Search and Adverse Selection*

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Abstract

This paper explores a dynamic model of adverse selection in which trading partners receive noisy information. A monopolistic buyer wants to procure service. Seller's cost depend on the buyer's type. The buyer contacts sellers sequentially and enters into a bilateral bargaining game. Each seller observes the buyer's offer. In addition, each seller observes a noisy signal. Contacting sellers (search) is costly. We characterize equilibrium when search cost become small. In the limit, the price will depend in a simple way on the curvature of the signal distribution. If signals are sufficiently strong, the limit outcome is equivalent to the full information outcome. (The equilibrium is separating and prices are equal to the true cost.) If signals are weak, the limit outcome is equivalent to an outcome with no information. (The equilibrium is pooling and prices are equal to ex ante expected cost.)

Away from the limit, a dynamic model of adverse selection with noisy information has several natural implications for the correlation between duration, quality, and prices. Most importantly, in many equilibria it will be the "lemons" that stay in the market for a long time, while good types trade fast. This is in accord with stylized facts about the housing or the labor market.

Very preliminary and Incomplete. Appendix not included

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1 Introduction

The paper looks at dynamic model of adverse selection. An agent that we call the buyer samples sequentially alternative trading partners, sellers, for a transaction that involves information asymmetry. The buyer knows his own characteristics (type), while sellers receive signals about it. The cost of the sellers depend on the buyer's characteristics. Signals are imperfect and the buyer has an incentive to search for a seller who received a favorable signal (indicating low cost). Sellers take this behavior into account when interpreting their own information. The main objective of this paper is to understand how the combination of search activity and information asymmetry affects prices and welfare. We identify a particular form of the winner's curse in the search environment which is more severe than the winner's curse in an auction, in a sense to be made precise. We show that this implies that the conditions to achive information aggregation in a search environment are more stringent than an auction.

Our main result concerns a situation in which search cost become small, i.e., we are looking at a limit. When search costs are small, equilibrium price can be characterized completely by the curvature of the tail of the signal distribution. We say that an equilibrium involves complete pooling if good and bad buyers trade at the same prices. An equilibrium is perfectly revealing if buyers get the same price they would get with perfect information. We show that, when search costs are small, equilibrium is perfectly revealing if and only if there are arbitrarily informative signals[1] *and* the tail of the distribution of the signals is sufficiently thick. If arbitrarily informative signals do not exist or if the tail is not thick enough, the limit equilibrium involves complete pooling, i.e., prices paid by the buyer are independent of individual characteristics. In particular, despite the fact that sellers receives potentially quite strong signals about the buyer's type, in equilibrium no information is transmitted, and the outcome is equivalent to the outcome in which there are no signals at all. The reason for this negative result is excessive search of the bad types, diminishing the value of information and excerbating the winner's curse for the seller.

Whether or not information is perfectly revealed implies whether or not equilibrium is efficient. Except for signal distributions that are degenerate or have arbitrarily thin tails, equilibrium with small search cost is efficient if and only if the equilibrium is separating. We also discuss the relation between welfare and information revelation in an extension, where the efficient allocation depends on the personal characteristics of the buyer.

We compare our result to a setting in which a buyer can commit to a procurement auction and we look at the case in which the number of bidders become large. In a procurement auction the limiting outcome never involves total pooling. Furthermore, if arbitrarily informative signals exist,

[1] Signals exist that are so informative that they are arbitrarily close to reveale the state.

the limiting outcome will be perfectly informative as shown by Milgrom (1979) and Wilson (1977).[2] In contrast, in a model with search, the outcome can involve complete pooling even with arbitrarily informative signals. The main difference is that, with an auction, a buyer can commit to sample only a fixed number of sellers and buy from the seller with the lowest bid. In a search model, the buyer cannot commit to sample more sellers (or commit to truthfully report the number of sellers sampled before.)

Beyond auction theory, the relation between the strength of signals and information revelation by limit equilibria is analyzed in models of Herding and Voting. Duggan and Martinelli (2001) find that the existence of arbitrarily informative signals are a necessary and sufficient condition for information aggregation in a model of voting in juries when the size of the jury increases. Smith and Sorenson (2000) show that with social learning, herds on the correct action must occur if signals are arbitrarily informative (and weak conditions that that often suffice.)

Our model is this: A buyer searches sequentially among sellers to obtain a service. The value of the service to the buyer is commonly known. The buyer incurs a cost $s > 0$ ("search cost") to sample a seller. A seller's cost of providing the service, c_w, is the same for all sellers and it depends on an underlying state $w \in \{L, H\}$ with $c_H > c_L$. The state w is known to the buyer but not to the sellers. We shall call w also the *type* of the buyer.

At the beginning of every sampling round, the buyer draws one seller at a cost s. The seller receives a signal that is correlated with the state. The signal is jointly observed by the buyer and the seller. Then, the buyer and the seller bargain over the terms of trade, to be described below. If they reach an agreement and trade, the game is over. If they do not trade and if the buyer chooses to proceed, the next round starts according to the same rule and the buyer samples another seller at cost s.

The bargaining process that takes place after a seller is sampled by the buyer is a critical part of the model. Due to the information asymmetry, we cannot use the simple surplus sharing solutions that are common in the search literature with symmetric information. A simple surplus sharing rule is characterized by a number $\beta \in [0, 1]$ such that the buyer receives a share β of the surplus. With complete information, a surplus sharing rule is equivalent to a game in which, with probability β, the buyer has all bargaining power and makes a take-it-or-leave-it price offer to the seller (and with probability $(1 - \beta)$ the seller makes such an offer).

We extend this simple game to a setting with asymmetric information and interdependent valuations. We assume that the buyer has all the bargaining power and offers a *mechanism* which

[2] Note that we are looking at a model in which individual characteristics matter; Pesendorfer and Swinkels (2000) show that information aggregation is possible under weaker conditions for characteristics common to many buyers (e.g., the common value of stocks).

the seller can either accept or reject. If the mechanism is accepted, an allocation (trading probability and price) is implemented, depending on the reported type of the buyer. Thus, we model bargaining as a principal-agent problem, with the buyer being the *informed principal* as in Myerson (1983), proposing a trading mechanism to the seller (agent). Since the principal has information that affects the preferences of the agent, the mechanism proposal game is a signaling game. In general, the game suffers from large multiplicity of equilibria due to the freedom of specifying beliefs off the path. We therefore employ a number of refinements. Given these refinements, we characterize the set of equilibrium mechanisms. The identified mechanisms are interim efficient and the full surplus of trade is extracted by the buyers.[3] We show that the mechanism can be implemented as the outcome of a price proposal game between the buyer and the seller. Modelling the interaction as a mechanism proposal game has the advantage that we can concentrate ourself on pure strategy equilibria (thanks to the inscrutability principle). In a price proposal game, the price offer and acceptance strategies are generally mixed.

While the buyer can commit for the current period, he cannot commit not to trade in future periods. The buyer can also not provide evidence about the number of sellers already sampled, let alone provide evidence about their signals. (The buyer would have an incentive to commit to sample only a finite number of sellers and/or the buyer would like to truthfully communicate the number of sampled sellers, provided he has sampled only a few.) Equilibrium would be more efficient if the buyer could fully commit.

Our main result concerns the limit of the equilibrium outcomes when s becomes small. Let F_w denote the distribution of the sellers beliefs in state w, conditional on their signal. We show that in the limit of every equilibrium the two types of buyers will trade at a price equal to the true costs c_w if and only if the appropriately defined tail of F_w is thick enough.

More formally, we show that an appropriately transformed tail of the signal distribution can be approximated by an exponential distribtion function. Concentrating on the tail of the signal distribution for a low cost buyer, F_L, the parameter $\lambda \in [0, 1]$ of the approximating exponential distribution directly determines the equilibrium price. If $\lambda = 0$, (if the tail is thick), the low cost buyer will trade a price equal to cost c_L; If $\lambda \in \left(0, \frac{1}{2}\right)$, the limit price will be between c_L and ex ante expected cost; the limit price is strictly increasing in λ. If $\lambda \geq \frac{1}{2}$, the limit price is equal to prior expected cost.

We analyze the relation between information aggregation and welfare. In our base model, welfare is only affected by the accumulated search costs (buyers will purchase the good in every

[3]By allocating bargaining power randomly and allowing a seller to be the proposer of a mechanism with some probability as well, we could capture situations with intermediate degrees of bargaining power as well.

equilibrium and price are welfare irrelevant transfers). We show that accumulated search cost become zero in the limit, if the limit is separating (because then almost no bad buyer searches). However, accumulated search cost stay positive in all (partial) pooling equilibria, except when the tail of the signal distribution is arbitrarily thin.

In an extension (not included in the current submission), we consider buyers with heterogeneous willingness to pay. In the efficient allocation, buyers with a low willingness to pay should receive service if and only if their type is good (only then is the cost of the service smaller than the valuation by the buyer). Whether or not the limit allocation is efficient depends on whether sellers can distinguish the types of the buyer. If the limit involves complete pooling (if signals are weak), the outcome is efficient.

Importantly, in this extension, smaller search cost can have a negative impact on welfare. With smaller search cost, bad buyers engage in more search and separation is harder to achieve. This is contrast with standart models of search in which smaller search cost increase welfare (directly) by increasing the match quality and (indirectly) by reducing the negative impact local market power.

In another extension (not included in the current submission), we consider a more structured search: a buyer first samples a small set of friends, before sampling strangers. Friends and strangers make different inference about the type of the buyer upon encounter. It takes stronger signal for a stranger to be willing to trade with a buyer than for a friend, i.e., strangers are more distrusting. We compare this result to the situation of an entrepreneur of a start-up company looking for an early investor. Convincing a friend (a member of an extended social network) to invest into a project seems much easier than convincing a stranger who is not socially connected to the entrepreneur.

We analyse limit equilibria for tractability. When search costs are not small, we cannot rule out multiplicity of equilibrium. For example, when good buyer sample more, sellers become more optimistic, making search more valuable. We discuss this in a separate section. We also illustrate the use of our refinements (for the principal agent game) in two lemmas following the main result. We show that we can get separating equilibria even without arbitrarily informative signals; however, such equilibria will involve (Pareto) dominated trading mechanisms. We also show that we can get pooling equilibria even if signals have a thick tail; however, such equilibria are supported by beliefs that fail devinity.

We discuss a numer of potential extensions. Most prominently, one can assume that the seller's signal is not observed by the buyer. As another extension, the buyer would learn his own type from either the signals or the rejection decisions by sellers (if the buyer does not observe sellers' signals.)

2 The Model

A buyer searches sequentially among sellers to obtain a service. To have a story in mind, one may think of a procurement scenario in which the buyer is seeking to fix a problem (repair or cure) and samples service providers sequentially to obtain bids[4]. The value of the service is commonly known and denoted by u. The buyer incurs a cost $s > 0$ ("search cost") to sample a seller and engage in bargaining. A seller's cost of providing the service, c_w, is the same for all sellers and it depends on an underlying state $w \in \{L, H\}$ with $c_H > c_L$. The prior probabilities of L and H are g_L and g_H respectively. The state w is known to the buyer but not to the sellers. We shall call w also the *type* of the buyer. The value of the service u is sufficiently larger than $c_H + s$ (the cost of the service and the cost of finding a seller) so that both types of the buyer would like to participate. We also assume that $s > c_H - c_L$, otherwise search never pays.

At the beginning of every sampling round, the buyer draws one seller at a cost s. The seller receives a signal $x \in [a, b] \subset [0, 1]$ that is correlated with the state. The distribution of x given w is F_w. We assume that F_w is atomless and F_w satisfies the montone likelihood ratio property and a low signal is indicative of the low state. The buyer observes the signal of the seller. Then, the buyer offers a direct mechanism M to the seller. The seller can either accept or reject the mechanism. If the mechanism is accepted, the buyer reports his type and the mechanism implements the prescribed allocation. If the mechanism is accepted and if trade happens, the game stops. If either the mechanism is rejected or if the mechanism prescribes no trade, the buyer can choose to stop the game.[5] If the buyer chooses to proceed, the next round starts according to the same rule and the buyer samples another seller at cost s.

If the buyer transacts at a price p after having sampled n sellers, his payoff is $u - p - ns$. The payoff of the seller who agreed to the transaction is $p - c_w$. The payoff of all other sellers is zero. The realized surplus is $u - c_w - ns$.

A collection of strategies - the mechanism offer M, acceptance decision A, and reporting decision R - and beliefs β of the seller is called a constellation σ. A direct mechanism M is a vector $[p_L, q_L, p_H, q_H]$, where p_w, q_w are the trading price and the trading probability conditional on a

[4] Alternatively, one may reverse the roles of what we call buyer and sellers to obtain an even more standard story of sale of an object of uncertain quality w.

[5] A seller always accepts a price above c_H. Since $u > c_H + s$, this implies that it is always worthwhile to continue and the buyer will never stop sampling. We therefore do not include a stopping decision in the formal analysis.

report $R = w$.[6] A typical mechanism will be

$$M = \left[c_L, q^C, c_H, 1\right],$$

which implies the following: If the seller accepts M and if the buyer reports a type H , then trade happens at a price equal to c_H with probability one; if the buyer reports a type L, then trade happens at a price equal to c_L with probability $q^C \leq 1$. In general, $M(x, w)$ describes the mechanism offered by a buyer w if the seller has a signal x and a mechanism offer strategy is $M(\cdot,\cdot) : \{L, H\} \times X \to \mathbb{R}^4_+$.[7] Given a mechanism offer M, the seller believes that the probability of the high state is $\beta(M, x)$, so beliefs are $\beta(\cdot,\cdot) : X \times \mathbb{R}^4_+ \to [0, 1]$. $A(M, x)$ describes the acceptance decision by a seller of type x, $A(\cdot,\cdot) : \mathbb{R}^4_+ \times X \to [0, 1]$, where A is the acceptance probability. If the mechanism is accepted, $R(w, x, M)$ describes the report conditional on the state w, the signal x, and the offered mechanism M, $R : \{L, H\} \times X \times \mathbb{R}^4_+ \to \{L, H\}$. We will define an equilibrium as a constellation in which strategies are mutually *optimal* and beliefs are *consistent*. By the inscrutability principle, there is no loss of generality in assuming that both buyers offer the same mechanism, the mechanism is accepted, and reports are truthful. Let us define these requirement precisely.

Given a constellation σ describing the behavior of the other players and their beliefs, expected payoffs of the buyer who samples a seller with signal x and who uses strategy M, R is recursively defined. The payoff is the probability of trading with the current seller times the expected profit conditional on trading plus the expected continuation payoff if no trade happens minus the search costs:

$$\begin{aligned} U_w(M, R, x, \sigma) &= A(M(x), x)\, q^{M(x)}_{R(w,x,M(x))} \left(u - p^{M(x)}_{R(w,x,M(x))}\right) \\ &\quad + \left(1 - A(M(x), x)\, q^{M(x)}_{R(w,x,M(x))}\right) \int_x U_w(M, R, x, \sigma) - s \end{aligned}$$

where q^M_R is the trading probability in mechanism M given report R and similarly for p^M_R. Let $V_w(\sigma)$ be the expected payoff of the buyer who uses the strategies prescribed by σ. The payoff of a seller with signal x who accepted an offer M is equal to the expected profit from the contract conditional on the high cost and the low cost buyer, respectively, weighted by the relative probabilities

$$\pi(M, x, \sigma) = \beta(M, x)\, q^M_{R(H,x,M)} \left(p^M_{R(H,x,M)} - c_H\right) + (1 - \beta(M, x))\, q^M_{R(L,x,M)} \left(p^M_{R(L,x,M)} - c_L\right).$$

[6] The price is paid conditional on trading, i.e., the expected transfer given a mechanism M and a report R is $t_R = q_R p_R$. This is without loss of generality relative to specifying transfers if the trading probability is positive whenever transfers are nonzero. This will the case in equilibrium.

[7] We disregard all measurability issues throughout the paper,e.g., we do not restrict the set of mechanisms by requiring $M(\cdot)$ to be measureable.

Let $\theta(\sigma)$ denote the ratio of the number of sellers who are sampled in expectation,

$$\theta(\sigma) = \frac{E[\#\text{ of sellers sampled}|w = L, \sigma]}{E[\#\text{ of sellers sampled}|w = H, \sigma]}.$$

If the low cost buyer L samples many more sellers than the high cost buyer, a seller who is sampled should update towards the low state. Let $\beta_0(x, \theta)$ denote the "interim" belief of a seller with signal x who is sampled by a buyer (β_0 does not condition on the price offer). The belief the seller depends on the relative prior likelihood of the types $\frac{g_L}{g_H}$, the relative likelihood of the the signals, $\frac{dF_L(x)}{dF_H(x)}$ and the relative likelihood of being sampled $\theta(\sigma)$. As shown in the appendix, the interim belief of the seller can be defined as

$$\beta_0(x, \theta) = \frac{1}{1 + \frac{g_L}{g_H} \frac{dF_L(x)}{dF_H(x)} \theta}.$$

(We implicitly assume an infinite number of sellers and the probability of being sampled follows an improper (uniform) prior. We derive β_0 as the limit of a Bayesian update with finitely many sellers when the number of sellers becomes large.) Note that the belief of a sampled seller depends on the equilibrium only through the ratio θ. In equilibrium, both buyers will offer the same mechanism and thus, the beliefs of a seller following an on-equilibrium mechanism offer are just $\beta(x, M) = \beta_0(x, \theta)$. Off-equilibrium beliefs are not restricted.

The acceptance decision by the seller is sequentially rational if he plans to accept mechanisms that lead to strictly positive profits and if he plans to reject mechanisms that lead to negative profits, given his beliefs $\beta(M, x)$, i.e.,

$$A^*(M, x) = \begin{cases} 1 & \text{if} \quad \pi(M, x, \sigma^*) > 0 \\ 0 & \text{if} \quad \pi(M, x, \sigma^*) < 0 \end{cases}.$$

By the inscrutability principle (Myerson, 1983), we can restrict attention to equilibria in which both types of the buyer offer the same, direct mechanism, the mechanism is accepted, and reports are truthful. Every equilibrium outcome of a larger game in which the buyer can offer more complex mechanism is equivalent to an outcome of an equilibrium in which both types of buyers offer the same, direct mechanism that is incentive compatible and individually rational. We will therefore drop the dependency of the offer strategy $M(w, x)$ on w during the analysis.

Note that we define strategies to be history independent, i.e., the buyer can condition his mechanism offer only on his own type and the signal of the seller and the reporting strategy may depend in addition on the offered mechanism. The seller's acceptance strategy depends only on the signal and on the mechanism offer (since sellers do not observe anything else). Our basic equilibrium definition is therefore essentially that of a Markov Perfect equilibrium:

Definition 1 *A constellation σ^* is an inscrutable equilibrium if*

1. $M^*(x,w)$ and R^* are *optimal,* $M^*, R^* \in \arg\max U_w(M,R,x,\sigma)$.
2. β^* is derived from *Bayes Rule* whenever applicable.
3. A^* is *sequentially rational.*
4. M^* is *accepted* and *reporting is truthful*, $A^*(M^*,x) = 1$ and $R^*(w,x,M^*) = w$.
5. The equilibrium is *inscrutable*, $M^*(x,H) = M^*(x,L)$.

As indicate before, we impose refinements. We discuss the implications of these refinements in the discuss section.

Beliefs following an off equilibrium mechanism offer M' satisfy "*Divinity*" if they put (weakly) higher probability on a type of buyer who is strictly better off if M' is accepted (rather than trading at the equilibrium mechansim).[8] With $U_w^*(x)$ denoting the equilibrium payoff, let $U_w(M',x)$ be the payoff to buyer w if the mechanism M' is accepted,

$$U_w\left(M',x\right) = q_R^{M'}\left(u - p_R^{M'}\right) + \left(1 - q_R^{M'}\right)V_w - s,$$

given optimal reporting. Beliefs $\beta(x,M')$ satisfy divinity given σ if

$$\begin{aligned}
\beta\left(x,M'\right) &\geq \beta_0(x,\theta) \text{ if } U_H\left(M',x\right) > U_w^*(x) \\
\beta\left(x,M'\right) &\leq \beta_0(x,\theta) \text{ if } U_L\left(M',x\right) > U_L^*(x)
\end{aligned}$$

Divinity (rather than refinements like D1/D2) is used because it makes the construction of equilibrium easier; for example, assigning the belief $\beta_0(x,\theta)$ off the equilibrium path would ensure that an equilibrium satisfies Divinity. Note that $\beta(x,M') = \beta_0(x,\theta)$ whenever both buyers strictly prefer M' to the equilibrium mechanism.

Divinity (as well as most of the other refinements) for signaling games relies on a single crossing condition on preferences. The condition does hold in our setup if the expected payoff of the low cost buyer is higher than the expected payoff of the high cost buyer. We restrict attention to equilibria in which the payoffs $V_H(\sigma)$ and $V_L(\sigma)$ are ordered in this way. Thus, we rule out a class of pooling equilibria in which both types of buyers trade at the same price.[9]

[8] This is NOT the original definition of Divinity and therefore put into quotation marks.

[9] Intuitively, this restriction makes it harder to find pooling equilibria and thus strengthens the result that separation is unlikely.

Divinity in the current definition implies that equilibrium mechnisms must be *undominated*. We state this as an extra requirement for transparency. A mechanism M is undominated if there is no other mechansim M'' such that both types of the buyer seller strictly prefer M'' to M and seller's expected profits under M' are strictly higher than under M, given the interim belief β_0. (Of course, if both types of buyers strictly prefer a mechanism M'' to M, then divinity requires that the belief of the seller is equal to β_0. If seller's profits are positive, sequential rationality requires him to accept a mechanism. Therefore, divinity implies that equilibrium mechanisms must be undominated.)

Here is the equilibrium definition which we will use most of the time. Whenever we use the term equilibrium without qualification, we mean an undominated, monotone equiilibrium:

Definition 2 A *constellation* σ^* *is an undominated, montone equilibrium if* σ^* *is an unscrutable equilibrium and if*

1. $M^*(x)$ is *undominated* given σ for all x.
2. $\beta^*(x, M)$ satisfies *Divinity given* σ for all x and M.
3. Payoffs are *monotone*, $V_L(\sigma^*) > V_H(\sigma^*)$.

3 Existence and Preliminary Observations

In this section we discuss and show existence of an undominated, monotone equilibrium. We also characterize the set of mechanisms $M(x)$ that satisfy the equilibrium definition for given continuation payoffs $V_w(\sigma)$ and interim beliefs $\beta_0(x, \theta(\sigma))$.

Given a ratio θ, the expected cost of a seller with signal x is

$$E_0[c|x, \theta] = \beta_0(x, \theta) c_H + (1 - \beta_0(x, \theta)) c_L.$$

A subscript zero refers to the evaluation of the expectation at the "interim belief," accounting for the information contained in the signal x and being sampled, but *not* accounting for the information contained in the mechanism offer.

We show that in any equilibrium, the mechanism that is offered in any given buyer-seller pair must maximizes the payoff of the L buyer, subject to feasibility constraints (the mechanism should be weakly profitable for the seller, reporting should be truthful, and the H buyer should not prefer to reveal his type and trade at a price equal to high cost c_H). Furthermore, every equilibrium is equivalent (in terms of expected prices, number of expected searches and payoffs) to an equilibrium that is characterized by three numbers: $\mathbf{x} = \left[x^*, x^{**}, q_L^C\right]$. With $Ec = E_0\left[c|x,\theta\right]$ denoting the interim expected cost of a seller, the mechanism that is offered to a seller with signal x is given by

$$M(x) = \begin{cases} [1, \quad Ec, \quad 1, \quad Ec] & \text{If} \quad x \leq x^* \\ \left[q_L^C, \quad c_L, \quad 1, \quad c_H\right] & \text{if} \quad x \in (x^*, x^{**}) \\ [\,0, \quad c_L, \quad 0, \quad c_H] & \text{if } x \geq x^{**}. \end{cases} \tag{1}$$

Thus, if the signal is low, $x \in [0, x^*]$, both buyers trade at the same price equal to the interim expected cost. If the signal is high, $x \geq x^{**}$, the trading probability is zero. The trading probability is positive if the signal is intermediate, $x \in (x^*, x^{**})$, but, of course, the trading probability at the lower price c_L cannot be one; otherwise, the mechanism would not be incentive compatible. Instead, q_L^C will make the H buyer just indifferent between trading at c_H with probability one and trading at c_L with probability q_L^C,

$$q_L^C(V_H, V_L, \theta) = \begin{cases} \frac{u - c_H - V_H}{u - c_L - V_H} & \text{if} \quad u - c_H - V_H > 0 \\ 0 & \text{if } u - c_H - V_H \leq 0 \end{cases}$$

The trading probability at c_L in the intermediate region (x^*, x^{**}) is positive only if $u - c_H - V_H > 0$; otherwise, it is not possible to make the H buyer indifferent. The cutoff x^* is always strictly positive while x^{**} can be one. The cutoff x^* corresponds to a signal such that the L buyer is indifferent between trading at a price equal to the expected cost of the seller and trading at the price c_L with probability q^C:

$$x^*(V_H, V_L, \theta) : u - E_0\left[c|x^*, \theta\right] = q_L^C\left(u - c_L\right) + \left(1 - q_L^C\right) V_L,$$

The cutoff x^{**} can be anything in $[x^*, 1]$.

The next lemma states that every equilibrium is equivalent to one in which the mechanism is as described before:

Lemma 1 *Given any equilibrium σ^* with payoffs $V_H(\sigma^*)$ and $V_L(\sigma^*)$, and ratio $\theta(\sigma^*)$. Then there is an equilibrium σ^{**} in which the offered mechanism is described by some $\mathbf{x} = \left[x^*, x^{**}, q_L^C\right]$, with $x^* = x^*(V_H, V_L, \theta)$, $x^{**} \geq x^*$, and $q_L^C = q_L^C(V_H, V_L, \theta)$ such that with $Ec = E_0\left[c|x, \theta\right]$*

$$M^{**}(x) = \begin{cases} [1, \quad Ec, \quad 1, \quad Ec] & \text{If} \quad x < x^* \\ \left[q_L^C, \quad c_L, \quad 1, \quad c_H\right] & \text{if} \quad x \in (x^*, x^{**}) \\ [\,0, \quad c_L, \quad 0, \quad c_H] & \text{if } x > x^{**}. \end{cases}$$

And σ^{**} leads to the same payoffs and ratio as σ^*.

We also show that equilibrium exists:

Theorem 1 *Equilibrium exists.*

4 Price Setting Alternative

Suppose that, instead of offering a mechanism, the buyer offers a price which is either accepted or rejected. A pricing strategy is a cumulative distribution $P_w(\cdot, x)$ on $[c_L, u]$ and an acceptance strategy is a probability of accepting any price, $\alpha(p, x)$. Let $\beta(p, x)$ denote the posterior following a price offer p. A constellation is a collection $[P_w, A, \beta]$ and denoted by σ^P. Given σ^P, we can again define the ratio of the expected number of searches, $\theta(\sigma^P)$ and expected payoffs $V_w(\sigma^P)$. Let $\beta(x, \theta(\sigma^P))$ be the interim belief of a seller who is sampled with signal x. In an equilibrium constellation, prices and acceptance strategies must be mutually optimal, a price is accepted if

$$p \geq \beta(p, x) c_H + (1 - \beta(p, x)) c_L,$$

and beliefs are consistent.

We now define a canonical equilibrium. Recall that in a typical model of search, bargaining is specified by assuming that one side receives all the rents from a match or that the rents are shared between the buyer and the seller according to some parameter. We cannot extend this immediately to our model, because there are two types of buyer who have conflicting interests. A constellation is called a canonical equilibrium if, within each match, the rent is fully extracted by the L buyer, subject to incentive compatibility and individual rationality constraints.

A constellation σ^P is a canonical equilibrium if, given continuation payoffs $V_w(\sigma^P)$ and interim beliefs $\beta(x, \theta(\sigma))$, the equilibrium of the pricing game within each pair is the best equilibrium for the L buyer as follows: An equilibrium of the pricing game consists of distributions P_w, acceptance decision $A(p)$, and beliefs $\beta(p)$ such that

$$\begin{aligned} p &\in suppP_w(x) \quad \text{if: } A(p)(u - p - V_w) = \max_p A(p)(u - p - V_w) \\ A(p) &= 1 \text{ if } \beta(p) c_H + (1 - \beta(p)) c_L < p \\ A(p) &= 0 \text{ if } \beta(p) c_H + (1 - \beta(p)) c_L > p \\ \beta(p) &= \frac{\beta(x, \theta) dF_H}{\beta(x, \theta) dF_H + (1 - \beta(x, \theta)) dF_L} \end{aligned}$$

An equilibrium P_w^*, A^*, β^* is the best equilibrium for the L buyer if there is no other equilibrium P_w', A', β' such that for any $p^* \in suppP_L^*(x)$ and $p' \in suppP_L'(x)$ the L buyer would be strictly better off with P_w', A', β',

$$A'\left(p'\right)\left(u-p\prime-V_L\right) > A^*\left(p^*\right)\left(u-p^*-V_L\right).$$

A constellation σ^P is a canonical equilibrium if for all x, $P_w(\cdot,x)$, $A(\cdot,x)$, $\beta(x,\theta)$ is the best equilibrium for the L buyer as defined above.

5 Main Result

The question is to what extent is information revealed in equilibrium when s is small. The extent of revelation is captured here by the price paid by the L buyer when s is small. If the price that the L buyer pays is close to c_L and (therefore) the price that the H buyer pays is close to c_H, revelation is maximal. Recall that the literature on auctions considered a related question. It inquired to what extent the equilibrium price in a common values auction reflects the correct information when the number of bidders is made arbitrarily large (Wilson(1977) and Milgrom(1979)). Milgrom's result translated to an auction version of our model is that the price approaches the true value iff $\lim_{x\to a}\frac{f_L(x)}{f_H(x)}=\infty$. That is, when there are signals that are exceedingly more likely when the true state is L than when it is H. In our model the number of bidders is endogenous. The counterpart of increasing the number of bidders in our model is reduction of the sampling cost s. The following proposition claims that in our model revelation requires even stronger requirements on the quality of the signals.

The buyers care only about the distribution of posteriors, not about the distribution of signals per se. Without loss of generality, signals are therefore normalized such that the posterior probability of the high state is equal to x, i.e., for all x,

$$x=\frac{\frac{1}{2}f_H(x)}{\frac{1}{2}f_H(x)+\frac{1}{2}f_L(x)}.$$

Rewriting shows that this requires $\frac{f_H}{f_L}=\frac{x}{1-x}$. Therefore, the distribution $F_L(\cdot)$ determines $F_H(\cdot)$ and we can concentrate on characterizing $F_L(\cdot)$, the distribution of signals from the viewpoint of the $L-$Buyer. In addition, sellers' posteriors can be expressed very as a function of the signal and the relative number of searches as shown below.

Signals $x>0$ are not revealing. If there are no revealing signals for the low state, the limit with $s\to 0$ involves complete pooling. Let $E_L[p|\sigma_k]$ be the expected price paid by the L buyer in expectation in an equilibrium σ_k, given s_k. We say that the limit of a sequence of equilibria σ_k^* involves complete pooling if $E_L[p|\sigma_k^*] \to g_H c_H + g_L c_L$.

Theorem 2 *Suppose the suppport of F_L is $[a,b] \subset [0,1]$. If $a > 0$, then the limit involves complete pooling at the ex ante expected price, i.e.,$E_L[p|\sigma_k^*] \to g_H c_H + g_L c_L$.*

Proof: Take a sequence of constellations σ_k for $s_k \to 0$. Let $V_{kw} = V_w(\sigma_k)$ and $\Delta_k = u - c_H - V_{kH}$. In general, a subscript k denotes parameters of the constellation σ_k (like x_k, θ_k, etc.). We distinguish three cases according to whether or not Δ_k is positive, zero, or negative when k is large (if the sign of Δ_k does not converge, the analyis is for an arbitrary convergent subsequence which is sufficient for the conclusion). We will only consider the first case, $\Delta_k < 0$, here. The other cases are appendicized. For k large enough, in equilibrium $\Delta_k < 0$.

Case 1: $\Delta_k < 0$ for all k large enough. Then $x_k^{**} = x_k^*$ and both buyers search for a seller with a signal $x \leq x_k^*$. The ratio of the number of searches is

$$\theta_k = \frac{F_H(x_k^*)}{F_L(x_k^*)}.$$

The cutoff x_k^* is determined by indifference of the L buyer between trading at the expected cost of a seller with this signal and continuing search

$$x_k^* : E_0[c|x_k^*, \theta_k] - \int_a^{x_k^*} E_0[c|x, \theta_k] \frac{dF_L(x)}{F_L(x_k^*)} = \frac{s_k}{F_L(x_k^*)}.$$

The cutoff x_k^* must converge to the lower bound of the support, a. Otherwise, search cost on the right hand side converge to zero, while the expected saving from search on the left hand side would be positive: Since x_k^* is bounded away from a, the ratio θ_k is bounded away from the extremes, 0 and ∞. Hence, sellers with different signals will offer different price.

Let $x_k^* \to a$. Then the ratio becomes equal to the inverse likelihood ratio,

$$\begin{aligned} \lim \theta_k &= \lim \frac{F_H(x_k^*)}{F_L(x_k^*)} \\ &= \lim \frac{f_H(x_k^*)}{f_L(x_k^*)} = \frac{a}{1-a}. \end{aligned}$$

The expected price at which the L buyer is trading is

$$\begin{aligned}
\lim E_L\left[p|\sigma_k^*\right] &= c_L + \lim \int_a^{x_k^*} \frac{1}{1+\frac{1-x}{x}\frac{g_L}{g_H}\theta_k}\left(c_H - c_L\right)\frac{dF_L\left(x\right)}{F_L\left(x_k^*\right)} \\
&= c_L + \frac{1}{1+\frac{1-a}{a}\frac{g_L}{g_H}\frac{a}{1-a}}\left(c_H - c_L\right) = g_L c_L + g_H c_H = E_0 c.
\end{aligned}$$

Hence, if $\Delta_k < 0$ for all k, the limit involves complete pooling. QED

The intuition is this: When search cost are small, both buyers search for sellers with the most favorable signals close to the lower bound a. The resulting ratio of the number of searches is

$$\lim \theta_k = \frac{a}{1-a}.$$

Of course, this is just the inverse likelihood ratio of the signals

$$\lim \frac{f_L\left(x_k^*\right)}{f_H\left(x_k^*\right)} = \frac{1-a}{a}.$$

Intuitively, if a high cost buyer is less likely to generate a signal close to the lower bound a, high cost buyers are searching even more. Hence, the informational content of the signals at the lower bound is just balanced by the informational content of being sampled.

We ask now whether the limit will be separating with a continuous signal distribution if its support includes zero, ie., if signals can be arbitrarily close to zero and therefore, signals can be arbitrarily informative. As noted before, in auctions it has been shown that the existence of such signals is sufficient for revelation of the state in the limit. As we will now see, this is not the case with search. The limit does not need to involve information revelation. Indeed, we will see that even with arbitrarily informative signals the limit can involve complete pooling. Thus, in a search model, the outcome can be very uninformative even in the presence of almost perfect information.

The intuition is this: If the limit is separating, the L buyer trades at c_L while the H buyer trades at c_H. It can be shown that the accumulated search cost of the L buyer must become zero. Hence, the L buyer must be able to find prices close to c_L at almost no cost. However, the search cost for the H buyer must be strictly positive. As we have seen before, this is not possible if the support of the signal distribution is bounded away from zero. Our main result shows that something similar happens when the support of the signal distribution is too thin near zero.

We will first look at equilibria in which the surplus of the H buyer is non-positive, $\Delta_k \leq 0$. In such equilibria, the cutoff x_k^* must converge to zero. This is intuitive: The L buyer can otherwise search for signals x close to zero, ensuring trade at a price close to c_L at almost no cost.

The incentive of the L buyer will depend strongly on the shape of the conditional distribution $\frac{dF_L(x)}{F_L(x_k^*)}$ on the left tail $(0, x_k^*)$. If this conditional distribution has a "thick" tail and puts a high mass on signals strictly below x_k^*, search will be more valuable (because the average seller will offer a strictly better deal) than in the case of a "thin" tail, when the conditional distribtion puts high mass on signals very close to x_k^* itself (because the average seller $x \leq x_k^*$ will offer almost the same deal as x_k^*). We therefore introduce a way of characterising the limiting tail distribution.

Given a constellation σ_k, the cutoff x_k^* is determined by indifference of the L buyer

$$E_0\left[c|x_k^*, \theta_k\right] - E_0\left[c|x \leq x_k^*, \theta_k, L\right] = \frac{s_k}{F_L\left(x_k^*\right)}.$$

The assumption $\Delta_k \leq 0$ implies that the H buyer has a weak incentive to not trade at c_H but rather incure search cost and find some $x \leq x_k^*$,

$$c_H - E_0\left[c|x \leq x_k^*, \theta_k, H\right] - \frac{s_k}{F_H\left(x_k^*\right)} \geq 0.$$

We are interested under which conditions the above inequality holds in the limit.

We can use the indifference condition of the L buyer to substitute s_k out of the expression. Furthermore, to do the substitution, we utilize two algebraic manipulations. First, let C_k be the likelihood ratio $\frac{f_L(x_k^*)}{f_H(x_k^*)}\theta_k$ at the cutoff seller,

$$E_0\left[c|x_k^*, \theta_k\right] = \frac{1}{1 + \frac{dF_L(x^*)}{dF_H(x^*)}\theta} = \frac{1}{1 + C_k}.$$

The price at the cutoff seller converges to prior expected cost if $C_k \to 1$. The price converges to $c_L = 0$ if $C_k \to \infty$. If $C_k \to \bar{C} \in (1, \infty)$, the limit price at the cutoff seller is in between. Second, we will do a change of variables. For each x_k^*, we map the interval $(0, x_k^*]$ into $[0, \infty)$ via the continuous transformation

$$t\left(x, x_k^*\right) = \frac{x_k^* - x}{x x_k^*}$$

which defined as the solution to $x = \frac{x_k^*}{1 + x_k^* t\left(x, x_k^*\right)}$. So, $t\left(x^*, x^*\right) = 0$ and $\lim_{x \to 0} t\left(x, x^*\right) = \infty$. We restrict attention to a class of exponential functions with parameter λ, $F_L\left(x\right) = e^{-\lambda \frac{1}{x} + \lambda}$. This class has the property that the induced distribtution of the variable t is independent of the cutoff x_k^*,

$$F_L^{x_k^*}\left(t\right) = 1 - \frac{e^{-\lambda\left(\frac{x^*}{1+x^*t}\right)^{-1} + \lambda}}{e^{-\lambda \frac{1}{x^*} + \lambda}} = 1 - e^{-\lambda t}\ .$$

Now, we substitute out and pass the limit into the integral:

$$
\begin{aligned}
\lim c_H - E_0\left[c|x \leq x_k^*, \theta_k, H\right] - \frac{s_k}{F_H\left(x_k^*\right)} &\geq 0 \\
\Leftrightarrow \lim \int_0^{x_k^*} \left(\left(1 - \frac{1}{1+\frac{(1-x)}{x}\theta_k}\right)\frac{x}{1-x} - \frac{1}{1+\frac{(1-x_k^*)}{x_k^*}\theta_k} + \frac{1}{1+\frac{(1-x)}{x}\theta_k}\right) \frac{dF_L(x)}{F_H\left(x_k^*\right)} &\geq 0 \\
\Leftrightarrow \lim \frac{x^* F_L\left(x_k^*\right)}{F_H\left(x_k^*\right)} \int_0^\infty \left(\frac{C_k}{1 - x_k^* + \left(1 + x^*\left(t-1\right)\right)C_k} - \frac{tC_k}{\left(1+C_k\right)\left(1 - x_k^* + \left(1 + x^*\left(t-1\right)\right)C_k\right)}\right)\lambda e^{-\lambda t} dt &\geq 0 \\
\Leftrightarrow \frac{1}{\left(1+\bar{C}\right)^2} \int_0^\infty \left(1 + \bar{C} - t\right)\lambda e^{-\lambda t} dt &\geq 0 \\
\Leftrightarrow \frac{1}{\left(1+\bar{C}\right)^2} \left(1 + \bar{C} - \frac{1}{\lambda}\right) &\geq 0
\end{aligned}
$$

The H buyer has an incentive to search if and only if the above inequality holds. We can use this to characterize equilibria with $\Delta_k \leq 0$. (To characterize equilibria $\Delta_k > 0$, we need to take care of q_k^C which makes the limit expressions more complicated.) Under which conditions $\Delta_k < 0$ for all k large enough? If $\Delta_k < 0$, the equilibrium must involve complete pooling in the limit: both buyers search and the expected price must be equal to the prior expected price (the search cost of the L buyer converge to zero; hence, the expected price conditional on $x \leq x_k^*$ must be equal to the price at the cutoff type for indifference.) Hence $C_k \to \bar{C} = 1$. Inspecting the limit expression shows that this is the case only if

$$
1 + 1 - \frac{1}{\lambda} \geq 0 \Leftrightarrow \lambda \geq \frac{1}{2}.
$$

Hence, we will get an equilibria with $\Delta_k < 0$ for all k only if $\lambda \geq \frac{1}{2}$.

Now, under which conditions $\Delta_k = 0$ for all k large enough? $\Delta_k = 0$ requires that

$$
1 + \bar{C} - \frac{1}{\lambda} = 0.
$$

Hence, $\Delta_k = 0$ for k large only if the limit price is $\bar{p} = \frac{1}{1+\bar{C}} = \lambda$. And hence, $\Delta_k = 0$ only if $\lambda \leq \frac{1}{2}$ (otherwise, $\bar{p} > \frac{1}{2}$, which contradicts seller's zero profits.)

(In the appendix we show that $\Delta_k > 0$ for all k can be true only if $\lambda = 0$. Of course, $\Delta_k > 0$ implies that the H buyer does not search while the L buyer searches for a seller with a signal close to zero. Hence, the limit is revealing, $C_k \to \infty$ if $\Delta_k = 0$. Conversely, $\lambda = 0$ implies that the distribution F_L has all mass at $x = 0$, hence the limit is trivially revealing.)

The above observations imply a complete characterization of equilibrium. If $\lambda > \frac{1}{2}$, then it must be that $\Delta_k < 0$ for k large enough and hence, the limit price must be $\bar{p} = \frac{1}{2}$. This follows because it cannot be that $\Delta_k < 0$ for k large enough (this requires $\lambda = 0$) and it cannot be that $\Delta_k = 0$ (this would imply $\bar{p} > \frac{1}{2}$, a contradiction). If $\lambda \in \left(0, \frac{1}{2}\right)$, the only possibility is $\Delta_k = 0$ and if $\lambda = 0$, the only possibility is $\bar{p} = 0$. We have chosen an arbitrary converging subsequence of prices p_k (a converging subsequence of C_k). But, since the limit price is independent of the choice of the subsequence, the sequence itself must converge to the same limit.

We can characterize equilibrium prices by λ. Let $E_L\left[p|\sigma_k\right]$ be the expected price paid by the L buyer in expectation in an equilibrium σ_k. We call the limit of a sequence of equilibria σ_k^* revealing if $E_L\left[p|\sigma_k^*\right] \to c_L$. Recall, $c_L = 0$ and $c_H = 1$, and $prob\left\{w = H\right\} = prob\left\{w = L\right\} = \frac{1}{2}$. The theorem is proven as a corollary of the more general Theorem 4.

Theorem 3 *Fix some distribution $F_L(x) = e^{-\lambda\frac{1}{x}+\lambda}$ and some sequence $\{s_k\}$, $s_k \to 0$. Let σ_k^* be a sequence of equilibria given s_k. Then the limit price paid by the L buyer is*

$$\lim E_L\left[p|\sigma_k^*\right] = \begin{cases} 0 & \text{if} \quad \lambda = 0 \\ \lambda & \text{if} \quad \lambda \in \left(0, \frac{1}{2}\right) \\ \frac{1}{2} & \text{if} \quad \lambda \geq \frac{1}{2}. \end{cases}$$

Thus, revelation in the search model with small s requires that there are signals that separate L from H even in a more pronounced way than in the large auction model. When both models the signals that make L exceedingly more likely are needed to counteract the winner's curse. This difference between the strengths of the requirement in the two models owes to the somewhat different form of the winner's curse in these models. As explained before, in the search model the winner's curse is produced both by the larger expected number of sellers who participate in the bidding (like in the auction) and by the worsened distribution that a sampled seller is facing due to the longer search duration of the H type.[10]

We can generalize the theorem to a larger set of distribution function. What we need for our proof technique is to ensure that we can pass the limit into the integral in the inequality. For this, we need that the stretched tail $F_L^{x^*}$ converges to a fixed distribution when x_k^* vanishes to zero. Any sequence $x_k^* \to 0$, defines a sequence of distributions $F_L^{x_k^*}(t)$ of t on $[0, \infty)$. The original tail $\frac{F_L(x)}{F_L\left(x_k^*\right)}$ on $[0, x_k^*]$ is called **regular** if the sequence of corresponding distributions $F_L^{x_k^*}(t)$ on $[0, \infty]$ converges to a limit F_L^*. (The limit F_L^* does not need to be a cumulative distribution function itself.) The set of distributions F_L which are regular is Φ,

[10] [Conclusion, Appendix and Literature to be added.]

$$\Phi = \left\{ \begin{array}{c} F_L(\cdot) : \exists\, F^*(t) \equiv 1 - \lim_{x^* \to 0} \frac{F_L\left(\frac{x^*}{1+x^* t}\right)}{F_L(x^*)} \in [0,1]\,, \ \ \forall t \\ \text{and } F^*(t) \leq 1 - \frac{F_L\left(\frac{x^*}{1+x^* t}\right)}{F_L(x^*)},\ \forall t \in (0,\infty),\ \forall x^*. \end{array} \right\}$$

Note that by Helley's selection theorem, all distribution functions have a pointwise convergent subsequence, so a limit as in line one exists for all distributions. The second line of the definition is a technical condition which is needed for the proof. It ensures that the integral of a linear function converges. A generic example of functions $F_L \in \Phi$ is $F_L(x) = e^{-k\frac{1}{x}+k}$, since $\lambda(F_L) = k$ and

$$F_L^{x_k^*}(t) = 1 - \frac{e^{-k\left(\frac{x^*}{1+x^* t}\right)^{-1}+k}}{e^{-k\frac{1}{x^*}+k}} = 1 - e^{-kt} \quad \forall x_k^*.$$

Every tail $F^*(t)$ for $F \in \Phi$ is exponential: (The appendix contains the proof.)

Lemma 2 *If $F \in \Phi$, then the limit tail $F^*(\cdot)$ is exponential, i.e., for some $\lambda \in [0,\infty]$,*

$$F^*(t) = 1 - e^{-\lambda t}.$$

The lemma is immediate if the limit $F^*(t)$ is constant at 0 or constant at 1. In these cases, $\lambda = 0$ and $\lambda = \infty$, respectively. Many distributions will have such a degenerate limit. If F^* is not constant, then it must have a stationarity property, since it must be independent of the cutoff x^*. This property requires that F^* is exponential. Therefore, we can define a mapping

$$\lambda : \Phi \to [0,\infty]\,,$$

which assigns a hazard rate $\lambda(F_L)$ to each distribution $F_L \in \Phi$.

Theorem 4 ***(Main Result.)*** *Fix some distribution $F_L \in \Phi$ and some sequence $\{s_k\}$, $s_k \to 0$. Let σ_k^* be a sequence of equilibria given s_k. Then the limit price paid by the L buyer is*

$$\lim E_L[p|\sigma_k^*] = \begin{cases} 0 & \textit{if} \quad \lambda(F_L) = 0 \\ \lambda & \textit{if} \quad \lambda(F_L) \in \left(0,\frac{1}{2}\right) \\ \frac{1}{2} & \textit{if} \quad \lambda(F_L) \geq \frac{1}{2}. \end{cases}$$

Distributions which have a positive density at zero are shown to be such that $\lambda(F_L) = 0$. Likewise, distributions for which the density is not falling quickly are such that $\lambda(F_L) = 0$:

Lemma 3 *Let $F_L(\cdot)$ be continuously differentiable. If $f_L(0) > 0$, then $\lambda(F_L) = 0$. More generally, $\lambda(F_L) = 0$ if*

$$\lim_{x \to 0} \frac{f_L(\alpha x)}{f_L(x)} > 0 \text{ for any } \alpha \in (0,1).$$

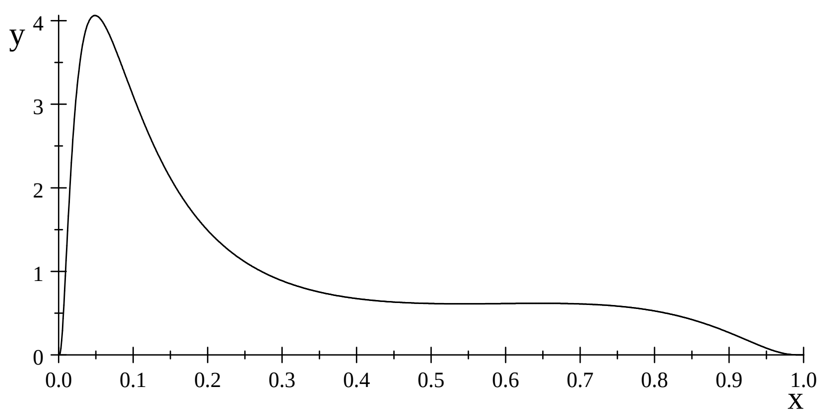
Example: Suppose, signals are generated by two normal distributions, depending on the state:

$$\begin{aligned} f_L(s) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}s^2} \\ f_H(s) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(s-1)^2} \end{aligned}$$

Then, I normalize signals so that they are equal to the posterior, denoted $x \in [0,1]$, $x(s) = \frac{f_L(s)}{f_H(s)+f_L(s)}$. The resulting density of posteriors in the low state is given by

$$f_L(\beta) = \sqrt{2} e \frac{\exp\left(-\frac{1}{2}\left(\ln\left(-\frac{1}{\beta}(\beta-1)\right)-1\right)^2\right)}{4\sqrt{\pi}\beta^4 - 8\sqrt{\pi}\beta^3 + 4\sqrt{\pi}\beta^2} \left(e^{-1} - 2\beta e^{-1} + \beta^2 + \beta^2 e^{-1}\right)^2,$$

with graph



Numerical calculation suggests that $\lim_{x \to 0} \frac{f_L(\alpha x)}{f_L(x)} = 0$ for all α but still $\lambda(F_L) = 0$. The example shows that even if the original tail $\frac{f_L(x)}{F_L(x_k^*)}$ vanishes to zero for almost all $x < x_k^*$, the stretched tails is, in fact, quite thick.

Furthermore:

Theorem 5 *The limit is efficient, $g_H V_H + g_L V_L = g_H(u - c_H) + g_L(u - c_L)$, if and only if either $\lambda(F_L) = 0$ or $\lambda(F_L) = \infty$.*

Thus, the limit is only efficient if either the limit is separating or signals are extremely weak. (An earlier example of an efficient limit with weak signals was the case with $a > 0$.)

5.1 Duration on the Market

Is it the good or the bad buyer who samples more sellers, i.e., who stays in the market for a longer duration? In general, the relation between the type of the buyer and the duration is ambiguous. In the special case with boundedly informative signals and small search costs, however, there is a clear prediction: the bad type will sample longer. An implication of this is that, conditional on observing a transaction between the buyer and the n'th seller, the probability of the buyer being bad is increasing in n. Therefore, the expected price paid by the buyer is increasing in n.

The observation that bad types sample longer and receive a lower price in expectation is in accord with empirical observations about adverse selection in markets, see for example the discussionin Gonzales and Shi (2008). XX

In our model, duration on the market is unobservable to the seller. For example, in procurement, sellers do not know how many other sellers have been called already. However, many markets are "thin" and trading opportunities arrive slowly over time. In such a market, duration can be observed (e.g., in the labor market, the time since the last employement is observabel and in the housing market the time on the realtor's listing service is reported.) Furthermore, there can be crude measures of duration. In a procurement situation, a buyer might have a group of preferred sellers who he contacts. Or an entrepreneur has a network of potential investors. Thus, social distance from the seller might serve as a measure of time on the market.

Let us assume that there are two types of sellers, "friends" and "strangers". The buyer samples friends first and turns to strangers only after not trade has taken place. To keep it simple, we assume that the number of friends is random. This will make the problem stationary. The first seller is a friend with probability γ. Furthermore, if the last seller sampled was a friend, the next seller will be a friend with probability γ and a stranger with probability $(1 - \gamma)$. If the last seller was a stranger, the next seller will be a stranger for sure. Sellers know their relationship to the buyer. They can infer from their relationship something about how many other sellers have been sampled already but they do not observe their order.

The mechanism offer now conditions on the seller's signal x and the nature of the relationship, $r \in \{fr, st\}$. Likewise, sellers' beliefs condition on x and r. Since the problem is stationary by construction, the buyer's continuation payoff (and hence the optimal offer) does not depend on time. Let σ denote a market constellation, defined in the natural manner (with $M(w, x, r)$ denoting the mechanism offer by type w to a seller with signal x and relation r, and so on for the other components). Let $V_w(\sigma, fr)$ be the payoff of a buyer who has last sampled a friend (or who has just entered).

A seller's belief who is a friend of the buyer depends on the ratio of the expected number of friends sampled (and likewise for a stranger's belief)

$$\theta(\sigma, fr) = \frac{E[\#\ fr|L, \sigma_k]}{E[\#\ fr|H, \sigma_k]} \qquad \theta(\sigma, st) = \frac{E[\#\ st|L, \sigma_k]}{E[\#\ st|H, \sigma_k]}.$$

We do not repeat the definition of equilibrium. When signals are boundedly informative and search cost are sufficiently small, the following characterizes the equilibrium. There are two cutoffs x^*_{fr} and x^*_{st}, such that the buyers trade with a seller who is a friend if and only if $x \le x^*_{fr}$ (the buyers trade with a seller who is a stranger if and only if $x \le x^*_{st}$.) The price is equal to interim expected cost. The ratios can be calculated to be[11]

$$\theta(\sigma, fr) = \frac{F_L\left(x^*_{fr}\right)\left(F_H\left(x^*_{fr}\right)\gamma + 1 - \gamma\right)^2}{F_H\left(x^*_{fr}\right)\left(F_L\left(x^*_{fr}\right)\gamma + 1 - \gamma\right)^2} \qquad \theta(\sigma, st) = \frac{\left(1 - \frac{\gamma F_L\left(x^*_{fr}\right)}{1-\left(1-F_L\left(x^*_{fr}\right)\right)\gamma}\right) F_H\left(x^*_{st}\right)}{\left(1 - \frac{\gamma F_H\left(x^*_{fr}\right)}{1-\left(1-F_H\left(x^*_{fr}\right)\right)\gamma}\right) F_L\left(x^*_{st}\right)}$$

we have $\beta(x, r, \theta_r) = \left(1 + \theta_r \frac{x}{1-x}\right)^{-1}$ and $E[c|x, r, \theta] = \beta c_H + (1-\beta) c_L$. We get

$$\begin{aligned} U_L(\sigma^*, r) &= F_L(x^*_r)\left(u - E[c|x \le x^*_r, r, \theta]\right) + \left(1 - F_L(x^*_r)\right) V_L(\sigma^*, r) \\ V_L(\sigma^*, fr) &= \gamma U_L(\sigma^*, fr) + (1-\gamma) U_L(\sigma^*, st) - s \quad \text{and} \quad V_L(\sigma^*, st) = U_L(\sigma^*, st) - s. \end{aligned}$$

The cutoff types are determined by indifference

$$x^*_r : u - E[c|x^*_r, r, \theta] = V_L(\sigma^*, r).$$

The bad buyer samples more strangers and hence, the interim beliefs of strangers are more pessimistic than the interim beliefs of the friends

$$\beta(x, stranger, \theta_r) > \beta(x, friend, \theta_r).$$

[11] From $\sum_{i=1}^{\infty} i\left(\gamma F\left((1-F)\gamma\right)^{i-1}\right) = \frac{\gamma F}{(F\gamma + 1 - \gamma)^2}$

This follows from

$$
\begin{aligned}
\frac{\left(1-\frac{\gamma F_L\left(x^*_{fr}\right)}{1-\left(1-F_L\left(x^*_{fr}\right)\right)\gamma}\right)F_H\left(x^*_{st}\right)}{\left(1-\frac{\gamma F_H\left(x^*_{fr}\right)}{1-\left(1-F_H\left(x^*_{fr}\right)\right)\gamma}\right)F_L\left(x^*_{st}\right)} &< \frac{\frac{1-\left(1-F_L\left(x^*_{fr}\right)\right)\gamma-\gamma F_L\left(x^*_{fr}\right)}{1-\left(1-F_L\left(x^*_{fr}\right)\right)\gamma}}{\frac{1-\left(1-F_H\left(x^*_{fr}\right)\right)\gamma-\gamma F_H\left(x^*_{fr}\right)}{1-\left(1-F_H\left(x^*_{fr}\right)\right)\gamma}} \\
&= \frac{1-\gamma+F_L\left(x^*_{fr}\right)\gamma-\gamma F_L\left(x^*_{fr}\right)}{1-\gamma+F_H\left(x^*_{fr}\right)\gamma-\gamma F_H\left(x^*_{fr}\right)}\frac{1-\left(1-F_H\left(x^*_{fr}\right)\right)\gamma}{1-\left(1-F_L\left(x^*_{fr}\right)\right)\gamma} \\
&= \frac{1-\left(1-F_H\left(x^*_{fr}\right)\right)\gamma}{1-\left(1-F_L\left(x^*_{fr}\right)\right)\gamma}
\end{aligned}
$$

and hence

$$
\begin{aligned}
\theta\left(\sigma,fr\right)=\frac{F_L\left(x^*_{fr}\right)\left(F_H\left(x^*_{fr}\right)\gamma+1-\gamma\right)^2}{F_H\left(x^*_{fr}\right)\left(F_L\left(x^*_{fr}\right)\gamma+1-\gamma\right)^2} &= \frac{F_L\left(x^*_{fr}\right)}{F_H\left(x^*_{fr}\right)}\frac{F_H\left(x^*_{fr}\right)\gamma+1-\gamma}{F_L\left(x^*_{fr}\right)\gamma+1-\gamma}\frac{1-\left(1-F_H\left(x^*_{fr}\right)\right)\gamma}{1-\left(1-F_L\left(x^*_{fr}\right)\right)\gamma} \\
&> \frac{1-\left(1-F_H\left(x^*_{fr}\right)\right)\gamma}{1-\left(1-F_L\left(x^*_{fr}\right)\right)\gamma}>\theta\left(\sigma,st\right).
\end{aligned}
$$

This implies that a stranger's interim expected cost is higher than a friend's interim expected cost, conditional on seeing the same signal. Hence, buyer like friends and having sampled a friend as the last seller is good news, i.e.,

$$V_L\left(\sigma^*,fr\right)>V_L\left(\sigma^*,st\right).$$

In the case of boundedly informative signals and low search cost we conclude that: A buyer gets a better deal if trading with a friend since friends are more trusting (more optimistic) than strangers. Bad types are dealing more often with strangers than good types.

6 Extension: Heterogeneous Buyers

Suppose buyers are heterogeneous with a willingness to pay of u_S or u_B, $u_B>u_S$. We assume that $u_B-c_H>s$, so that the high valuation buyer wants to participate and we assume that $u_S-c_L>s$. However, trade is not efficient for the low valuation, high cost buyer, $u_S-c_H<0$. In particular, we assume that the low valuation type is not willing to pay the prior expected cost of the seller, $u_S-\left(g_Lc_L+g_Hc_H\right)<0$. If the equilibrium outcome is complete pooling, this type is better off not participating. Therefore we introduce a prior decision stage: before sampling sellers, upon observing his own type, a buyer can decide to either start sampling sellers or to opt out. This decision is non-reversable, i.e., once the buyer decided to sample sellers, he cannot opt out again; once the buyer opted out, he cannot re-enter.

The type of a buyer is now denoted by $(w,i) \in \{(L,S),(L,B),(H,S),(H,B)\}$. We assume that the willingness to pay and the cost are independent. Let g_B and g_S denote the probabilities of u_B and u_S, respectively. Then the probability of the type (w,i) is just $g_w g_i$, e.g., the probability of $(H,S) = g_H g_S$.

While we will introduce heavy machinerie (we will define equilibrium and introduce notation parallel to the case with homogeneous buyers), the main idea is quite simple: For given search cost, the trading outcome will be almost identical to the full information outcome when signals are sufficiently informative. With full information, the high cost-low valuation type (H,S) does not enter, while the other types trade at prices equal to the costs of serving them. This outcome is efficient. However, for the same signal distribution as before, when search cost become small, equilibrium will involve complete pooling. Therefore, the expected trading price will be $E_0 c = g_H c_H + g_L c_L$ and, hence, the low cost-low valuation buyer can no longer trade, so that the outcome becomes inefficient. We construct this example to show that, with adverse selection, lower search cost can decrease welfare. The reason is simple: search cost inhibit excessive search so that signals have value and allow different cost types to be separated.

Heterogeneous valuations are easily dealt with because, conditional on entry, the optimal offer of a buyer does not depend on his willingness to pay. Instead, the buyer only trades off the price he has to pay today vs the price he would have to pay in the future plus the expected search cost. The willingness to pay does not enter this trade off. (This would be different, of course, if we would model search cost by discounting.)

Let $e \in \{0, 1\}$ denote the entry decision (with 0 denoting "no entry"). We now define equilibrium with heterogeneous buyers. As mentioned before, this is simplified by the fact that conditional on entry, the preferences of the buyer will not depend on his willingness to pay. Therefore, we can restrict attention to equilibria in which the strategy of the buyer, conditional on entry, depends only on his cost type w. In particular, buyers report only their cost type to the direct mechanism. A constellation σ is extended to include the entry decision, $e((w,i)) \in \{enter, \text{not enter}\}$. Conditional on entry, equilibrium payoffs are $U_{w,i}(M,R,x,\sigma)$, where

$$U_{w,i}(M,R,x,\sigma) = u_i - E[p|w,\sigma] - sE[\#|w,\sigma].$$

Definition 3 *A constellation σ^* is an inscrutable equilibrium if*

1. $e^*((w,i))$, $M^*(x,w)$ and R^* are *optimal,* $M^*, R^* \in \arg\max U_{w,i}(M,R,x,\sigma)$. Entry $e(w,i) =$ "not enter" if $U_{w,i} < 0$ and $e(w,i) =$"enter" if $U_{w,i} > 0$

2. β^* is derived from *Bayes Rule* whenever applicable.

3. A^* is *sequentially rational.*

4. The equilibrium is *inscrutable*: $M^*(x,H) = M^*(x,L)$, M^* is *accepted* and *reporting is truthful,* $A^*(M^*,x) = 1$ and $R^*(w,x,M^*) = w$.

5. $\beta^*(x,M)$ satisfies *Divinity given* σ for all x and M and Payoffs are *monotone*, $V_{Li}(\sigma^*) > V_{Hi}(\sigma^*)$.

Beliefs now take into account the entry decision. Note that, by $u_B - c_H > s$, $e(w,B) = 1$ for $w \in \{L,H\}$:

$$\beta(x,\theta,e) = \frac{g_H(g_B + g_S e(H,S))}{g_H(g_B + g_S e(H,S)) + \theta \frac{f_L(x)}{f_H(x)} g_L(g_B + g_S e(H,S))}.$$

Let $E_0[c|x,\theta,e]$ be the interim expected cost of a sampled seller.

Then equilibrium mechanisms are characterized by

Lemma 4 *Given any equilibrium σ^* with payoffs $V_{Hi}(\sigma^*)$ and $V_{Li}(\sigma^*)$, and ratio $\theta(\sigma^*)$. Then there is an equilibrium σ^{**} in which the offered mechanism is described by some $\mathbf{x} = [x^*, x^{**}, q_L^C]$, with $x^* = x^*(V_H, V_L, \theta)$, $x^{**} \geq x^*$, and $q_L^C = q_L^C(V_H, V_L, \theta)$ such that with $Ec = E_0[c|x,\theta,e]$*

$$M^{**}(x) = \begin{cases} [1, \quad Ec, \quad 1, \quad Ec] & \text{If} \quad x < x^* \\ [q_L^C, \quad c_L, \quad 1, \quad c_H] & \text{if} \quad x \in (x^*, x^{**}) \\ [\,0, \quad c_L, \quad 0, \quad c_H] & \text{if } x > x^{**}. \end{cases}$$

And σ^{**} leads to the same payoffs and ratio as σ^*.

We concentrate on the case with boundedly informative signals, because in this case equilibrium predictions are unique. We discuss the case with unboundedly informative signals later. Clearly, with sufficiently informative signals, equilibrium will be almost identical to the full information outcome:

Lemma 5 *Given any s, there is some distribution F_L^s with support $[a,b]$, $0 < a < b < 1$, such that in every equilibrium entry is efficient, $e(H,S) = 0$, and search cost are small, $sE[\#|w,\sigma] \leq 2s$. Prices are almost revealing, $E[p|L,\sigma] \leq c_L + 2s$ and $E[p|L,\sigma] \geq c_H - 2s$*

This implies that for small search cost, payoffs are almost first best efficient. When the signals are informative as defined in the last lemma (signals are distributed according to F_L^s), trading surplus is

$$\begin{aligned} & \sum_{(w,i)} g_w g_i \left(e\left(w,i\right) \left(u_i - c_w - sE\left[\#|w,\sigma\right]\right)\right) \\ \geq \ & g_S g_L \left[u_S - c_L\right] + g_B \left[u_B - g_H c_H - g_L c_L\right] - 4s. \end{aligned}$$

The lemon's outcome refers to the case where buyers either trade at the first seller at prior expected cost or do not enter at all. This outcome corresponds to a situation in which sellers receive no information except for the price offer, similar to the original lemon's model. Then surplus in the lemon's outcome is

$$g_B \left[u_B - g_H c_H - g_L c_L - s\right]$$

When search cost are small, the fact that signals are boundedly informative implies that prices will not depend on the cost of serving the buyer and equal prior expected cost, $E_0 c = g_H c_H + g_L c_L$. This is an immediate implication of Theorem 2. If the price does not depend on the willingness to pay, then buyer with a low valuation will be priced out of the market. Hence, the outcome is inefficient:

Let $\{s_k\}$ be a sequence of decreasing, vanishing search cost, $s_k \to 0$. Let signals be distributed according to $F_L^{s_1}$ such that, given the highest search cost s_1, the outcome is almost efficient:

Lemma 6 *Given $F_L^{s_1}$ with support $[a,b]$, $a > 0$. When $s_k \to 0$, prices are equal to prior expected cost, $E\left[p|w,\sigma_k\right] \to g_L c_L + g_H c_H$ and the surplus of the equilibrium outcomes σ_k converges to the inefficient lemon's outcome,*

$$\sum_{(w,i)} g_w g_i \left(e_k\left(w,i\right)\left(u_i - c_w - s_k E\left[\#|w,\sigma_k\right]\right)\right) \to g_B \left[u_B - g_H c_H - g_L c_L\right].$$

7 Conclusion

8 Appendix

8.1 Proof of Lemma 1

We will proof the lemma by a sequence of auxilliary lemmas.

We say a mechanism M is *feasible* relative to continuation payoffs V_w and beliefs β_0 if the mechanism is individually rational for the buyer (better than no trade), truthful reporting is incentive compatible, and if it is individually rational for the seller to accept the mechanism, given truthful reports. Suppose a buyer w has sampled a seller with signal x and suppose the continuation payoff of the buyer is V_w. Then reporting truthfully is optimal if

$$q_w^M \left(u - p_w^M\right) + \left(1 - q_w^M\right) V_w \geq q_{w'}^M \left(u - p_{w'}^M\right) + \left(1 - q_{w'}^M\right) V_w, \quad \forall w' \in \{L, H\}.$$

Acceptance of a mechanism is profitable if expected profits are positive

$$\beta_0 q_H^M \left(p_H^M - c_H\right) + (1 - \beta_0)\, q_L^M \left(p_L^M - c_L\right) \geq 0.$$

Given continuation payoffs $V_H(\sigma), V_L(\sigma)$ and beliefs $\beta_0(x, \theta(\sigma))$, let $\hat{M}(x, (V_H, V_L, \theta))$ be the set of feasible mechanisms that maximize the expected payoff of the low cost buyer, subject to the constraint that the high cost buyer has an incentive to offer the mechanism as well:

$$\begin{aligned}
\hat{M}(x, \sigma) \quad = \quad & \arg \max_{[p_L, q_L, p_H, q_H]} \left(q_L (u - p_L) + (1 - q_L) V_L(\sigma)\right) \\
& s.t. \;\; \text{M is feasible given } V_w(\sigma) \text{ and } \beta_0(x, \theta) \\
& \qquad q_H (u - p_H) + (1 - q_H) V_H(\sigma) \geq u - c_H \quad \text{(IR H-Buyer)}
\end{aligned}$$

The last inequality implies that the H buyer does not want to deviate from M to a mechanism that prescribes trade at c_H for sure. We will argue that this is sufficient to ensure that H does not find any other mechanism profitable, given suitably chosen beliefs by the seller.

A mechanism M can be part of an equilibrium only if it maximizes the payoff of the L buyer as defined before:

Lemma 7 *If σ^* is an equilibrium constellation with payoffs $V_H(\sigma), V_L(\sigma)$ and ratio $\theta(\sigma)$, then $M(x) \in \hat{M}(x, (V_H, V_L, \theta))$ for all x.*

The intuition behind the lemma is as follows. First, sellers must not receive positive profits. Otherwise, the low cost buyer could use the slack in the seller's IR constraint to make himself better off as shown in the proof. Second, suppose the theorem would not hold and there would be some feasible mechanism M' such that the low cost buyer prefers the allocation (p'_L, q'_L) strictly to the allocation in the original mechanism (p_L, q_L). If the high cost buyer strictly prefers the allocation from M' to the one from M as well, then moving from M to M' would make both types of buyers strictly better off. Increasing the prices slightly would not change this, but would strictly increase the profit of the seller. Thus, M' would *dominate* M and hence M cannot be part of an equilibrium. If, on the other hand, the high cost buyer prefers the original mechanism M (weakly) over M', then, following the deviation to M', Divinity requires that the seller updates towards the low cost buyer. Again, increasing prices slightly (in particular, increasing p_L) would imply that the seller's expected profits from accepting M' are strictly postive and thus, the deviating mechanism M' would be accepted. Thus, M' would be a profitable deviation for the low cost buyer.

Proof: We will be slightly informal in the usage of ε - changes. (I can introduce the complete ε-δ arguments.)

Step 1: Sellers' profits must be zero. Suppose in an equilibrium σ^* there is some x and some $M^*(x)$ such that $\pi(x, M^*) > 0$. Then $q_L^{M^*}$ must be positive. (Otherwise, if $q_L = 0$, then, for profits to be positive, p_H must be strictly larger than c_H. But then the H buyer could deviate to M' with $p_H = c_H + \varepsilon$, ε small, which would be accepted. This contradicts $\pi > 0$, since ε is arbitrary.) Given $q_L^{M^*}$ is positive, $q_H^{M^*}$ must be positive as well by incentive compatibility for the H buyer. Now, we construct again a deviation to a mechanism M': Choose p'_L and q'_L such that a) the high cost buyer H is indifferent between $\left(p_L^{M^*}, q_L^{M^*}\right)$ and (p'_L, q'_L) and b), $p'_L < p^{M^*}$ and c) $M' = \left[p'_L, q'_L, p_H^{M^*}, q_H^{M^*}\right]$ is strictly profitable for the seller given his prior β_0- since M^* was strictly profitable, decreasing p_L only slightly keeps M' strictly profitable. (To ensure truthful reporting by the H buyer, one can decrease p_H slightly.) Given the single crossing property, the L buyer strictly prefers M' to M^* while, by definition,the H buyer (weakly) prefers M^*, thus, $\alpha_L(M', x, \sigma) < 1 < \alpha_H(M', x, \sigma) = \infty$. Therefore, divinity requires that $\beta(M', x) \leq \beta_0(x, \sigma)$. Since M' is strictly profitable, the seller accepts the offer, $A(M', x, \sigma) = 1$, by sequential rationality. Since M' makes L strictly better off than $M^*(x)$, M^* is not an optimal offer and cannot be part of the equilibrium. Thus, assuming positive profits for the seller leads to a contradiction.

Step 2. Suppose, contrary to the claim of the theorem that there is some feasible M' such that the low cost buyer is strictly better off than with the equilibrium mechanism $M^*(x)$ where M' and M^* both satisfy the constraints of the maximization program. (This is the only way in which the Theorem can fail. Every equilibrium mechanism M^* must satisfy the constraints of the maximization program.)

2a): Suppose M' makes *both* buyers strictly better off if accepted (and reports are truthful). Then, M'' defined by increasing the prices slightly relative to M', $M'' = [p'_L + \varepsilon_L, q'_L + \varepsilon_L, p'_H + \varepsilon_H, q'_H + \varepsilon_H]$, is strictly profitable for the seller while being strictly incentive compatible for the buyers (by choosing suitable values for ε_L and ε_H small enough). Thus, M'' dominates M^* and M^* is not part of an undominated equilibrium.

2b): Suppose M' makes the low cost buyer strictly better off while making the H buyer weakly worse off. By assumption, H is weakly better off with M^* and therefore, $\alpha_H(M', x, \sigma) = \infty$. By assumption, M' makes L strictly better off and therefore, $\alpha_L(M', x, \sigma) < 1$. Thus, the posterior of the seller must be $\beta(x, M') \leq \beta_0(x, \sigma)$. Finally, there will be some feasible M'' close to M' (see below for its construction) such that M'' makes L better off than M^* but not H, while M'' is strictly profitable for the seller at β_0 (and in particular at $\beta(x, M'') \leq \beta_0$). Therefore, M'' is accepted by the seller and, hence, M'' is a profitable deviation for L.

M'' can be constructed as follows. First, suppose the incentive compatibility constraint for the L buyer does not bind in M'. Then, M'' can be found by increasing p'_L slightly while keeping q'_L constant. The incentice compatibility contraints for the H buyer still hold (a forteriori) and M'' is strictly profitable after the price increase. If the incentive compatibility constraint for the L buyer does bind, the incentive compatibility constraint of the H buyer does not bind (by the single crossing condition). Then, M'' can be found by increasing p'_L slightly while increasing q'_L to keep L indifferent between (p'_L, q'_L) and the new allocation (q''_L, p''_L). QED

The next lemma characterizes $\hat{M}$. Define Δ to be the surplus for the H buyer, $\Delta = (u - c_H) - V_H$. Define a cutoff x^* and a probability q_L^C

$$x^*(V_H, V_L, \theta) \begin{cases} = 1 & \text{if} \quad u - c_H - \max\{\Delta, 0\} > V_L \\ \in [0,1] \quad \text{s.t.} & u - E[c|x,\theta] - \beta \max\{\Delta, 0\} = V_L \end{cases},$$
$$q_L^C(V_H, V_L, \theta) \begin{cases} = \frac{u - c_H - V_H}{u - c_L - V_H} & \text{if} \quad u - c_H - V_H > 0 \\ = 0 & \text{if } u - c_H - V_H \leq 0 \end{cases}.$$

where x^* is well defined because the left hand side of the equality is (strictly) increasing in β.[12]

Lemma 8 *A mechanism $M(x)$ is in $\hat{M}(x, (V_H, V_L, \theta))$ if and only if for $x < x^*$, $M(x) = [1, \ \ E_0[c|x,\theta], \ \ 1, \ \ E_0[c|x,\theta]]$ and for all $x > x^*$,*
a) if $\Delta > 0$, then $M(x) = \left[q_L^C, \ \ c_L, \ \ 1, \ \ c_H\right]$

[12] One can show that whenever $x > x^*$,

$$u - Ec < q_L^C(u - c_L) + \left(1 - q_L^C\right) V_L$$

(by rewriting the function $X(q_L)$ in the appendix).

b) if $\Delta < 0$*, then* $M(x) = [0, \quad p_L, \quad 0, \quad p_H]$
c) if $\Delta = 0$*, then* $M(x) = [0, \quad p_L, \quad q_H, \quad c_H]$ *for any* $q_H \in [0,1]$*.*

Note: For $x < x^*$ the mechanisms $M(x)$ is (incentive) efficient. This is not true when x is strictly larger but close to x^*. When $x \cong x^*$ the L buyer is almost indifferent between trading at Ec with probability one and trading at c_L with probability q_L^C. However, the H buyer is strictly worse off when the mechanism changes from trading at Ec to trading at c_H. (The seller receives always zero profits.)

Proof: The L Buyers maximization problem can be written

$$\begin{aligned}
& \max q_L (u - p_L) + (1 - q_L) V_L \\
\text{s.t} \quad & (1-\beta) q_L (p_L - c_L) + \beta q_H (p_H - c_H) = 0 \quad \text{(IR Seller)} \\
& q_L (u - p_L) + (1 - q_L) V_H = q_H (u - p_H) + (1 - q_H) V_H \qquad \text{(IC H Buyer)} \\
& q_L (u - p_L) + (1 - q_L) V_H \geq u - c_H \quad \text{(IR H Buyer)}
\end{aligned}$$

We know the zero profit (IR seller) constraint is binding. The IC H Buyer constraint must be binding as well, otherwise increasing q_L and/or decreasing p_L would make the L Buyer strictly better off (it cannot be that $q_L = 1$ and $p_L = c_L$ at the same time, thus, the proposed change keeps the seller's IR constraint). A solution to the maximization problem exists by continuity of the objective function. Let M be a solution. We want to characterize M.

We consider three cases:

Case 1: $\Delta = u - c_H - V_H > 0$. The H buyer is strictly better off trading at c_H rather than continuing search. At the optimal solution, it has to be that $q_H = 1$. (see Note 1).

We can use $q_H = 1$ to rewrite the maximization problem (see Note 2):

$$\max_{q_L} X(q_L) \equiv q_L u - (1-\beta) q_L c_L + \beta u - \beta q_L u - \beta (1 - q_L) V_H - \beta c_H + (1 - q_L) V_L$$
$$\text{st.} \;\; q_L (u - p_L) + (1 - q_L) V_H \;\; \geq \;\; u - c_H$$

The objective function X is now linear in q_L and the derivative is

$$\begin{aligned}
\frac{\partial}{\partial q_L} X &= u - (1-\beta) c_L - \beta u + \beta V_H - V_L \\
&= u - (1-\beta) c_L - \beta c_H + \beta c_H - \beta u + \beta V_H - V_L \\
&= (u - Ec) - \beta \Delta - V_L
\end{aligned}$$

The derivative of X is **positive** if $x < x^*$,

$$(u - Ec) - \beta\Delta > V_L.$$

and a positive derivative implies that we get $q_L = 1$. At $q_L = 1$, the ic constraint for the H buyer requires $q_L \leq q_H = 1$ (given the single crossing condition, H cannot trade at a lower price and lower probability); then the seller's IR constraint requires $p_L = p_H = Ec$. Hence

$$M(x) = [1, Ec, 1, Ec] \qquad x < x^*$$

Thus, the $IR\ H$ Buyer constraint does not bind, $q_L(u - p_L) + (1 - q_L)V_H = u - Ec \geq u - c_H$.

The derivative of X is **negative** if $x > x^*$, since at x^*, $u - E[c|x^*] - \max\{\beta\Delta, 0\} = V_L$ and $\beta(x)$ is strictly increasing in x,

$$(u - Ec) - \beta(x)\Delta < V_L \quad , \quad x > x^*.$$

In this case we get a corner solution because the H IR constraint will bind at $q_L = 0$ (this is immediate by inspection of the constraint and the hypothesis of the case). Given $q_H = 1$, a binding IR constraint implies $p_H = c_H$ (by inspection of the buyer's IR constraint and the H buyer's IC constraint.) Rewriting the H buyers' IC constraint and using the seller's IR (zero profit) constraint implies (see Note 3):

$$p_L = c_L,\, p_H = c_H, \quad q_L^C = \frac{(u - c_H - V_H)}{(u - c_L - V_H)}$$

where $q_L^C > 0$ by the hypothesis of the case. So, the mechanism must be

$$M(x) = \left[\frac{(u - c_H - V_H)}{(u - c_L - V_H)}, c_L, 1, c_H\right] \qquad x > x^*.$$

To check: Rewriting the linear function $X(q_L)$ at q_L^C shows that

$$\begin{aligned}
X\left(q_L^C\right) &\equiv q_L^C u - (1-\beta)\, q_L^C c_L + \beta u - \beta q_L^C u - \beta\left(1 - q_L^C\right)V_H - \beta c_H + \left(1 - q_L^C\right)V_L \\
&= q_L^C(u - c_L) + \left(1 - q_L^C\right)V_L + \beta\left(u - c_H - V_H - q_L^C(u - c_L - V_H)\right) \\
&= q_L^C(u - c_L) + \left(1 - q_L^C\right)V_L + \beta\left(u - c_H - V_H - \frac{(u - c_H - V_H)}{(u - c_L - V_H)}(u - c_L - V_H)\right) \\
&= q_L^C(u - c_L) + \left(1 - q_L^C\right)V_L.
\end{aligned}$$

And since $X(1) = u - Ec$ and $X(0) = \beta\Delta + V_L$, we have

$$X\left(q_L^C\right) = q_L^C(u - c_L) + \left(1 - q_L^C\right)V_L \geq u - Ec = X(1)$$

if and only if $\beta\Delta + V_L \geq u - Ec$.

Case 2: $\Delta = u - c_H - V_H < 0$. (This is simple but needs expansion.) Note that we can no longer assume that $q_H = 1$. Suppose $x < x^*$. Then $(u - E_0 c) > V_L$. Then, the following mechanism satisfies the constraints of the maximization program

$$[1, E_0 c, 1, E_0 c],$$

and since $(u - E_0 c) > V_L$, the L buyer strictly prefers the above mechanism to any mechansim with a trading probability of zero. Thus, we are looking for the best mechanism with $q_L > 0$. The buyer IC contraint and the seller's zero profit conditon imply that every mechanism in which $q_L > 1$ must specify $p_H = p_L = Ec$: By incentive compatibility for the H buyer, $q_H \geq q_L$ and (therefore) $p_H \geq p_L$. We can write the price p_H as the average of trading with probability q_L at the price p_L and trading with probability d at some higher price e,

$$q_H = q_L + d \text{ and } p_H = \frac{q_L}{q_L + d} p_L + \frac{d}{q_L + d} e.$$

Since $V_H < u - c_H$, the H buyer IC constraint implies that $e < c_H$ if $d > 0$. Otherwise, if $e \geq c_H$

$$\begin{aligned} q_H (u - p_H) + (1 - q_H) V_H &= q_L (u - p_L) + d (u - e) + (1 - q_L - d) V_H \\ &\leq q_L (u - p_L) + d (u - c_H) + (1 - q_L - d) V_H \\ &< q_L (u - p_L) + (1 - q_L) V_H \end{aligned}$$

Profits of the seller are

$$\pi = (1 - \beta) q_L (p_L - c_L) + \beta q_H (p_H - c_H) = q_L (p_L - Ec) + d\beta (e - c_H)$$

and so, whenever $(e - c_H) < 0$ and $d \geq 0$, profits are zero only if $(p_L - Ec) \geq 0$. Clearly, the best mechanism with $p_L \geq Ec$ is the mechanism prescribing trade at Ec with probability one. Thus

$$M(x) = [1, Ec, 1, Ec] \quad \text{ if } x < x^*.$$

Suppose $x > x^*$. Then $(u - E_0 c) < V_L$. The preceeding reasoning implies that for every mechanism with $q_L > 0$, $p_L \geq Ec$. But since $(u - E_0 c) < V_L$, the L buyer would rather not trade and hence any mechanism must have $q_L = 0$. Given $q_L = 0$, $p_L \geq c_H$ if $q_H > 0$; however, the H buyer would rather not trade than trade at $p \geq c_H$ and hence, $q_H = 0$. Hence, every mechanism must be

$$M = [0, p_L, 0, p_H] \quad \text{ if } x > x^*.$$

Case 3: $\Delta = u - c_H - V_H = 0$. Again, if $q_L > 1$, then $p_L = Ec$. Therefore, if $x < x^*$ and $(u - E_0 c) > V_L$

$$M(x) = [1, Ec, 1, Ec] \qquad \text{if } x < x^*.$$

If $x > x^*$, then $(u - E_0 c) > V_L$ implies that $q_L = 0$. Since $\Delta = 0$ implies that the H buyer is indifferent between trading at c_H and not trading, we get

$$M(x) = [0, c_L, q_H, c_H] \qquad \text{if } x > x^*.$$

Note 1: We want to show that if $\Delta > 0$, then $q_H = 1$. The seller's IR constraint implies that it must be that $q_H > 0$ (since otherwise trading at c_H for sure would be better). Given $q_H > 0$, $p_H \leq c_H$. Given $V_L < u - c_L - s$, it must be the case that $u - p_L > V_L$, i.e., trading is strictly better for the L buyer. This follows since otherwise $M' = [\varepsilon, c_L, 1, c_H]$ would be strictly better for the L buyer and M' satisfies the constraints of the maximization program for ε small enough, given $u - c_H > V_H$. Thus, $q_L > 0$ and $p_L < u - V_L$.

If $q_L > 0$ then, by $p_H \leq c_H$, it must be that $p_L \geq c_L$ for profitability. Suppose $q_H < 1$. Given the mechanism $M = [q_L, p_L, q_H, p_H]$ we can construct a mechanism $M' = [q_L + \Delta, p_L, 1, q_H p_H + (1 - q_H) c_H]$. This mechanism is (weakly) profitable by feasibility of M and $p_L \geq c_L$. For Δ small enough, the mechanism M' is incentive compatible for the H buyer: Since M satisfies the IC constraint of the H buyer, the H buyer prefers (q_H, p_H) to (q_L, p_L). By $\Delta > 0$, the H buyer strictly prefers $(1, q_H p_H + (1 - q_H) c_H)$ to (q_H, p_H). Hence, we can choose Δ small enough such that the H buyer prefers $(1, p_H + (1 - q_H) c_H)$ to $(q_L + \Delta, p_L)$ so M' satisfies H IC. The L buyer prefers M' strictly to M for any $\Delta > 0$ given $u - p_L > V_L$.

Note 2: Given $q_H = 1$, the H-IC constraint becomes:

$$\begin{aligned} q_L (u - p_L) + (1 - q_L) V_H &= u - p_H \\ \Rightarrow p_H &= u - q_L (u - p_L) - (1 - q_L) V_H \end{aligned}$$

and the zero profit condition becomes

$$\begin{aligned} (1 - \beta) q_L (p_L - c_L) + \beta 1 (p_H - c_H) &= 0 \\ \Rightarrow (1 - \beta) q_L (p_L - c_L) + \beta (u - q_L (u - p_L) - (1 - q_L) V_H - c_H) &= 0 \\ (1 - \beta) q_L p_L - (1 - \beta) q_L c_L + \beta u - \beta q_L u + \beta q_L p_L - \beta (1 - q_L) V_H - \beta c_H &= 0 \\ q_L p_L - (1 - \beta) q_L c_L + \beta u - \beta q_L u - \beta (1 - q_L) V_H - \beta c_H &= 0 \\ - (1 - \beta) q_L c_L + \beta u - \beta q_L u - \beta (1 - q_L) V_H - \beta c_H &= -q_L p_L \end{aligned}$$

So the objective function is

$$
\begin{aligned}
& q_L u - q_L p_L + (1-q_L) V_L \\
& q_L u - (1-\beta) q_L c_L + \beta u - \beta q_L u - \beta (1-q_L) V_H - \beta c_H + (1-q_L) V_L
\end{aligned}
$$

Note 3: At the corner solution,

$$p_H = u - q_L (u - p_L) - (1-q_L) V_H = c_H$$

and thus, the seller's zero profit condition

$$
\begin{aligned}
(1-\beta) q_L (p_L - c_L) + q_H \beta (p_H - c_H) &= 0 \\
(1-\beta) q_L (p_L - c_L) + 0 &= 0 \\
&\Rightarrow p_L = c_L
\end{aligned}
$$

and rewriting the seller's IC constraint:

$$
\begin{aligned}
u - q_L (u - p_L) - (1-q_L) V_H &= c_H \\
u - q_L (u - c_L) - (1-q_L) V_H &= c_H \\
u - V_H - c_H &= q_L (u - c_L - V_H) \\
&\Rightarrow q_L = \frac{u - V_H - c_H}{u - c_L - V_H}
\end{aligned}
$$

QED

We have not characterized mechanisms at the point x^*. But note that the behavior at x^* does not affect equilibrium if the signals distribution does not contain atoms.

The problem is that at $x = x^*$ the number of possible combinations for equilibrium mechanisms becomes quite large. As shown in the appendix, we can characterize equilibrium via a probability α such that if $\Delta > 0$, the mechanism is a mixture of $\left[q_L^C,\ c_L,\ 1,\ c_H\right]$ with probability α and $[1,\ E_0[c|x,\theta],\ 1,\ E_0[c|x,\theta]]$ with probability $(1-\alpha)$. The resulting (expected) trading probabilities and prices for given α are

$$
\begin{aligned}
q_L^\alpha &= \alpha q_L^c + (1-\alpha) 1, \quad p_L^\alpha = \alpha c_L + (1-\alpha) Ec \\
q_H^\alpha &= 1, \quad p_H^\alpha = \alpha c_H + (1-\alpha) Ec.
\end{aligned}
$$

Lemma 9 *At $x = x^*$, a mechanism $M(x)$ is in $\hat{M}(x, (V_H, V_L, \theta))$ if and only if*
a) if $\Delta > 0$, then $M(x) = [q_L^\alpha, \quad p_L^\alpha, \quad 1, \quad p_H^\alpha]$, any $\alpha \in [0,1]$
b) if $\Delta \leq 0$, then $M(x) = [q_L, \quad E_0[c|x,\theta], \quad q_L, \quad E_0[c|x,\theta]]$, any $q_L \in [0,1]$

Note: The mechanisms $M(x)$ at $x = x^*$ can be implement as outcomes of a price setting game with a public randomization device.

Proof: We can rewrite the binding IC and seller IR constraints from Note 2 as:

$$\begin{aligned} p_H &= u - q_L u - (-q_L p_L) - (1-q_L) V_H \\ p_H(q_L, V_H) &= u - q_L u - (-(1-\beta) q_L c_L + \beta u - \beta q_L u - \beta(1-q_L) V_H - \beta c_H) - (1-q_L) V_H \quad (2) \\ p_L(q_L, V_H) &= (1-\beta) c_L - \beta u / q_L + \beta u + \beta(1-q_L) V_H / q_L + \beta c_H / q_L \quad (3) \end{aligned}$$

Let α be defined via

$$\begin{aligned} q_L &= \alpha q_L^C + (1-\alpha) \\ \alpha\left(1 - q_L^C\right) &= 1 - q_L \Rightarrow \alpha = \frac{1-q_L}{1-q_L^C} \end{aligned}$$

and using the definition of q_L^C:

$$\begin{aligned} \alpha &= \frac{(1-q_L)(u-c_L-V_H)}{(u-c_L-V_H)-(u-c_H-V_H)} = \frac{(1-q_L)(u-c_L-V_H)}{c_H-c_L} \\ \Rightarrow 1-\alpha &= \frac{q_L - q_L^C}{1-q_L^C} = \frac{q_L(u-c_L-V_H)-(u-c_H-V_H)}{(u-c_L-V_H)-(u-c_H-V_H)} \\ 1-\alpha &= \frac{q_L(u-c_L-V_H)-(u-c_H-V_H)}{c_H-c_L} \end{aligned}$$

then

$$\begin{aligned} p_L &= \frac{1}{q_L}(\alpha c_L + (1-a) Ec) \\ &= \frac{\alpha q_L^C}{q_L} c_L + \frac{(1-\alpha)}{q_L}(\beta c_H + (1-\beta) c_L) \\ &= \frac{\alpha q_L^C}{q_L} c_L + \frac{(1-\alpha)}{q_L} c_L + \frac{(1-\alpha)}{q_L} \beta(c_H - c_L) \\ &= c_L + \frac{1}{q_L} \frac{q_L(u-c_L-V_H)-(u-c_H-V_H)}{c_H-c_L} \beta(c_H-c_L) \\ &= c_L + (u-c_L-V_H)\beta - \beta(u-c_H-V_H)/q_L = p_L(q_L, V_H) \end{aligned}$$

and similarly for p_H,

$$\begin{aligned}
p_H &= (1-a)\,Ec + \alpha c_H \\
&= \beta c_H + (1-\beta)\,c_L + \alpha\,(c_H - \beta c_H - (1-\beta)\,c_L) \\
&= \beta c_H + (1-\beta)\,c_L + \alpha\,((1-\beta)\,c_H - (1-\beta)\,c_L) \\
&= \beta c_H + (1-\beta)\,c_L + \alpha\,(1-\beta)\,(c_H - c_L) \\
&= \beta c_H + (1-\beta)\,c_L + \frac{(1-q_L)\,(u - c_L - V_H)}{c_H - c_L}\,(c_H - c_L)\,(1-\beta) \\
&= \beta c_H + (1-\beta)\,c_L + (1-q_L)\,(u - c_L - V_H)\,(1-\beta) \\
&= \beta c_H + (1-\beta)\,c_L + (1-q_L)\,(u - c_L - V_H) - \beta\,(1-q_L)\,(u - c_L - V_H) \\
&= p_H\,(q_L, V_H)
\end{aligned}$$

QED

8.2 Proof of Existence, Theorem

We will now state a couple of claims that together proof existence of equilibrium. First, we introduce some notation. Define the set $\Xi^S = \left\{\mathbf{x} \in [0,1]^3\,|x^* \leq x^{**}\right\}$ and using

$$q_w^{\min} : u - \frac{s}{q_w^{\min}} = u - c_H - s.$$

and define

$$W^S = \left\{(V_H, V_L, \theta) \in [u - c_L - s,\ u - c_H - s]^2 \times \mathbb{R}_+ |\ \theta \in \left[q_H^{\min}, \left(q_L^{\min}\right)^{-1}\right]\right\}.$$

where we bounded θ to avoid problems at zero and infinity. We define the cutoff x^{**} via

$$X^{**}\,(V_H, V_L, \theta) = \begin{cases} 1 & \text{if} \quad u - c_H - V_H > 0 \\ [x^*, 1] & \text{if} \quad u - c_H - V_H = 0 \\ 0 & \text{if } u - c_H - V_H < 0 \end{cases}$$

and we get a correspondence $\Xi\,(V_H, V_L, \theta) \equiv x^* \times X^{**} \times q_L^C\,(V_H, V_L, \theta)$ which maps $W^S \rightrightarrows \Xi^S$. A vector $\mathbf{x}$ defines per period trading probabilities:

$$\begin{aligned}
q_L\,(\mathbf{x}) &= \max\left\{F_L\,(x^*) + q_L^C\,(F_L\,(x^{**}) - F_L\,(x^*))\,, q_L^{\min}\right\}, \\
q_H\,(\mathbf{x}) &= \max\left\{F_H\,(x^*) + (F_L\,(x^{**}) - F_L\,(x^*))\,, q_H^{\min}\right\}.
\end{aligned}$$

Also, $\mathbf{x}$ defines expected payoffs $W(\mathbf{x}) = V_L(\mathbf{x}) \times V_H(\mathbf{x}) \times \theta(\mathbf{x})$ via

$$\begin{aligned}
\hat{V}_L(\mathbf{x}) &= u - \int_0^{x^*} E[c|x,\theta] \frac{f_L(x)}{q_L} dx - \frac{q_L^C (F_L(x^{**}) - F_L(x^*))}{q_L} c_L - \frac{1}{q_L} s, \\
\hat{V}_H(\mathbf{x}) &= u - \int_0^{x^*} E[c|x,\theta] \frac{f_H(x)}{q_H} dx - \frac{F_H(x^{**}) - F_H(x^*)}{q_H} c_H - \frac{1}{q_H} s, \\
\theta(\mathbf{x}) &= \frac{q_H(\mathbf{x})}{q_L(\mathbf{x})}.
\end{aligned}$$

and $V_L(\mathbf{x}) = \max\left\{\hat{V}_L(\mathbf{x}), u - c_H - s\right\}$ and $V_H(\mathbf{x}) = \max\left\{\hat{V}_H(\mathbf{x}), u - c_H - s\right\}$. If $q_w(\mathbf{x}) = q_w^{\min}$, payoffs are minimal $V_w(\mathbf{x}) = u - c_H - s$. This will ensure that the bound on q_w is not binding in the fixed point.

Claim 6 $W \times \Xi : \Xi^S \times W^S \rightrightarrows W^S \times \Xi^S$.

Proof: It is immediate that $\Xi : W^S \rightrightarrows \Xi^S$. For $W : \Xi^S \rightrightarrows W^S$, note that $\theta(\mathbf{x}) \in \left[q_H^{\min}, \left(q_L^{\min}\right)^{-1}\right]$ by force. $\hat{V}_L(\mathbf{x}) \geq u - c_H - s$ by force. $\hat{V}_L(\mathbf{x}) \leq u - c_L - s$ by inspection: If $q_L(\mathbf{x}) > \left(q_L^{\min}\right)$, then the expected price (the second and third term in the definition of $\hat{V}_L(\mathbf{x})$) is at least c_L, so payoffs are at most $u - c_L - s$. If $q_L(\mathbf{x}) = \left(q_L^{\min}\right)$, then $\frac{s}{q_L(\mathbf{x})} = c_H + s$ implies that $\hat{V}_L(\mathbf{x}) \leq u - c_H - s < u - c_L - s$. QED

Claim 7 *The mapping $W \times \Xi$ is convex valued and upper hemi continuous.*

Proof: First, Ξ: The component X^{**} is convex valued by definition and upper hemi continuous by inspection. Continuity of q_L^C follows by inspection as well (noting that every $V_H \in W^S$ is bounded away from $u - c_L$). Continuity of $x^*(V_H, V_L, \theta)$ (recall

$$x^*(V_H, V_L, \theta) = \begin{cases} 1 & \text{if} \quad u - c_H - \max\{u - c_H - V_H, 0\} > V_L \\ x \in [0,1] \quad \text{s.t.} & u - E[c|x,\theta] - \beta \max\{u - c_H - V_H, 0\} = V_L \end{cases} \quad)$$

follows by inspection, once we rewrite,

$$\begin{aligned}
u - E[c|x,\theta] - \beta \max\{u - c_H - V_H, 0\} &= V_L \\
u - (\beta(x,\theta) c_H + (1 - \beta(x,\theta)) c_L) - \beta(x,\theta) \max\{u - c_H - V_H, 0\} &= V_L \\
u - c_L - \beta(x,\theta)(c_H - c_L + \max\{u - c_H - V_H, 0\}) &= V_L.
\end{aligned}$$

Any solution $x \in (0,1)$ (if it exists) is continuous in the parameters, since the belief $\beta(x,\theta) = \frac{x}{x + (1-x)\theta}$ is strictly increasing in x and the term in brackets is strictly positive by $u - c_H - V_H \leq s$

(from $V_H \in W^S$) and $c_H - c_L > s$ (by assumption on s). It is never the case that $x = 0$ is a solution since $V_L \in W^S$ implies $V_L < u - c_L$.) If the solution $x = 1$, the solution is again continuous in the parameters.

Second, W: The component $\theta(\mathbf{x}) = \frac{q_H(\mathbf{x})}{q_L(\mathbf{x})}$ is continuous, since $q_L(\mathbf{x}) \geq q_L^{\min} > 0$ by definition. The component $\hat{V}_L(\mathbf{x})$ is continuous (and likewise V_H. (recall, if $\hat{V}_L(\mathbf{x}) \geq u - c_H - s$ does not bind, then

$$\hat{V}_L(\mathbf{x}) = u - \int_0^{x^*} E[c|x,\theta] \frac{f_L(x)}{q_L(\mathbf{x})} dx - \frac{q_L^C (F_L(x^{**}) - F_L(x^*))}{q_L(\mathbf{x})} c_L - \frac{1}{q_L(\mathbf{x})} s)$$

Again, $q_L(\mathbf{x})$ is bounded away from 0 by definition. Therefore, inspections reveals that the "expected price" (the second and third term) is continuous in $\mathbf{x}$. Note that when $F_L(x^*) + q_L^C (F_L(x^{**}) - F_L(x^*)) < q_L^{\min}$, the "expected price" can be less than c_L. QED

Claim 8 *If $(\mathbf{x}^*, V_L^*, V_H^*, \theta^*) \in W \times \Xi(\mathbf{x}^*, V_L^*, V_H^*, \theta^*)$, then there exists an equilibrium σ^* with payoffs V_L^*, V_H^* and a mechanism characterized by $\mathbf{x}^*$.*

Proof: We construct σ^*. We first construct the on-equilibrium parameters: Let σ_1^* be some constellation such that $M^*(x)$ is given by $\mathbf{x}^*, \theta^*$:

$$M^*(x) = \begin{cases} [1, \quad Ec, \quad 1, \quad Ec] & \text{If} \quad x \leq x^* \\ [q_L^C, \quad c_L, \quad 1, \quad c_H] & \text{if} \quad x \in (x^*, x^{**}) \\ [\,0, \quad c_L, \quad 1, \quad c_H] & \text{if } x \geq x^{**}. \end{cases}$$

and let $A^*(x, M^*(x)) = 1$ (this satisfies optimality for the seller, since M^* is weakly profitable) and let $\beta(x, M^*) = \beta_0(x, \theta^*)$ (The fact that the prior satisfies divinity was the reason for choosing this refinement over stronger one, like D1.) Also, let $R^*(x, M^*(x), w) = w$. Reporting truthfully will be optimal once we show that for the constructed constellation σ_1, continuation payoffs are given by $V_w(\sigma_1^*) = V_w^*$ and $V_H^* \leq V_L^*$. We will also show that $\theta(\sigma_1^*) = \theta^*$. For this we need to show that the solution is not on the boundary, where we artificially set $q_w(\mathbf{x}) = q_w^{\min}$ and $V_w^*(\mathbf{x}) = u - c_H - s$.

First, $q_L(\mathbf{x}^*) > q_L^{\min}$. Suppose $q_L(\mathbf{x}^*) = q_L^{\min}$. Then $V_L^* = u - c_H - s$ (by $\frac{s}{q_L^{\min}} = c_H + s$). Since $V_H^* \in W^S$ implies $V_H^* \geq u - c_H - s$, $x^* = 1$. However, if $x^* = 1$, then $q_L(\mathbf{x}^*) \geq F_L(x^*) = 1$. Thus, $q_L(\mathbf{x}^*) = q_L^{\min}$ leads to a contradiction. If $q_H(\mathbf{x}^*) = q_H^{\min}$, then $V_H^* = u - c_H - s$ and hence $x^{**} = 1$ by definition. Thus $q_H(\mathbf{x}^*) = F_H(x^{**}) = 1$, contradicting $q_H(\mathbf{x}^*) = q_H^{\min}$.

Second, $V_L^* > u - c_H - s$. If $V_L^* = u - c_H - s$, then $x^* = 1$, and hence

$$\hat{V}_L(\mathbf{x}) = u - \int_0^1 E[c|x,\theta^*] \frac{f_L(x)}{1} dx - \frac{1}{1} s > u - c_H - s$$

where the inequality follows from $\theta^* > 0$. Similarly, $V_H^* = u - c_H - s$ implies $x^{**} = 1$. If $F_L(x^*) = 0$, then

$$V_L^*(\mathbf{x}) = u - c_L - \frac{1}{q_L^C} s$$

and from $q_L^C = \frac{u - c_H - V_H^*}{u - c_L - V_H^*}$, $V_L^*(\mathbf{x}) = u - c_L - \frac{u - c_L - V_H^*}{s} s = V_H^* = u - c_H - s$. Contradicting our earlier finding. Since $F_L(x^*) > 0$ implies $F_H(x^*) > 0$ (recall, no atom at $x = 0$),

$$\hat{V}_H(\mathbf{x}) = u - \int_0^{x^*} E[c|x, \theta] \frac{f_H(x)}{1} dx - \frac{1 - F_H(x^*)}{1} c_H - \frac{1}{1} s > u - c_H - s$$

using $\int_0^{x^*} E[c|x,\theta] \frac{f_H(x)}{1} dx < c_H$, from $\theta^* > 0$, again. Thus, $V_L^* = V_L(\sigma_1^*)$ and $V_H^* = V_H(\sigma_1^*)$ and $\theta^* = \theta(\sigma_1^*)$ as claimed. Finally, $V_L^* > V_H^*$ follows since otherwise, $x^* = 1$. At $x^* = 1$, the expected price paid by the L buyer is strictly lower than the expected price paid by the H buyer, contradicting $V_L^* < V_H^*$. Given $V_L^* > V_H^*$, incentive compatibility of $M^*(x)$ follows from $M^*(x) \in \hat{M}(x, (V_H, V_L, \theta))$.

Now we construct the off equilibrium part. Let $R^*(x, M', w) = w$ whenever incentive compatible. Let us restrict attention to incentive compatible mechanisms (any not incentive compatible mechanisms is equivalent to some incentive compatible mechanism). Given R^*, set $A^*(M', x) = 0$, for all M' that are weakly non-profitable at the prior,

$$(1 - \beta_0(x, \theta^*)) q_L^{M'} \left(p_L^{M'} - c_L\right) + \beta_0(x, \theta^*) q_H^{M'} \left(p_H^{M'} - c_H\right) \leq 0.$$

and set $\beta(x, M') = \beta_0$ for all unprofitable M'. Thus, A^* is sequentially rational for these mechanisms, no buyer wants to deviate and offer them, and $\beta(x, M') = \beta_0$ satisfies devinity

Set $\beta(x, M') = 0$ if trading according to M' is worse for the H buyer than either trading at c_H or continuing without trading. Suppose the H buyer strictly prefers to trade at c_H (the other case follows similarly). Set $A(x, M') = 0$ if M' is weakly unprofitable at $\beta(x, M') = 0$. If M' is strictly profitable at $\beta(x, M') = 0$, then the following mechanism is strictly profitable at $\beta(x, M'') = \beta_0$,

$$M'' = \left[q_L^{M'}, p_L^{M'}, 1, c_H\right]$$

and M'' incentive compatible: since M' was incentive compatible, the H buyer preferred $\left(q_H^{M'}, p_H^{M'}\right)$ over $\left(q_L^{M'}, p_L^{M'}\right)$, and $(1, c_H)$ over $\left(q_H^{M'}, p_H^{M'}\right)$ by assumption. Thus, M'' is incentive compatible as well. Hence, M'' satisfies the constraints of the maximization problem for $\hat{M}$ and the L buyer must prefer the maximizer $M^*(x)$ over $M''(x)$ and, by equivalence, prefers $M^*(x)$ over $M'(x)$. Furthermore, $M^*(x)$ is strictly better than M' for the H buyer by our starting hypothesis. Hence, no buyer prefers to deviate to M'.

Any other off equilibrium mechanism M' that is feasible and profitable satisfies the constraints of the maximization problem for $\hat{M}$. Thus, the L buyer prefers M^* to M' and therefore, setting $\beta(x, M') = 1$ satisfies devinity. Since M' satisfies the contraint for the H buyer, $p_H^{M'} \leq c_H$ and hence, M' must be weakly unprofitable at $\beta(x, M') = 1$ and we can set $A(x, M') = 0$ without violating sequential rationality. QED

Existence is an immediate consequence of the above claims. Claim 1 and 2 imply that a fixed point of $W \times \Xi$ exists by Kakutani's theorem. Claim 3 showed how to construct an equilibrium from that fixed point. QED

8.3 Proof of Theorem 2

Case 2: $\Delta_k = 0$ for all k large enough. Again $x_k^* \to 0$. If $x_k^* \to a$, then the difference between the cutoff seller and the expected price becomes zero. This is because the signal will converge to $a \in (0, 1)$,

$$\begin{aligned}
& \lim E_0\left[c|x_k^*, \theta_k\right] - \int_a^{x_k^*} E_0\left[c|x, \theta_k\right] \frac{dF_L(x)}{F_L\left(x_k^*\right)} \\
= & \lim \frac{1}{1 + \frac{1 - x_k^*}{x_k^*}\theta_k} c_H - \int_a^{x_k^*} \frac{1}{1 + \frac{1-x}{x}\theta_k} c_H \frac{dF_L(x)}{F_L\left(x_k^*\right)} \\
= & \lim \frac{1}{1 + \frac{1-a}{a}\theta_k} c_H - \frac{1}{1 + \frac{1-a}{a}\theta_k} c_H = 0.
\end{aligned}$$

Hence, indifference by the L buyer implies,

$$\begin{aligned}
\lim E_0\left[c|x_k^*, \theta_k\right] c_H - \int_a^{x_k^*} E_0\left[c|x, \theta_k\right] c_H \frac{dF_L(x)}{F_L\left(x_k^*\right)} &= \lim \frac{s_k}{F_L\left(x_k^*\right)} \\
0 &= \lim \frac{s_k}{F_L\left(x_k^*\right)}.
\end{aligned}$$

So, expected search cost for the L buyer are zero. $\Delta_k = 0$ requires

$$\begin{aligned}
\lim c_H - \int_a^{x_k^*} E_0\left[c|x, \theta_k\right] c_H \frac{dF_H(x)}{F_H\left(x_k^*\right)} &= \lim \frac{s_k}{F_H\left(x_k^*\right)} \\
&= \lim \frac{s_k}{F_L\left(x_k^*\right)} \frac{F_L\left(x_k^*\right)}{F_H\left(x_k^*\right)} = \lim \frac{s_k}{F_L\left(x_k^*\right)} \frac{1-a}{a} \\
&= 0
\end{aligned}$$

Since the lower bound is not zero, if L can search at almost no cost for $x \in [a, x_k^*]$, H can do so as well. Since H should be indifferent between trading at c_H and searching, this would require that the expected price conditional on searching becomes c_H. However, since the ratio is bounded

$$\begin{aligned}\theta_k &= \frac{F_H\left(x_k^*\right)+\left(1-F_H\left(x_k^{**}\right)\right)}{F_L\left(x_k^*\right)} \\ &\geq \frac{F_H\left(x_k^*\right)}{F_L\left(x_k^*\right)}=\frac{a}{1-a},\end{aligned}$$

the expected price becomes

$$\lim \int_a^{x_k^*} \frac{1}{1+\frac{1-x}{x}\theta_k} c_H \frac{dF_H(x)}{F_H\left(x_k^*\right)} \leq \frac{1}{1+\frac{1-a}{a}\frac{a}{1-a}} c_H=\frac{1}{2} c_H.$$

And hence, it would pay for the H buyer to search, rather than accept c_H.

Case 3: $\Delta_k>0$ for all k large enough. The additional problem relative to Case 2 is that, when $\Delta_k>0$, the cutoff type is determined differently and the expected number of searches of the L type also depend on the trading probability q_k^C when $x>x_k^*$. We will show that the influence of the additional probability q_k^C becomes negligible.

First, the same arguments as before imply (a forteriori) that $x_k^* \rightarrow a$. Now, clearly it must be the case that it does not pay for the H buyer to try and trade at c_L,

$$\begin{aligned}\frac{s_k}{q_k^C\left(1-F_H\left(x_k^*\right)\right)} &> c_H-c_L \\ &\Rightarrow \frac{s_k}{q_k^C}>c_H-c_L.\end{aligned} \tag{4}$$

If $\Delta_k>0$, only the L buyer is searching. If the per period probability of trading of the L buyer becomes zero, the expected cost of any seller become equal to c_L,

$$\text{if } F_L\left(x_k^*\right)+q_k^C\left(1-F_L\left(x_k^*\right)\right) \rightarrow 0 \text{ then } \theta_k \rightarrow \infty \text{ and } \beta_0\left(x, \theta_k\right) \rightarrow 0,\ \forall x \in(0,1).$$

Hence, the expected payoff of the L buyer must be

$$V_{kL} \rightarrow u-c_L.$$

And therefore, expected accumulated search cost become zero,

$$\frac{s_k}{F_L\left(x_k^*\right)+q_k^C\left(1-F_L\left(x_k^*\right)\right)} \rightarrow 0.$$

By (4), the trading probability $q_k^C\left(1-F_L\left(x_k^*\right)\right)$ becomes zero fast relative to s_k, while the overall trading probability $F_L\left(x_k^*\right)+q_k^C\left(1-F_L\left(x_k^*\right)\right)$ must be large relative to s_k. Hence, the probability of trading at c_L, conditional on trading at all, must become zero,

$$\frac{q_k^C\left(1-F_L\left(x_k^*\right)\right)}{F_L\left(x_k^*\right)+q_k^C\left(1-F_L\left(x_k^*\right)\right)}\to 0.$$

Thus, as claimed at the beginning of this case, the probability q_k^C does not affect the equilibrium outcome in the limit. In particular, the H buyer can search for a seller with $x\leq x_k^*$ at almost no cost in the limit,

$$\begin{aligned}\lim\frac{s_k}{F_L\left(x_k^*\right)}\frac{F_L\left(x_k^*\right)}{F_H\left(x_k^*\right)} &= \lim\frac{s_k}{F_L\left(x_k^*\right)}\frac{1-a}{a}\\ &= 0\end{aligned}$$

where the second line from the fact that the search cost for the L buyer become zero. Of course, if the H buyer could also search for $x\leq x_k^*$ at almost no cost, his payoffs cannot be lower than $u-c_H$, since the expected cost of the seller with $x\leq x_k^*$ becomes equal to c_L.

Hence, when k is small, it cannot be the case that either $\Delta_k=0$ or $\Delta_k>0$. So, it must be the case that we are in case 1, $\Delta<0$, for which we found the expected trading price to be equal to the prior, as claimed. QED

8.4 Proof of Theorem 4

A complete proof of the theorem follows. Substituting the indifference condition of the L buyer we find $I\left(x_k^*,\theta_k\right)$

$$\begin{aligned}I\left(x_k^*,\theta_k\right)F_H\left(x_k^*\right) &= c_H F_H\left(x_k^*\right)-\int_0^{x_k^*}E_0\left[c|x,\theta_k\right]dF_H\left(x\right)-s_k\\ &= c_H F_H\left(x_k^*\right)-\int_0^{x_k^*}E_0\left[c|x,\theta_k\right]dF_H\left(x\right)-\left(E_0\left[c|x_k^*,\theta_k\right]F_L\left(x_k^*\right)-\int_0^{x_k^*}E_0\left[c|x,\theta_k\right]dF_L\left(x\right)\right)\\ &= \int_0^{x_k^*}\left(c_H-E_0\left[c|x,\theta_k\right]\right)dF_H\left(x\right)-\int_0^{x_k^*}\left(E_0\left[c|x_k^*,\theta_k\right]-E_0\left[c|x,\theta_k\right]\right)dF_L\left(x\right)\end{aligned}$$

and using that

$$dF_H\left(x\right)=\frac{x}{1-x}dF_L\left(x\right)$$

we get that $I(x_k^*, \theta_k)$ has the sign of

$$\int_0^{x_k^*} \left((1 - E_0[c|x,\theta_k]) \frac{x}{1-x} - E_0[c|x_k^*,\theta_k] + E_0[c|x,\theta_k] \right) dF_L(x)$$
$$\int_0^{x_k^*} \left(\left(1 - \frac{1}{1 + \frac{(1-x)}{x}\theta_k} \right) \frac{x}{1-x} - \frac{x_k^*}{x_k^* + (1 - x_k^*)\theta_k} + \frac{1}{1 + \frac{(1-x)}{x}\theta_k} \right) dF_L(x)$$

Let C_k be the likelihood ratio $\frac{f_L(x_k^*)}{f_H(x_k^*)}\theta_k$ at the cutoff seller,

$$E_0[c|x_k^*,\theta_k] = \frac{1}{1 + \frac{dF_L(x^*)}{dF_H(x^*)}\theta} = \frac{1}{1+C_k}.$$

And note that

$$\frac{f_L(x^*)}{f_H(x^*)}\theta_k = \frac{1-x}{x^*}\theta_k = C$$

implies that

$$\theta_k = C_k \frac{x_k^*}{1-x_k^*}$$

We asssume that the price $E_0[c|x_k^*,\theta_k]$ at the cutoff seller converges (along some subsequence); let $\bar{C} = \lim C_k$. We use C_k to rewrite the integral,

$$\begin{aligned}
& \int_0^{x_k^*} \left(\frac{x}{1-x} - \frac{x_k^*}{x_k^* + (1-x_k^*) C_k \frac{x_k^*}{1-x_k^*}} + \frac{x}{x + (1-x) C_k \frac{x_k^*}{1-x_k^*}} \left(1 - \frac{x}{1-x} \right) \right) dF_L(x) \\
= & \int_0^{x_k^*} \left(\left(1 - \frac{1}{1 + \frac{(1-x)}{x} C_k \frac{x_k^*}{1-x_k^*}} \right) \frac{x}{1-x} - \frac{x_k^*}{x_k^* + (1-x_k^*) C_k \frac{x_k^*}{1-x_k^*}} + \frac{1}{1 + \frac{(1-x)}{x} C_k \frac{x_k^*}{1-x_k^*}} \right) dF_L(x) \\
= & \int_0^{x_k^*} \left(\left(1 - \frac{1}{1 + \frac{(1-x)}{x} C_k \frac{x_k^*}{1-x_k^*}} \right) \frac{x}{1-x} - \frac{1}{1+C_k} + \frac{1}{1 + \frac{(1-x)}{x} C_k \frac{x_k^*}{1-x_k^*}} \right) dF_L(x) \\
= & \int_0^{x_k^*} \left(\frac{\frac{(1-x)}{x} C_k \frac{x_k^*}{1-x_k^*}}{1 + \frac{(1-x)}{x} C_k \frac{x_k^*}{1-x_k^*}} \frac{x}{1-x} - \frac{1}{1+C_k} + \frac{1}{1 + \frac{(1-x)}{x} C_k \frac{x_k^*}{1-x_k^*}} \right) dF_L(x)
\end{aligned}$$

Now, we rewrite the integral, by using

$$x(t) = \frac{x^*}{1 + x^* t}$$

which implies that

$$\frac{x(t)}{1-x(t)}=\frac{\frac{x^*}{1+x^*t}}{1-\frac{x^*}{1+x^*t}}=\frac{x^*}{1+x^*t-x^*}=\frac{x^*}{1+x^*(t-1)}$$

and recall that $F^{x_k^*}(t)\equiv 1-\frac{F_L(x(t))}{F_L\left(x_k^*\right)}$ so that

$$f^{x_k^*}(t)=-\frac{f_L(x(t))}{F_L\left(x_k^*\right)}x'(t).$$

Substituting into the integral and rewriting further yields[13]

$$\begin{aligned}
&= \int_0^{x_k^*}\left(\frac{\frac{(1-x)}{x}C_k\frac{x_k^*}{1-x_k^*}}{1+\frac{(1-x)}{x}C_k\frac{x_k^*}{1-x_k^*}}\frac{x}{1-x}-\frac{1}{1+C_k}+\frac{1}{1+\frac{(1-x)}{x}C_k\frac{x_k^*}{1-x_k^*}}\right)dF_L(x)\\
&= -\int_0^{\infty}\left(\frac{\frac{1+x^*(t-1)}{x^*}C_k\frac{x_k^*}{1-x_k^*}}{1+\frac{1+x^*(t-1)}{x^*}C_k\frac{x_k^*}{1-x_k^*}}\frac{x^*}{1+x^*(t-1)}-\frac{1}{1+C_k}+\frac{1}{1+\frac{1+x^*(t-1)}{x^*}C_k\frac{x_k^*}{1-x_k^*}}\right)(f_L(x(t)))\,x'(t)\,dt\\
&= \int_0^{\infty}\left(\frac{\frac{1+x^*(t-1)}{1-x_k^*}C_k}{1+\frac{1+x^*(t-1)}{1-x_k^*}C_k}\frac{x^*}{1+x^*(t-1)}-\frac{1}{1+C_k}+\frac{1}{1+\frac{1+x^*(t-1)}{1-x_k^*}C_k}\right)F_L(x_k^*)\,f^{x_k^*}(t)\\
&= \int_0^{\infty}\left(\frac{(1+x^*(t-1))\,C_k}{1-x_k^*+(1+x^*(t-1))\,C_k}\frac{x^*}{1+x^*(t-1)}-\frac{1}{1+C_k}+\frac{1-x_k^*}{1-x_k^*+(1+x^*(t-1))\,C_k}\right)F_L(x_k^*)\,f^{x_k^*}(t)\\
&= \int_0^{\infty}\left(\frac{C_kx^*}{1-x_k^*+(1+x^*(t-1))\,C_k}+\frac{(1-x_k^*)(1+C_k)-(1-x_k^*+(1+x^*(t-1))\,C_k)}{(1+C_k)\left(1-x_k^*+(1+x^*(t-1))\,C_k\right)}\right)F_L(x_k^*)\,f^{x_k^*}(t)\\
&= x^*F_L(x_k^*)\int_0^{\infty}\left(\frac{C_k}{1-x_k^*+(1+x^*(t-1))\,C_k}-\frac{tC_k}{(1+C_k)\left(1-x_k^*+(1+x^*(t-1))\,C_k\right)}\right)f^{x_k^*}(t)
\end{aligned}$$

Hence

$$I(x_k^*,\theta_k)=\frac{x^*F_L(x_k^*)}{F_H\left(x_k^*\right)}\int_0^{\infty}\left(\frac{C_k}{1-x_k^*+(1+x^*(t-1))\,C_k}-\frac{tC_k}{(1+C_k)\left(1-x_k^*+(1+x^*(t-1))\,C_k\right)}\right)f^{x_k^*}(t)$$

[13] To rewrite the nominator we use
$(1-x_k^*)(1+C_k)-(1-x_k^*+(1+x^*(t-1))\,C_k)=$
$(1-x_k^*)(1+C_k)-(1-x_k^*)-(1+x^*(t-1))\,C_k=$
$(1-x_k^*)\,C_k-(1+x^*(t-1))\,C_k=$
$(1-x_k^*)\,C_k-C_k-x^*tC_k+x^*C_k=$
$-x^*tC_k$

Note that

$$\begin{aligned}\lim \frac{x^* F_L\left(x_k^*\right)}{F_H\left(x_k^*\right)} &= \lim \frac{F_L\left(x_k^*\right)}{f_H\left(x_k^*\right)} + \frac{x^* f_L\left(x_k^*\right)}{f_H\left(x_k^*\right)} \\ &= \lim \frac{F_L\left(x_k^*\right)}{f_H\left(x_k^*\right)} + \left(1 - x_k^*\right) \geq 1. \end{aligned} \tag{5}$$

The integrand converges pointwise

$$\begin{aligned} & \lim_{x_k^* \to 0} \frac{C_k}{1 - x_k^* + \left(1 + x^*\left(t-1\right)\right) C_k} - \frac{tC_k}{\left(1 + C_k\right)\left(1 - x_k^* + \left(1 + x^*\left(t-1\right)\right) C_k\right)} \\ = \;& \frac{\bar{C}}{1 - 0 + \left(1 + 0\right)\bar{C}} - \frac{t\bar{C}}{\left(1 + \bar{C}\right)\left(1 - 0 + \left(1 + 0\left(t-1\right)\right)\bar{C}\right)} \\ = \;& \frac{\bar{C}}{1 + \bar{C}} - \frac{t\bar{C}}{\left(1 + \bar{C}\right)\left(1 + \bar{C}\right)} \\ = \;& \frac{1}{\left(1 + \bar{C}\right)\left(1 + \bar{C}\right)}\left(1 + \bar{C} - t\right) \end{aligned}$$

and if $C_k \geq 1$ (as we will show later), then for $t > 1$, the integrand is bounded from below by a linear function,

$$\begin{aligned} & \frac{C_k}{1 - x_k^* + \left(1 + x^*\left(t-1\right)\right) C_k} - \frac{tC_k}{\left(1 + C_k\right)\left(1 - x_k^* + \left(1 + x^*\left(t-1\right)\right) C_k\right)} \\ \geq \;& -\frac{tC_k}{\left(1 + C_k\right)\left(1 - x_k^* + \left(1 + x^*\left(t-1\right)\right) C_k\right)} \\ = \;& -\frac{tC_k}{\left(1 + C_k\right)\left(1 + C_k + x_k^*\left(C_k\left(t-1\right) - 1\right)\right)} \\ \geq \;& -\frac{tC_k}{\left(1 + C_k\right)\left(1 + C_k\right)}. \end{aligned}$$

The lower bound will allow us to conclude that the integrand does not diverge to infinite fast on its right tail than the mass converges to zero: Suppose $\lambda \in (0, \infty)$. Fix any $T >> 1$. From the observation before and from the way Φ was defined (the limit dominates the elements of the sequence),

$$\begin{aligned} 0 \;&\geq\; \int_T^\infty \left(\frac{C_k}{1 - x_k^* + \left(1 + x^*\left(t-1\right)\right) C_k} - \frac{tC_k}{\left(1 + C_k\right)\left(1 - x_k^* + \left(1 + x^*\left(t-1\right)\right) C_k\right)}\right) f^{x_k^*}(t) \equiv \varepsilon(T) \\ &\geq\; -\int_T^\infty \frac{tC_k}{\left(1 + C_k\right)\left(1 + C_k\right)} e^{-\lambda t} \end{aligned}$$

and for $T \to \infty$, the latter term is zero (since the exponential dominates the linear function). Hence, the value of the intergral on (T, ∞) is $\varepsilon(T)$ and with $T \to \infty$, $\varepsilon(T) \to 0$. Thus, the limit of the integral is

$$\begin{aligned}
& \lim \int_0^\infty \left(\frac{C_k}{1 - x_k^* + (1 + x^*(t-1)) C_k} - \frac{tC_k}{(1 + C_k)\left(1 - x_k^* + (1 + x^*(t-1)) C_k\right)} \right) f^{x_k^*}(t) \\
= \; & \lim \int_0^T \left(\frac{C_k}{1 - x_k^* + (1 + x^*(t-1)) C_k} - \frac{tC_k}{(1 + C_k)\left(1 - x_k^* + (1 + x^*(t-1)) C_k\right)} \right) f^{x_k^*}(t) \\
& + \lim \int_T^\infty \left(\frac{C_k}{1 - x_k^* + (1 + x^*(t-1)) C_k} - \frac{tC_k}{(1 + C_k)\left(1 - x_k^* + (1 + x^*(t-1)) C_k\right)} \right) f^{x_k^*}(t) \\
= \; & \lim_{T\to\infty} \lim \int_0^T \left(\frac{C_k}{1 - x_k^* + (1 + x^*(t-1)) C_k} - \frac{tC_k}{(1 + C_k)\left(1 - x_k^* + (1 + x^*(t-1)) C_k\right)} \right) f^{x_k^*}(t) - \lim_{T\to\infty} \varepsilon(T) \\
= \; & \lim_{T\to\infty} \int_0^T \lim \left(\frac{C_k}{1 - x_k^* + (1 + x^*(t-1)) C_k} - \frac{tC_k}{(1 + C_k)\left(1 - x_k^* + (1 + x^*(t-1)) C_k\right)} \right) \lim f^{x_k^*}(t) \\
= \; & \lim_{T\to\infty} \int_0^T \frac{\bar{C}}{\left(1 + \bar{C}\right)^2} \left(1 + \bar{C} - t\right) \lambda e^{-\lambda t} dt \\
= \; & \frac{\bar{C}}{\left(1 + \bar{C}\right)^2} \left(1 + \bar{C} - \frac{1}{\lambda}\right).
\end{aligned}$$

Suppose $\lambda \in (0, \infty)$. We now characterize equilibria with $\Delta_k \leq 0$. (To characterize equilibria $\Delta_k > 0$, we need to take care of q_k^C.) Under which conditions $\Delta_k < 0$ for all k large enough? If $\Delta_k < 0$, the equilibrium must involve complete pooling in the limit: both buyers search and the expected price must be equal to the prior expected price (the search cost of the L buyer converge to zero; hence, the expected price conditional on $x \leq x_k^*$ must be equal to the price at the cutoff type for indifference.) Hence $C_k \to \bar{C} = 1$. Furthermore, $\Delta_k < 0$ requires $I(x_k^*, \theta_k) > 0$ for all k. Inspecting the limit expression shows that his is the case only if

$$1 + 1 - \frac{1}{\lambda} \geq 0 \Leftrightarrow \lambda \geq \frac{1}{2}.$$

Hence, we will get a pooling equilibrium only if $\lambda \geq \frac{1}{2}$. Now, under which conditions $\Delta_k = 0$ for all k large enough? $\Delta_k = 0$ requires that

$$1 + \bar{C} - \frac{1}{\lambda} = 0.$$

Hence, $\Delta_k = 0$ for k large only if the limit price is

$$\bar{p} = \frac{1}{1+\bar{C}} = \lambda.$$

And hence, $\Delta_k = 0$ only if $\lambda \leq \frac{1}{2}$ (otherwise, $\bar{p} > \frac{1}{2}$, which contradicts seller's zero profits.)

Importantly, we get a separating equilibrium in the limit ($C_k \to \infty$) with $\Delta_k = 0$ only if $\lambda = 0$. If $C_k \to \infty$ and $\lambda > 0$, then the integral converges to one, observing that

$$\lim \int_0^T \left(\frac{C_k}{1 - x_k^* + (1 + x^*(t-1)) C_k} - \frac{tC_k}{(1+C_k)\left(1 - x_k^* + (1 + x^*(t-1)) C_k\right)} \right) f^{x_k^*}(t)$$
$$+ \lim \int_T^\infty \left(\frac{C_k}{1 - x_k^* + (1 + x^*(t-1)) C_k} - \frac{tC_k}{(1+C_k)\left(1 - x_k^* + (1 + x^*(t-1)) C_k\right)} \right) f^{x_k^*}(t)$$

and for any ε, there is a k' and T large enough such that for all k larger than that, the integral is at least

$$1 - \varepsilon + \int_T^\infty (1 - \varepsilon - \varepsilon t) f^{x_k^*}(t)$$

and, given T large enough and $\lambda \in (0, \infty)$, the intergral must be at least $1 - 2\varepsilon$. Hence, if $\Delta_k = 0$ for all k, then the integral is bounded away from zero for large enough k, contradicting $I(x_k^*, \theta_k) = 0$ for all k.

Finally, under what conditions $\Delta_k > 0$ for all k? Note that x_k^* is determined by

$$\begin{aligned} & u - E[c|x_k^*, \theta_k] - \beta_0(x_k^*, \theta_k)\Delta_k \\ = \; & u - \frac{1}{F_L(x_k^*) + \left(1 - F_L(x_k^*)\right) q_k^C} \left[F_L(x_k^*) E[p|x \leq x_k^*] + (1 - F_L(x_k^*)) q_k^C \quad c_L + s_k \right] \end{aligned}$$

Rewriting

$$\begin{aligned} \frac{s_k}{F_L(x_k^*) + \left(1 - F_L(x_k^*)\right) q_k^C} &= \Bigg[E[c|x_k^*, \theta_k] - \frac{F_L(x_k^*)}{F_L(x_k^*) + \left(1 - F_L(x_k^*)\right) q_k^C} E[p|x \leq x_k^*] - \frac{(1 - F_L(x_k^*)) q_k^C}{F_L(x_k^*) + \left(1 - F_L(x_k^*\right.} \\ s_k &= F_L(x_k^*)\left[E[c|x_k^*, \theta_k] - E[p|x \leq x_k^*, L]\right] + (1 - F_L(x_k^*)) q_k^C E[c|x_k^*, \theta_k] + \beta_0(x_k^*, \theta_k \\ \Rightarrow \quad 1 &= \frac{F_L(x_k^*)\left[E[c|x_k^*, \theta_k] - E[p|x \leq x_k^*, L]\right]}{s_k} + \frac{(1 - F_L(x_k^*)) q_k^C \beta_0(x_k^*, \theta_k) c_H + \beta_0}{} \end{aligned}$$

and since $q_k^C \leq s_k$ and $\Delta_k \leq s_k$ we have

$$1 \geq \frac{F_L(x_k^*)\left[E[c|x_k^*, \theta_k] - E[p|x \leq x_k^*, L]\right]}{s_k} + (1 - F_L(x_k^*)) \beta_0(x_k^*, \theta_k) c_H + \beta_0(x_k^*, \theta_k)\left(F_L(x_k^*) + (1 - F_L(x_k^*)) q_k^C\right)$$

which implies by $\beta_0(x_k^*,\theta_k)\to 0$ that the last terms are of vanishing order relative to the first term, hence

$$s_k = F_L(x_k^*)\left[E[c|x_k^*,\theta_k]-E[p|x\le x_k^*,L]\right]+\gamma_k F_L(x_k^*)\left[E[c|x_k^*,\theta_k]-E[p|x\le x_k^*,L]\right]$$

with $\gamma_k\to 0$.

The incentive for the H buyer to search is at least

$$\begin{aligned}
& c_H - E[p|x\le x_k^*] - \frac{s_k}{F_H(x_k^*)} \\
\le\; & c_H - E[p|x\le x_k^*] - \frac{F_L(x_k^*)}{F_H(x_k^*)}\left[E[c|x_k^*,\theta_k]-E[p|x\le x_k^*]\right] - \gamma_k\frac{F_L(x_k^*)}{F_H(x_k^*)}\left[E[c|x_k^*,\theta_k]-E[p|x\le x_k^*]\right] \\
\le\; & I(x_k^*,\theta_k) - \gamma_k\frac{F_L(x_k^*)}{F_H(x_k^*)}\left[E[c|x_k^*,\theta_k]-E[p|x\le x_k^*]\right].
\end{aligned}$$

And note that, whenever $I(x_k^*,\theta_k)$ does not diverge, $\gamma_k\frac{F_L(x_k^*)}{F_H(x_k^*)}\left[E[c|x_k^*,\theta_k]-E[p|x\le x_k^*]\right]$ must vanish to zero.

Suppose $\Delta_k>0$ and $\lambda>0$. Then, $C_k\to\infty$ and, since $\Delta_k>0$, we have $x_k^{**}=1$ and

$$\begin{aligned}
C_k &= \theta_k\frac{1-x_k^*}{x_k^*} \\
&= \frac{1}{F_L(x_k^*)}\frac{1-x_k^*}{x_k^*}
\end{aligned}$$

and we can use this to rewrite

$$\begin{aligned}
I(x_k^*,\theta_k) &= \frac{x^*F_L(x_k^*)}{F_H(x_k^*)}\int_0^\infty\left(\frac{C_k}{1-x_k^*+(1+x^*(t-1))C_k}-\frac{tC_k}{(1+C_k)\left(1-x_k^*+(1+x^*(t-1))C_k\right)}\right)f^{x_k^*}(t) \\
&= \frac{1}{F_H(x_k^*)}\int_0^\infty\left(\frac{1}{1-x_k^*+(1+x^*(t-1))C_k}-\frac{t1}{(1+C_k)\left(1-x_k^*+(1+x^*(t-1))C_k\right)}\right)f^{x_k^*}(t)
\end{aligned}$$

and with $\lambda\in(0,1)$,

$$\begin{aligned}
\lim I(x_k^*,\theta_k) &= \lim\frac{1}{F_H(x_k^*)(1+C_K)}-\frac{1}{\lambda(1+C_k)^2F_H(x_k^*)} \\
&= \lim\frac{1}{F_H(x_k^*)(1+C_K)}\left(1-\frac{1}{\lambda(1+C_k)}\right)
\end{aligned}$$

and from

$$F_H(x_k^*)(1+C_K) = \frac{F_H(x_k^*)}{F_L(x_k^*)}\frac{1-x_k^*}{x_k^*} + F_H(x_k^*)$$

and from (5) we get

$$\begin{aligned}
\lim \frac{F_H(x_k^*)}{F_L(x_k^*)}\frac{1-x_k^*}{x_k^*} + F_H(x_k^*) &= \lim \frac{F_H(x_k^*)}{F_L(x_k^*)\,x_k^*} + 0 \\
&= \lim \frac{f_H(x_k^*)}{F_L(x_k^*)}\frac{1}{x_k^*} \leq 1
\end{aligned}$$

and so

$$\lim I(x_k^*, \theta_k) \geq \lim \left(1 - \frac{1}{\lambda(1+C_k)}\right) = 1.$$

Hence, $I(x_k^*, \theta_k)$ does not diverge and the incentive for the H buyer to search when $\Delta_k > 0$ and $\lambda \in (0, \infty)$ is at least

$$\lim I(x_k^*, \theta_k) - \gamma_k Z_k \geq 1 - 0 > 0.$$

Thus, $\Delta_k > 0$ for all k only if $\lambda = 0$. QED

8.5 Proof of Lemma

We know that when $\lambda > \frac{1}{2}$, $\Delta_k < 0$ for all k (since $\Delta_k = 0$ requires $\lambda \in \left(0, \frac{1}{2}\right)$ while $\Delta_k > 0$ requires $\lambda = 0$. We want to know the limit of $I(x_k^*, \theta_k)$,

$$I(x_k^*, \theta_k) = \frac{x^* F_L(x_k^*)}{F_H(x_k^*)} \int_0^\infty \left(\frac{C_k}{1 - x_k^* + (1 + x^*(t-1))C_k} - \frac{tC_k}{(1+C_k)\left(1 - x_k^* + (1 + x^*(t-1))C_k\right)}\right) f^{x_k^*}(t)$$

and using that

$$\begin{aligned}
C_k &= \theta_k \frac{1-x_k^*}{x_k^*} \\
&= \frac{F_H(x_k^*)}{F_L(x_k^*)}\frac{1-x_k^*}{x_k^*}
\end{aligned}$$

we get

$$I(x_k^*, \theta_k) = \frac{1}{1-x_k^*} \int_0^\infty \left(\frac{1}{1 - x_k^* + (1 + x^*(t-1))C_k} - \frac{t}{(1+C_k)\left(1 - x_k^* + (1 + x^*(t-1))C_k\right)}\right) f^{x_k^*}(t)$$

and by $C_k \to 1$,

$$\begin{aligned}
\lim I\left(x_k^*, \theta_k\right) &= \lim \frac{1}{1-x_k^*} \int_0^\infty \left(\frac{1}{1-x_k^* + \left(1+x^*\left(t-1\right)\right) C_k} - \frac{t}{\left(1+C_k\right)\left(1-x_k^* + \left(1+x^*\left(t-1\right)\right) C_k\right)} \right) f^{x_k^*}(t) \\
&= \int_0^T \left(1 - \frac{t}{(2)(2)}\right) f^*(t) + \lim \varepsilon_k(T) \\
&= 1 - \frac{1}{\lambda 4}.
\end{aligned}$$

QED

8.6 Sketch of Proof for Conjecture 2

Sketch: Let $\bar{p} \in (c_L, g_H c_H + g_L c_L)$. Such a $\bar{p}$ exists. Given s_k small enough, let x_k^p be such that $c_H - \bar{p} = \frac{s_k}{F_H\left(x_k^P\right)}$. Take $x_k^{**} \in \left(x_k^P, 1\right)$ such that with

$$\theta_k = \frac{F_H\left(x_k^P\right) + \left(F_H\left(x_k^{**}\right) - F_H\left(x_k^P\right)\right)}{F_L\left(x_k^P\right)},$$

we have

$$c_H - \frac{1}{F_H\left(x_k^P\right)} \int_a^{x_k^P} \frac{x}{x + (1-x)\theta_k} c_H dF_H(x) = \frac{s_k}{F_H\left(x_k^P\right)}.$$

Such an x_k^{**} exists: With $x^{**} = 1$, $\theta_k \to \infty$, and so $\frac{1}{F_H\left(x_k^P\right)} \int_a^{x_k^P} \frac{x}{x+(1-x)\theta_k} c_H dF_H(x) \to c_L < \bar{p}$. With $x^{**} \to x_k^P$, $\theta_k \to 0$, so $\frac{1}{F_H\left(x_k^P\right)} \int_a^{x_k^P} \frac{x}{x+(1-x)\theta_k} c_H dF_H(x) \to c_H > \bar{p}$. Let $\bar{\theta}$ be the limit of θ_k (a subsequence of θ_k if necessary), then

$$\int_a^{x_k^P} \frac{x}{x + (1-x)\theta_k} c_H dF_H(x) \to \frac{a}{a + (1-a)\bar{\theta}} c_H = \bar{p}.$$

Note that the expected price for the L buyer is the same,

$$\int_a^{x_k^P} \frac{x}{x + (1-x)\theta_k} c_H dF_L(x) \to \frac{a}{a + (1-a)\bar{\theta}} c_H = \bar{p}.$$

Since $a > 0$,

$$\begin{aligned}
\lim_{x_k^P \to a} \frac{s_k}{F_L\left(x_k^P\right)} &= \lim \frac{F_H\left(x_k^P\right)}{F_L\left(x_k^P\right)} \frac{s_k}{F_H\left(x_k^P\right)} \\
&= \frac{a}{1-a} \frac{s_k}{F_H\left(x_k^P\right)} = \frac{a}{1-a}\left(c_H - \bar{p}\right).
\end{aligned}$$

Thus, the L buyer strictly prefers to search, rather than accept c_H

$$c_H - \frac{1}{F_L\left(x_k^P\right)} \int_a^{x_k^P} \frac{x}{x+(1-x)\theta_k} c_H dF_L(x) \to c_H - \bar{p} < \frac{a}{1-a}(c_H - \bar{p}) \leftarrow \frac{s_k}{F_L\left(x_k^P\right)}$$

from $a < \frac{1}{2}$. Also, for s_k small enough, the L buyer strictly prefers to trade at $x = x_k^P$,

$$\frac{x_k^P}{x_k^P + \left(1 - x_k^P\right)\theta_k} c_H - \int_a^{x_k^P} \frac{x}{x+(1-x)\theta_k} c_H dF_L(x) \to 0 < \frac{a}{1-a}(c_H - \bar{p}) \leftarrow \frac{s_k}{F_L\left(x_k^P\right)}. \tag{6}$$

Now, for s_k small enough, a constellation with mechanism offer

$$M_k^P(x) \begin{cases} [0, p, 0, p] & \text{if} \quad x > x_k^{**} \\ [0, 0, 1, c_H] & \text{if} \quad x \in \left(x_k^P, x_k^{**}\right) \\ [1, Ec, 1, Ec] & \text{if} \quad x < x_k^P \end{cases}$$

will be part of an equilibrium. The crucial observation is this: For $x < x_k^P$, the L buyer offers the optimal mechanism. For some $x > x_k^P$, the L buyer would prefer a mechanism with $[1, Ec, 1, Ec]$ rather than not trading (by continuity, following (6)). However, given $\Delta_k = u - c_H - V_H = 0$, whenver the L buyer prefers a mechanism with $q_L > 0$ to not trading, the H buyer strictly prefers the mechanism as well. Thus, devinity has not bite. QED