Dynamic Bilateral Trading in Networks*

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Abstract

I study a *dynamic market-model* where a set of agents, located in a *network* that dictates who can trade with whom, engage in *bilateral trading* for a *single object* under *asymmetric information* about the private values. My equilibrium characterization provides new insights into how economic networks shape trading outcomes. Traders who link otherwise disconnected areas of the trading network become intermediaries. They pay the object at their resale values but, if they have a high value, they consume and extract a positive rent. All other traders, except for the initial owner of the object, make zero profit. The object travels along a chain of intermediaries before someone consumes it. Intermediaries who are located later in the trading chain have a lower probability of acquiring the object, but they pay lower prices for it. Compounding, early intermediaries gain a payoff advantage over late ones. Adding links to the network increases downstream competition and it is beneficial to the initial owner. However, it has ambiguous effects on the other traders and *may* be detrimental to total welfare, when information is asymmetric. More generally, inefficient outcomes are possible if information is not complete *and* the network is not fully connected.

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1 Introduction

I consider a *finite-horizon dynamic model* where a set of agents located in an exogenous *network* engage in *bilateral trading* for a *single object*, initially owned by one of them. Traders are risk neutral and have a *high or low monetary evaluation* for the object, which is *private information*. Values are drawn from the same support and are independently but possibly asymmetrically distributed. In each round the owner of the good can either consume the good, make a take-it-or-leave-it offer to a single agent located in his neighborhood, or wait one round. If an offer is made and accepted, the terms of trade are enforced. Thereafter, a new round of trading starts. The game ends if someone consumes the object or a known deadline is reached. Traders do not discount the future and all actions are publicly observed.

This model provides a novel framework to investigate how the existence of a network of potential transactions, in which agents are embedded, affect their trading and bargaining outcomes under asymmetric information. In particular, the model captures well the main features of *over-the-counter* (OTC) trading in financial instruments. OTC refers to bilateral trading between buyers and sellers, as opposed to trading in centralized exchanges.[1] Products such as credit default swaps, forward rate agreements, and exotic options are almost always traded in this manner. These instruments are often associated with a specific maturity date and can change hands several times before reaching a consumer. Furthermore, bilateral transactions are not guaranteed by the stock exchange, and are subject to counter-party risk. Therefore, bonds of trusts between trading firms are particularly important, and represent the main source of the trading network structure in which OTC transactions take place.[2]

A number of insights emerge from the characterization of a set of *perfect Bayesian equilibria* for the market game outlined above. Two types of *active* traders arise endogenously in each equilibrium, *final customers* and *intermediaries*.[3] Final customers receive only offers at a high price, that they accept only when they have a high value for the object, and leave them with zero profit. Intermediaries acquire the object at lower prices, equal to their resale values. They resell the object in their neighborhood if their value is low, while they consume

[1]The notational amount of OTC derivatives outstanding at the end of 2008 exceeded 500 trillion dollars, according to statistics from the Bank for International Settlements.

[2]See Allen and Babus (2009) for a survey of papers on financial networks. This model could be applied also to international trading networks (e.g. see Rauch (1999), (2001) and Casella and Rauch (2002))

[3]A trader is active if he takes at least one action with positive probability along the equilibrium path.

it and make a positive profit, if their value is high. Whether a trader becomes an intermediary, a final customer or remains inactive is determined jointly by the prior information and the network structure. Traders on the periphery of the network and high value traders tend to become final customers, while players that are essential to provide access to valuable areas of the network or have a low expected value become intermediaries (see subsection 5.1). This is in line with a large body of empirical evidence illustrating that bridging two areas of an economic network that would otherwise be disconnected provides a payoff advantage. See for example the analysis of *structural holes* in Burt (1992).

In equilibrium, the object travels along a *chain* of intermediaries who make offers to final customers, until consumption takes place. Intermediaries acquiring the object later have a smaller probability of realizing a profit upon consumption. Though, the price that they pay is lower, because the resale values decrease as rounds pass, every time an offer is rejected. In general, the first effect dominates the second, and intermediaries who acquire the object earlier in the trading chain are better off than those who acquire it later (see subsection 5.2).[4]

Ex-post efficiency is attained under complete information (see section 3), when the network is fully connected, or if players are ex-ante identical. However, in general, the interplay between asymmetric information and network structure can generate inefficiencies (see subsection 5.4). In fact some high value player might remain inactive and never receive an offer, even if all other traders have a low value for the object.

Comparative statics show that the initial owner is better off when the network becomes more connected. More generally, an increase in connectivity will increase, *ceteris paribus*, the resale value of each trader. However, for traders other than the initial owner, an increase in the resale value has ambiguous effects. It may harm an intermediary, because it increases the price that he has to pay for the object. It may be beneficial to a final customer, if it changes him in to an intermediary. Moreover, I show with an example that an increase in connectivity can also represent a source of inefficiency (see subsection 5.3).[5]

This paper is a first attempt to introduce asymmetric information into a dynamic model

[4]Intermediaries can also receive other offers earlier in the game, before they acquire the good for resale, at a price that they will only accept if they have a high value. When this is the case, a clear-cut payoff ranking, favoring early over late intermediaries, is available only if traders are ex-ante identical.

[5]In addition, subsection 6.1 analyzes networks with a large number of traders and subsection 6.2 characterize optimal networks.

of exchange in networks. Its main contribution is to the literature on the impact of network structures on market outcomes, a topic which has not been explored exhaustively.[6] Kranton and Minehart (2001) was perhaps the first paper to provide a non-cooperative model of exchange in a network. They study networks of buyers and sellers under asymmetric information. Sellers use auctions to sell their goods, after which no further trade takes place. Therefore, intermediation and resale do not figure in their model. Blume et al. (2009) study a two stage game of complete information, where intermediaries, sellers and buyers coexist. Intermediaries choose bid and ask prices to offer to sellers and buyers to whom they are connected. Traders accept or reject their offers. Nava (2009) develops a static model of Cournot competition in networks, under complete information. Players who buy sell and retail are determined endogenously in equilibrium. In his model prices increase along the supply chain because each trader has local market power.[7] In contrast with these three papers, the approach to trade in my paper is dynamic and the market model is fully decentralized (i.e. only bilateral negotiations are allowed).

The most closely related dynamic model is Gale and Kariv (2007). They analyze trade in a network under complete information. Their main result, that an efficient outcome is attained in the long run, is in line with my full information finding in proposition 1 (see section 3).[8]

The present paper is also related to the literature on decentralized markets with matching and bargaining under complete information, initiated by Rubinstein and Wolinsky (1985). Calvo-Armengol (2003) and Corominas-Bosh (2004) were the first to consider an environment where the matching technology is constrained by a network structure.[9] In contrast to this set of papers, in my paper the analysis is not restricted to bipartite buyer-seller networks, and there is both incomplete information and resale of goods in equilibrium.

[6]As Jackson (2008) puts it in a section on networked markets "[...] there is much left to be learned in this extensive area of application where network play such a central and critical role."

[7]Kakdade et al. (2005) and (2004) adopt a static and centralized competitive equilibrium perspective. In their models traders are price takers and prices are defined by market clearing conditions.

[8]My work is also related to Jehiel and Moldovanu (1999). They study dynamic resale processes with externalities under complete information, in a setting where everyone can trade with everyone else. Zheng (2002) and Calzolari and Pavan (2006), on the other hand, analyze models including resale under asymmetric information. Their focus is non mechanism design and their informational structure is more general. However, they do not address the limitations imposed by a network structure.

[9]Recent contributions to this area include Polanski (2007a), Abreu and Manea (2009) and Manea (2009). Furthermore, Polansky (2007b) considers a model of information pricing in networks, with random matching.

2 Model

The economy consists of a set of *traders*, $N \equiv \{1, \ldots, n\}$, and two types of goods: *money*, which is distributed to all players in large quantity, and a *single indivisible object*, initially owned by agent 1. Each trader i is risk neutral and has a binary *private monetary evaluation* for the good, v_i, normalized to be either *zero* or *one*. Therefore, if trader i consumes the good with probability x_i and expects to pay m_i, then his utility is $x_i v_i - m_i$. The *common prior* probability that $v_i = 1$ is denoted $\pi_i \in [0, 1]$, and values are assumed to be *independently distributed*.

Traders are located in a *connected* and *undirected network* $G = (N, E)$, where the set of players coincides with the set of *vertices* N, and $E \subseteq 2^{N \times N}$ is the set of *edges* between pairs of players. Existence of an edge between two players means that trade between them is possible. Traders i and j are neighbors if $\{i, j\} \in E$ (also written $ij \in E$). A network is *undirected* if $ij \in E$ implies that $ji \in E$. A *path* from i to j in G is a non-empty graph where the set of vertices is $\{i, b_1, \ldots, b_m, j\} \subseteq N$ and the set of edges is $\{ib_1, b_1b_2, \ldots, b_mj\} \subseteq E$. I refer to a path by the ordered sequence of its vertices $b(i, j) = (i, b_1, \ldots, b_m, j)$, and refer to the *length of the path* by the cardinality of the set of edges. A network G is *connected* if there is a path between every pair of players.[10]

The game consists of a finite number of rounds T. Agents do not discount the future and all actions taken by all traders are observed by everyone in the network. Each round t develops in a number of stages. Denote by $s^t \in N$ the owner of the good at the beginning of round t. First, s^t can make a take-it-or-leave-it offer to one of his neighbors, or make no offer. In the former case, I denote the chosen neighbor i^t and the price asked $p^t \in [0, 1]$. Second, if s^t makes offer (i^t, p^t), then i^t decides whether to accept or reject it. If the offer is accepted, s^t transfers the object to i^t and receives a payment p^t from i^t. Finally, at the end of the round, the current owner of the object decides whether or not to consume it. If the object is consumed, the game ends. Otherwise, the game proceeds to round $t + 1$. The game also ends if it reaches the end of round T. Of course, mixed strategies are allowed at all stages. Everything but private values is common knowledge, including the network structure.

[10]The assumption of connectedness is without loss of generality. No activity occurs in an area of the network that is disconnected from the initial owner of the object.

A triple $\langle G, \boldsymbol{\pi}, T \rangle$ is a *network trading game*, which is an extensive form games with observed actions, independent types and a common prior.[11] The adopted solution concept is *perfect Bayesian equilibrium*. Informally, a perfect Bayesian equilibrium is a *strategy profile and a belief system* such that the strategies are sequentially rational given the belief system, and the belief system is consistent with Bayesian updating, wherever possible, given the strategy profile. See Fudenberg and Tirole (1991) for a formal definition.[12]

Some of the assumptions I make can be relaxed. First, most of the results extend to the case in which, in case of sale, the seller must bear a transaction costs, which is edge specific and depends on the identity of the buyer and the seller. From here on, therefore, it is intended that each result, unless otherwise specified in a footnote, will be valid in a setting with transaction costs, perhaps after minor modifications. Second, discounting can be accommodated easily within a finite horizon model, and, again, most of the result would hold with only slight modifications. Third, full observability of prior actions is not strictly necessary for the survival of the specific equilibria that I construct. In fact, because generically each player play a pure strategy along the equilibrium path, knowing the round of the game in which an offer is received and the identity of the seller making that offer will provide a sufficient statistic of all past actions.

3 Efficiency and Complete Information Games

This section develops the benchmark case of trading games played under *complete information*. As a preliminary step, I present the standard notion of efficiency tailored to this environment. If the profile of values is known, a *feasible outcome* is an allocation of the goods

[11] This is a Bayesian game of incomplete information and, for each player, the only two types *à la* Harsanyi are the payoff types, either zero or one.

[12] The notion of Perfect Bayesian Equilibrium (PBE), which is extensively used in economics, imposes as assumptions about the belief systems a number of propositions which appear in Kreps and Wilson (1982) treatment of Sequential Equilibirum (SE) as consequences of a single consistency requirement. I adopt the notion of PBE for two reasons. First, there are some technical difficulties in extending the sequential equilibrium notion to infinite games. Second, the set of SE and the set of PBE would coincide in my environment with only two possible types if the set of possible prices was discrete (see Fudenberg and Tirole (1991)). Therefore I conjecture that, if SE was appropriately defined, the two sets would coincide also in the limit (see Fudenberg and Levine (1986)).

to the players that is achievable within T rounds of sequential trade. An outcome is *Pareto efficient* (or simply *efficient*) if it is feasible and there is no alternative feasible outcome that would make at least one trader strictly better off without making any other trader worse off. Because utility is linearly transferable through monetary exchanges, an efficient outcome is one where the object is allocated to a value one player, if any, who can be reached by the object in at most T rounds of trade.

The first and second fundamental theorems of welfare economics hold trivially in this economy, since competitive equilibrium outcomes and efficient outcomes coincide.[13] The next proposition states that, even when trading is decentralized, an efficient outcome is always attained under complete information. The proof is in appendix B.

Proposition 1 (Equilibrium in the Full Information Game)**.** *Assume that values are known, $\pi_i \in \{0, 1\}$ for all $i \in N$. In every network trading game $\langle G, \boldsymbol{\pi}, T\rangle$ every sub-game perfect equilibrium outcome is efficient.*

The equilibrium takes the following form. Player 1, also hereinafter referred to as the *initial owner*, consumes the good if $v_1 = 1$, or no other player with has value one can be reached in less than T rounds of trade. Otherwise, denote by O^T the set of traders other than 1 who have value one and can be reached in less than T rounds. For each $i \in O^T$ and, for each path $(1, b_1, \ldots, b_m, i)$ with length less than T, there is a subgame perfect equilibrium that takes the following form. Trader 1 sells the good to b_1, b_1 sells the good to b_2 and so on, until player $i \in O^T$ buys the good from b_m and consumes it. The price along the trading path is constant and equal to one. Every trader other than the initial owner makes zero profit

The main insight from the analysis is that a network structure does not generate inefficiency *per se*, if information is complete. This is in line with other work on networked economies in settings without informational asymmetries (e.g. Blume et al. (2009) and Gale and Kariv (2007)).[14]

[13] Competitive equilibria outcome can be computed in the standard way, allowing for the technical limitations imposed by the network structure.

[14] In general, as Gomes and Jehiel (2005) analysis implies in a related environment, in the absence of consumption externalities, dynamic processes of social and economic interaction under complete information tend to converge in the long run to an efficient outcome.

4 Trading Equilibria

Let us now focus on the case where information on values is asymmetric. The next theorem is the first main result of this paper.

Theorem 1 (Equilibrium Existence)**.** *A perfect Bayesian equilibrium exists for each network trading game* $\langle G, \boldsymbol{\pi}, T \rangle$.[15]

The proof is constructive and follows the backward induction logic (see appendix A, where transaction costs are explicitly considered). It proceeds in three steps. First a unique equilibrium for a game starting in the last round T is constructed, for every possible owner in T and for any state of beliefs. Second one or more equilibria for an arbitrary game starting in round $t-1$ are constructed for each possible owner and profile of beliefs, assuming that the set of equilibria for a game starting in t has been computed. Finally, an equilibrium for the whole game is constructed by induction. Observe that equilibrium uniqueness can not be guaranteed generically, unless the set of possible networks is appropriately restricted.

The next example clarifies the equilibrium construction algorithm. The exposition includes a number of observations related to the general properties of trading equilibria, which are instrumental to the discussion in section 5.

Example 1 (Equilibrium Construction)**.** *Players* $\{1,2,3,4\}$ *are located in a network, where* $E = \{12, 13, 23, 24\}$ *and* $\boldsymbol{\pi} = \{0, \pi, 1/8, 1/2\}$ *(see Figure 1). Assume that* $T = 3$ *and that player 1 is the initial owner. The game is solved backward, starting with round* T*.*

Round T. *Take any arbitrary history that led to round* T*, and assume that* $\boldsymbol{\mu}^T$ *is the state of beliefs and the owner is player 1, that is* $s^T = 1$*. Since* T *is the last round, both players* 2 *and* 3 *will accept every price* $p^T \leq 1$ *if they have value one, while they will reject every price greater than zero if they have value zero.*

Therefore, in equilibrium, player 1 offers the object at price $p^T = 1$ *to player 3 if* $\mu_3^T > \mu_2^T$*. Otherwise, he offers the object at* $p^T = 1$ *to player 2. Player 1's expected payoff is* $\max\{\mu_3^T; \mu_2^T\}$ *and all other players make zero profit. An analogous argument can be made to obtain an equilibrium when* s^T *is either 2, 3 or 4.*

[15] Equilibrium existence,to my knowledge, is not guaranteed by existing results because the game is dynamic and the action space is not finite.

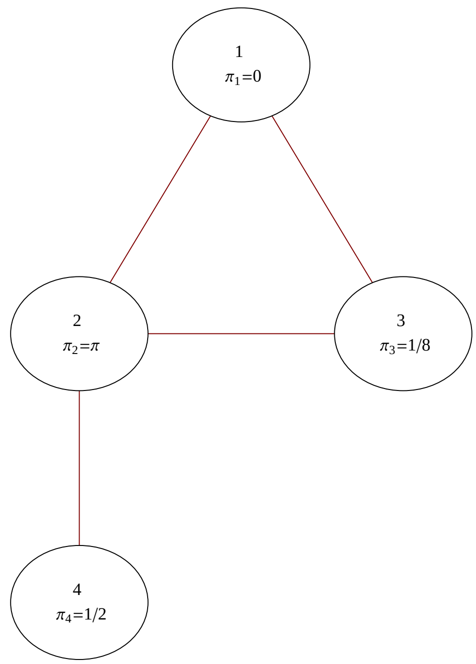


Figure 1: Trading Network in Example 1

Round T-1. *Let $\boldsymbol{\mu}^{T-1}$ be the state of beliefs at the beginning of round $T-1$ and assume first that $s^{T-1}=1$. Hereafter, I can assume that $\mu_{s^t}=0$, because if $v_{s^t}=1$, then s^t would have consumed the good instead of putting it up for sale. In general, I can make the following two observation.*

Observation 1. *In any round t, it is a dominant strategy for a trader i who acquires the good, to consume it only if $v_i=1$, and to put it up for sale only if $v_i=0$.*

Observation 2. *It follows from Bayesian updating that, whenever a trader puts the good for sale, he must have value zero. That is, $\mu^t_{s^t}=0$ for all t and s^t.*

Player 1 can make an offer to either 2 or 3, or make no offer, in which case he remains the owner in round T and beliefs do not change. To obtain an equilibrium it is necessary to pin down the acceptance strategies of players 2 and 3, for any price that may be offered *to them. The analysis in round T shows that rejecting an offer in $T-1$ will provide zero utility to 2 and 3, because the owner in round T will again be player 1. Therefore, if they have value one, it is a best reply for 2 and 3 to accept any price $p^{T-1} \leq 1$.*

If they have value zero, a best reply for 2 and 3 is to accept any price below or equal to their value from reselling the object in period T. The resale value of a player i in round t, denoted $\mathcal{V}_i^t$, depends on the network configuration and on the state of beliefs. If player i accepts an offer in round $T-1$ and puts the object up for sale, he is signalling a value zero and therefore $\mu_i^T = 0$. The beliefs about the seller and the other players remain unchanged. Therefore $\mathcal{V}_2^T = \max\{\mu_3^{T-1}; \mu_4^{T-1}\}$ and $\mathcal{V}_3^T = \mu_2^{T-1}$. This logic applies throughout the game, as stated in the two observations below.

Observation 3. *In equilibrium a trader i with $v_i = 0$ accepts an offer in round t if and only if the price offered p^t is equal to or below his resale value in round $t+1$, denoted $\mathcal{V}_i^{t+1}$.*

Observation 4. *In equilibrium, a buyer i with $v_i = 1$ will always accept any price $p^t \leq \mathcal{V}_i^{t+1}$, or otherwise he will reveal his type and in equilibrium will make zero profit for sure.*

Having fixed the strategies of the potential buyers, the following simple observations apply to the strategies for sellers.

Observation 5. *If i's acceptance probability is constant in the interval $[p_l, p_h]$, then it is never optimal for a seller to offer the object to i at any price $p^t \in [p_l, p_h]$ such that $p^t \neq p^h$.*

Observation 6. *In any round t, it can never be optimal for a seller s^t to offer the object too a trader i at a price below i's resale value, that is $\mathcal{V}_i^{t+1}$.*

Therefore, if player 1 is the owner in round $T-1$, he only needs to consider four possible offers, in addition to not making any offer:

(i) Player 1 asks $p^{T-1} = 1$ to player 2. *In this case player 2 accepts if and only if $v_2 = 1$. In case of a refusal player 1 assumes that $\mu_2^T = 0$ and in round T offers $p^T = 1$ to player 3. The expected payoff for player 1 is: $\mu_2^{T-1} + (1 - \mu_2^{T-1})\mu_3^{T-1}$. The expected payoff for 2 and 3 is zero.*

(ii) Player 1 asks $p^{T-1} = 1$ to player 3. *In this case player 3 accepts if and only if $v_3 = 1$. In case of a refusal player 1 offers $p^T = 1$ to player 2 in round T. The expected payoff for player 1 is: $\mu_3^{T-1} + (1 - \mu_3^{T-1})\mu_2^{T-1}$. The expected payoff for 2 and 3 is zero.*

(iii) Player 1 asks $p^{T-1} = \mathcal{V}_2^T$ to player 2. *Player 2 accepts for sure and his expected profit is* $1 - \mathcal{V}_2^T$ *if he has value one and zero otherwise. Player* 1 *obtains* $\mathcal{V}_2^T$.

(iv) Player 1 asks $p^{T-1} = \mathcal{V}_3^T$ to player 3. *In this case player* 1 *obtains* $\mathcal{V}_3^T$. *Player 3 obtains* $1 - \mathcal{V}_3^T$ *if* $v_3 = 1$ *and zero otherwise.*

Player 1 *is indifferent between offers (i) and (ii). Moreover, offer (iv) is always dominated by offer (i) and (ii). Finally, not making any offer is also strictly dominated by (i) and (ii). Therefore, an optimal strategy for player 1 is either (iii) or either of (i) and (ii), depending on the state of beliefs. To sum up, an equilibrium is obtained for the continuation game starting in* $T-1$ *with* $s^{T-1} = 1$. *Analogously, an equilibrium can be obtained when the seller at* $T-1$ *is another player than 1.*

ROUND $T - 2 = 1$. *Let us now analyze the entire game. Therefore, let* $\boldsymbol{\mu}^1 = \boldsymbol{\pi}$ *and* $s^1 = 1$. *The analysis is developed under two assumptions. Under assumption* A *the expected value of player 2,* $\pi_2 = \pi$, *is relatively low, while under assumption* B *it is relatively high.*

Assumption A: $1/2 > 1/8 + 7/8\pi$. *First, we need to compute the equilibrium acceptance strategies for traders 2 and 3. Consider* 2 *first. If* $v_2 = 0$, *according to observation 3, player* 2 *accepts every price below or equal to* $\mathcal{V}_2^{T-1} = 1/2 + 1/16 = 9/16$.[16] *If* $v_2 = 1$, *by accepting an offer at a price* p^1 *he gets* $1 - p^1$, *whereas if he rejects the offer he gets a payoff that can be computed by looking at round* $T-1$, *for the case where* 1 *is the owner in* $T-1$, μ_2^{T-1} *is determined according to Bayesian updating and* $\mu_{-2}^{T-1} = \mu_{-2}^{T-2}$. *Let* $V_2^{T-1}(\mu_2^{T-1})$ *be this payoff. In this case* $V_2^{T-1}(\mu_2^{T-1}) = 1/2$ *for all* $\mu_2^{T-1} \leq \pi_2$, *because it is always optimal in round* $T-1$ *for player 1 to offer player 2 a price of* $1/2$. *In fact, under assumption* A, $1/2 > 1/8 + 7/8\pi$, *which is the maximum that player 1 could achieve by making a different offer in* $T-1$. *Therefore an optimal strategy for player 2 is to accept every* $p^1 \leq 9/16$ *and reject higher prices.*

Next, consider player 3. If $v_3 = 0$, *player 3 accepts every price below or equal to his resale value* $\mathcal{V}_3^{T-1} = \max\{\pi, 1/2\} = 1/2$, *where the second equality follows from assumption A.*[17] *If*

[16] This is the expected payoff that player 2 obtains in the game starting at round $T-1$, with $s^{T-1} = 2$ and $\boldsymbol{\mu}_{-2}^2 = \boldsymbol{\pi}_{-2}, \mu_2^2 = 0$. In this subgame player 2 resells at price one to 3 and 4.

[17] This is the expected payoff that he will obtain in the game starting at round $T-1$, with $s^{T-1} = 3$ and $\boldsymbol{\mu}_{-3}^2 = \boldsymbol{\pi}_{-3}, \mu_3^2 = 0$. In this subgame trader 3 asks price 1/2 to 2, who then resells to 4 at price one.

$v_3 = 1$, $V^{T-1}(\mu_3^{T-1}) = 0$ *for any* μ_3^{T-1} *because player 3 will obtain no offer in round* $T-1$. *Therefore, it is optimal for player 3 with* $v_3 = 1$ *to accept every price* $p^1 \leq 1$.

Equilibrium path under assumption A. *In round one player 1 offers* $p^1 = 1$ *to player 3, who accepts only if* $v_3 = 1$, *and thereafter consumes. In case of a refusal, player* 1 *offers* $p^2 = 1/2$ *to player 2, who accepts regardless of his value, and consumes if* $v_2 = 1$. *If* $v_2 = 0$, *in the last round, player* 2 *asks for a price* $p^3 = 1$ *to player* 4, *who accepts only if* $v_4 = 1$.

Assumption B: $1/2 < \pi$. *The acceptance strategy of player 3 is computed as under Assumption A. The only difference is that the resale value of* 3 *will now be equal to* π.

Instead, when $v_2 = 1$ *player 2 will not play a pure strategy after an history in which he obtains an offer in round* $T-2$. *To see this, note that, under assumption B, the best strategy for player* 1 *in round* $T-1$ *if he believes that player 2 has value one with probability* π, *is to ask* $p^{T-1} = 1$ *from player* 3 *and, if refused, to ask* $p^T = 1$ *from player* 2*; hence,* $V_2^{T-1}(\pi) = 0$. *On the other hand, if player* 1 *in round* $T-1$ *believes that player 2 has value zero, when in reality player* 2 *has value one, the expected payoff of* 2 *is* $V_2^{T-1}(0) = 1/2 > V_2^{T-1}(\pi)$. *It follows that it can not be part of an equilibrium for player 2 with* $v_2 = 1$ *to mimic a value zero trader and refuse all prices above* $\mathcal{V}_2^{T-1} = 9/16$. *If he adheres to this strategy he obtains zero profit, whereas accepting a price* $p \in (9/16, 1)$ *would provide a positive payoff. Moreover, it cannot be part of an equilibrium for* 2 *with* $v_2 = 1$ *to accept with probability one offers above* $9/16$, *because if he rejects, then the seller would believe that he has value zero and 2 would find this profitable.*[18]

Therefore, in this case the acceptance strategy of player 2 *with* $v_2 = 1$ *must involve mixing. In particular, player 2 accepts any price below or equal to* $9/16$, *and he randomizes his acceptance decision for all prices in* $(9/16, 1]$, *in such a way that, upon refusal, the seller changes his belief about 2 to* $\mu_2^{T-1} = 3/7$. *However, for this to be a best reply player 2 must be indifferent between accepting and rejecting an offer at a price* p^1 *in that interval. Therefore, the payoff of player 2, upon refusing price* p^1, *must be equal to* $1 - p^1$. *This can happen because if player 1 assumes* $\mu_2^{T-1} = 3/7$, *then he is indifferent between two courses of action starting in* $T-1$*: either asking price* $p^2 = 1/2$ *from 2, or asking price* $p^{T-1} = 1$ *from 3 and in case of refusal asking price* $p^T = 1$ *from 2 in round* T. *Therefore, he can randomize*

[18] Note that player 2 cannot refuse a price below or equal to 9/16 as otherwise he would signal that he has value one.

between the two options, as a function of the price posted to 2 in $T-2$ in such a way that player 2 becomes indifferent between accepting and rejecting at $T-2$.[19]

Equilibrium path under assumption B. *Player 1 will ask a price of one from player 3 in round $T-2$, who will accept only if he has value one. In case of a refusal player 1 will ask a price of one from player 2 in round $T-1$. In round $T-1$ player 2 will accept any price below or equal to 1.*

It should be emphasized that the outcome of the game under assumption A is ex-post efficient, while it is inefficient under assumption B. In fact, in the latter case, player 4 never gets an offer, even if he happens to be the only player in the game with value one. Therefore, in contrast to the complete information case, ex-post efficiency is not guaranteed under incomplete information. This point is discussed in more detail in section 5.4.

Even though in the above example it is assumed that $T = 3$, increasing the number of rounds to four or more will not change the set of equilibrium payoffs. The idea is that, because learning is irreversible, the number of payoff relevant offers that can be made in any equilibrium of any game, before everyone learns everything, is finite. Therefore, the equilibria constructed according to Theorem 1 are not sensitive to the time horizon of the game, when this is sufficiently long.[20]

Proposition 2 (Time Stability of Equilibria)**.** *For each given G and $\boldsymbol{\pi}$ there exist a number of rounds T^* such that, for each $T \geq T^*$ the set of equilibrium payoffs in the game $\langle G, \boldsymbol{\pi}, T \rangle$, computed according to the equilibrium construction algorithm in appendix A, coincides with the set of equilibrium payoffs in the game $\langle G, \boldsymbol{\pi}, T^* \rangle$.*

This property is exploited in the next section. I refer to it either by using T^* rather than T in the definition of a network trading game, or by stating that the number of rounds is sufficiently large.

[19] This example also shows that *Markov equilibria* will not always exist in network games, as sometimes the seller's future price will have to depend on the present one, which does not affect the continuation payoffs in the game that starts in the following round.

[20] This property is common to other settings with a fixed deadline and no discounting (see e.g. Jehiel and Moldovanu (1999)).

5 Equilibrium Analysis

This section characterizes the main properties of equilibrium outcomes. From hereon, I focus on equilibria where sellers play pure strategies *along the equilibrium path*. Such equilibria always exist, while equilibria where sellers randomize their decisions along the equilibrium path are non-generic. In fact, they can be eliminated by introducing transaction costs and thereafter appropriately restricting the set of possible priors and transaction costs, without reducing the dimensionality of the set where these parameters are defined.[21]

In what follows, I discuss first how the location of a trader in the trading network determines the role that he will play in the trading process and the terms of trade he will face. Next, I provide results on the distribution of payoffs among traders and then I perform some comparative statics. Finally, I discuss efficiency under incomplete information. All proofs are in appendix B.

5.1 Final Customers and Intermediaries

In a given equilibrium, a trader is *active* if he takes at least one action with positive probability. Player 1 is always active and makes a strictly positive expected profit. Active traders other than the initial owner can be divided in two classes: final customers and intermediaries.

Final customers are active traders who only get offers at price one. This implies that final customers never acquire the object if they have value zero, and their payoff is always zero.

Intermediaries are active traders who, at some point in the game, get an offer such that they can buy the object, even if they have value zero, in order to resell it in their neighborhood. In contrast to the complete information case, under asymmetric information an intermediary with value one obtains a positive expected profit. An intermediary with value zero, instead, makes zero profit, because no seller will ever ask him to pay a price below his equilibrium resale value (see observations 3 and 6).[22]

[21] The analysis could be extended to equilibria where sellers mix within offers along the equilibrium path without much difficulties. However an additional machinery would be needed to present the results.

[22] The fact that only intermediaries extract a positive rent is a consequence of the binary value assumption. In a more general setting, final customers would be also able to extract a lower but still positive information rent, as standard in asymmetric information environments.

To summarize, the following Table 1 reports the sign of the interim payoff (value one in the second and value zero in the third column) and the ex-ante payoff (fourth column) for player 1, final customers and intermediaries.

Table 1: Interim and Ex-ante Equilibrium Payoffs

Roles ($\pi_i > 0$)	$U_i(1)$	$U_i(0)$	U_i
Initial Owner	1	≥ 0	> 0
Intermediary	≥ 0	0	≥ 0
Final Customer	0	0	0

The distinction between intermediaries and final customers is exhaustive for each equilibrium of network trading games, as stated in the following proposition.

Proposition 3. *In every equilibrium of a network trading game $\langle G, \boldsymbol{\pi}, T \rangle$, every active player is either the initial owner, an intermediary or a final customer.*

The idea of as follows. Whenever a player is not an intermediary, he will only obtain offers that he will accept exclusively if he has value one. However, if this is the case, the last of these offers must be at price one, otherwise that seller could improve his profit. Therefore, anticipating this, previous sellers will also make offers at price one.

Whether a player is either active or not and whether he is either a final customer or an intermediary are determined endogenously in equilibrium and will depend on the complex interaction of the exogenous variables.[23] While it is difficult to derive simple conditions that identify the roles of traders in arbitrary trading network games, there are two classes of traders, isolated traders and bottleneck traders, whose equilibrium role in the trading network can be more easily characterized.

A player $i \neq 1$ is an *isolated trader* of G if he is connected to only one player, that is $ij \in E$ for only one $j \in N \setminus i$. For isolated traders I can establish the following proposition.

Proposition 4 (Isolated Traders)**.** *In every equilibrium of any game $\langle G, \boldsymbol{\pi}, T \rangle$, an isolated trader i with $\pi_i > 0$ is either a final customer or an inactive trader.*

[23] The fact that players with intermediating roles arise endogenously is in contrast with other models of intermediation, where intermediaries are exogenously assigned to their role, as for example in Rubinstein and Wolinsky (1987), and Blume et al. (2009).

The idea is that players who are at the perimetry of the network will never be useful to intermediate the good to some other area of the network. Therefore no trader will have no interest in selling the object to them at prices lower than one.

A *bottleneck trader* is a trader $i \neq 1$ whose presence is necessary for the network to remain connected. Formally, let G_{-i} be the network obtained by removing node i and all his incident edges from G.[24] Player i is a bottleneck trader in G if the network G_{-i} is not connected. Now, let G^1_{-i} denote the largest connected subgraph of G_{-i} which includes player 1. Let $G^i = G - G^1_{-i}$ be the connected subgraph of G which contains i and is obtained by deleting all vertices in G^1_{-1} and their incident edges.[25] Call $\widetilde{U}_i(0)$ the maximum equilibrium payoff of i with $v_i = 0$ in $\langle G^i, \boldsymbol{\pi}, T^* \rangle$, assuming that i is the initial owner of an object.[26]

Proposition 5 (Bottleneck Traders)**.** *Consider a network trading game $\langle G, \boldsymbol{\pi}, T^* \rangle$, and suppose that i is a bottleneck trader of G. If $\widetilde{U}_i(0) > \pi_i$, then in every equilibrium where i is an active player, i is an intermediary.*

Proposition 5 states that a sufficient condition for an *active* trader to be an intermediary is that the profit that he can extract from reselling to the part of the network for which he provides monopolistic access is greater than his own expected value. When this is the case, a player selling to a bottleneck trader i will always prefer to demand a price equal to the resale value of i, and have this offer accepted for sure, rather than asking a price of one, and selling only if i has value one.

The concept of *structural hole*, introduced by Burt (1992), refers to the absence of connections within two groups of agents in a social network. Burt's argument is that individuals who fill structural holes, by offering connection between otherwise separated groups, obtain important advantages, in economic and social terms. My analysis provides a foundation for such advantage, explaining how individuals who are essential for connecting a valuable part of the trading network to the initial owner may extract a larger rent than other individuals.[27]

[24]Write $G_{-i} = (N \setminus \{i\}, E_{-i})$, where $E_{-i} = \{\{j, j'\} \in E : j, j' \neq i\}$. In general, a network G' is a subset of G if it is obtained from G by removing a set of players and their incident edges.

[25]In example 1, player 2 is a bottleneck trader. $G^1_{-i} = (\{1, 3\}, \{13\})$ and $G^i = (\{2, 4\}, \{24\})$.

[26]Observe that T^* is always defined specifically for the game under consideration.

[27]The analysis of structural holes in Goyal and Vega-Redondo (2007) adopts a surplus sharing rule that provides an exogenous advantage to players who have an intermediating role. Hence, they consider a network formation game and focus on whether equilibrium networks include players who fill structural holes.

While bottleneck and isolated traders do not exhaust all possible types of players in arbitrary trade networks, a player is either an isolated or a bottleneck trader when the trading network is a *tree*, that is a graph where every pair of players is connected via a unique path. The implications of Proposition 5 are illustrated in the following example.

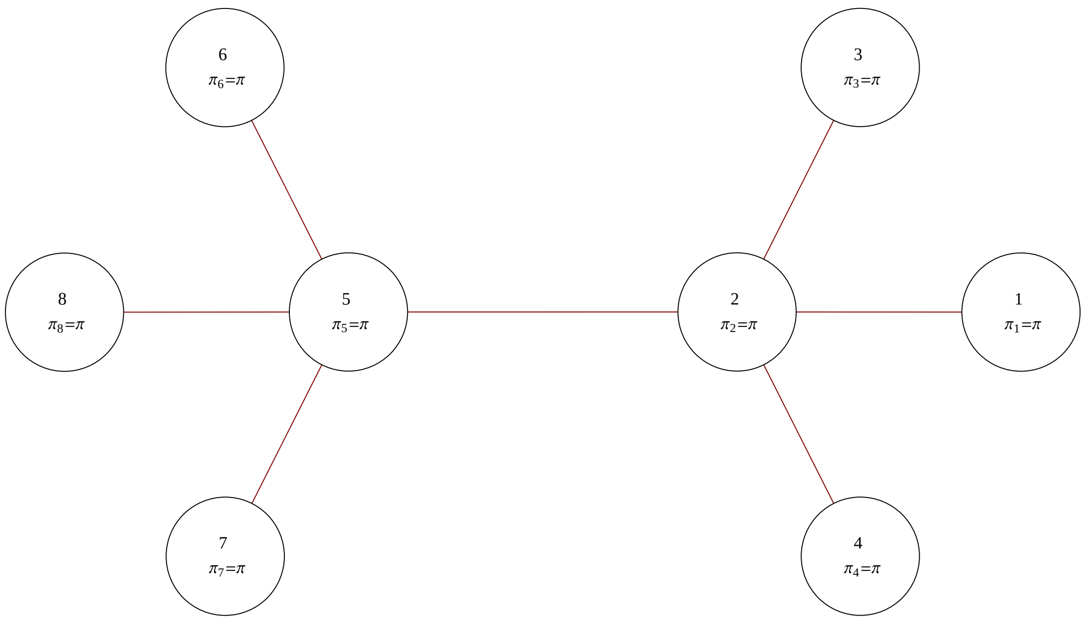


Figure 2: Trading Network in Examples 2 and 3

Example 2 (Isolated Traders and Bottleneck Traders in a tree)**.** *Assume that players* $1, 2, \ldots, 8$ *are located in the network depicted in Figure 2, that* $T \geq 7$*, player* 1 *is the initial owner, and* $1 > \pi > 0$*. The set of isolated traders is* $\{3, 4, 6, 7, 8\}$*, while* $\{2, 5\}$ *is the set of bottleneck traders. Note that* $\widetilde{U}_2(0) = 1 - (1-\pi)^5 > \pi$ *and* $\widetilde{U}_5(0) = 1 - (1-\pi)^3 > \pi$*. Therefore if player 2 and 5 are active players in equilibrium, then they are intermediaries.*

It is not difficult to see that in any network that is a tree, if the number of rounds is sufficiently large, (i) all players connected to player 1 will be active, and (ii) a bottleneck trader will be active if and only if all other bottleneck traders in the unique path going from player 1 to him are active intermediaries.[28]

[28]In example 2, trader 2 is active because he is connected to trader 1. Therefore trader 5 is also active.

5.2 Payoff Ranking for Intermediaries

This section investigates how the location of an intermediary in the trading chain affect his payoff, when he has value one.[29] By *trading chain* I mean the ordered sequence of intermediaries that receive offers with positive probability, *at prices equal to their resale values*.[30] The question is whether an intermediary who makes his first appearance early in the trading chain achieves a higher payoff than an intermediary who appears later, or viceversa.

Two countervailing effects are present. First, intermediaries who obtain offers in later rounds will get them with a lower probability than intermediaries who get offers earlier in the game. Second, offers received by intermediaries in later rounds will be at lower prices, as formally stated in the next proposition.

Proposition 6 (Decreasing Prices)**.** *Consider a trading network game $\langle G, \boldsymbol{\pi}, T\rangle$. Let i^t and i^{t+k} indicate two intermediaries along the trading chain, selling the object in round t and $t+k$, with $k \in \mathbb{N}$. Let p^{t-1} and p^{t+k-1} be the price that i^t and i^{t+k} pay for the object, if they receive offers, in rounds $t-1$ and $t+k-1$. In equilibrium, $p^{t-1} \geq p^{t+k-1}$ holds with equality if and only if no other player accepts with positive probability an offer from round t to $t+k$ at a price greater than p^{t+k-1}.*[31]

The price of the object decreases over time because, as rounds of trade take place, it becomes known that there are fewer traders who are potentially interested in consuming the object. This, in turn, reduces the resale value of the object, which is the price that intermediaries are asked to pay. More formally, assume that i^t and i^{t+k} are two consecutive intermediaries in the trading chain and let $X(i^t) = \{x^t, x^{t+1}, \ldots, i^{t+k}\}$ be the ordered set of players to which i^t makes offers from round t to $t+k-1$. Let α_j indicate the probability that player $j \in X(i^t)$ accepts his offer. The price at which an intermediary i^t acquires the

[29] Recall that final customers and intermediaries with value zero always make zero profit.

[30] Recall that we consider equilibria where sellers adopt a pure strategy along the equilibrium path and therefore the chain is well defined.

[31] Suppose that transaction costs were present and equal to τ for each edge of the network. In this case the relation within prices in round $t-1$ and $t+k-1$ would be $p^{t-1} + \tau \geq p^{t+k-1}$. That is, transaction costs tend, *ceteris paribus*, to reduce earlier prices as the total amount of the expected transaction costs that remains to be paid must be decreasing in time.

good in round $t-1$ is equal to his resale value, that is:

$$p^{t-1} = \alpha_{x^t} p^t + (1-\alpha_{x^t})\alpha_{x^{t+1}} p^{t+1} + \cdots + p^{t+k-1} \prod_{j \in X(i^t) \setminus i^{t+k}} (1-\alpha_j)$$

Because the seller is rational the sequence of prices must be weakly decreasing and we can conclude that $p^{t-1} \geq p^{t+k-1}$.

The decrease in price and the reduced probability of obtaining an offer are countervailing forces, but the second dominates the first. In particular, the decrease in price is sufficient to compensate intermediaries for the reduced probability of getting an offer induced by the event that some final customers, who get earlier offers, have value one and consume the object. Nevertheless, it does not compensate the later intermediary for the reduction in probability due to the event that other intermediaries who intervene earlier in the trading chain may themselves consume the object if they have value one.[32]

However, it it not possible to conclude from this observation alone that a clear-cut interim payoff ranking exists for intermediaries with value one. In fact, an intermediary could also receive an offer earlier in the game (i.e. before the round in which he acquires the object at his resale value), at a price that he is supposed to accept only if he has value one. This early offer will not change his interim payoff, because a rational seller will keep him indifferent between accepting and rejecting. However, it will reduce the payoff of intermediaries who have earlier positions than him in the trading chain, but receives their offers after he has obtained the early offer. Therefore, when value are known but the game has not started yet, a payoff ranking among intermediaries with value one participating in the trading chain can not be established in general.

Instead, when all agents are ex-ante identical, an offer made early to some intermediary will at most equalize his payoff to the level of that of the intermediary preceding him in the trading chain. Therefore, as formally stated in the following proposition, it is possible to establish that intermediaries who are earlier in the trading chain will obtain payoff greater or, at worst, equal to the intermediaries that come later.[33]

[32]This phenomenon is easy to see if two intermediaries acquire the object one immediately after the other with no other players receiving offers in between. They must be paying the same price, but the second one is worse off than the first, as he only gets the good if the previous one has value zero.

[33]The result is robust to the presence of small transaction costs and the introduction of discount rates

Proposition 7 (Payoff Ranking for Intermediaries)**.** *Consider a network trading game* $\langle G, \boldsymbol{\pi}, T\rangle$ *where* $\pi_i = \pi_j > 0$ *for all* $i, j \in N$*. Take any equilibrium of the game and let* i^t *and* i^{t+k} *indicate two intermediaries along the equilibrium path, selling the object for the first time in round* t *and* $t+k$*, with* $k \in \mathbb{N}$*, then* $U_{i^t}(1) \geq U_{i^{t+k}}(1)$*.*

This result shows that the heterogeneity in outcomes generated by the network structure is not limited to that arising from the different roles of traders in the network (i.e. intermediaries or final customers). The following example illustrates the result.

Example 3 (Intermediaries'Payoff ranking in a tree)**.** *Consider again the environment in example 2. It can be checked that the equilibrium path is the following: Player* 1 *first asks a price* $p^1 = 1-(1-\pi)^5$ *to intermediary* 2*. If* $v_2 = 0$*, player* 2 *ask a price of one to final customer* 3 *and* 4*, and then asks a price* $p^4 = 1-(1-\pi)^3$ *to intermediary* 5*. In fact, if player* 5 *has value zero, he asks a price of one to his final customers* 6 *and* 7 *and* 8*. Therefore, conditional on both having value one, the relation between the interim payoff of 2 and 4 is* $U_2(1) = (1-\pi)^5 > U_4(1) = (1-\pi)^6$*.*

In general, whenever the network is a tree it is possible to rank the payoff between any two traders who are on the same path to the initial owner, even if traders are ex-ante heterogeneous. In this case, in fact, it is not possible for an intermediary who is later in the trading chain to receive an offer before an intermediary who comes earlier. Therefore, for any two given intermediaries with non zero expected value lying on the same path from the initial owner, we can conclude that the one closer to the initial owner will obtain a strictly higher payoff, when he has value one, than the one who is more far away.

5.3 Comparative Statics

It is now possible to perform some comparative statics on how equilibrium payoffs change in response to changes in the exogenous variables. Since my main focus is on the effects of the network structure on trading outcomes, I will examine changes in connectivity only. I say that the network $G' = (N', E')$ is *more connected* than $G = \{N, E\}$ if $N = N'$ and $E \subset E'$. First, consider the initial owner of the object.

close to one. Furthermore, when players are asymmetric, it is always possible to make the proposition valid for traders who are sufficiently far away in the trading chain.

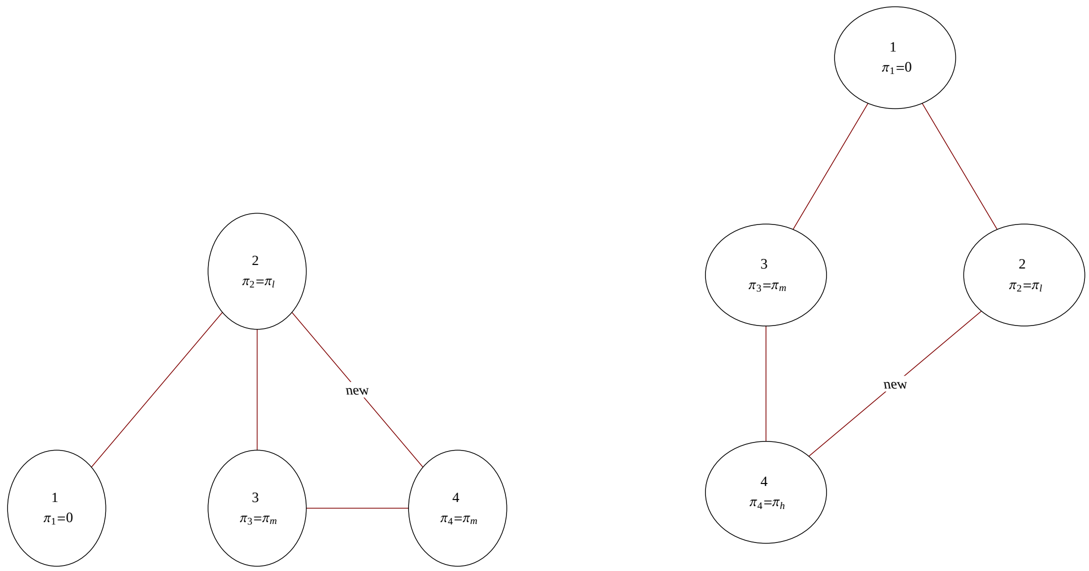
Proposition 8 (Comparative Statics: Initial Owner)**.** *Let $\langle G, \boldsymbol{\pi}, T^* \rangle$ and $\langle G', \boldsymbol{\pi}, T^* \rangle$ be two games which differ only because G' is more connected than G. Then for every equilibrium of $\langle G, \boldsymbol{\pi}, T^* \rangle$ there exists an equilibrium of $\langle G', \boldsymbol{\pi}, T^* \rangle$ where the initial owner achieves a higher or equal expected payoff.*[34]

The intuition behind the result is that an increase in connectivity accentuates downstream competition between intermediaries. In contrast, there is no analogous result for traders other than the initial owner. In particular, a change in connectivity could have both positive and negative effects on them. The reason is that an increase in connectivity that increases the resale value of a player will be beneficial to him if it changes him from being a final customer to being an intermediary, but it will have a negative effect if he is already an intermediary, because it will increase the price that he pays for the object. This is illustrated by the following example.

Consider the two networks depicted in Figure 3 and let the label "*new*" on an edge indicate the extra edge that is added to the graph. Consider the equilibrium outcome *before* and the *after* the addition of the *new* edge. In case (a) the equilibrium utility of player 2 when $v_2 = 1$ decreases from $U_2(1) = 1 - \pi_m$ to $U_2(1) = (1 - \pi_m)^2$ when the extra edge is added. That is, player 2 acquires the good at a higher price after the introduction of the new edge. Instead, in case (b) the equilibrium utility of player 2 when $v_2 = 1$ increases from $U_2(1) = 0$ to $U_2(1) = 1 - \pi_h$ when the extra edge is added. In this case player 2 is a final customer initially and becomes an intermediary after the introduction of the new edge, because 1 prefers to route the good via the lowest expected value trader.

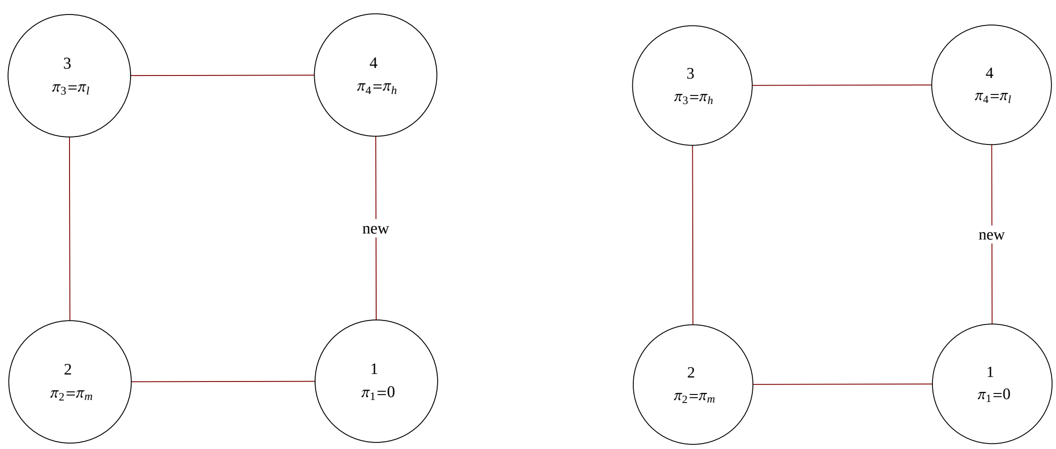
The effect of an increase in connectivity on total welfare can sometimes be negative. This phenomenon has been referred to in the literature on transportation networks as *Braess paradox*. To see this point consider the two networks depicted in figure 4. In case (a) the equilibrium is ex-post efficient before the introduction of the new edge, while it is inefficient thereafter. In fact player 1 will only offer the object at price 1 to players 2 and 4 and player 3 will never receive an offer, even if he happens to be the only player in the network with value one. In case (b) the introduction of a new edge makes the outcome of the trading game ex-post efficient, as in the new equilibrium all players receive offers with positive probability.

[34] This result will not hold if there are large transaction costs or the number of rounds is not sufficiently large.



(a) $U_2(1) \downarrow$ (b) $U_2(1) \uparrow$

Figure 3: Change in own connectivity - $\pi_h > \pi_m > \pi_l$



(a) $\sum_{i \in N} U_i \downarrow$ (b) $\sum_{i \in N} U_i \uparrow$

Figure 4: Braess Paradox - $\pi_h > \pi_m > \pi_l$

5.4 Efficiency under Incomplete Information

An *outcome* of the trading game under incomplete information is a mapping from the profile of values into the set of possible allocations of the goods in the economy. Following Holmstrom and Myerson (1983), I say that an outcome is *ex-post Pareto efficient* (or simply *efficient*) if it is feasible, and there is no alternative outcome such that (i) all players are weakly better off for all profiles of values, and (ii) there is at least one profile of values for which at least one player is strictly better off.

In contrast to the full information case, incomplete information limits the range of network trading games in which an efficient outcome is achieved. First, because it is not clear in advance who has a value of one, inefficiencies may arise if the number of rounds is insufficient for all players to be reached with an offer before the deadline.[35] It is not surprising, however, that *time constraints* generate inefficiencies. Therefore, let us consider a setting where the deadline is sufficiently far away.

In this case, a network where everyone is connected to the initial owner always induces an efficient outcome. For example, see figure 5 for the lower bound on the set of efficient networks. In particular, the initial owner extracts all the available surplus by making an offer at price one to all players in sequence, until the object is sold.[36]

In general, because in equilibrium no one will refuse an offer, unless he gets a later offer at a lower price, a *necessary and sufficient* condition for an equilibrium to implement an efficient outcome is that every player has a positive probability of receiving at least one offer along the equilibrium path. A violation of this condition represents a second source of inefficiency, which I refer to as *market power*. To illustrate this point consider again the example 1 under assumption B. An inefficient outcome ensues in the example 1 under assumption B because player 4 never gets an offer in equilibrium. In fact, player 1 prefers to exploit his market power and *play tough* by asking a price of one from player 2, therefore running the risk of not selling at all, rather than asking a price that player 2 could afford if he had value zero.[37]

[35] In fact, suppose that player i with $\pi_i > 0$ never gets an offer in equilibrium but he could be reached within T rounds of trade. An efficient outcome is not achieved whenever $v_i = 1$ but $v_j = 0$ for all $j \neq i$.

[36] Not surprisingly, if transaction costs were present at each edge only a network where everyone were connected to the initial owner would be efficient. Because transaction costs are paid only in case of sale, the initial owner will minimize the expected cost by making early offers to players with low transaction costs.

[37] Note that there is no separating equilibrium where player 1 asks a higher price in the first round and

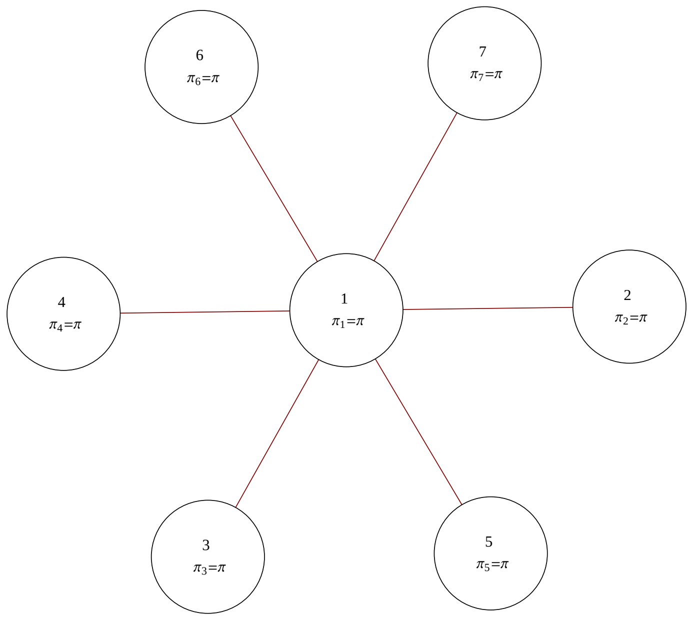


Figure 5: Star Network, n=7

Inefficiencies arise, in general, when there exist one or a group of players who jointly represent a bottleneck in some area of the network, and who all receive offers at price one. This suggests that inefficiencies will tend to disappear in networks that are very well connected or where everyone is ex-ante similar. In particular, if the number of trading rounds is sufficiently large the following can be established.

Proposition 9 (Efficiency in Homogeneous Networks)**.** *In every game* $\langle G, \boldsymbol{\pi}, T^* \rangle$*, where* $\pi_i = \pi_j$ *for all* $i, j \in N$ *there exists an efficient equilibrium.*

The idea is that, even in the worst case scenario for attaining efficiency, where there is just one player i who grants exclusive access to a single isolated player j, the expected payoff

a lower price later. In fact, whenever player 2 anticipates that a lower price will be offered in later rounds he will always refuse the current offer. However the situation might be different if traders discounted the future. In this case the seller might reach a "compromise" with the value one buyer rather than taking a tough stance. But, this would clearly not eliminate inefficiencies for all values of the parameters (see Fudenberg and Tirole (1983)).

to i from reselling to the isolated trader j only, is at least equal to his own expected value. Therefore, i will receive an offer at his resale value and, in turn, will be able to make an offer at price one to j (see proposition 5). Note that this result is not robust to the introduction of small transaction costs, or small differences in the expected value of players. However, if transaction costs and ex-ante trader heterogeneity are introduced, efficiency can be restored by making the network more finely connected. In fact, in general, the higher the connectivity of the network, the higher will be the chance of an efficient outcome.

6 Extensions

In this section I consider two extensions to the standard model. First, I study equilibria in large networks. Second, I compute optimal network structures, for the initial owner and for the other traders considered jointly. Proofs are in appendix B.

6.1 Large Networks

Consider a setting where agents are ex-ante identical, that is $\pi_i = \pi > 0$ for all i, and suppose that the number of traders grows large. Will the payoff to the initial owner converge to one in the limit? My results show that as the number of players increases, the payoff of the initial owner converges to one in most cases, but not always, in a sense that I will made more formal soon.[38] First, to see that a payoff of one is not guaranteed, suppose that the network grows forming a *path*, starting from player 1, with player 2 connected to 1, player 3 only to player 2, and so on. In this case, the ex-ante equilibrium payoff of player 1 will be constant and equal to $\pi + (1-\pi)\pi$. All buyers will make positive ex-ante profit, which will decrease for players farther away from the initial owner, and converge to zero in the limit.

More generally, I can prove that the equilibrium profit of the initial owner converges to one if and only if the network architecture E is such that the number of equilibrium final customers grows with the set of traders N. In order to formalize this idea, some further

[38]This is contrast with (i) the case of complete information, where the expected payoff converges to one as the probability that there is no player with value one converges to zero, and (ii) the standard optimal auction framework, where the payoff to the seller converges to the upper bound of the support of values as the number of players grows (see Myerson (1981) and Riley and Samuelson (1981)).

definitions are required. First, define a countably infinite *sequence of graphs* $\{\boldsymbol{G}\}^n$ as a sequence of connected networks $G^1, G^2, \ldots$ where (i) $G^1 \subset G^2 \subset \ldots$ and (ii) G^1 includes only player 1, G^2 includes $1, 2$, and so on. Assume that $\{\boldsymbol{G}\}^n$ admits a limit. Next, let $\mathcal{S}(G)$ be the set of subgraphs of G that are trees. For any graph $G' \in \mathcal{S}(G)$, let $l(G')$ indicate the number of isolated traders in G' (i.e. the number of leaves in the tree G'). Finally, let $G^*(G) = \arg\max_{G' \in \mathcal{S}(\mathcal{G})} l(G')$ be the subtree with the highest number of isolated traders.

Proposition 10 (Limit Payoffs for $n \to \infty$)**.** *For each $G^n \in \{\boldsymbol{G}\}^n$, let $\widehat{U}_1^n$ be the highest ex-ante payoff of player one across all equilibria in the network game $\langle G^n, \boldsymbol{\pi}, T^* \rangle$, with $\pi_i = \pi_j > 0$ for all $i, j \in N$. Then,* $\lim_{n\to\infty} \widehat{U}_1^n = 1$ *if and only if* $\lim_{n\to\infty} l(G^*(G^n)) \to \infty$.[39]

The idea behind the proof is that, whenever the number of isolated traders is finite in the subtree of G with the maximum of isolated traders, then the number of players who are not arranged in a path (i.e. are connected to more than two traders) will also be finite. Therefore, because it is not possible to extract a surplus from the intermediaries lying consecutively in a path, the number of players from which a surplus can be extracted remains finite and the surplus collected by the initial owner is bounded away from one in the limit.

It is possible to measure the set of graphs in which the initial owner makes a payoff of one by embedding the space of possible network into a probability space. One way to do this is as follows. Let us assume that for fixed $N = \{1, \ldots, n\}$, the set of edges E is determined randomly, with each possible edge $\{i, j\}$ belonging to E with probability $0 < p < 1$, independently of other edges (often referred to as Erdős-Rényi model). Once the probability space $\mathcal{G}(n, p)$ has been constructed, it is now possible to ask about the limit probability, as n tends to infinity and p remains constant, that the number of leaves in all subtrees of G remains finite. It is relatively easy to show that that, for each $k \in \mathbb{N}$, $\lim_{n\to\infty} \Pr\{l(G^*(G^n) > k)\} \to 1$.[40] Therefore, within this natural probability space, for almost all network configurations, the initial owner's profit tends to one as the number of traders grows.

[39] The presence of transaction costs would definitively bound below one the profit of the initial owner.

[40] In fact, for every constant p and $k \in \mathbb{N}$ almost every graph in $\mathcal{G}(n, p)$ is k-connected, where a graph G is said to be k-connected if there is no set of $k - 1$ vertices whose removal disconnects the graph (see Lemma 11.3.2 in Diestel (2005)). Therefore, by Lemma 1 in the proof of Proposition 10, there must be a subtree with an infinite number of leaves in a k-connected graph with $k > 1$.

6.2 Optimal Network Structures

I will now keep the set of players N and the prior $\boldsymbol{\pi}$ fixed, and compute the optimal network structure E (i.e. the set of edges), from the point of view of player 1, and and from that of the other traders jointly considered. Finding optimal networks is interesting in thinking about modeling network formation and, furthermore, it provides upper bounds for the payoff of the initial owner and the joint payoff for other players.

It is easy to see that the optimal network for the initial owner is the *star network*, where $\{12, 13, \ldots, 1n\} \subseteq E$, or any other network that includes the star network as a subgraph (see Figure 5). In this case, all outcomes are ex-post efficient (i.e. the surplus is maximized) and the entire surplus generated is collected by the the initial owner.

Instead, as formalized in the next proposition, the network structure that maximizes the ex-ante utilitarian welfare of all traders other than the initial owner is a *path*, possibly not including all traders, with player 1 at one end and some other trader appropriately chosen at the other.

Proposition 11 (Optimal Networks)**.** *Fix N, and assume that $\pi_i > 0$ for all $i \in N$. The network structure E for which the game $\langle G, \boldsymbol{\pi}, T^* \rangle$ has an equilibrium that maximizes the sum of ex-ante payoffs of traders other than 1 is a path. Player 1 is at one end and at the other end is player*

$$j^* = arg \max_{j \in N \setminus 1} \left[1 - \prod_{\{i \in N \setminus \{1,j\} | \pi_i \leq \pi_j\}} (1 - \pi_i) \right] (1 - \pi_j).$$

Included in the path, in no particular order between 1 and j^*, are all other players with $\pi \leq \pi_{j^*}$. The remaining players are disconnected from the network.[41]

In the optimal network every connected trader other than j^* is an intermediary and pays

[41] When transaction costs are positive the optimal network structure that is jointly preferred by the buyers is not easy to characterize. To see this consider the following example. Let $T > 4$ and $N = \{1, 2, 3, 4\}$. Assume that $\pi_2 = \pi_3 = \pi_4 = \pi$, whereas $\pi_1 = 0$. Also, let τ be the transaction cost for each edge. In this case, a network where all players are in a path provides zero payoff to all buyers, because, due to transaction costs, the initial owner will always ask a price of 1 from his neighbor. However, in a network where $E = \{12, 23, 24\}$ traders 2,3 and 4 achieve a joint payoff of $\pi\{1 - (1 - \tau)[1 - (1 - \pi)^2]\}$. In fact, 1 will ask to 2 a price of $(1 - \tau)[1 - (1 - \pi)^2]$, and 3 will demand one to 3 and 4.

the price π_{j^*}.[42] Including a player i with $\pi_i > \pi_{j^*}$ is not profitable because, even though it provides positive payoff to j^*, it increases the price paid by all players other than j^* who are already in the path (e.g. this follows trivially for example if $\pi_i = 1$).

7 Conclusions

In this paper I develop a theory of decentralized trading in networks under asymmetric information. I show how traders' payoffs are shaped in equilibrium by the complex interplay of incomplete information and network architecture. In particular, traders that provide monopolistic access to some valuable area of the network become intermediaries and, in contrast to the case of complete information, are able to extract a positive rent. Furthermore, I show that it is the joint effect of asymmetric information and poorly connected networks that produces inefficiencies. In fact, when information is complete or the network is highly connected (e.g. if everyone is connected to the initial owner) inefficiencies disappear.

The model has been developed under a number of non-trivial restrictions. In particular, I assume that traders have two possible values only, that all traders know the network structure and observe all actions, that there is a deadline to the negotiations and that traders do not discount the future. While relaxing these assumptions would seem worthwhile and might provide some new insights, my main results appear robust to changes to these hypotheses.

In terms of future research, I see two promising avenues. First, to consider network formation games assuming that subsequent interactions are modeled using equilibria of network trading games. Second, to approach the problem from a mechanism design perspective. In particular, to allow traders to negotiate more flexibly with agents in their neighborhood. Condorelli and Galeotti (n.d.) made some progress in this direction. In this paper, which is work in progress, we consider a setting almost identical to the present one, with the only difference that, in each round, the owner of the good can run a second price auction with a reserve price, in which all traders in his neighborhood are allowed to participate. Remarkably, we observe that, in some cases, the initial owner will do better from bilateral bargaining

[42] The result is extended easily to cover the case where the resulting network must connect all traders in N. In this case, a path would still be optimal. At one end would be player 1 and at the other end would be the trader, different from 1, with the highest expected value.

than running an auction, assuming that he can not set individualized reserve prices.[43] To illustrate this point consider the following four-players example depicted in figure 6. Let $N = \{1, 2, 3, 4\}$, $\boldsymbol{\pi} = \{0, \pi, 0, \pi\}$, and $E \equiv \{12, 13, 34\}$. In a bilateral bargaining game, player 1 is able to achieve $1 - (1 - \pi)^2$. Instead, by running an auction the initial owner can at most obtain π.[44]

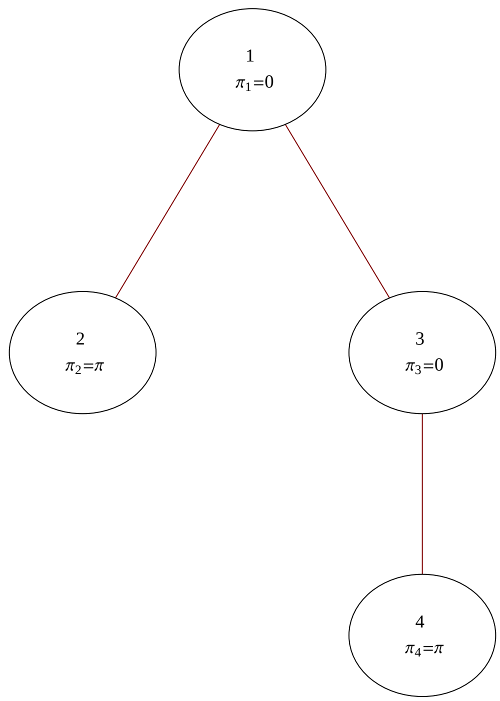


Figure 6: Auctions vs Bilateral Trading

[43]If a seller can exclude bidders from joining the auction or set individualized reserve prices, then he can always replicate the outcome of bilateral bargaining through an auction.

[44]This is what he gets if he set a reserve price of π or a reserve price of 1, as setting any other reserve price would provide an inferior payoff.

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Appendix A: Proof of Theorem 1

The proof shows how to construct one or more equilibria for each trading game. I will assume that there is an edge specific transaction cost $0 \leq \tau(i, j) < 1$ which the seller must bear only in case of sale. I first construct a unique equilibrium for a game starting in the last round T, for any possible owner and state of beliefs. Thereafter I show how to construct one or more equilibria in a generic time $t-1$ for each possible owner, profile of beliefs, and past public history, using the equilibria that I have computed for time t. By induction, I construct a set of equilibria for the whole game. A set of examples is provided to facilitate the reader.

Call $h \in \mathcal{H}$ an arbitrary history of the associated *perfect information game* (which I also call a *public history*) and call $\mathcal{Z}$ the set of terminal histories. Histories in the original imperfect information game are elements of the set $\{\emptyset\} \cup (\mathcal{H} \times \{0,1\}^n)$. An information set for player i is a set of elements in $\mathcal{H} \setminus \mathcal{Z} \times \{0,1\}^n$ that takes the following form: $I_i(h, v_i) = \{(h, v_i, v_{-i}) : v_{-i} \in \{0,1\}^{n-1}\}$. The profile of strategies specifies a behavioral strategy at each information set of each player in the game. In this setting with common prior, independent types and observed action, a system of beliefs specifies for all non terminal public histories h a profile of common posterior probabilities $\boldsymbol{\mu}(h) = (\mu_1(h), \ldots, \mu_n(h))$, where $\mu_i(h)$ indicates the probability that player i has value one (I will often write, omitting to mention the history, $\boldsymbol{\mu} = \langle \boldsymbol{\mu}_{-i}, \mu_i \rangle$).

Consumption decisions.

As a first step, I can fix from now the optimal consumption decisions in the entire game (i.e. after any public history): all buyers who obtain the good will consume it as soon as possible if and only they have value one. That is, for any public history h:

$$c_i^t(v_i)[h] \quad = \quad \begin{cases} 1 & \textit{if } v_i = 1 \\ 0 & \textit{otherwise} \end{cases} \tag{1}$$

This is an optimal strategy because: (i) no one can obtain in any equilibrium strictly more than one; (ii) consuming is a weakly dominated strategy for a player with value zero as he can not obtain less than zero by trying to resell. The following is an important consequence of this strategy in terms of belief updating: after any public history, a player i who does not consume the good will signal that he has value zero.

Equilibrium in round T.

As planned, let's consider first an arbitrary game starting at time T, with s^T in the role of the owner and with state of beliefs being $\boldsymbol{\mu}^T$ at the beginning of the round (i.e. at a point in which the seller has already signalled that he has value zero by not consuming in $T-1$), after public history h^T. Let's start backward within the round and consider the strategy of buyers in the neighborhood of s^T upon obtaining an offer at price p^t. Because T is the last round it is straightforward to determine an optimal strategy for all buyers after any public history:

$$a_i^T(p^T, v_i)[h] = \begin{cases} 1 & \mathit{if}\ p^T \leq 1\ \mathit{and}\ v_i = 1 \\ 0 & \mathit{otherwise} \end{cases}$$

Next, let's consider a seller. The probability that an offer from s^T at price $p^T \leq 1$ is accepted is equal to $E_{v_i}[a_i^T(p^T, v_i)] = \mu_i^T$ for each $i \in \mathcal{N}_{s^T}$. Therefore, in round T, the value for s^T from offering the good to player $i \in \mathcal{N}_{s^T}$ at price p^T is:

$$E_{v_i}[a_i^T(p^T, v_i)][p^T - \tau(s^T, i)] + \{1 - E_{v_i}[a_i^T(p^T, v_i)]\}v_{s^T} = \mu_i^T[p^T - \tau(s^T, i)] + (1 - \mu_i^T)v_{s^T}$$

Formally, I can allow for the possibility that s^T does not make an offer without modifying the formula above by assuming that there exists a fictitious player 0 (with $\mu_i^T = 0$) connected to all players, that will always rejects any offer. Selling to player 0 is a dominant strategy for a seller with value one. Therefore consider a seller with $v_{s^T} = 0$. By inspection of the objective function I conclude that a seller with $v_{s^t} = 0$ sets $p^T = 1$ and $i = arg\max_{i \in \mathcal{N}_s} \mu_i^T[1 - \tau(s^T, i)]$.

If the seller has more than one optimal offer there will be a multiplicity of equilibria, which, for the case of round T are all payoff equivalent for all players. However, it is easy to see that ties are non-generic. Denote by $V_i^T(s^T, \boldsymbol{\mu}^T, v_i)$ the payoff of player i with value v_i. The value for a trader who is a seller is $V_i^T(i, \boldsymbol{\mu}^T, 0) = \max_{j \in \mathcal{N}_i} \mu_j^T(1 - \tau(s^t, i))$ or $V_i^T(i, \boldsymbol{\mu}^T, 1) = 1$, while the value to all other players $i \neq s^T$ (including those not connected to s^T) is $V_i^T(s^T, \boldsymbol{\mu}^T, v_i) = 0$.

From an equilibrium in t+1 to an equilibrium in t

Let's now consider the game at the beginning of round t, with a generic owner s^t, arbitrary state of beliefs $\boldsymbol{\mu}^t$ and past history h^t. Define *a profile of continuation payoffs* as a list of possible equilibrium payoffs (i.e. a payoff for each type of each player) in a continuation game (starting at a certain time t, with arbitrary owner s^t and beliefs $\boldsymbol{\mu}^t$). While for a game that starts in round T there is a unique profile of equilibrium payoffs, in a general continuation game multiple payoffs vectors can in principle arise due to multiple equilibria. Therefore, the set of continuation profiles is a correspondence assigning to each round, state of beliefs and owner a set of profiles of continuation payoffs. Let $\mathcal{P}(t, s, \boldsymbol{\mu})(i, v)$ be the set of possible continuation payoffs for a player i with value v in a game starting in round t with seller s and beliefs $\boldsymbol{\mu}$.

In order to construct an equilibrium in the entire game, I will need to operate an equilibrium selection that might in principle depend on past history (not only, as the Markov property would require,on the state of beliefs and on the identity of the current seller). Therefore, I introduce the notation $V_i^t(s^t, \boldsymbol{\mu}^t, v_i)[h]$ to indicate the continuation payoff of a player i with value v_i, when the owner is s^t and beliefs are $\boldsymbol{\mu}$, according to a given profile in $\mathcal{P}(t, s^t, \boldsymbol{\mu}^t)$, after a specific public history h including all actions up to the beginning of round t. Throughout this section I will be assuming that $\mathcal{P}(t+1, s, \boldsymbol{\mu})$ is well defined in $t+1$ for each s and $\boldsymbol{\mu}$.[45]

Let's now proceed backward within round t. I have already discussed consumption decisions (see strategy (1)). Therefore let's consider first a buyer i connected to s^t.

A buyer with value zero.

Let's consider the strategy of a *buyer with value zero* after history $[h^t, (i, p^t)]$ including all events up to his information set. Set $a^t = 1$ and define $V_i^{t+1}(i, \langle \boldsymbol{\mu}_{-i}^t, 0\rangle, 0)[h^t, (i, p^t), a_i^t]$, hereinafter also referred to as $\mathcal{V}_i^{t+1}$, by making a selection from $\mathcal{P}(t+1, i, \langle \boldsymbol{\mu}_{-i}^t, 0\rangle)$. To simplify the analysis I make this selection independently from the entire past history. $\mathcal{V}_i^{t+1}$

[45] The proof that the equilibrium correspondence is indeed always well defined (not only in round T) will follow once the induction hypothesis has been established.

is the expected value that player i could make by selling the good in round $t+1$ or later.[46]

Because, in the equilibrium we are going to construct, no one will ever ask to player i a price below his expected profit from reselling the good, refusing an offer to buy always provides zero continuation utility to a player with value zero. Therefore, the following strategy is optimal for a buyer with value zero:

$$a_i^t(p^t,0)[h^t,(i,p^t)] \quad = \quad \begin{cases} 1 & \textit{if } p^t \leq \mathcal{V}_i^{t+1} \\ 0 & \textit{otherwise} \end{cases} \tag{2}$$

A buyer with value zero refuses any offer below, and accepts any offer above, his expected resale value, computed taking in to account that if he puts for sale the good, then everyone knows that he has value zero.

Bayesian updating.

Assume that $a_i^t(p^t,1)[h^t,(i,p^t)]$ is the strategy played by a buyer i with value one, after history h^t. When the rejection of an offer by player i is observed, the beliefs about players other than i remains unchanged (i.e. $\boldsymbol{\mu}_{-i}^{t+1} = \boldsymbol{\mu}_{-i}^{t}$). Taking in to account strategy (2) of a player with value zero along the same public history, updating in case of refusal from player i proceeds as follows:

$$\mu_i^{t+1} \quad = \quad \begin{cases} 1 & \textit{if } p^t \leq \mathcal{V}_i^{t+1} \\ \frac{\mu_i^t(1-a_i^t(p^t,1)[h^t])}{\mu_i^t\left[1-a_i^t(p^t,1)[h^t]\right]+(1-\mu_i^t)} & \textit{if } p^t > \mathcal{V}_i^{t+1} \end{cases} \tag{3}$$

This is standard Bayesian updating, except that I are fixing also an out of equilibrium belief. That is, even out of equilibrium, if someone refuses an offer at a price below $\mathcal{V}_i^{t+1}$ I assume that the deviator has value one. The motivation is that equilibria supported otherwise do not satisfy the intuitive criterion requirement (see Cho and Kreps (1987)): deviating for a player with value zero is a strictly dominated strategy, while a player with value one could benefit from rejecting an offer below the resale value, assuming that he could convince the seller that he has not value one for sure (see Example 4 later).

[46]Note that the buyers knows that the seller has value zero and so the proposal of the seller can not signal anything that i does not know

A buyer with value one.

Next, consider a *buyer with value one*, connected to s^t. If he accepts the offer from s^t he gets $1-p^t$ and the game ends, because he immediately consumes the good. In case he refuses, he gets his future continuation utility, as determined by the equilibrium selected from the the set $\mathcal{P}(t+1, s^t, \langle \boldsymbol{\mu}^t_{-i}, \mu_i^{t+1}\rangle)$, with μ_i^{t+1} determined according to Bayesian updating, for the particular public history at stake.

First, let's operate an arbitrary selection from the possible continuation equilibria in order to define uniquely the payoff of player i in two particular instances: (i) when he refuses an offer and beliefs remains unchanged, i.e. $V_i^{t+1}(s^t, \langle \boldsymbol{\mu}^t_{-i}, \mu_i^t\rangle, 1)[h^t, (i, p^t), a_i^t]$, with $a_i^t = 0$ and (ii) when he refuses an offer and everyone believes that he has value zero $V_i^{t+1}(s^t, \langle \boldsymbol{\mu}^t_{-i}, 0\rangle, 1)[h^t, (i, p^t), a_i^t]$, with $a_i^t = 0$. Henceforth I assume that these *two* continuation equilibria *do not* depend on the price offered in round t.

It is a straightforward consequence of the assumptions on belief updating that refusing any price p^t such that $p^t \leq \mathcal{V}_i^{t+1}$ is a dominated strategy for a player with value one. In fact, a refusal would signal that the buyer has value one, and in equilibrium no one would ever make him an offer at a price below one (i.e. $\mathcal{P}(t+1, s, \langle \boldsymbol{\mu}_{-i}, 1\rangle)(i, 1) = \{0\}$ for all s and $\boldsymbol{\mu}_{-i}$ if $s \neq i$). Therefore, let's now restrict attention to the case where $1 \geq p^t > \mathcal{V}_i^{t+1}$. I consider four cases in turn, that exhaust all possibilities.

Case 1. First, consider the case where both $1 - V_i^{t+1}(\mu_i^t) \leq \mathcal{V}_i^{t+1}$ and $1 - V_i^{t+1}(0) \leq \mathcal{V}_i^{t+1}$ (note that $1 - V_i^{t+1}(\mu)$ is the price at which player i is indifferent between accepting and rejecting an offer given that $\mu_i^{t+1} = \mu_i$ in case of refusal). A best reply for a buyer with value one in round t is the following:

$$a_i^t(p^t, 1)[h^t, (i, p^t)] = \begin{cases} 1 & \text{\textit{if} } p^t \leq \mathcal{V}_i^{t+1} \\ 0 & \textit{otherwise.} \end{cases} \tag{4}$$

If buyer i behaves as above, the refusal of a price above $\mathcal{V}_i^{t+1}$ leaves beliefs unchanged and provides him with $V_i^{t+1}(\mu_i^t)$, which is greater than the payoff from accepting the offer.[47]

[47] There exists no other best reply in pure strategies, but there might exist a class of mixed strategies that are best replies, where the buyer randomize his acceptance for prices above $\mathcal{V}_i^{t+1}$. This does not happen in example 1 but it can happen if there exist a $0 < \mu_i' < \mu_i^t$ such that $V_i^{t+1}(\mu_i')$ can be selected lower than $1 - \mathcal{V}_i^{t+1}$ (see example 6). This remark applies to the next three cases as well.

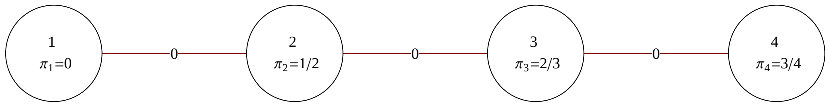

Figure 7: Trading network in examples 4 and 5

Example 4. *Players 1 ($\pi_1 = 0$), 2 ($\pi_2 = 1/2$), 3 ($\pi_3 = 2/3$) and 4 ($\pi_3 = 3/4$) are located in a line and $\tau(i, j) = 0$ for all i and j (see Figure 7 below). Assume that $s^1 = 1$ and $T = 3$. In round one player 2, no matter his value, refuses any price strictly above $\mathcal{V}_2^2 = 3/4$, because $1 - V_2^2(\pi_2) = 1 - V_2^2(0) = 2/3$.*

Case 2. Next, assume that $V_i^{t+1}(\mu_i^t) = V_i^{t+1}(0) < 1 - \mathcal{V}_i^{t+1}$. In this case, a best reply for a buyer with value one is the following:

$$a_i^t(p^t, 1)[h^t, (i, p^t)] \quad = \quad \begin{cases} 1 & \text{if } p^t \leq 1 - V_i^{t+1}(0) \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Accepting a price strictly above $1 - V_i^{t+1}(0)$ can never be profitable, as by refusing player i can guarantee himself $V_i^{t+1}(0)$. Refusing a price between $\mathcal{V}_i^{t+1}$ and $1 - V_i^{t+1}(0)$ is not convenient either, as by refusing player i mimics a value zero player and obtains $V_i^{t+1}(0)$.

Example 5. *Players 1 ($\pi_1 = 0$), 2 ($\pi_2 = 1/2$), 3 ($\pi_3 = 2/3$) and 4 ($\pi_4 = 3/4$) are located in a line and $\tau(i, j) = 0$ for all i and j (see Figure 1 above). Assume that $s^1 = 1$ and $T = 2$. I have that $V_2^2(\pi_2) = V_2^2(0) = 0$. Therefore, in round one, player 2 with $v_2 = 1$ accepts any price $p^1 \leq 1 - V_2^T(\pi_2) = 1$.*

Case 3. Next, consider the case where $V_i^{t+1}(\mu_i^t) > V_i^{t+1}(0)$ and also $1 - V_i^{t+1}(0) > \mathcal{V}_i^{t+1}$. The following is a best reply:

$$a_i^t(p^t, 1)[h^t, (i, p^t)] \quad = \quad \begin{cases} 1 & \text{if } p^t \leq 1 - V_i^{t+1}(0) \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Accepting a price above $1 - V_i^{t+1}(0)$ can never be convenient, as by refusing beliefs are left unaltered and i obtains $V_i^{t+1}(\mu_i^t)$. Further, refusing a price below $1 - V_i^{t+1}(0)$ is not convenient because it provides at most $V_i^{t+1}(0)$.[48] As shown in the following example, the presence of a time constraints (or transaction costs) is crucial for the possibility that a player benefitting from being considered with high expected value.

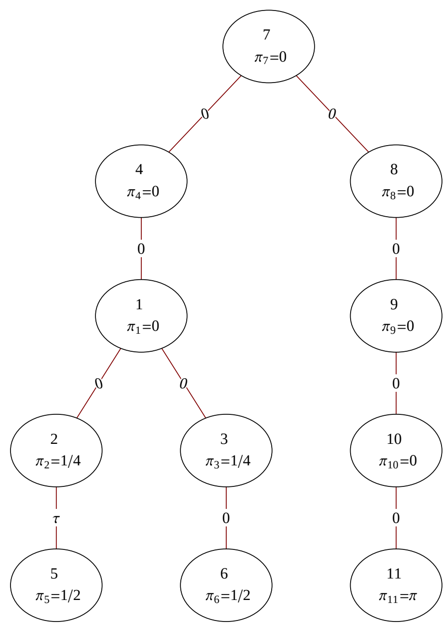

Figure 8: Trading network in example 3

[48]In this case, along with the possible mixed equilibria, there exists also another best reply in pure strategies:

$$a_i^t(p^t,1)[h^t] \quad = \quad \begin{cases} 1 & \textit{if } p^t \leq \max\{\mathcal{V}_i^{t+1}; 1 - V_i^{t+1}(\mu_i^t)\} \\ 0 & \textit{otherwise.} \end{cases} \tag{7}$$

This is optimal because refusing a price below the max above will lead the seller into believing either that the buyer has value 1 for sure or that the buyer has value zero for sure. In both case the buyer is worse off by refusing. On the other hand, refusing a price above that maximum provides no signal about player i and therefore guarantees him $V_i^{t+1}(\mu_i^t)$ in the continuation of the game. Note that the one I are considering in the main text is the one that is best for the seller, while the one presented in this footnote his best for the buyer.

Example 6. *A set of 11 players is located in a network (see figure 8). I assume that* $\frac{3}{4} - \frac{\tau}{4} < \pi < \frac{49}{64} - \frac{15\tau}{64}$*,* $s^1 = 1$*,* $T = 7$ *and* τ *is very small (such that the interval where* π *is defined is non-empty). Observe that* $V_2^2(\pi_2) = \frac{3}{8} - \frac{3(1-\tau)}{16}$*, because in round* two *player 1 will make an offer to player 2 at price* $1 - V_2^2(\pi_2)$ *that he will only accept if he has value one. In this case player one makes a profit of* $\frac{49}{64} - \frac{15\tau}{64}$*. However,* $V_2^2(0) = 0$*, because in this case player 1 in round 2 will sell to player 4 at price* π *(the payoff that player 1 obtains by selling to 3 is* $\frac{3}{4} - \frac{\tau}{4}$*). Finally, note that* $\mathcal{V}_2^2 = \frac{3}{4} - \frac{\tau}{4}$*, as 2 will immediately resell to 1 if he obtains the good. Therefore, in round 1, player 2 will accept any offer below* $1 - \frac{3}{8} - \frac{3(1-\tau)}{16}$*.*

Case 4. Finally, consider the case where $V_i^{t+1}(\mu_i^t) < V_i^{t+1}(0)$ and $1 - V_i^{t+1}(\mu_i^t) > \mathcal{V}_i^{t+1}$. Finding a best reply in this last case is slightly more complicate as an equilibrium in pure strategies will not exist.[49]

First, let's write $V_i^{t+1}(\beta)$ to denote $V_i^{t+1}(s^t, \langle \boldsymbol{\mu}_{-i}^t, \beta\rangle, 1)[h^t, (i, p^t), a_i^t]$. Next, call $\phi_i^t = \max\left\{1 - \mathcal{V}_i^{t+1}; V_i^{t+1}(0)\right\}$. I can show that there exists a $\mu_i^* \in (0, \mu_i^t)$ such that for all $\eta \in [0,1]$ I have that $\{\eta\phi_i^t + (1-\eta)V_i^{t+1}(\mu_i^t)\} \in \mathcal{P}(t+1, s^t, \langle \boldsymbol{\mu}_{-i}^t, \mu_i^*\rangle)(i, 1)$.[50] In order to construct an equilibrium I will assume that the selected continuation equilibrium in $t+1$ when $\mu_i^{t+1} = \mu_i^*$, depends not only on i and the previous public history, but also on p^t. In particular, I ask that: $V_i^{t+1}(\mu_i^*) = \eta(p^t)\phi_i^t + (1 - \eta(p^t))V_i^{t+1}(\mu_i^t)$, where $\eta(p^t) = \frac{(1 - V_i^{t+1}(\mu_i^t) - p^t)}{\phi_i^t - V_i^{t+1}(\mu_i^t)}$, (i.e. the probability that continuation ϕ_i^t ensues, $\eta(p^t)$, is such that $\eta(p^t)\phi_i^t + [1 - \eta(p^t)]V_i^{t+1}(\mu_i^t) = 1 - p^t$). The

[49] For a contradiction suppose there is a $p^* \geq \mathcal{V}_i^{t+1}$ such that all offers strictly above p^* are refused, while those below or equal to p^* are accepted. First, try to set any $\mathcal{V}_i^{t+1} \leq p^* < 1 - V_i^{t+1}(\mu_i^t)$. Because $1 - p^* > V_i^{t+1}(\mu_i^t)$, there exists small ϵ such that $1 - (p^* + \epsilon) > V_i^{t+1}(\mu_i^t)$. This implies that accepting price $p^* + \epsilon$ is preferred to refusing it (as beliefs remain unchanged upon refusal), which contradicts the statement. So, consider $p^* \geq 1 - V_i^{t+1}(\mu_i^t)$. Because $V_i^{t+1}(0) > V_i^{t+1}(\mu_i^t)$, there exists ϵ such that $V_i^{t+1}(0) - \epsilon > V_i^{t+1}(\mu_i^t)$. This implies that there exists a price $p^* - \epsilon \leq p^*$ that a player with value one would like to refuse, given that he will induce the beliefs that he has a value zero, which contradicts the statement.

[50] To see this point observe that if $V_i^{t+1}(\mu_i^t) < V_i^{t+1}(0)$ then player one must be an intermediary making a positive profit when $\mu^{t+1} = 0$, but must receive an offer at price one if $\mu^{t+1} = \mu^t$. Because the payoff correspondence is piecewise constant and payoffs are upper hemi-continuous with respect to the priors it must be the case that both continuation are possible for some μ_i^*. Furthermore, if both are possible at μ_i^*, some seller must be indifferent between two courses of actions and therefore by randomizing within the two he can produce any payoff for i in the specified interval.

following is an optimal strategy for a player with value one:

$$a_i^t(p^t,1)[h^t,(i,p^t)] \;=\; \begin{cases} 1 & \text{if } p^t \leq 1-\phi_i^t \\ \frac{\mu_i^t-\mu_i^*}{\mu_i^t(1-\mu_i^*)} & \text{if } 1-\phi_i^t \leq p^t < 1-V_i^{t+1}(\mu_i^t) \\ 0 & \text{if } p^t > 1-V_i^{t+1}(\mu_i^t) \end{cases} \tag{8}$$

First, observe that it is clearly optimal to reject any price above $1-V_i^{t+1}(\mu_i^t)$. Furthermore, note that it is optimal to accept any price below $1-\phi_i^t$, as refusing will provide at most $V_i^{t+1}(0)$. Finally consider the behavior of player one when the price is interval $(1-\phi_i^t, 1-V_i^{t+1}(\mu_i^t)]$. Given the strategy above, by rejecting the buyer induces belief μ_i^*. Therefore, given the continuations specified as a function of p^t, the buyer is indifferent between accepting and rejecting for each price in that interval, and therefore he can mix between acceptance and rejection with arbitrary probability.

Note that along the equilibrium path a seller will always ask price equal to $1-V_i^{t+1}(\mu_i^t)$ to a player that randomizes with constant probability in the interval $(1-\phi_i^t, 1-V_i^{t+1}(\mu_i^t)]$. Therefore, along the equilibrium path, sellers will have no need to randomize in the future as $\eta(p^t)=0$.

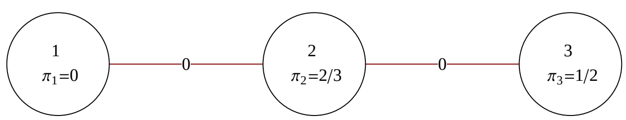

Figure 9: Trading network in example 4

Example 7. *Players 1* ($\pi=0$), *2* ($\pi=2/3$), *3* ($\pi=1/2$) *are located in a line and* $\tau(1,2)=\tau(2,3)=0$ *(see Figure 9 above). Assume that* $s^1=1$ *and* $T=3$. *I have that* $1-\mathcal{V}_2^2=1/2>0=V_2^2(\pi_2)$ *and* $V_2^2(0)=1/2$. *At* $T-2$ *player* 2 *with* $v_2=1$ *plays as follows: he accepts any price below* $\mathcal{V}_i^{T-1}$ *with probability one and any price below 1 with probability* $a(p^{T-2},1)=1/2$ *(i.e.* $\mu_i^*=1/2$*). A rejection from player 2 in round one induces belief* $\mu_2^2=1/2$ *in round two. At* $t=2$ *player* 1 *is now indifferent between selling at price* $p^2=1$ *or* $p^2=1/2$ *and I assume that he randomizes between the two with probability* $\eta(p^1)=2(1-p^1)$. *In equilibrium player* 1 *do not sells in round one (or asks for a price of 1) and asks for a price* $p=1$ *in the second round, while player 2 accepts with probability for sure in round two and, if he gets an offer, with probability* $1/2$ *in round one.*

The seller.

Let's turn to the seller. His utility from making an offer to player i at price p^t is equal to:

$$E_v[a_i^t(p^t, v_i)][p^t - \tau(s^t, i)] + \{1 - E_v[a_i^t(p^t, v_i)]\}V_{s^t}^{t+1}(s^t, \boldsymbol{\mu}^{t+1}, v_{s^t})[h^t, (i, p^t), a_i^t] \tag{9}$$

First, observe that, given the strategy of each type of buyer, both the beliefs at $t+1$ and the expected probability of acceptance are well defined for each price and buyer. Second, observe that $V_{s^t}^{t+1}(s^t, \mu^{t+1}, v_{s^t})[h^t, (i, p^t), a_i^t]$ (with $a^t = 1$) is well defined for all histories where the seller has made an offer to some buyer in his neighborhood, as I have already operated a selection in continuation equilibria for all possible profiles of beliefs that can arise at $t+1$. I'm left with the need to operate an arbitrary equilibrium selection for the case where the seller does not make any offer, in order to define $V_{s^t}^{t+1}(s^t, \mu^t, v_{s^t})[h^t, (i, p^t), a^t]$.

Because the acceptance strategy of all buyers connected to s^t is piecewise constant and continuous from the left in p^t, the space of relevant choices is finite and the maximization of seller's utility is well defined and has one solution (or more in case of ties). Therefore I have obtained one equilibrium, or more if the seller has more than one optimal choice.

Constructing continuation values.

By varying the selections in continuations made I can populate $\mathcal{P}(t, s, \boldsymbol{\mu})$, which will be used in the analysis of round $t-1$. It is important to observe that $\mathcal{P}(t, s, \boldsymbol{\mu})$ can contain more than one payoff values for some players other than the seller, even if s^t has a unique optimal choice. This will happen if some of the sellers in rounds from $t+1$ to T are indifferent to some of their actions. It follows that in case of ties, the action taken by the seller in time t will be selected during the analysis of round $t-1$ or even earlier rounds, as the next example shows.

Example 8. *Players 1 ($\pi_1 = 0$), 2 ($\pi_2 = \pi$), 3 ($\pi_3 = 1/5$), 4 ($\pi_4 = 1/2$) and 5 ($\pi_5 = 1/4$) are located in a line as in Figure 4 and τ is very small. Assume that $s^1 = 1$ and $T = 5$ and $\pi > 1/4$. Note that in round 2, $V_2^2(\pi) = 0$ while $V_2^2(0) = 3/4(1-1/2)(1-1/5) = 3/10$ and $\mathcal{V}_2^2 = 5/8$. Therefore, if player 2 gets an offer in round one, he accepts any price below $7/10$ (i.e. $\max\{5/8; 7/10\}$) for sure and any price above with probability $4/3 - 1/(4\pi)$ as to induce $\mu_2^* = 1/4$. It is understood by the selection that if 2 received an offer in round one (e.g. at*

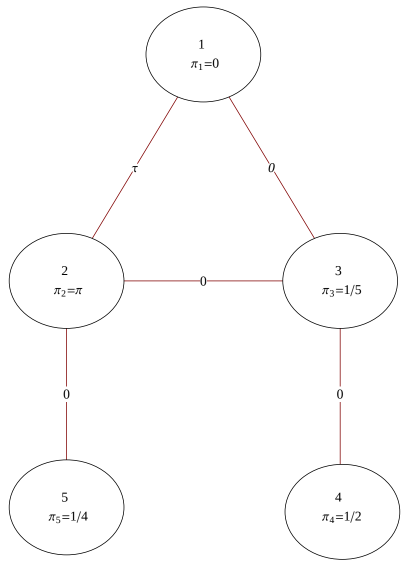

Figure 10: Network in Example 5

$p^1 = 3/4$*), then, while in round 2 player 1 would find in his interest to sell to player 3, the latter in turn after selling to 4, would be indifferent between selling to 2 at price* $p^4 = 1$ *and* $p^4 = 1/4$ *and therefore will play as established in the analysis of round one.*

Equilibrium in the entire game

By progressing backward from round T, by induction, I can complete the construction and arrive at an equilibrium for the game starting in round 1. Naturally, by employing different selections from continuations payoffs I can generate a large multiplicity of equilibria. The following example discusses multiple equilibria and thereafter the next section comments on the fact that this type of multiplicity in non-generic.

Example 9. *Players 1* ($\pi_1 = 0$)*, 2* ($\pi_2 = 1/2$)*, 3* ($\pi_3 = 3/4$)*, 4* ($\pi_4 = 1/2$) *and 5* ($\pi_5 = 3/4$) *are located in a line and* $\tau(i, j) = 0$ *for all* i *and* j *(see figure 11). Assume that* $s^1 = 1$ *and* $T = 3$*.*

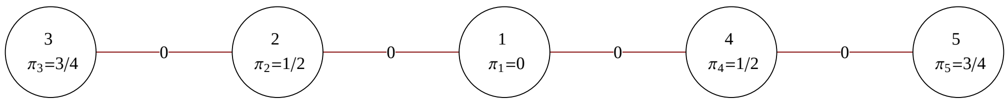

Figure 11: Network in Example 6

Let's compute the strategies of players 2 and 4 in round one when the seller is player 1. To do this, I need to compute $\mathcal{P}(2,1,\boldsymbol{\mu})$ first. As far as $\mu_2 \leq \pi_2$ and $\mu_4 \leq \pi_4$, $\mu_3 = \pi_3$ and $\mu_5 = \pi_5$ there are only two possible (pure) optimal offers for player 1 in round two. He either sells to player 2 or to player 4 at price $3/4$. In the first case, if both player 2 and 4 have value one, they make respectively a profit of 1/4 and 0. Instead, in the second case, they make 0 and $1/4$. Therefore I can conclude that $[0,1/4] \in \mathcal{P}(2,1,\langle 0,\mu_2,\pi_3,\mu_4,\pi_5\rangle)(2,1)$ and $[0,1/4] \in \mathcal{P}(2,1,\langle 0,\mu_2,\pi_3,\mu_4,\pi_5\rangle)(4,1)$ for all $\mu_2,\mu_4 \in [0,\pi_2]\times[0,\pi_4]$. Furthermore, $\mathcal{P}(2,i,\langle 0,\mu_2,\pi_3,\mu_4,\pi_5\rangle)(i,1) = 3/4$ for $i = 2,4$ and $\mu_2,\mu_4 \in [0,\pi_2]\times[0,\pi_4]$.

I have the following cases, depending on how I operate a selection of continuation payoffs (assume that I make the same selection for each $\mu_i \in [0,\pi_i]$):

- *First, suppose that I select $V_2^2(0) = 1/4$ and $V_4^2(0) = 0$. According to the selection rules I set $V_2^2(\mu) = 1/4$ for all $\mu \in [0,\pi_2]$ and $V_4^2(\mu) = 0$ for $\mu \in [0,\pi_2]$. Also, $\mathcal{V}_i^2 = 3/4$ for $i = 2,4$ and $\mu_{-i} \leq \pi_{-i}$ (if $i = 2$ then $-i = 4$ and viceversa). Therefore, player 2 in round one refuses any offer with $p^1 > 3/4$, while player 4 accepts any price $p^1 \leq 1$. It is implied by the selected continuations that in round one player one will first make an offer at price 1 to player 4 (that will be accepted if and only if 4 has value one) and, in case of refusal, he will, in round two, sell to player 2 at $p^2 = 3/4$, being indifferent between selling to 2 and 4. Player 2 will accept.*

- *Second, suppose that I select $V_2^2(0) = 0$ and $V_4^2(0) = 1/4$. Then, the analysis is as above but the roles of 2 and 4 are reversed.*

- *Third, suppose, that $V_2^2(0) = 1/4$ and $V_4^2(0) = 1/4$. In this case both players believe that they will receive an offer in the continuation of the game whenever they get an offer in round one. This selection is possible, because it is conditional on two different public histories. Therefore they will both refuse any price above $3/4$. Here, in round one, player 1 asks a price $p = 3/4$ to any of the players in round 1 and, according to* *the selected continuations, repeat the offer in round two. Therefore his payoff will be lower in this third case than in the first and second.*

- *Fourth, assume that* $V_2^2(0) = 0$ *and* $V_4^2(0) = 0$*. In this case both players foresee that if they receive an offer today, they will not obtain another one in the second round. Therefore, they accept in round one any price below or equal to one. The payoff of player 1 will be equal to that in case 1 and 2.*

- *Finally any other combination of continuations in* $[0, 1/4]$ *is also possible. For example, both players may believe that player 1 will make in round two an offer to one or the other with equal probability.*

It is easy to see, at this point, that by perturbing the beliefs or introducing transaction costs, the multiplicity disappears. In fact, whenever, for example $\pi_5 = 3/4 + \epsilon$*, player 1 will find optimal to sell to 4 in round two. This same result can be obtained by introducing a small transaction cost for one of the links between 1 and 3.*

However another type of multiplicity that arises from multiple best replies from buyers can be illustrated using case 4. Assume that the selection is the following for player 2: $V_2^2(0) = V_2^2(0) = 1/4$ *but* $V_2^2(\pi_2/2) = 1 - p^t$*. Then player 2 will accept for sure any price below 3/4 but may also accept with probability* $2/3$ *the prices between 3/4 and 1 (inducing belief* $\pi_2/2 = 1/4$*).*

Finally, another possibility of refinement deserves mention: I might want to require that the selected continuation is independent from the identity of the player who gets the offer, hereby eliminating the cases 3 and 4 above and other similar cases.

Deterministic Trading Chains

The previous example shows that, even along the equilibrium path, the seller can sometimes be indifferent between two or more different offers. This produces a number of equilibria in which he mixes between two or more offers. It is easy to see that by introducing transaction costs, and by randomly perturbing the priors and transaction costs, the occurrence of a tie has probability zero. Therefore equilibria where the seller randomizes are non generic.

Appendix B: Proofs of Propositions in Section 5 and 6

Proof of Proposition 1. Let's consider a setting with transaction costs. Let the cost of a path $b(i,j)$ be the total transaction cost needed for carrying the good from i to j along $b(i,j)$. For example, if $b(i,j) = (i, b_1, \ldots, b_m, j)$ the cost of the path is equal to $\tau(i, b_1) + \sum_{k=1}^{m-1} \tau(b_k, b_{k+1}) + \tau(b_m, j)$. We denote by $b_k^*(i,j)$ the path between i and j of at most length k who generates the minimum cost among all paths $b(i,j)$ of length at most k. If the path exists let $c_k^*(i,j)$ be the cost of $b_k^*(i,j)$. If such path does not exist, we set $c_k^*(i,j) = \infty$. Let $O = \{i \in N \mid v_i = 1\}$. If O is not empty then an outcome is efficient if and only if player $i^* = \arg\min_{i \in O} c_T^*(1,i)$ consumes the good, $c_T^*(1,i^*) < 1$ and the good reaches i^* from 1 along path $b_T^*(1,i^*)$. If O is empty or $c_T^*(1,i^*) > 1$ then an outcome is efficient if some player j consumes the good and the good reaches j from 1 along a zero cost path.

I will now show that all subgame perfect equilibria of the game attains efficient outcomes. The game with perfect information can be solved by backward induction. First, whenever the set O is empty or there is no $i^* = \arg\min_{i \in O} c_T^*(1,i)$ such that $c_T^*(1,i^*) < 1$, in every round of the game, every trader will be willing to pay at most zero for the good. Therefore, due to transaction costs the only equilibria are those where the goods moves around edges with zero transaction cost.

Next assume that there exists $i^* = \arg\min_{i \in O} c_T^*(1,i)$ with $c_T^*(1,i^*) < 1$. For simplicity assume that both the player i^* and the path $b_T^*(1,i^*)$ that minimizes the transaction costs is unique and takes exactly T rounds. This makes the argument more sharp, but of course the proof extends easily to cases where ties are present. By reasoning backward it can shown that, in round one when player 1 is the seller, player $b_1 \in b_T^*(1,i^*)$ has the highest willingness to pay for the good, equal to $1 - c_k^*(i,j) + \tau(1, b_1)$, within all players connected to 1. Moreover, the willingness to pay of every trader is reduced whenever the seller waits one round. The willingness to pay is determined by solving the game backward, assuming that there are $T-1$ rounds and that b_1 is the owner of the good. Therefore player 1 will sell to him at that price, realizing a net profit of $1 - c_k^*(i,j)$, which corresponds to the total surplus. In round two, the new owner b_1 sells the good to $b_2 \in b_T^*(1,i^*)$ at a price $p^2 = 1 - c_T^*(1,i^*) + \tau(1,b_1) + \tau(b_1,b_2)$, but makes zero profit due to transaction costs. In round T player $b^m \in b_T^*(1,i^*)$ sells to good to player i^* at price one. □

Proof of Proposition 2. Because I focus on equilibria where traders do not take randomized decision along the equilibrium path, each equilibrium payoff profile in a network trading game can be identified by the ordered set of sellers, offers that are made and acceptance decisions along the equilibrium path:

$$\{(1, i^1, p^1, a^1), (s^2, i^2, p^2, a^2), \ldots, (s^T, i^T, p^T, a^T)\}$$

.

I define a *payoff producing* offer as an offer (s, i, p, a), made in some round t, from trader s to i, such that (i) $\mu_i^t > 0$, (ii) $p > \mathcal{V}_i^{t+1}$, and (iii) $a_i^t(p, 1) > 0$. That is a payoff producing offer is one made at a price above his resale value, to a player who has value one with positive probability, and that would be accepted with positive probability by that player.

First, I show that the number of payoff producing offers that can be made in any equilibrium, computed according to the algorithm in the appendix A is finite. If an offer is payoff producing and $p > \mathcal{V}_i^{t+1}$, it must be that $a_i^t(p, 0) = 0$. Therefore, given that $a_i^t(p, 1) > 0$ the posterior belief about any player who refuses an equilibrium offer must strictly decrease, that is $\mu_i^t > \mu_i^{t+1}$. In particular, according to the equilibrium characterization, if we are in cases 1-3 of appendix A, $a_i^t(p, 1) = 1$ and therefore $\mu_i^{t+1} = 0$. In the first case, no other payoff producing offer can be made to player i. Otherwise, if we are in case 4, $a_i^t(p, 1) = \frac{\mu_i^t - \mu_i^*}{\mu_i^t(1-\mu_i^*)}$. In this case, because there are no ties, $\mu^* < \mu_i^t$ and $\mu^{t+1} = \mu^*$. Because sellers are rational they will always ask price equal to the upper bound of the support in which the buyer is mixing, given that the buyer randomizes with constant probability in the interval. In the continuation game, the buyer will play a pure strategy. If he mimics a type zero player then the offer made to him is not payoff payoff producing. If he plays a separating strategy, then is value is fully revealed at the subsequent offer. Therefore, I conclude that the number of payoff producing offers that can be made along the equilibrium path is finite.

With this in mind, it is possible to show that the number of offers that can influence payoffs, call them *payoff relevant*, must be finite. In fact, the offer other than payoff producing, which serve to transfer the object within intermediaries, all made at a price equal to resale values, are of two types. They might be irrelevant for payoffs, if both the seller and the buyer have $\mu_i = 0$ and no other offer has been made in between. They are relevant otherwise. Therefore, in general, there is a finite number of payoff relevant offer that can be made, before $\mu_i = 0$ for all i.

Next, take an ordered sequence of payoff relevant offers

$$s_{pro} = ((s^1, i^1, p^1, a^1), (s^2, i^2, p^2, a^2) \dots (s^n, i^n, p^n, a^n))$$

where the index does not indicate the timing but only the order. Each sequence is finite, and the set of all such sequences, denoted S_{pro}, is finite. Next, I can show that, absent discounting, any two equilibrium e and e' of two network trading games, that differ possibly only on the number of rounds, must produce the same profile of interim payoffs if the sequence $s_{pro}(e)$ of payoff relevant offers of e, is identical to the sequence in e', denoted $s_{pro}(e')$. This is easy to see, because all the remaining offers, which are not payoff relevant, involve a sale from a player with value zero, who acquired at his resale value to another player with value zero who will have the same resale value.

Because the number of possible equilibrium payoffs profiles is finite, and each can be obtained in a finite game, there must be a T^* such that all profiles can be attained in games with $T < T^*$. Let's fix this number as the reference number of rounds. To conclude the proof, I need to show (I) that for any equilibrium e of a trading game with $T > T^*$ rounds there exist an equilibrium e' in a game with T^* rounds such that the profile of interim utilities for all types of all players are the same and (II) the converse that for any equilibrium e of a game with $T = T^*$ there exist an equilibrium e' in the same game with $T > T^*$. To show (I) note that, because the payoffs can be entirely recovered from the set of payoff relevant actions, in the equilibrium e there must be a number of redundant trading actions that would not change the equilibrium if removed. To show (II) note that it must be an equilibrium for player one to wait until the number of rounds left is T^* and then play as in e'. □

Proof of Proposition 3. I need to prove that no player who only receives offers that would be accepted if he has value one only, will ever receive any offer at a price strictly below one. In fact, any player who receives an offer that he will accept also if he has value zero is classified as an intermediary. Therefore, suppose there is a player i who only receives offers that would be accepted if he has value one only. Then there must be a last round in which i receives an offer. This offer will be at price one, if the seller selling to i is rational. It follows that also the previous offer made to i must be at price one. In fact, the previous seller anticipates that in equilibrium i has no other opportunity to acquire the good at a price lower than one and so i must be willing to pay up to one for it. Therefore, by the same argument repeated backward, all offers made to trader i will have a price equal to one. □

Proof of Proposition 4. I show that the no player s^t connected to an isolated player i with $\pi_i > 0$ will ever ask to i a price below one in equilibrium, in any round t. Suppose the game is in round t, the owner is s^t and belief are $\boldsymbol{\mu}^t$. Furthermore, assume that t is the first round in which s^t is willing to make an offer to i, so that $\mu_i^t = \pi_i > 0$. In fact, if no such round t and seller s^t exists, then i must be inactive. Let $V_{s^t}^{t+1}(s^t, \langle \boldsymbol{\mu}_{-i}^t, 0\rangle, 0)$ be the expected value of s^t obtainable from selling the good at the beginning of time $t+1$, assuming that player i has value zero. Note that this must be greater or equal to $V_i^{t+1}(i, \langle \boldsymbol{\mu}_{-i}^t, 0\rangle, 0)$, that is the resale value of player i once he gets the good. In fact he can only immediately resell to s^t. Furthermore, note that there must be a time $t^* \geq t$ such that

$$V_{s^t}^{t^*}(s^t, \langle \boldsymbol{\mu}_{-i}^t, 0\rangle, 0) = V_{s^t}^{t+1}(s^t, \langle \boldsymbol{\mu}_{-i}^t, 0\rangle, 0) > V_i^{t^*}(i, \langle \boldsymbol{\mu}_{-i}^t, 0\rangle, 0)$$

because the sale from i back to s^t wastes one round. Therefore, it follows that at t^* no offer that i may have an interest in making will ever be accepted by a player i with value zero (i.e. with price $p^{t^*} \geq V_{s^t}^{t^*}(s^t, \langle \boldsymbol{\mu}_{-i}^t, 0\rangle, 0)$). Now consider that in round t player i with value one will refuse a price of one only if he expects to receive an offer with a lower price in the future. However, this will never happen. In fact, it follows from proposition 3 that he will only receive offers at price one. Because I have seen that s^t can achieve $\pi_i + (1-\pi_i)V_{s^t}^{t^*}(s^t, \langle \boldsymbol{\mu}_{-i}^t, 0\rangle, 0)$ by waiting until t^*, then it is clear that he will never make an offer at price below one to player i earlier. $\square$

Proof of Proposition 5. Consider any equilibrium of $\langle G, \boldsymbol{\pi}, T\rangle$ where i is an active player. Since i is a active he get an offer with positive probability from some player $j \in G_{-i}^1$ in some round t, with beliefs being $\boldsymbol{\mu}^t$. The most that player j can achieve by asking a price that only a type one of i accepts, is $\pi_i + (1-\pi_i)V_j^{t+1}(j, \langle \boldsymbol{\mu}_{-i}^t, 0\rangle, 0)$. Note that whenever a type one accepts the offer it must be the case that he does not get a later offer at lower price. Therefore, the value $V_j^{t+1}(j, \langle \boldsymbol{\mu}_{-i}^t, 0\rangle, 0)$ does not include any resale to i at a price lower than 1 later on. Therefore, $V_j^{t+1}(j, \langle \boldsymbol{\mu}_{-i}^t, 0\rangle, 0)$ will be also equal to the resale value of i in $G_{-i}^1 \cup i$ when beliefs are $\langle \boldsymbol{\mu}_{-i}^t, 0\rangle, 0)$. In fact, if two players with value zero are connected (note that $\mu_j^t = 0$ because j is reselling), time is sufficiently large, and transaction costs are zero they must make the same profit by reselling the good.

Next, observe that, at any point in time, i will accept for sure a price below or equal to his resale value. If time is sufficiently large, the resale value of i (when acquires the good for the first time), at time t with beliefs $\langle \boldsymbol{\mu}_{-i}^t, 0\rangle$ will be greater or equal to:

$$\widetilde{U}_i(0) + (1 - \widetilde{U}_i(0))V_j^{t+1}(j, \langle \boldsymbol{\mu}_{-i}^t, 0\rangle, 0).$$

Therefore, it follows that whenever $\widetilde{U}_i(0) > \pi_i$ asking for a price that only a type one of the player would ever accept can not be optimal for player j. $\square$

Proof of Proposition 6. It is sufficient to prove the statement for any two successive intermediaries (i.e two intermediaries such that no other intermediary is within them in the trading chain). The price at which an intermediary i^t acquires the good in round $t-1$ is equal to his resale value $\mathcal{V}_i^t$, computed when he is a seller in round t, with beliefs being $\boldsymbol{\mu}^t = \langle \boldsymbol{\mu}_{-i}^{t-1}, 0\rangle$. Let $X(i^t) = \{x^t, \ldots, x^{t+k-1}\}$ be the ordered set of players to which i^t makes, in equilibrium, an offer that is accepted with positive probability. Assume that only $i^{t+k} = x^{t+k-1}$ accepts an offer with probability one (otherwise i^{t+k} would not be the successive intermediary after i^t). I now compare the price paid by i^t, that is $p^{t-1} = \mathcal{V}_i^t$, with the one paid by i^{t+k}, that is $p^{t+k-1} = \mathcal{V}_{i^{t+k}}^{t+k}$. Let α_j indicate the probability that player $j \in X(i^t)$ accepts his offer. I know that, along the equilibrium path:

$$p^{t-1} = \alpha_{x^t} p^t + (1 - \alpha_{x^t})\alpha_{x^{t+1}} p^{t+1} + \cdots + \mathcal{V}_{i^{t+k}}^{t+k} \prod_{j \in X(i^t) \setminus x^{t+k-1}} (1 - \alpha_j)$$

Next, observe that, because the seller is rational, the sequence of prices must be weakly decreasing. Therefore I can conclude that $p^{t-1} \geq p^{t+k-1}$, with strict inequality unless no player before i^{t+k} accepts with positive probability an offer of i^t at a price strictly $\mathcal{V}_{x_k}^{t+k}$. This happens for example if $X(i^t) = \{i^{t+k}\}$. $\square$

Proof of Proposition 7. It is sufficient to prove the statement for any two successive intermediaries (i.e two intermediaries such that no other intermediary is within them in the trading chain). The price at which an intermediary i^t acquires the good in round $t-1$ is equal to his resale value $\mathcal{V}_i^t$, computed when he is a seller in round t, with beliefs being $\boldsymbol{\mu}^t = \langle \boldsymbol{\mu}_{-i}^{t-1}, 0\rangle$. Let $X(i^t) = \{x^t, \ldots, x^{t+k-1}\}$ be the ordered set of players to which i^t makes, in equilibrium, an offer that is accepted with positive probability. Assume that only $i^{t+k} = x^{t+k-1}$ accepts an offer with probability one (otherwise i^{t+k} would not be the successive intermediary after i^t). Let α_j indicate the probability that player $j \in X(i^t)$ accepts his offer in equilibrium.

Consider two cases. First, suppose that both i_t and i_k received no other offer in the past. I know that:

$$p^{t-1} = \alpha_{x^t} p^t + (1-\alpha_{x^t})\alpha_{x^{t+1}} p^{t+1} + \cdots + \mathcal{V}^{t+k}_{i^{t+k}} \prod_{j \in X(i^t) \setminus x^{t+k-1}} (1-\alpha_j)$$

Next, observe that (i) the maximum probability with which an offer is accepted (but not for sure) is achieved if a player value one accepts with probability one, and (ii) the maximum price that could be charged is one. Therefore, I can write:

$$\mathcal{V}^t_{i^t} \leq 1 - \prod_{j \in X(i^t) \setminus x^{t+k-1}} (1-\mu^t_j) + \mathcal{V}^{t+k}_{x^{t+k-1}} \prod_{j \in X(i^t) \setminus x^{t+k-1}} (1-\mu^t_j) \tag{10}$$

where I use the fact that only the belief of a player that receives an offer can be updated passing to the next round. Observe that (10) holds with equality only if $p^{t+j-1} = 1$ for all $j \in \{1, \ldots, k-1\}$ or if $X(i^t) = \{i^{t+k}\}$. By rewriting, I get

$$1 - \mathcal{V}^t_{i_t} \geq (1 - \mathcal{V}^{t+k}_{x_{k-1}}) \prod_{j \in X(i^t) \setminus x_{t+k-1}} (1-\mu^t_j).$$

Therefore, evaluating the payoff of the two players at round $t-1$ (or at any prior round) the proposition holds with strict inequality, assuming that $\mu^{t-1}_{i^t} > 0$:

$$U_{i^t}(1) = 1 - \mathcal{V}^t_{i^t} > U_{i^{t+k}}(1) = (1 - \mathcal{V}^{t+k}_{x^{k-1}})(1-\mu^t_{i^t}) \prod_{j \in X(i^t) \setminus x_{t+k-1}} (1-\mu^t_j).$$

Next, suppose that any of the players gets another offer in round $t-2$ or earlier that he will accept with some positive probability but not for sure (otherwise this would contradict the statement that i^t and i^{t+k} sells for the first time in rounds t and $t+k$). First, if i^t got another offer, this could only, in principle, increase his payoff, while the payoff of i^{t+k} remains unchanged and so the proposition continue to hold. Therefore, as a worst case, assume that i^{t+k} only got another offer earlier in the game. Because the seller who made the offer to i^{t+k} was rational, and the offer accepted with positive probability, the price that has been asked must made a type one player of i^{t+k} indifferent between accepting and rejecting (in which case i^{t+k} expects to receive another offer in $t+k$). It follows that the payoff of i^{t+k} will not change if he gets an earlier offer. However, the payoff of i^{t+k} will be reduced and in particular $U_{i^t}(1)$ will be discounted by the probability with the the offer is accepted by i^{t+k}, that happens at most with probability $\pi = \mu^t_{i^t}$. It follows that in this case the proposition will hold at worst with equality. $\square$

Proof of Proposition 8. Let's focus on the resale value of player one, that is $U_1(0)$. In fact, if he has value one, is payoff will be equal to one always, no matter the set of edges. In an equilibrium for the original game, let $X(1) = \{x^1, \dots, x^k\}$ be the ordered set of players to which 1 makes an offer that is accepted with positive probability.

First, suppose that all players in $X(1)$ are final customers. Therefore assume that the probability with which a $j \in X(1)$ accepts an offer, α_j, is equal to the prior probability that j has value one, and the price asked is equal to one. The resale value of 1 is:

$$U_1(0) = \pi_{x^1} + (1 - \pi_{x^1})\pi_{x^2} + \cdots + \pi_{x^k} \prod_{j \in X(1) \setminus x^k} (1 - \pi_j) = 1 - \prod_{j \in X(1)} (1 - \pi_j)$$

In this case, the introduction of a new edge can only make player 1 better off. In fact, he can always replicate the same set of offers, possibly in a different order, after the introduction of a new edge. The point here is that the introduction of a new edge can not prevent player 1 to treat someone as a final customer, if he finds in his interest to do so.

Next, assume that $x^x = i^2$ is the an intermediary, to which player 1 sells with probability one, and that there are, in addition to final customers, possibly other players, who are later intermediaries in the set $X(1)$. The resale value of one is:

$$U_1(0) = \pi_{x^1} p^1 + (1 - \pi_{x^1})\pi_{x^2} p^2 + \cdots + \mathcal{V}_{i^2}^{k+1} \prod_{j \in X(1) \setminus i^2} (1 - \pi_j)$$

Note that, by induction, assuming that the statement of the proposition holds, the resale value of player i^2 must increase thanks to introduction of the edge, assuming that the set of players to which player 1 makes an offer stay the same. Therefore, for this reason, if we can show that player one can replicate the same set of offers, we can prove that he must be weakly better off. This is so because time is sufficiently large. In this case he can order the final customers and the other intermediaries to which he makes offer that they accept only if they have value one in such a way that, if he finds in his interest to do so, he will be able to replicate the set of offers. Therefore the initial owner will obtain at least the same payoff when the network becomes more connected. □

Proof of Proposition 9. For an inefficiency to arise, some player i must not be receiving an offer with positive probability in equilibrium. In fact, because agents are forward looking and offers are at prices below or equal to one, no player with value one will refuse an offer unless he expects to get a subsequent offer with positive probability.

Therefore, for a contradiction suppose that there exists no equilibrium in the trading game where player i gets an offer with positive probability. There are three possible cases to consider: (i) i is connected to some intermediary (or to player 1), (ii) i is connected to some final customer and (iii) i is only connected to inactive traders. First assume that i is connected to some player j that is an intermediary (or to player 1). In this case, if time is sufficiently large player j will always find profitable to make some offer to i . Instead, suppose that i is connected a final customer. In this case the condition in Proposition 5-(II) holds, at worst with equality, and therefore there will be an equilibrium in which the final customer is an intermediary and the argument of the previous point applies. Therefore, consider finally the case where i is connected only to inactive players. In this case the entire proof can be applied to him and show that he must be receiving an offer, unless is not connected to any active player. Proceeding by induction it become clear that every player must be active and an efficient equilibrium must exist. □

Proof of Proposition 10. To prove sufficiency one can note that $1-(1-\pi)^{l(G^*(G^n))}$ represents a lower bound to the payoff that can be obtained by player one. In fact, $l(G^*(G^n)$ represents the number of isolated traders that will be asked a price of one that they will accept if they have value one. Therefore as $l(G^*(G^n)) \to \infty$ I have that $\widehat{U}_1^n \to 1$. To prove necessity, I need the following auxiliary Lemma which shows that number of traders from which a surplus can be extracted by player 1 is finite, if all subtrees of the limit graph $\lim_{n\to\infty} G^n$ presents only a finite number of isolated traders (i.e. leafs).

Lemma 1. *Any infinite tree that has a finite number of leafs must have at most a finite number of vertices with degree three or more.*

Proof. I will establish a contradiction by showing that any tree with an infinite number of vertices of degree three or more, must have an infinite number of leafs. First, note that that any tree with a finite number of leafs must be locally finite (i.e. every vertex must have a finite number of neighbors). Then, observe that by König's Lemma, any infinite graph must include at least a path of infinite length. Next, assume that at least one infinite path, say $v_1, v_2, \dots$ includes an infinite number of vertices of degree three or more. This must be the case because if no infinite path contains an infinite number of vertices of degree three or more, than the number of leafs is not finite (otherwise the number of leaf-to-leaf paths would be finite and some of them will necessarily contain an infinite number of vertexes with degree three or more).

Since the tree is locally finite, let $k \geq 3$ be the upper bound on the degrees of vertices included in the path $v_1, v_2, \ldots$. The following algorithm proves the Lemma by contradiction showing how to construct a sub-sub tree with an infinite number of leafs.

Algorithm As initial conditions let $x = 2$ (i.e. the index of the player used in each round of the algorithm), let $l(G) = 0$ (i.e. the leaf counter at each round) and let $L(G) =$.

Step 1. Take v_x. It has either degree 2 (I write $d(v_x) = 2$) or $2 < d(v_x) \leq k$. If $d(v_x) = 2$ set $x := x + 1$ and restart with step 1. If $2 < d(v_x) \leq k$ then go to step 2.

Step 2. v_x is connected to v_{x+1} and to to v_{x-1}. Therefore, there are other y players connected to him, with $1 \leq y \leq k - 2$. If any of them is outside the path $v_1, v_2, \ldots$ then increase the leaf counter by one $l(G) := l(G) + 1$, put the selected player in $L(G)$, set $x := x + 1$ and go back to Step 1. Otherwise, go to Step 3.

Step 3. If none of the y players which are inside the path is a successor of v_x than set $x := x + 1$ and go back to step 1. If there is a player which is a successor of v_x, say v_z, two cases are possible: (i) there is still an infinite number of players in the path following v_y or (ii) this number is finite. In case (i), if $v_{x+1} \notin L(G)$, put v_{x+1} in $L(G)$ and increase the leaf counter $l(G) := l(G) + 1$. Then, in any case (i.e. even if $v_{x+1} \in L(G)$) go back to step 1 with $x := y$. In case (ii), if $v_{x+1} \notin L(G)$, put v_{x+1} in $L(G)$ and increase the leaf counter $l(G) := l(G) + 1$. Then note that there must still be an infinite number of player between v_x and v_y, therefore, restart the argument in Step 1 by setting $v_x = v_y$ but now going backward with the indexes. Do this also if $v_{x+1} \in L(G)$.

This process must identify an infinite number of leafs, as the cases where a leaf is not selected are at most finite, given that the tree is locally finite and the number of vertexes with degree three or more is infinite (observe that the cases in which a leaf is not selected because it has been a leaf before are at most finite as no vertices can get more than k connections). *Q.E.D.*

As a result, any infinite graph G where every sub-tree has a finite number of leafs must have at most a finite number of vertices with degree three or more. Furthermore, the Lemma also implies that there is an infinite number of players with degree two that must be connected one to each other in a chain and that the number of players outside the said chain must remain finite. This implies that there is only a finite number of intermediaries from which player 1 can extract a surplus in the game. Therefore,one sees that the profit of the initial owner must be bounded away from one. □

Proof of Proposition 11. Assume first that all players in N must be connected in G. First, observe that Player 1 can always achieve a minimum ex-ante profit of $\pi_1+(1-\pi_1)\max_{i\in N\setminus 1}\pi_i$ by asking to one of his neighbor a price of $\max_{i\in N\setminus 1}\pi_i$. In fact, because a player with such expected value exists, this must be (at least) the resale value of some player connected to 1 if the number of rounds is sufficiently large, when there are no transaction costs. Next, observe that a chain where player 1 is at one end and player $i_{(1)} = arg\max_{i\in N\setminus 1}\pi_i$ is at other end (and all other players in between) has an equilibrium that maximize the total surplus available in the network (i.e. the sum of ex-ante utility), that is $1-(1-\pi)^n$. Therefore, because it also achieves the lower bound in ex-ante utility collected by player 1, it must achieve a maximum for the joint welfare of the other player, i.e. $\sum_{i=2}^n U_i$. Note that this has been established for an arbitrary number of traders connected in G. Therefore, I know that the optimal network when only a subset of N is connected (if a player other than 1 is disconnected he makes zero ex-ante utility) must be also a chain constructed as above, with the player with the highest expected value at one end. It remains to be checked that the set of players included in the line is determined as in the proposition. First, note that there is no gain in excluding players with expected values lower than the player at the end of the chain. Therefore, it is easy to see that j^* is selected exactly as the j that maximize the joint surplus of the traders other than player 1, which is:

$$\left[1-\prod_{\{i\in N\setminus\{1,j\}|\pi_i\leq\pi_j\}}(1-\pi_i)\right](1-\pi_j).$$

□