Satisfaction Approval Voting

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Overview

Satisfaction approval voting (SAV) is a voting system applicable to *multiwinner* elections (e.g., to a council or legislature). It uses an *approval ballot*, whereby voters can approve of as many candidates as they like (no rankings).

A voter's *satisfaction score* is the fraction of his or her approved candidates who are elected. If *k* candidates are to be elected, SAV chooses the set of *k* candidates that maximizes the *sum* of all voters' satisfaction scores. SAV has several desirable features:

• It is independent of the number of candidates a voter approves of—it works equally well for voters who are discriminating and not-so-discriminating in their choices.

• It tends to elect a more "representative" set of candidates than approval voting (AV)—in fact, SAV and AV may elect disjoint subsets.

• In the 2003 election of the Game Theory Society, SAV would have elected a more representative council.

• It can be applied to party-list systems, wherein it gives parties approximate proportional representation (PR).

• Because SAV favors larger parties, it gives parties an incentive to share support, form alliances, or even merge—perhaps into as few as two broad coalitions—and renders them responsive to voter preferences.

SAV: Voting for Individual Candidates

Proposition 1. *The winning subsets under SAV and AV may be disjoint.*

Example 1a: 9 voters, 3 candidates, <u>1 winner</u>

5 voters: *ab*
4 voters: *c*

AV outcome: {*a*} or {*b*} (**5** votes each)
SAV outcome: {*c*}, because its satisfaction score (*s*) is greater than that of {*a*} or {*b*}:

$s(a) = s(b) = 5(1/2) + 4(0) = \mathbf{2\tfrac{1}{2}}$
$s(c) = 5(0) + 4(1) = \mathbf{4}$

Whereas *a* or *b* gives 5 voters partial satisfaction of ½, *c* (a disjoint subset) gives 4 voters full satisfaction of 1.

Example 1b: Same as Example 1a, except <u>2 winners</u>

$s(a, b) = 5(1) + 4(0) = \mathbf{5}$
$s(a, c) = s(b, c) = 5(\tfrac{1}{2}) + 4(1) = \mathbf{6\tfrac{1}{2}}$

Note that {*a*, *b*} fully satisfies only a bare majority of 5 voters, whereas {*a*, *c*} or {*b*, *c*} either partially or fully satisfy *all* 9 voters. Thus, the latter outcome not only maximizes total voter satisfaction but also is the more "representative" outcome.

Proposition 2. *If there are k candidates to be elected, the winners under SAV are the k candidates whose individual satisfaction scores are the highest*.

In Example 1a, $s(a)$ = **2½**, $s(b)$ = **2½,** and $s(c)$ = **4**. Thus, when $k = 1$, $\{c\}$ is the winner; when $k = 2$, $\{a, c\}$ or $\{b, c\}$ is the winner. Each winning set comprises the 1 or 2 candidates with the highest satisfaction scores.

The *additivity* of candidate satisfaction scores follows from the fact that voter satisfaction is rooted in a relative rather than an absolute measure. The contribution of each voter to a candidate's score is $1/n$, where n is the number of candidates approved of by the voter.

We do *not* assume that each voter equally divides 1 vote among his or her approved candidates. This is a *consequence* of SAV: Summing voters' fractional approvals for each candidate determines which candidates win under SAV—those with the largest satisfaction scores—rendering SAV winners easy to determine.

The *representativeness of an outcome* is the number of voters who approve of at least one winner.

Proposition 3. *Neither SAV nor AV may give the most representative outcome*.

Example 2: 12 voters, 5 candidates, k = 2 winners

4 voters: *ab*
4 voters: *acd*
3 voters: *ade*
1 voter: *e*

AV outcome: $\{a, d\}$ (**11** and **7** votes, respectively)
SAV outcome: $\{a, d\}$, because

$s(a) = 4(½) + 7(1/3) =$ **4 1/3**
$s(b) = 4(½) = 2$
$s(c) = 4(1/3) = 1\ 1/3$
$s(d) = 7(1/3) =$ **2 1/3**
$s(e) = 3(1/3) + 1(1) = 2$

But $\{a, d\}$ is *not* the most representative outcome, because the voter who bullet votes for *e* receives no satisfaction.

By contrast $\{a, e\}$ gives *some* representation to all 12 voters—each approves at least one of the winners—but it does not maximize total voter satisfaction.

SAV *usually* represents at least as many, and often more, voters than AV, making it more representative.

This is so because the candidates that tend to benefit under SAV often have distinctive appeals that attract bullet voters. But these voters may not be so numerous as to give candidates more approval than that received by mainstream candidates, who win under AV but represent fewer voters.

The Game Theory Society Election

In 2003, the Game Theory Society used AV for the first time to elect 12 new council members from a list of 24 candidates. (The council comprises 36 members, with 12 elected each year to serve 3-year terms.) The 161 voters approved an average 9.8 candidates and a median 10.

Two of the 12 AV winners would not have been elected under SAV. Each set of winners is given below—ordered from most approved on the left to the least approved on the right—with differences between those who were elected under AV and those who would have been elected under SAV underscored:

AV: 111111111111000000000000
SAV: 111111111010110000000000

The AV winners who came in 10th (70 votes) and 12th (69 votes) would have been displaced under SAV by the candidates who came in 13th (66 votes) and 14th (62 votes).

The SAV outcome is more representative, because the elected subset under SAV represented all but 2 of the 161 voters, whereas the elected subset under AV failed to give representation to 5 of the 161 voters.

Although the SAV outcome is more representative, neither is the best possible: There are several subsets of only 8 candidates who would represent all 161 voters—but they do not maximize total voter satisfaction (nor are they the most approved candidates).

SAV: Voting for Political Parties

In most parliamentary democracies, voters vote for political parties—not candidates—which win seats in a parliament in proportion to the number of votes they receive.

Under SAV, voters would *not* be restricted to voting for one party but could vote for as many parties as they like.

Because no party typically wins a majority of seats, it would seem that voters would have an incentive to vote for multiple parties to try to (i) ensure that a favorite *coalition* of parties wins a majority of seats in order to (ii) enable it to become the governing coalition.

Example 3: 3 parties, 11 voters, 3 seats are to be filled

Bullet Voting

5 voters: party A
4 voters: party B
2 voters: party C

Party i's *quota*, q_i, is its proportion of votes times the number of seats to be apportioned:

$q_A = (5/11)(3) \approx 1.364$
$q_B = (4/11)(3) \approx 1.091$
$q_C = (2/11)(3) \approx 0.545$

Under SAV, we assume that each party nominates a number of candidates equal to its *upper quota* (i.e., its quota rounded up), so *A*, *B*, and *C* nominate 2, 2, and 1 candidates, respectively—2 more than the number of candidates to be elected.

SAV finds apportionments of seats to parties that (i) maximize total voter satisfaction and (ii) are *monotonic*: A party that receives more votes than another cannot receive fewer seats.

In Example 3, there are two monotonic apportionments —(2, 1, 0) and (1, 1, 1) to parties (*A*, *B*, *C*)—giving *s* values of

$s(2, 1, 0) = 5(1) + 4(½) + 2(0) = 7$
$s(1, 1, 1) = 5(½) + 4(½) + 2(1) = 6½$

Apportionment (2, 1, 0) maximizes *s*, giving

- 5 *A* voters satisfaction of 1 for getting their 2 nominees elected;

- 4 *B* voters satisfaction of ½ for getting 1 of their 2 nominees elected; and

- 2 *C* voters satisfaction of 0 because *C*'s nominee is not elected.

Notice that a voter's satisfaction is the fraction of his or her party's *nominees* that are elected. Each party will get either its upper quota or *lower quota* (i.e., its quota rounded down) of nominees elected.

Multiple-Party Voting

If a voter votes for multiple parties, his or her approval is equally divided among all his or her approved parties.

In Example 3, suppose parties *B* and *C* reach an agreement on policy issues, and their 4 and 2 supporters, respectively, approve of both parties. The 5 party *A* supporters continue to vote for just *A*.

Now *B* and *C* receive a total of 6(½) = 3 votes, which are equally divided between them, making the quotas of the three parties the following:

$$q_A = (5/11)(3) \approx 1.364$$
$$q_B = (3/11)(3) \approx 0.818$$
$$q_C = (3/11)(3) \approx 0.818$$

These quotas allow for the three monotonic apportionments —shown on the left sides of the equations below—which yield the following satisfaction scores for each apportionment:

$$s(2, 1, 0) = 5(1) + 4(\tfrac{1}{2}) + 2(\tfrac{1}{2}) = 5(1) + 6(\tfrac{1}{2}) = 8$$
$$s(2, 0, 1) = 5(1) + 4(\tfrac{1}{2}) + 2(\tfrac{1}{2}) = 5(1) + 6(\tfrac{1}{2}) = 8$$
$$s(1, 1, 1) = 5(\tfrac{1}{2}) + 4(1) + 2(1) = 8\tfrac{1}{2}$$

Now the SAV apportionment is (1, 1, 1). Compared with apportionment (2, 1, 0) earlier with bullet voting, *A* loses a seat, *B* stays the same, and *C* gains a seat. Thereby *B* and *C* ensure themselves of a majority of seats that only *A* previously obtained.

Proposition 4. *SAV gives the same apportionment as the Jefferson/d'Hondt apportionment method with a quota restriction.*

Of the five so-called divisor methods of apportionment (Balinski and Young, 1982/2001), Jefferson/d'Hondt most favors large parties.

Unlike (unrestricted) Jefferson/d'Hondt, SAV apportionments satisfy upper quota, because parties cannot nominate, and therefore cannot receive, more seats than their quotas rounded up.

Because Jefferson/d'Hondt apportionments always satisfy lower quota (Balinski and Young, 1982/2001), SAV apportionments satisfy quota (i.e., both lower and upper).

A Paradox

Proposition 4 notwithstanding, the supporters of *B* and *C* may *not* approve of each other's party, because B does not *individually* benefit from doing so.

Therefore, despite the fact that *B* and *C* supporters can together ensure themselves of a majority of seats through mutual approval, they may still go their separate ways.

A possible way around this paradox is for *B* and *C* to become one party, reducing the party system to just two parties. Because the combination of *B* and *C* has more supporters than *A* does, this combined party would win a majority of seats.

Insofar as SAV encourages compromises that reflect voter preferences, PR systems are likely to become less fractious and more responsive, enhancing their stability.

Conclusions

1. SAV is applicable to multiwinner elections. It uses an approval ballot—whereby voters can approve of as many candidates or parties as they like—but they are not elected based on the number of approval votes they receive.

2. SAV measures the satisfaction of a *voter* by the fraction of his or her approved candidates that are elected. The set of candidates that maximizes the sum of voter satisfaction scores is selected.

3. This measure is independent of the number of candidates a voter approves of—it works equally well for voters who approve of few or many candidates—and so, in a sense, mirrors a voter's *personal* tastes. SAV may elect a completely different set of candidates from AV.

4. The satisfaction score of a *candidate* is the sum of the satisfaction contributions received from all voters. This is $1/n$ from each voter who approves of him or her, where n is the number of candidates approved of by the voter.

5. These equal contributions of voters to candidates make the winning candidates those with the highest individual satisfaction scores, rendering SAV outcomes easy to compute.

6. SAV tends to elect candidates that give more voters either partial or complete satisfaction—and thus representation—than does AV.

7. Because bullet voting is risky when voting for individual candidates (a voter's satisfaction score will be either 0 or 1), a risk-averse voter may be inclined to approve of multiple candidates.

8. When SAV is applied to party-list systems in parliamentary democracies, the satisfaction score of a voter is the fraction of each party's *nominated* candidates—its quota rounded up—the voter approves of who are elected.

9. The number of seats apportioned to a party is never greater than a party's quota rounded up. SAV mimics the Jefferson/d'Hondt divisor method with a quota requirement, which favors larger parties.

10. Individually, parties may be hurt when their supporters approve of other parties. Collectively, however, they may be able to increase their combined seat share by forming coalitions—whose supporters approve of all parties in it—or even merging.

11. The coordination of policies and the formation of coalitions may reduce the party system to two broad left-of-center and right-of-center parties, or coalitions of parties.

12. Alternatively, a third moderate party might emerge (e.g., Kadima in Israel) that peels away supporters from the left and the right. This seems all very democratic, making coalitions fluid and responsive to voter sentiment.

13. More coordination by the parties would give voters a better idea of what to expect when they decide which parties to support, compared with the situation today when voters can never be sure about what parties will join in a governing coalition and what its policies will be.

14. Because SAV makes it easier for voters to know what parties to approve of, and for party coalitions that reflect voter interests to form, SAV should lead to more informed voting and more responsive government.