ... > \theta_q = y_k > \theta_{q+1} > ... > \theta_n$.[6] Then:

$$\begin{aligned} \theta_1 > ... > \theta_{q-1} > y_k \geq r^q_{t,m-j-k+1} \\ \theta_{q+1} \geq r^{q+1}_{t,m-j-k+1} > r^q_{t,m-j-k+1} \\ ... \\ \theta_n \geq r^n_{t,m-j-k+1} > r^{n-1}_{t,m-j-k+1} \\ \theta_{n+2} < \theta_{n+1} < r^{n+1}_{t,m-j-k+1} \end{aligned}$$

   Thus, in Ψ^1_{t+2}, $\{\theta_1, ..., \theta_n\}$ must get assigned whereas $\{\theta_{n+2}, ...\}$ do not. θ_{n+1} may or may not get assigned: if he does we are back to C_2, and if he doesn't we are back to C_1.

[6] If y_k loses at $t+2$, then the set of buyers who get assigned at $t+2$ are identical in both problems and we remain in C_2.

This algorithm has to stop in finite times, and whenever it stops the realized surplus difference is weakly increasing in v. Furthermore, since the states in which the surplus difference is strictly increasing in v is of a strictly positive measure, we conclude that $b^{j,1}_{t,m}(v, v'_{<v})$ is strictly increasing in v.

Proof for Theorem 3

In period $t \leq T-1$ let $y_1 > ... > y_j > r^j_{t,m}$ and $y_{j+1} < r^{j+1}_{t,m}$. First, consider a pivotal bidder's incentive to play one shot deviation and focus on the problem of y_1 for illustrational purpose. If everyone follows the prescribed strategy, the top j bidders will win the jth auction and pay y_{j+1}'s bid for the jth unit at the end of t. Suppose $y_1 \geq r^{j+1}_{t,m}$ and $y_{j+1} \geq r^j_{t,m}$ so that y_1's equilibrium payoff can be written as:

$$\begin{aligned}
\Pi(y_1) &= y_1 - (y_{j+1} - \delta[W^{m-j}_{t+1}(y_{\geq j+1}) - W^{m-j}_{t+1}(y_{\geq j+2})]) \\
&> y_1 - \delta[W^{m-j}_{t+1}(y_{\geq j+1}) - W^{m-j-1}_{t+1}(y_{\geq j+2}) - (W^{m-j}_{t+1}(y_{\geq j+1}) - W^{m-j}_{t+1}(y_{\geq j+2}))] \\
&= y_1 - \delta[W^{m-j}_{t+1}(y_{\geq j+2}) - W^{m-j-1}_{t+1}(y_{\geq j+2})]
\end{aligned}$$

By losing in the current period, y_1 could participate in future auctions instead and change his payoffs accordingly. Since the off-equilibrium strategy from period $t+1$ onward is independent of bidder's behavior at time t, y_1's one shot deviation has no impact on other bidders' strategies and beliefs in all subsequent auctions. By assumption that $y_1 \geq r^{j+1}_{t,m}$,

$$\begin{aligned}
\Pi(y_1) &> y_1 - \delta[W^{m-j}_{t+1}(y_{\geq j+2}) - W^{m-j-1}_{t+1}(y_{\geq j+2})] \\
&\geq \delta[W^{m-j}_{t+1}(y_1, y_{\geq j+2}) - W^{m-j-1}_{t+1}(y_{\geq j+2})] - \delta[W^{m-j}_{t+1}(y_{\geq j+2}) - W^{m-j-1}_{t+1}(y_{\geq j+2})] \\
&= \delta[\underbrace{W^{m-j}_{t+1}(y_1, y_{\geq j+2})}_{(1)} - \underbrace{W^{m-j}_{t+1}(y_{\geq j+2})}_{(2)}] \qquad (3.2)
\end{aligned}$$

We claim that (3.2) is y_1's expected payoff from future auctions. By assumption that $y_{j+1} \geq r^j_{t,m}$, y_{j+1} will win the object instead and leaves the seller with $m-j$ units for sales from period $t+1$ onward. As a result, (1) is the maximum expected surplus evaluated at the beginning of period $t+1$ if y_1 bids according to the prescribed strategy in all subsequent auctions, and let s be a future state in which he wins. Furthermore, let k_s be the number of remaining units at the beginning of state s, q_s out of which are sold to the top q_s buyers, including y_1. Then,

$$\begin{aligned}
y^{q_s}_s &> r^{q_s}_{s,k_s} \\
y^{q_s+1}_s &< r^{q_s+1}_{s,k_s}
\end{aligned}$$

If $y_s^{q_s+1} \geq r_{s,k_s}^{q_s}$, then q_s units are sold in the absence of y_1, and

$$\begin{aligned}(1)-(2)|s &= \delta^{s-t+1}[y_1 + \delta W_{s+1}^{k_s-q_s}(y_s^{\geq q_s+1}) - y_s^{q_s+1} - \delta W_{s+1}^{k_s-q_s}(y_s^{\geq q_s+2})] \\ &= \delta^{s-t+1}[y_1 - b_{s,k_s}^{q_s,2}(y_s^{q_s+1})(y_s^{\geq q_s+2})]\end{aligned}$$

Similarly, if $y_s^{q_s+1} < r_{s,k_s}^{q_s}$, $q_s - 1$ units are sold in the absence of y_1, and

$$\begin{aligned}(1)-(2)|s &= \delta^{s-t+1}[y_1 + \delta W_{s+1}^{k_s-q_s}(y_s^{\geq q_s+1}) - \delta W_{s+1}^{k_s-q_s+1}(y_s^{\geq q_s+1})] \\ &= \delta^{s-t+1}[y_1 - b_{s,k_s}^{q_s,1}(y_s^{q_s+1})(y_s^{\geq q_s+2})]\end{aligned}$$

Therefore, $(3.2) = \mathbb{E}_t[\sum_s \delta^{s-t}(y_1 - b_{s,k_s}^{q_s}(y_s^{q_s+1})(y_s^{\geq q_s+2}))]$, which is exactly the expected payoff y_1 could achieve he bids according to the prescribed strategy in all subsequent auctions. Thus, y_1 prefers to win the auction now than in future. Use this type of argument for any realization of y_1 and y_{j+1}, we conclude that any pivotal bidder in period t has no incentive to play one shot deviation and lose the current auction.

Now consider any non-pivotal bidder's incentive to play one-shot deviation and overbid in the current auction, and without losing generality focus on y_{j+1}'s problem. According to the analysis above, if y_{j+1} uses the equilibrium strategy, he loses in the current period and earns

$$\Pi(y_{j+1}) = \delta[W_{t+1}^{m-j}(y_{j+1}, y_{\geq j+2}) - W_{t+1}^{m-j}(y_{\geq j+2})]$$

from the subsequent auctions. Furthermore, assume that $y_j < r_{t,m}^{j+1}$ so that only $\{y_1, ..., y_{j-1}, y_{j+1}\}$ get assigned if y_{j+1} overbids and wins in period t. Since $y_j \geq r_{t,m}^{j}$, y_{j+1}'s payoff from overbidding becomes:

$$\begin{aligned}y_{j+1} - b_{t,m}^{j,2}(y_j)(y_{\geq j+2}) &= y_{j+1} - [y_j + \delta W_{t+1}^{m-j}(y_{\geq j+2}) - \delta W_{t+1}^{m-j}(y_j, y_{\geq j+2})] \\ &= -(y_j - y_{j+1}) + \delta[W_{t+1}^{m-j}(y_j, y_{\geq j+2}) - W_{t+1}^{m-j}(y_{j+1}, y_{\geq j+2})] + \Pi(y_{j+1}) \\ &< \Pi(y_{j+1})\end{aligned}$$

where the last inequality follows Lemma 4 (*iii*). Thus y_{j+1} has no incentive to overbid in order to win the current auction. Moreover, this conclusion is robust to the realization of y_j and applies to any non-pivotal entrant in period t.

Finally, for any off-equilibrium path, let each bidder believe that all the other bidders use the equilibrium strategy and repeat the proof above.