On the Folk Theorem with One-Dimensional Payoffs and Different Discount Factors

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Abstract

Until now, proving the folk theorem in a game with three or more players required imposing restrictions on the dimensionality of the stage-game payoffs. Fudenberg and Maskin (1986) assume full dimensionality of payoffs, while Abreu, Dutta, and Smith (1994) assume the weaker NEU condition ("nonequivalent utilities"). In this note, we consider a class of n-player games where each player receives the same stage-game payoff, either zero or one. The stage-game payoffs therefore constitute a one dimensional set, violating NEU. We show that if all players have different discount factors, then for discount factors sufficiently close to one, any strictly individually rational payoff profile can be obtained as the outcome of a subgame-perfect equilibrium with public correlation.

Keywords: repeated games, folk theorem, different discount factors
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1 Introduction

For the folk theorem to hold with more than two players, it is necessary to have the ability to threaten any single player with a low payoff, while also offering rewards to the punishing players. In assuming full dimensionality of the interior of the convex hull of the set of feasible stage-game payoffs, Fudenberg and Maskin (1986) guarantee that those individual punishments and rewards exist. Abreu, Dutta, and Smith (1994) show that the weaker NEU condition ("nonequivalent utilities"), whereby no two players have identical preferences in the stage-game, is sufficient for the folk theorem to hold.

When the NEU condition fails, players that have equivalent utilities can no longer be individually punished in equilibrium. Wen (1994) introduces the notion of *effective minmax* payoff, which takes into account the fact that when a player is being minmaxed, another player with equivalent utility might unilaterally deviate and best respond. The effective minmax payoff of a player cannot

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be lower than his individual minmax payoff. Wen shows that the effective minmax is the lower bound on subgame-perfect equilibrium payoffs and establishes a folk theorem: when players are sufficiently patient, any feasible payoff vector can be supported as a subgame-perfect equilibrium, provided it dominates the effective minmax payoff vector.

As pointed out by Lehrer and Pauzner (1999), when players have different discount factors, the set of feasible payoffs in a two-player repeated game is typically larger and of higher dimensionality than the set of feasible stage-game payoffs. In a particular three-player game in which two players have equivalent utilities, Chen (2008) illustrates how with unequal discounting payoffs below the effective minmax may indeed be achieved in equilibrium for one of the players.

In this note, we explore the notion that unequal discounting restores the ability to punish players individually in an n-player game where all players have equivalent utilities. We find that a small difference in the discount factors suffices to hold a player to his individual minmax for a certain number of periods while still being able to reward the punishing players. For discount factors sufficiently close to one, any strictly individually rational payoff, including those dominated by the effective minmax payoff, can be obtained as the outcome of a subgame-perfect equilibrium with public correlation, restoring the validity of the folk theorem.

Although our result is shown in the case of games where all players have equivalent utilities, we conjecture that it extends to weaker cases of violation of NEU, as long as any two players with equivalent utilities have different discount factor. The intuition behind this conjecture is that following Abreu, Dutta, and Smith (1994) we could design specific punishments for each group of players with equivalent utilities and use the difference in discount factors within each group to enforce those specific punishments.

1.1 An Example

	L	R
T	1,1,1	0,0,0
B	0,0,0	0,0,0

C

	L	R
T	0,0,0	0,0,0
B	0,0,0	1,1,1

D

Figure 1: A stage game with one-dimensional payoffs

Consider the stage-game in Figure 1, where Player 1 chooses rows, Player 2 columns and Player 3 matrices. This stage-game is infinitely repeated and the players evaluate payoff streams according to the discounting criterion. When the players share a common discount factor $\delta < 1$, Fudenberg and Maskin (1986, Example 3) show that any subgame-perfect equilibrium yields a payoff of at least 1/4 (the effective minmax) to each player, whereas the individual minmax payoff of each player is zero.[1] The low dimensionality of the set of stage-game payoffs

[1] For example, when Player 1 plays T and Player 2 plays R, Player 3 gets a payoff of 0

weakens the punishment that can be imposed on a player as another player with equivalent utility can deviate and best respond. The inability to achieve subgame-perfect equilibrium payoffs in $(0, 1/4)$ means that the "standard" folk theorem fails in this case.

We show however that if all three players have different discount factors, there exists a subgame-perfect equilibrium in which the payoff to each player is arbitrarily close to zero, the individual minmax, provided that the discount factors are sufficiently close to one. Any payoff in the interval $(0, 1/4)$ can then be achieved in equilibrium, restoring the validity of the folk theorem in the context of this game.

1.2 Notation

We consider an n-player repeated game, where all players have equivalent utilities. We normalize payoffs to be in $\{0, 1\}$ and each player's individual minmax payoff is zero. We use public correlation to convexify the payoff set, although we argue later that this is not necessary. Players have different discount factors, and are ordered according to their patience level: $0 < \delta_1 < \cdots < \delta_{n-1} < \delta_n < 1$. We use an exponential representation of discount factors: $\forall\, i$, $\delta_i := e^{-\Delta \rho_i}$, where $\Delta > 0$ could represent the length of time between two repetitions of the stage game. As $\Delta \to 0$, all discount factors tend to one. The ρ's are strictly ordered: $0 < \rho_n < \cdots < \rho_2 < \rho_1$. We assume that the stage game has a (mixed) Nash equilibrium which yields a payoff $Q < 1$ to all players.[2]

We summarize our assumptions about the game and introduce a notation for the lowest subgame-perfect equilibrium payoff of a player i in the following definitions:

Definition 1. Let $\Gamma(\Delta)$ be the set of n-player infinitely repeated games such that:

A1. The set of stage-game payoffs is one-dimensional and all players receive the same payoff in $\{0, 1\}$.

A2. The stage game has a mixed-strategy Nash equilibrium which yields a payoff of $Q < 1$ to all players.

A3. Each player's pure action individual minmax payoff is zero.

A4. Players evaluate payoff streams according to the discounting criterion, and discount factors are strictly ordered: $0 < \delta_1 < \cdots < \delta_n < 1$, where $\delta_i := e^{-\Delta \rho_i}$.

Note that the stage game of Figure 1 satisfies assumptions A1 to A3 of Definition 1.

whether he plays C or D.

[2]For example in the game of Figure 1, the mixture $\left\{ \left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}\right) \right\}$ is a Nash equilibrium that yields a payoff of $1/4$.

Definition 2. We denote by a_i the lowest subgame-perfect equilibrium payoff of Player i in a game $G_\Delta \in \Gamma(\Delta)$.

For given discount factors, the existence of the $(a_i)_{i=1,\dots,n}$ is ensured by the compactness of the set of subgame-perfect equilibrium payoffs (see Fudenberg and Levine (1983, Lemma 4.2)).

1.3 Main Result and Outline of the Proof

Our main result, Theorem 1, states that for games in $\Gamma(\Delta)$, the lowest subgame-perfect equilibrium payoff of each player goes to zero (the common individual minmax payoff) as discount factors tend to one:

Theorem 1. *Consider an n-player infinitely repeated game $G_\Delta \in \Gamma(\Delta)$. Then $a_i \in O(\Delta)$ for all i.*[3]

Theorem 1 states that for discount factors sufficiently close to one (that is for Δ sufficiently close to zero), the lowest subgame-perfect equilibrium payoff of each player i, a_i, is arbitrarily close to zero. We do not provide a full characterization of the set of subgame-perfect equilibrium payoffs but note that any feasible and strictly individually rational payoff is a subgame-perfect equilibrium payoff. In recent work, Sugaya (2010) characterizes the set of perfect and public equilibrium payoffs in games with imperfect public monitoring when players have different discount factors, under a full-dimensionality assumption.

To prove Theorem 1, we first show that when stage-game payoffs are identical, the lowest subgame-perfect equilibrium payoffs are ordered according to the discount factors (Lemma 1). A player cannot have a lower lowest subgame-perfect equilibrium payoff than another player who is less patient. We then show that the lowest subgame-perfect equilibrium payoffs of the two most patient players (Player $n-1$ and Player n) are arbitrarily close to each other when discount factors tend to one (Lemma 2). This is done by explicitly constructing a subgame-perfect equilibrium of the repeated game.

In a similar way, we then construct a set of subgame-perfect equilibria (one for each player $i \in \{2,\dots,n-1\}$) (Lemma 3) and use those to bound the distance between the lowest subgame-perfect equilibrium payoffs of players i and $i-1$ (Lemma 4). We then show by induction that the lowest subgame-perfect equilibrium payoffs of *any* two players are arbitrarily close to each other as discount factors tend to one (Lemma 5). Finally we show that Player 1's lowest subgame-perfect equilibrium payoff can be made arbitrarily close to zero as discount factors tend to one (Lemma 6). We are then able to conclude and prove Theorem 1.

[3] That is, $\exists\, M \geq 0$ and $\Delta^* > 0$ such that $a_i \leq M \cdot \Delta$ for $\Delta \leq \Delta^*$.

2 Lowest Equilibrium Payoffs

2.1 Strategy Profiles and Incentive Compatibility Constraints

To prove Theorem 1, we explicitly construct several subgame-perfect equilibria of the repeated game. To do so, we consider strategy profiles that give a constant expected stage-game payoff between zero and one (using public correlation) to all players for a given number of periods, and then stage-game payoffs of one forever:

Definition 3. Let $\sigma(\mu, \tau, i)$ be the strategy profile such that:

(i) For τ periods, in each stage-game, players use a public correlating device to generate an expected payoff of μ. When the public correlating device generates a payoff of zero, players minmax Player i.

(ii) In all subsequent periods $t > \tau$, players play an action profile yielding a stage-game payoff of 1 to each player.

(iii) During the first τ periods, deviations by Player i are ignored. After that, if Player i deviates from the equilibrium path, players then play a subgame-perfect equilibrium which gives the lowest possible payoff to Player i, a_i.

(iv) If a deviation by Player $j \neq i$ occurs at any time, players then play a subgame-perfect equilibrium which gives the lowest possible payoff to Player j, a_j.

Assuming that the correlating device generates a payoff of zero at $t = 0$, a player $j \neq i$ will not have an incentive to deviate from $\sigma(\mu, \tau, i)$ if:[4]

$$(1 - \delta_j) + \delta_j a_j \leq \delta_j \left((1 - \delta_j^{\tau-1})\mu + \delta_j^{\tau-1} \right), \tag{1}$$

which can be rewritten as

$$\delta_j^\tau \geq \frac{1 - \delta_j + \delta_j a_j - \delta_j \mu}{1 - \mu}. \tag{2}$$

To prove Theorem 1, we show that there exists a "low" μ and a large τ such that for Δ sufficiently close to zero, $\sigma(\mu, \tau, i)$ is subgame perfect, that is, we show that (2) is satisfied for any $j \neq i$. To do so, we identify the player with the tightest incentive compatibility constraint as j_i^* and find the largest τ such that (2) is satisfied for Player j_i^* (Lemma 3). By a "low" μ we mean that μ must be close to a_{i-1}. To this end, we define a stage-game payoff μ_i that is slightly above a_{i-1}:

[4] Note that if a player has no incentive to deviate if the public correlating device generates a payoff of zero at $t = 0$, he will have no incentive to deviate in subsequent periods either.

Definition 4. For all $i \in \{1, ..., n\}$, let μ_i be such that:[5]

$$\mu_i = \begin{cases} a_{i-1} + \frac{1-\delta_1}{\delta_1} & \text{if } 2 \leq i \leq n, \\ 0 & \text{if } i = 1. \end{cases}$$

2.2 Proof of Theorem 1

In a first step towards Theorem 1 we now show that the lowest subgame-perfect equilibrium payoffs are ordered according to the discount factors (Lemma 1), and that Player n's lowest subgame-perfect equilibrium payoff is arbitrarily close to Player $n-1$'s for Δ close enough to zero (Lemma 2).

Lemma 1. *$\forall i \in \{2, \dots, n\}$, $a_{i-1} \leq a_i$.*

The proof of Lemma 1 is presented in Appendix A. The main idea is to find a stream of payoffs $(z_t)_{t=0,\dots,\infty}$ in $[0,1]^{\mathbb{N}}$ that minimizes Player i's average discounted payoff, given Player $i-1$ is guaranteed his lowest subgame-perfect equilibrium payoff at each stage. By definition, the resulting average discounted payoff for Player i cannot be greater than a_i. We show that the constraints imposed by Player $i-1$'s lowest subgame-perfect equilibrium payoff must all be binding and that $z_t = a_{i-1}$, $\forall t \geq 0$.

Lemma 2. *$|a_n - a_{n-1}| \in O(\Delta)$.*

Proof. Consider the strategy profile $\sigma(\mu_n, \infty, n)$, where $\mu_n = a_{n-1} + \frac{1-\delta_1}{\delta_1}$. We are going to show that this constitutes a subgame-perfect equilibrium.

First, note that in a period in which the public correlating device generates a payoff of one, no player has a one-shot profitable deviation. Secondly, because Player n is being minmaxed in a period in which the public correlating device generates a payoff of zero, he doesn't have a profitable one-shot deviation. Thirdly, because punishment phases consist of subgame-perfect equilibrium strategies, no players has a profitable one-shot deviation during one of those. Thus, to verify that $\sigma(\mu_n, \infty, n)$ is subgame perfect, we only need to check that players $i \leq n-1$ do not have profitable one-shot deviations when the public correlating device generates a payoff of zero.

A deviation from Player $i \leq n-1$ leads at most to a one-off gain of one followed by a payoff of a_i forever. Therefore, there is no one-shot profitable deviation if $(1-\delta_i) + \delta_i a_i \leq \delta_i \left(a_{n-1} + \frac{1-\delta_1}{\delta_1}\right)$, where the right-hand-side is the repeated game payoff to Player i if the public correlation device indicates a zero payoff action profile in that period. This inequality is always satisfied for $i \leq n-1$ as $a_i \leq a_{n-1}$ (Lemma 1) and as $\frac{1-\delta_i}{\delta_i} \leq \frac{1-\delta_1}{\delta_1}$.

By definition of a_n, and by Lemma 1, we have that $a_{n-1} \leq a_n \leq a_{n-1} + \frac{1-\delta_1}{\delta_1}$. We conclude the proof by noting that $a_n - a_{n-1} \leq \frac{1-\delta_1}{\delta_1}$ and that $\frac{1-\delta_1}{\delta_1} \in O(\Delta)$. □

[5] Note that for all i and for Δ sufficiently close to zero, $\mu_i \leq 1$. Indeed, $\mu_i \leq Q + \frac{1-\delta_1}{\delta_1} \to_{\Delta \to 0} Q < 1$.

We have shown that the lowest subgame-perfect equilibrium payoffs of the two most patient players are arbitrarily close as Δ tends to zero. The intuition behind this result is that all players can collude against Player n by minmaxing him whenever the public correlation device generates a payoff of zero. Since Player $n-1$ is the most patient of the colluding players and since lowest subgame-perfect equilibrium payoffs are ordered according to discount factors, his lowest subgame-perfect equilibrium will determine by how much Player n's equilibrium payoff can be pushed down.

We now show that the lowest subgame-perfect equilibrium payoffs of *any* two players are arbitrarily close to each other as Δ tends to zero (Lemma 5). We start by identifying bounds on Player $i>1$'s lowest subgame-perfect equilibrium payoff. To do this, we find the largest time $\tau \geq 1$ such that the strategy profile $\sigma(\mu_i, \tau, i)$ is a subgame-perfect equilibrium and compute its equilibrium payoff for Player i. We then show Lemma 5 by induction.

First, we introduce some useful notation. For every player $i \in \{1, \dots, n-1\}$, define

$$N_+^i := \{j > i : 1 - \delta_j + \delta_j a_j - \delta_j \mu_i > 0\}.$$

When proving that for a particular τ, $\sigma(\mu_i, \tau, i)$ is a subgame-perfect equilibrium, N_+^i should be thought of as the set of players for whom profitable deviations might exist depending on the value of τ. That is, N_+^i is the set of players for whom the right-hand side of (2) (when replacing μ with μ_i) is strictly positive. We will therefore chose τ to satisfy the non-deviation constraints of all players in N_+^i. When N_+^i is not empty, we identify the player from this set with the tightest constraint as j_i^* and we define $\widetilde{t_i}$ as follows:

$$j_i^* := \arg\min_{j \in N_+^i} \frac{\log\Big((1 - \delta_j + \delta_j a_j - \delta_j \mu_i) \,/\, (1 - \mu_i) \Big)}{\log \delta_j},$$

$$\widetilde{t_i} := \frac{\log\Big(\big(1 - \delta_{j_i^*} + \delta_{j_i^*} a_{j_i^*} - \delta_{j_i^*} \mu_i\big) \,/\, (1 - \mu_i) \Big)}{\log \delta_{j_i^*}}.$$

Let $t_i^* := \lfloor \widetilde{t_i} \rfloor$ be the largest integer smaller or equal than $\widetilde{t_i}$ and define $r_i \in (0,1)$ to be the fractional part of $\widetilde{t_i}$:

$$r_i := \widetilde{t_i} - t_i^*.$$

Note that t_i^* is the longest time τ such that j_i^* does not have a profitable one-shot deviation in $\sigma(\mu_i, \tau, i)$.

In Lemma 3 we show that t_i^* is well defined and arbitrarily large and that the strategy profile $\sigma(\mu_i, t_i^*, i)$ is indeed subgame perfect for Δ sufficiently close to zero.

Lemma 3. *Let $i \in \{2, ..., n-1\}$, and assume that $N_+^i \neq \emptyset$. Given j_i^*, t_i^* and μ_i, $\exists\, \Delta_i^* > 0$ such that for $\Delta \in (0, \Delta_i^*)$, $\sigma(\mu_i, t_i^*, i)$ constitutes a subgame-perfect equilibrium.*

Proof. For notational convenience, we omit the i subscript on j_i^*, $\widetilde{t}_i$, t_i^*, and r_i. First, recall that for Δ sufficiently close to zero, $\mu_i \leq 1$.[6] We now check that t^* is well defined. Note that $\exists\, \Delta_{ij} > 0$ and $\eta_{ij} < 1$ such that for $\Delta \leq \Delta_{ij}$, $\frac{1-\delta_j+\delta_j a_j-\delta_j\mu_i}{1-\mu_i} < \eta_{ij}$.[7] Because η_{ij} does not depend on Δ, this shows that $\lim_{\Delta\to 0}\widetilde{t} = \infty$ and ensures that $\exists\, \Delta_i^* > 0$ such that t^* is well defined and strictly positive for $\Delta \in (0, \Delta_i^*)$.

Because i is being minmaxed if the public correlating device generates a payoff of zero, i does not have a profitable one-shot deviation. Also, no player will have a profitable one-shot deviation during the punishment phases of $\sigma(\mu_i, t_i^*, i)$, as those are subgame perfect.

We now check that no player $j \neq i$ has a profitable one-shot deviation, that is, we check that (1) (when replacing μ with μ_i and τ with t^*) holds for all players $j \neq i$:

$$(1-\delta_j) + \delta_j a_j \leq \delta_j \left((1-\delta_j^{t^*-1})\mu_i + \delta_j^{t^*-1}\right). \tag{3}$$

We first check that (3) holds for players $j \leq i-1$ and then for players $j > i$:

(i) No deviation from player $j \leq i-1$: Note that because $\mu_i \in [0,1]$, we have that $\mu_i \leq \left(1-\delta_j^{t^*-1}\right)\mu_i + \delta_j^{t^*-1}$. In order to show that (3) holds, we can therefore show that $(1-\delta_j) + \delta_j a_j \leq \delta_j \mu_i$, which is equivalent to $\frac{1-\delta_j}{\delta_j} + a_j \leq a_{i-1} + \frac{1-\delta_1}{\delta_1}$. This inequality holds $\forall j \leq i-1$, as $\frac{1-\delta_j}{\delta_j} \leq \frac{1-\delta_1}{\delta_1}$ and $a_j \leq a_{i-1}$.

(ii) No deviation from player $j > i$: We can rearrange (3) to get

$$\delta_j^{t^*} \geq \frac{1-\delta_j+\delta_j a_j-\delta_j\mu_i}{1-\mu_i}. \tag{4}$$

First, note that if $j \notin N_+^i$ then j has no incentive to deviate as $\delta_j^{t^*} > 0 \geq \frac{1-\delta_j+\delta_j a_j-\delta_j\mu_i}{1-\mu_i}$. Now let $j \in N_+^i$. Since t^* has been chosen such that (4) is satisfied for player j^*, (4) is also satisfied for all other players in N_+^i, and no player $j \in N_+^i$ will have an incentive to deviate.

We conclude that for Δ sufficiently close to zero, $\sigma(\mu_i, t_i^*, i)$ is a subgame-perfect equilibrium. □

Note on public correlation. In Lemma 3, we show that $\sigma(\mu_i, t_i^*, i)$ is a subgame-perfect equilibrium and that t_i^* goes to infinity as Δ approaches zero. Instead of using the strategy $\sigma(\mu_i, t_i^*, i)$, which relies on public correlation, we can consider a deterministic strategy that alternates between $t_{i,1}^*$ zeros and $t_{i,2}^*$ ones, where

[6] See footnote 5.

[7] Since $a_j \leq Q$, $\frac{1-\delta_j+\delta_j a_j-\delta_j\mu_i}{1-\mu_i} \leq \delta_j \frac{Q-\mu_i}{1-\mu_i} + \frac{1-\delta_j}{1-Q-(1-\delta_1)/\delta_1}$. For any x in $[0,1)$, $\frac{Q-x}{1-x} \leq Q$, thus the right-hand-side of the previous inequality is bounded from above by $\delta_j Q + \frac{1-\delta_j}{1-Q-(1-\delta_1)/\delta_1}$, which tends to $Q < 1$ as Δ tends to zero.

$t^*_{i,1} + t^*_{i,2} = t^*_i$ and $t^*_{i,2}/t^*_i$ is arbitrarily close to μ_i, starting with a payoff of zero. This is possible because t^*_i goes to infinity. Intuitively, as Δ goes to zero, such a strategy will yield a payoff to any player arbitrarily close to the payoff from $\sigma(\mu_i, t^*_i, i)$, while having a period-zero incentive compatibility constraint less stringent than (3) since μ_i is promised on average over the first t^*_i periods and the first period payoff is a zero. This should ensure that Lemmas 3 and 4 still hold under such a deterministic strategy.

We now compute the payoff of player i from $\sigma(\mu_i, t^*_i, i)$ in order to bound the distance between a_i and a_{i-1}.

Lemma 4. $\forall i \in \{2, \dots, n-1\}$, *we have that either:*

(i) $\forall\, j > i$, $|a_j - a_{i-1}| \in O(\Delta)$, *or*

(ii) $|a_i - a_{i-1}| \in O(\Delta) + O\left(a_{j^*_i} - a_i\right)$, *where* $j^*_i > i$.

Proof. Again, for notational convenience, we omit the i subscript on j^*_i, t^*_i and r_i. If N^i_+ is empty we directly have an indication of the distance between a_j and a_{i-1} by noting that no player $j > i$ has an incentive to deviate from $\sigma(\mu_i, \tau, i)$, irrespective of τ: if $N^i_+ = \emptyset$, then $\forall\, j > i$, $0 \leq a_j - a_{i-1} \leq \frac{1-\delta_1}{\delta_1} - \frac{1-\delta_j}{\delta_j}$, which implies that $|a_j - a_{i-1}| \in O(\Delta)$.

Assume now that $N^i_+ \neq \emptyset$, so that $\sigma(\mu_i, t^*, i)$ is a subgame-perfect equilibrium. We now compute Player i's payoff from $\sigma(\mu_i, t^*, i)$ and compare it with his lowest subgame-perfect equilibrium payoff. The payoff to Player i from the strategy profile $\sigma(\mu_i, t^*, i)$ is:

$$
\begin{aligned}
(1-\delta_i^{t^*})\mu_i + \delta_i^{t^*} &= \mu_i + \delta_i^{t^*}(1-\mu_i) \\
&= \mu_i + \delta_i^{-r}\left(\frac{1-\delta_{j^*}+\delta_{j^*}a_{j^*}-\delta_{j^*}\mu_i}{1-\mu_i}\right)^{\frac{\rho_i}{\rho_{j^*}}}(1-\mu_i) \\
&\geq a_i,
\end{aligned}
$$

where the last inequality holds because a_i is i's lowest subgame-perfect equilibrium payoff. This inequality can be rewritten as

$$
\frac{a_i-\mu_i}{1-\mu_i} \leq \delta_i^{-r}\left(\frac{1-\delta_{j^*}+\delta_{j^*}a_{j^*}-\delta_{j^*}\mu_i}{1-\mu_i}\right)^{\frac{\rho_i}{\rho_{j^*}}-1}\left(\frac{1-\delta_{j^*}+\delta_{j^*}a_{j^*}-\delta_{j^*}\mu_i}{1-\mu_i}\right),
$$

where $\frac{\rho_i}{\rho_{j^*}} - 1 > 0$, as $i < j^*$. Recall from the proof of Lemma 3 that for $\Delta \leq \Delta_{ij^*}$, $(1-\delta_{j^*}+\delta_{j^*}a_{j^*}-\delta_{j^*}\mu_i)/(1-\mu_i) < \eta_{ij^*}$, where $\eta_{ij^*} < 1$ does not depend on Δ. For $\Delta \leq \Delta_{ij^*}$, we therefore have:

$$
\frac{a_i-\mu_i}{1-\mu_i} \leq \delta_i^{-r}\eta_{ij^*}^{\frac{\rho_i}{\rho_{j^*}}-1}\left(\frac{1-\delta_{j^*}+\delta_{j^*}a_{j^*}-\delta_{j^*}\mu_i}{1-\mu_i}\right).
$$

The previous inequality can be rewritten as:[8]

$$a_i - a_{i-1} \leq \frac{1-\delta_1}{\delta_1} + \delta_i^{-r} \eta_{ij^*}^{\frac{\rho_i}{\rho_{j^*}}-1} \delta_{j^*} \left(a_i - a_{i-1}\right) + \delta_i^{-r} \eta_{ij^*}^{\frac{\rho_i}{\rho_{j^*}}-1} \left(1 - \delta_{j^*} + \delta_{j^*} \left(a_{j^*} - a_i\right) - \delta_{j^*} \frac{1-\delta_1}{\delta_1}\right).$$

Because

$$\lim_{\Delta \to 0} \delta_i^{-r} \eta_{ij^*}^{\frac{\rho_i}{\rho_{j^*}}-1} \delta_{j^*} = \lim_{\Delta \to 0} \delta_i^{-r} \eta_{ij^*}^{\frac{\rho_i}{\rho_{j^*}}-1} = \eta_{ij^*}^{\frac{\rho_i}{\rho_{j^*}}-1} < 1,$$

there exists a $\widetilde{\Delta_i} \geq 0$ and an $R < 1$ such that for $\Delta \leq \widetilde{\Delta_i}$ we have:

$$a_i - a_{i-1} \leq \frac{1-\delta_1}{\delta_1} + R\left(a_i - a_{i-1}\right) + R\left(1 - \delta_{j^*} + \delta_{j^*}\left(a_{j^*} - a_i\right) - \delta_{j^*}\frac{1-\delta_1}{\delta_1}\right).$$

To conclude, note that $\frac{1-\delta_1}{(1-R)\delta_1} + \frac{R}{1-R}\left(1 - \delta_{j^*} - \delta_{j^*}\frac{1-\delta_1}{\delta_1}\right) \in O\left(\Delta\right)$, and that $\frac{R}{1-R}\delta_{j^*}\left(a_{j^*} - a_i\right) \in O\left(a_{j^*} - a_i\right)$, as $R < 1$ is a fixed constant. □

Recall that the difference between the two most patient players' lowest subgame-perfect equilibrium payoffs, a_n and a_{n-1}, is of order Δ (Lemma 2). Moreover in Lemma 4 we established a bound for the distance between a_{i-1} and the lowest subgame-perfect equilibrium payoff of a more patient player. We can now establish by induction that the lowest subgame-perfect equilibrium payoffs of *any* two players are arbitrarily close to each other as Δ tends to zero.

Lemma 5. *$|a_i - a_j| \in O(\Delta)$, $\forall\,(i, j)$.*

Proof. By Lemma 2, we know that this result is true for $i, j \in \{n-1, n\}$. We now prove this result by induction. Assume that $\forall i, j \geq k$, $|a_i - a_j| \in O(\Delta)$. Our aim is to show that $\forall i \geq k$, $|a_i - a_{k-1}| \in O(\Delta)$.

If the first statement of Lemma 4 holds, then we have that $\forall j > k$, $|a_j - a_{k-1}| \in O\left(\Delta\right)$. Moreover, $|a_k - a_{k-1}| \leq |a_k - a_j| + |a_j - a_{k-1}|$ for any $j > k$. By induction, $|a_k - a_j| \in O\left(\Delta\right)$, thus we have $|a_k - a_{k-1}| \in O\left(\Delta\right)$.

If the second statement of Lemma 4 holds then $\exists\, k^* > k$ such that $|a_k - a_{k-1}| \in O\left(\Delta\right) + O\left(a_{k^*} - a_k\right)$. From our induction hypothesis, $|a_{k^*} - a_k| \in O\left(\Delta\right)$, which implies that $|a_k - a_{k-1}| \in O\left(\Delta\right)$. Using the triangle inequality, $\forall i \geq k$, $|a_i - a_{k-1}| \leq |a_i - a_k| + |a_k - a_{k-1}| \in O\left(\Delta\right)$.

This shows that $\forall i, j \geq k-1$, $|a_i - a_j| \in O(\Delta)$. □

Finally, we show that the lowest subgame-perfect equilibrium payoff of Player 1 is arbitrarily close to zero as Δ tends to zero. This is done by using a proof similar to the one of Lemma 4, and considering the strategy profile $\sigma(0, t_1^*, 1)$.

Lemma 6. *$a_1 \in O\left(\Delta\right)$.*

[8] By canceling the $1 - \mu_i$ and adding and subtracting $\delta_{j^*} a_i$ inside the term in parentheses.

The proof of Lemma 6 is presented in Appendix B. We are now able to prove Theorem 1:

Prood of Theorem 1. From Lemma 5 and 6, we have that $\forall i \in \{1, \ldots, n\}$, $|a_i - a_1| \in O(\Delta)$ and $a_1 \in O(\Delta)$. Using the triangle inequality, $|a_i| \leq |a_i - a_1| + |a_1| \in O(\Delta)$. $\square$

3 Conclusion

We have restored the validity of the folk theorem in games where full dimensionality / NEU is violated and the stage-game payoff set is one-dimensional. The dimensionality typically required to punish and reward players individually is obtained through the assumption that all players have different discount factors, allowing them to trade payoffs over time.

A Proof of Lemma 1

To find a lower bound on player $i \geq 2$'s lowest subgame-perfect equilibrium payoff, we find a stream of payoffs $(z_t)_{t=0,\ldots,\infty}$ in $[0,1]^{\mathbb{N}}$ that minimizes player i's average discounted payoff, given player $i-1$ is guaranteed his lowest subgame-perfect equilibrium payoff at each stage. By definition, the solution to this minimization problem cannot yield a payoff to player i which is greater than his lowest subgame-perfect equilibrium payoff.

Formally, we solve the following minimization problem:

$$\min_{(z_t)_{t=0,\ldots,\infty} \in [0,1]^{\mathbb{N}}} (1-\delta_i) \sum_{t=0}^{\infty} \delta_i^t \, z_t \tag{5}$$

subject to

$$(1-\delta_{i-1}) \sum_{t=s}^{\infty} \delta_{i-1}^{t-s} \, z_t \geq a_{i-1}, \qquad \forall s \geq 0 \tag{6}$$

We show by induction that all constraints in (6) will be binding, which implies that $z_s = a_{i-1}$, $\forall s \geq 0$. Our induction hypothesis is that the constraints in (6) must bind for $s = 0, \ldots, \tau$ and therefore, that the minimization problem (5) subject to the constraints (6) can be rewritten as:

$$\min_{(z_t)_{t=\tau,\ldots,\infty} \in [0,1]^{\mathbb{N}}} \lambda_{\tau-1} \left(a_{i-1}, \delta_{i-1}, \delta_i\right) + (1-\delta_i) \left(\sum_{t=\tau+1}^{\infty} \delta_i^{\tau} \left(\delta_i^{t-\tau} - \delta_{i-1}^{t-\tau}\right) \, z_t \right) \tag{7}$$

subject to

$$(1-\delta_{i-1}) \sum_{t=s}^{\infty} \delta_{i-1}^{t-s} \, z_t \geq a_{i-1}, \qquad \forall s \geq \tau + 1 \tag{8}$$

where the function λ_τ is defined by $\lambda_0 \left(a_{i-1}, \delta_{i-1}, \delta_i\right) = (1-\delta_i) \frac{a_{i-1}}{1-\delta_{i-1}}$ and $\lambda_\tau \left(a_{i-1}, \delta_{i-1}, \delta_i\right) = \lambda_{\tau-1} \left(a_{i-1}, \delta_{i-1}, \delta_i\right) + (1-\delta_i) \, \delta_i^{\tau} + (\delta_i - \delta_{i-1}) \frac{a_{i-1}}{1-\delta_{i-1}}$.

Initialization: $\tau = 0$

The first constraint is the only constraint featuring z_0 and can be rewritten as $z_0 \geq \frac{a_{i-1}}{1-\delta_{i-1}} - \sum_{t=1}^{\infty} \delta_{i-1}^{t} \, z_t$. Moreover, z_0 enters with a positive coefficient in the objective function, therefore, the first constraint must be binding. The constraint is then used to eliminate z_0 from the objective function: the minimization problem (5) subject to (6) can therefore be written in the following way:

$$\min_{(z_t)_{t=1,\ldots,\infty} \in [0,1]^{\mathbb{N}}} (1-\delta_i) \left(\frac{a_{i-1}}{1-\delta_{i-1}} + \sum_{t=1}^{\infty} \left(\delta_i^t - \delta_{i-1}^t \right) \, z_t \right)$$

subject to

$$(1-\delta_{i-1}) \sum_{t=s}^{\infty} \delta_{i-1}^{t-s} \, z_t \geq a_{i-1}, \qquad \forall s \geq 1$$

This verifies (7) and (8).

Induction

We assume that our minimization problem can be rewritten as (7) subject to (8) for some $\tau > 1$. Because $\delta_i > \delta_{i-1}$, $z_{\tau+1}$ enters with a positive coefficient in the objective function and $z_{\tau+1}$ only appears in the constraint $z_{\tau+1} \geq \frac{a_{i-1}}{1-\delta_{i-1}} - \sum_{t=\tau+2}^{\infty} \delta_{i-1}^{t-(\tau+1)} \, z_t$, this constraint will be binding and the objective function can be rewritten by substituting for $z_{\tau+1}$ as follows:

$$\begin{aligned}
&\lambda_{\tau-1} \left(a_{i-1}, \delta_{i-1}, \delta_i\right) + (1-\delta_i) \left(\sum_{t=\tau+1}^{\infty} \delta_i^{\tau} \left(\delta_i^{t-\tau} - \delta_{i-1}^{t-\tau} \right) \, z_t \right) \\
&= \lambda_{\tau-1} \left(a_{i-1}, \delta_{i-1}, \delta_i\right) + (1-\delta_i) \left(\delta_i^{\tau} \left(\delta_i - \delta_{i-1}\right) \left(\frac{a_{i-1}}{1-\delta_{i-1}} - \sum_{t=\tau+2}^{\infty} \delta_{i-1}^{t-(\tau+1)} \, z_t \right) \right) \\
&\quad + (1-\delta_i) \sum_{t=\tau+2}^{\infty} \delta_i^{\tau} \left(\delta_i^{t-\tau} - \delta_{i-1}^{t-\tau} \right) \, z_t \\
&= \lambda_{\tau} \left(a_{i-1}, \delta_{i-1}, \delta_i\right) + (1-\delta_i) \sum_{t=\tau+2}^{\infty} \left(\delta_i^{\tau} \left(\delta_i^{t-\tau} - \delta_{i-1}^{t-\tau} \right) - \delta_i^{\tau} \left(\delta_i - \delta_{i-1}\right) \delta_{i-1}^{t-(\tau+1)} \right) z_t \\
&= \lambda_{\tau} \left(a_{i-1}, \delta_{i-1}, \delta_i\right) + (1-\delta_i) \left(\sum_{t=\tau+2}^{\infty} \delta_i^{\tau+1} \left(\delta_i^{t-(\tau+1)} - \delta_{i-1}^{t-(\tau+1)} \right) \, z_t \right),
\end{aligned}$$

where the first equality is obtained by substituting for $z_{\tau+1}$ and the other equalities are obtained by grouping the terms in z_t ($t \geq \tau + 2$) together. Thus (7) and (8) hold for $\tau + 1$ also.

This concludes the proof by induction and so all constraints in (6) must bind: $(1-\delta_{i-1}) \sum_{t=s}^{\infty} \delta_{i-1}^{t-s} \, z_t = a_{i-1}$, $\forall s \geq 0$. We now show that this implies that $z_s = a_{i-1}$, $\forall s \geq 0$. Consider the constraint for some $s \geq 0$:

$$\begin{aligned} a_{i-1} &= (1-\delta_{i-1}) \sum_{t=s}^{\infty} \delta_{i-1}^{t-s} z_t \\ &= (1-\delta_{i-1}) \left\{ z_s + \delta_{i-1} \sum_{t=s+1}^{\infty} \delta_{i-1}^{t-(s+1)} z_t \right\} \\ &= (1-\delta_{i-1}) \left\{ z_s + \frac{\delta_{i-1}}{1-\delta_{i-1}} a_{i-1} \right\}, \end{aligned}$$

where the last inequality holds because the constraint is binding for $s+1$. This implies that $z_s = a_{i-1}$, $\forall s \geq 0$.

Given the constraints imposed on stage-game payoffs by player $i-1$'s lower subgame-perfect equilibrium bound, the lowest average discounted payoff which can be given to player i is a_{i-1}. We therefore have $a_{i-1} \leq a_i$. □

B Proof of Lemma 6

We follow the same line of reasoning as in the proof of Lemma 3 and Lemma 4, using the strategy $\sigma(0, t_1^*, 1)$. As in Lemma 3, $\sigma(0, t_1^*, 1)$ is well defined and constitutes a subgame-perfect equilibrium. Again, for notational convenience, we omit the subscript 1.

The strategy profile $\sigma(0, t^*, 1)$ yields a payoff of $\delta_1^{t^*} = \delta_1^{-r} \left(1 - \delta_{j^*} + \delta_{j^*} a_{j^*}\right)^{\frac{\rho_1}{\rho_{j^*}}}$ to Player 1. Because a_1 is player 1's lowest subgame-perfect equilibrium payoff, we have

$$\begin{aligned} a_1 &\leq \delta_1^{-r} \left(1 - \delta_{j^*} + \delta_{j^*} a_{j^*}\right)^{\frac{\rho_1}{\rho_{j^*}}} \\ &= \delta_1^{-r} \left(1 - \delta_{j^*} + \delta_{j^*} a_{j^*}\right)^{\frac{\rho_1}{\rho_{j^*}} - 1} \big(1 - \delta_{j^*} + \delta_{j^*} \left(a_{j^*} - a_1\right)\big) \\ &\quad + \delta_1^{-r} \delta_{j^*} \left(1 - \delta_{j^*} + \delta_{j^*} a_{j^*}\right)^{\frac{\rho_1}{\rho_{j^*}} - 1} a_1. \end{aligned}$$

Because

$$\lim_{\Delta \to 0} \delta_1^{-r} \delta_{j^*} \left(1 - \delta_{j^*} + \delta_{j^*} a_{j^*}\right)^{\frac{\rho_1}{\rho_{j^*}} - 1} = \lim_{\Delta \to 0} \delta_1^{-r} \left(1 - \delta_{j^*} + \delta_{j^*} a_{j^*}\right)^{\frac{\rho_1}{\rho_{j^*}} - 1} \leq \eta_{1j^*}^{\frac{\rho_1}{\rho_{j^*}} - 1},$$

and $\eta_{1j^*}^{\frac{\rho_1}{\rho_{j^*}} - 1} < 1$ there exists an $R < 1$ and $\Delta_1^* \geq 0$ such that for $\Delta \leq \Delta_1^*$ we have

$$a_1 \leq R \bigg(1 - \delta_{j^*} + \delta_{j^*} \left(a_{j^*} - a_1\right)\bigg) + R a_1,$$

or

$$a_1 \leq \frac{R}{1-R} \bigg(1 - \delta_{j^*} + \delta_{j^*} \left(a_{j^*} - a_1\right)\bigg).$$

We know from Lemma 5 that $a_{j^*} - a_1 \in O(\Delta)$, which concludes the proof, as $R < 1$ does not depend on Δ. □

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