Strategic collusion in auctions with externalities*

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Abstract

We study a first price auction preceded by a negotiation stage, during which bidders may form a bidding ring. We prove that in the absence of external effects the all-inclusive ring forms in equilibrium, allowing ring members to gain the auctioned object for a minimal price. However, identity dependent externalities may lead to the formation of small cartels, as often observed in practice. Finally, we analyze cartels' efficiency in the presence of externalities.

1 Introduction

Auctions are known as a common trading mechanism. In order to suppress competition, increase the chances of winning and reduce the winning price, bidders may try to collude, namely to form a cartel or a bidding ring. A cartel in which all bidders participate may seem as the efficient way to operate, since it totally eliminates competition and may allow bidders to win the good for a minimal price. Nevertheless, the presence of externalities may introduce inefficiencies and disturb full cooperation.

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Collusion in private value auctions without externalities was already studied using the tools of mechanism design. McAfee and McMillan (1992) study first price auction with independent private values, showing that the all-inclusive bidder cartel is feasible, and that if transfers between cartel members are allowed then the collusive mechanism is efficient. Graham and Marshall (1987) and Mailath and Zemsky (1991) study collusion in second price auction with private values, and find that partial collusion is possible. (The latter consider heterogeneous bidders.) Marshall and Marx (2007) and Robinson (1985)[1] compare the resistance of first and second price auctions to collusion, finding the second price auction more vulnerable.

In practice, collusion in auctions and in auction-like situations is widely observed. Examples include Long-Island highway construction contracts (Porter and Zona (1993)), Ohio school milk procurements (Porter and Zona (1999)), Midwest seal coat contracts (Bajari and Ye (2003)), and U.S. oil and gas leases federal auctions (Hendricks, Porter and Tan (2008)).[2]

Yet however, full collusion is typically not observed. An obvious obstacle to collusion is due to the problem of information. Bidders are usually not familiar with the characteristics of their opponents, hence full cooperation is hard to achieve. As we wish to understand the limitations of negotiations and binding agreements in auctions, we separate between the information and commitment aspects. We study a market with complete information, raising the question: suppose that agents had full knowledge regarding the characteristics of their opponents, would the anticipated all-bidders cartel indeed form? Same approach of bidders' behavior analysis in auctions with complete information was taken by, e.g., Jehiel and Moldovanu (1996).

We follow a setup of a single good market with direct externalities, as in, e.g., Jehiel and Moldovanu (1996, 1999), Caillaud and Jehiel (1998), and Jehiel, Moldovanu and Stacchetti (1996). Each bidder in the market assigns a positive valuation to the auctioned good, which is the utility he derives if he wins the auction and consumes the good. Additionally, each bidder exerts identity dependent external effects on the others if consuming the good. Namely, each losing bidder gets a certain utility, which (possibly negative) value is determined as a function of the identity of the winner.

[1] Robinson (1985) considers also common value auctions.

[2] We find the latter example particularly interesting as the considered auctions took place in the years 1954-1970, when joint bidding ventures were legal in this market. (In late 1975, however, Congress passed prohibiting legislation.)

We study a first price auction[3] where the winner is obliged to consume the good, and no resale is allowed after the auction ends. Such an assumption is reasonable, for example, in state tenders where the winning firm has to carry out the project in question and cannot resell the execution rights to a third party (See, e.g., the South-Korean high-speed train case study in Jehiel and Moldovanu (1996)).

The auction is preceded by a negotiation stage in which bidders may form a bidding ring. As in, e.g., Bloch (1996), Bloch and Gomes (2006), Ray and Vohra (1999, 2001), we restrict our attention to a specific bargaining protocol. The bargaining protocol we consider takes the following form. One of the bidders is chosen by a chance move to be the collusion designer. He may then address any subset of the others, and offer them to form a cartel. He designates one of the members of the proposed cartel, possibly himself, as the representative of the cartel, or the designated cartel bidder. Finally, he specifies a contingent transfer scheme[4] which is implemented if the cartel wins the good. If all addressed agents accept the offer then the cartel forms, and all but the designated cartel bidder are committed to place an irrelevant bid in the auction. Otherwise, agents act independently in the auction.

A complementary approach omits the specification of the bargaining process and focuses instead on examining properties of the bargaining result, i.e., the admitted partition of the society (see, e.g., Ray and Vohra (1997)). For an application of this approach in auctions we may refer the interested reader to Biran and Forges (2010) who apply a core notion on auctions in the presence of direct external effects in order to study the stability of small cartels vs the all-inclusive bidder cartel without referring to the question of how a given cartel emerged.

For the sake of simplicity we will assume that non-relevant bidders, namely all cartel members but the designated cartel bidder, bid 0. However, our results hold for any configuration of irrelevant bids.[5] In practice, careful irrelevant bids may help the cartel concealing the collusion, as illustrated by

[3]Our results can be verified in a second price auction as well.

[4]Collusion case studies find that as transfer payments between cartel members are easily tracked, cartels tend to participate in several auctions, letting each member be the relevant bidder according to "the phases of the moon" (see, e.g., Porter and Zona (1999), Bajari and Ye (2003)). As we consider a single-auction setup, we shall use transfer payments as motivation for cartel members to cooperate.

[5]In a first price auction any placed bid which is under the bid of the cartel bidder is irrelevant.

the Ohio school milk case study in Porter and Zona (1999).[6]

The negotiation process which may lead to the formation of a bidding ring is analyzed from a non-cooperative point of view. That is, we analyze bidder behavior in sub-game perfect Nash equilibrium. In particular, in order to determine a reasonable transfer payment for an addressed agent i, the collusion designer compares i's equilibrium payoff if the bidding ring forms with i's payoff if the ring fails to form (see, e.g., Jehiel and Moldovanu (1996, 1999)). An alternative (cooperative) approach determines transfer payments between coalition members according to the coalition's value, possibly in view of the global partition of the society (see, e.g., Ray and Vohra (2001), Bloch (1996)[7]).

In order to simplify the discussion we restrict our attention to pure bidding strategies.[8] Such a restriction, however, calls into question the existence of an equilibrium bid in the auction. We handle this difficulty in appendix A providing a characterization of first price auction equilibrium bids in pure bidding strategies in the presence of externalities.

Our main results are the following. We start by studying, as a benchmark, an auction which takes place in the absence of externalities. Namely, every losing bidder is indifferent, in terms of his final utility, regarding the identity of the winner. Not surprisingly, we find that the primary intuition indeed holds. Bidders always form the grand coalition, represented by the bidder with the highest valuation. As a consequence they win the good for a minimal price, and the seller's surplus as a whole is divided between ring members through transfer payments.

Introducing external effects between agents changes the outcome dramatically. Externalities lead to a trade-off between reducing competition

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[6] Local dairies in Cincinnati formed a cartel in order to win a local school milk tender. (Note that in a tender, as opposed to an auction, an irrelevant bid is rather high.) As milk distribution costs raise with distance, Cincinnati dairies were expected to bid low in the local market and higher in distant markets. However, being part of a cartel in the local market, dairies different than the cartel bidder made irrelevant high bids. Same dairies participated also independently and non-cooperatively in distant market tenders, where they made competitive bids which ironically turned out to be lower than the bids they placed in the local market. Thus, providing evidence to collusion.

[7] Bloch (1996) does not even calculate explicitly the transfer payments between coalition members but assumes some fixed rule according to which coalition members share it's value.

[8] The results we are presenting hold for the non-restricted case as well, where mixed bidding strategies are allowed.

and compensating some of the participating agents via transfer payments. For instance, an agent may demand a high transfer payment if he anticipates a considerably low externality if participating in the collusion, compared to his anticipated utility if he declines to participate. Hence, in order to avoid paying high transfers, the collusion designer may find it optimal to exclude "demanding" agents from the considered bidding ring, although risking a tougher competition in the auction. We, therefore, identify externalities as a probable cause for the formation of small bidding rings instead of the all-inclusive one. We give an example of a market with externalities, where the collusion designer indeed forms a small cartel in a sub-game perfect Nash equilibrium.

We then move on to discuss the question of the identity of the bidding ring representative. As mentioned above, in the absence of externalities the bidder with the highest valuation optimally represents the all-bidder ring in the auction. The intuition is quite clear. The bidder with the highest valuation allows splitting the "largest pie" among cartel members. This intuition does not necessarily hold in the presence of externalities. We say that an agent is the cartel efficient member if the sum of his valuation and the externalities he exerts on other cartel members is maximal. In order to be able to split the "largest pie" the efficient bidder should represent the cartel. However, the efficient bidder may happen to exert terribly low externalities on agents outside the cartel. Such a threat on others translates into aggressive bids in the auction. In turn, yielding a high winning price and reducing the net benefit from collusion. The cartel may therefore find it optimal to be represented by an "inefficient" and less threatening agent.

Finally, we compare our results with Jehiel and Moldovanu (1996)'s strategic non-participation. They have proved that agents may find it optimal to commit not to participate in the auction just before it takes place. In this way, an agent eliminates himself from being a potential consumer, hence the externalities he may exert on others become irrelevant. Such a decision changes the market description and may lead to a different winner in the auction. For example, an agent may find it optimal not to participate if by doing so he anticipates that an agent he prefers as a consumer will win the good. We show that in our collusion game the collusion designer is strictly better off forming an appropriate cartel rather than choosing not to participate. Thus, allowing the designer not to participate is redundant in our model.

The paper takes the following structure: In section 2 we illustrate in an example the motivation for the formation of small cartels in the presence of external effects. In section 3 we present the model of the collusion game. In section 4 we analyze agents' behavior in the collusion game if no external effects exist, and prove that full collusion always emerges in equilibrium. In section 5 we introduce externalities and show that partial collusion may arise. In section 6 we demonstrate that in the presence of externalities a formed cartel may prefer to be represented in the auction by an inefficient member. Section 7 analyzes strategic non-participation in view of the collusion game. In section 8 we study possible extensions of our model. We obtain a characterization of equilibrium bids in first price auctions in markets with externalities, and of weakly dominated strategies in this setup in appendices A and B. Appndices C, D and E contain proofs of the main propositions of sections 4, 5 and 6 correspondingly.

2 Partial collusion in 4-player market example

Before we go on with the detailed analysis, let us demonstrate the motivation for our work with an example. Consider a market consisting of 4 competitive firms. A tender is organized in order to issue a single valuable production license in this market. Externalities are due to the pollution level that the winning firm is anticipated to cause.[9]

Firms F_C and F'_C are Conservative players who operate in the market for quite some time. Their estimated profit if winning the tender is rather low due to a relatively old technology they possess, which also causes quite a great deal of pollution. Therefore, if either F_C or F'_C wins the production license, it is likely to exert significantly negative externalities on the others. Firm F_G is a young player in this market, who acts under the banner of conserving the environment, and may therefore exert no externalities on the

[9]Partial collusion may be demonstrated in a 3-player market as well. However, the discussed tension between excluding over-demanding agents and decreasing competition is more obvious as the number of agents grows. Moreover, the symmetry which is considered in this example is not necessary for the formation of a small cartel. With respect to the notion of "externality matrix" which we introduce in the following section, partial collusion can be demonstrated in the non-symmetric 3-player market given by:

3	-4	0
0	1	0
0	-3	1

others if winning. Due to the high costs of its "Green" technology its profit if winning the contract is anticipated to be rather low. The last player, F_H, is a dynamic High-tech firm with a rather high valuation for the contract in question, who is anticipated to exert some mild externalities on the others if winning.

If no cartel forms and all firms participate in the tender non-cooperatively, F_G is assumed to win being ready to pay a high price for the license. That is as it fears the externality it might suffer if either F_C or F'_C wins the production license. The latter fail to compete the aggressive bid of F_G as they cannot afford paying such a high price for the license, given their low valuations. F_H on its side would rather let F_G win instead of paying an expensive price, anticipating that the latter would exert no externalities.

Let us consider a state of nature where F_G is drawn to be the collusion designer in the negotiation stage which precedes the auction. Full collusion should designate F_H as the cartel bidder, since it maximizes the total welfare (high profit, relatively low externalities). However, if indeed such a cartel is formed, both F_C and F'_C would demand a positive transfer as a compensation for the externality they are about to suffer, as opposed to the negotiation status-quo, where no cartel is formed and F_G wins the license, exerting no externalities.

Excluding the demanding F_C and F'_C from the proposed cartel increases the competition in the tender. Hence, a narrower cartel, consisting of F_G and F_H only, is risking high prices in the auction. However, F_C and F'_C are rather weak competitors due to their low profit, and therefore the threat they pose on the smaller cartel is rather tolerable. In such a setup "partial collusion" might be more profitable than "full collusion". Example 5.2 demonstrates the discussed scenario.

3 The model

The market consists of a seller S, $n \in \mathbb{N}$ potential buyers $B = \{B_1, B_2, \dots, B_n\}$, and one indivisible good. Each buyer B_i assigns a valuation π_i to the good in case he consumes it. An identity dependent externality $\alpha_{ij} \in \mathbb{R}$, $i \neq j$, is the utility to buyer B_j in case buyer B_i consumes the good. We refer to this setup of valuations and externalities as an $n \times n$ *matrix of externalities.*

We consider a first price auction in this market, which the seller organizes. All participants place simultaneously their non-negative bids. The highest positive bid which was placed, denoted p, wins[10]. The winner, denoted B_w, pays p to the seller, consumes the good, and gets his valuation, π_w. All other agents, $B_j \neq B_w$, get their corresponding externality, α_{wj}. We make the following assumptions:

- The winner must consume the good, and no resale to another agent is allowed.
- If the highest positive bid was placed by several agents (tie), each of them has an equal probability of winning the good.
- If all participants place a zero-bid, the good stays in the possession of the seller, and each agent gets a utility normalized to 0.

A 0 utility in case of no sale in the auction is chosen for the sake of simplicity. Alternatively, one may consider identity dependent externalities which agents receive in the case of no-sale in the auction, and normalize the matrix of externalities by subtracting from every column in the matrix the corresponding externality. As can be verified from the analysis we follow, such a normalization yields an equivalent market in terms of the final consumer, and the formed cartel. [11]

We assume that each agent in the market prefers consuming the good rather than either having some other agent consuming it, or leaving it in the possession of the seller, i.e. status-quo. Formally, for all B_i, $\pi_i > \alpha_{ji}$ for all $j \neq i$, and $\pi_i > 0$[12].

As in, e.g., Jehiel and Moldovanu (1996), we will assume the existence of a smallest money unit in the market, denoted ϵ, in order to avoid problems related to the existence of Nash equilibrium in the first price auction. In particular, bids, valuations and externalities are discrete with respect to this

[10] We do not let the agents win the good for free, in order to allow an extension of our model to an auction with a reserve price. With respect to the money unit ϵ, which is defined later in this section, one may consider the first price auction we discuss, as an auction with an ϵ-reserve-price.

[11] As an intuitive illustration consider the following. An agent will not make a bid which is greater than the difference between his valuation and the worst externality he may suffer. There exists an equilibrium point of the first price auction where the agent who maximizes this difference is the winner. The identity of the maximizer in a normalized matrix is the same as in the original one.

[12] Caillaud and Jehiel (1998) also assume that every agent prefers the status-quo, i.e. no sale, over a sale to another. Formally, they assume $\forall i, \alpha_{ji} < 0$ for all $j \neq i$.

money unit. For instance, if B_i places a bid equals to $b_i = k\epsilon$, and B_j wishes to overbid it, then B_j must bid at least $b_i + \epsilon = (k+1)\epsilon$.

We analyze markets where valuations and externalities are generic in the following sense.

Definition 3.1. We will refer to a market as *generic* , if the following holds:

- All valuations and externalities are linearly independent with respect to the set of coefficients $\{-1, 0, 1\}$. Namely, for any two sets of coefficients $\{\delta_i\}_{i=1}^n$ and $\{\delta_{ij}\}_{i \neq j}$ which take values in $\{-1, 0, 1\}$, if not all coefficients are null then $\sum_{i=1}^{n} \delta_i \pi_i + \sum_{i \neq j} \delta_{ij} \alpha_{ij} \neq 0$.

- Adding or subtracting up to $(n+2)\epsilon$ to any of the valuations and externalities maintains the linear independence. Namely, for any two sets of coefficients $\{\eta_i\}_{i=1}^n$, and $\{\eta_{ij}\}_{i \neq j}$ such that $-(n+2)\epsilon \leq \eta_i, \eta_{ij} \leq (n+2)\epsilon$, the valuations $\{\pi_i + \eta_i\}_{i=1}^n$, and externalities $\{\alpha_{ij} + \eta_{ij}\}_{i \neq j}$, are independent with respect to the set of coefficients $\{-1, 0, 1\}$.

The first price auction is preceded by a negotiation process between the potential bidders, in which they may agree to collude. One of the agents is chosen by a chance move to be the collusion designer. He may then address the others, and propose them a take-it-or-leave-it offer to form a cartel. He designates one of the proposed cartel members, who will make a relevant bid while the others place irrelevant bids. Finally, the collusion designer proposes a configuration of transfer payments. Let us stress out that the designated cartel bidder participates in the auction as an actual player, and no fictitious player which represents the cartel is added to the game[13]. We assume the following:

- The offer is observed by all agents in B.

- By accepting the offer, all addressed agents, but the designated cartel bidder, commit to make an irrelevant bid, namely 0, in the auction.

- If the offer is accepted, all members of the cartel commit to implement the transfer payment scheme if the designated cartel bidder indeed wins the auction[14].

[13]See, e.g., Hearinger (2004) who considers a game between "meta-players", where a "meta-player" stands for a coalition.

- Agents' responses are also observed by all agents in B.

Note that the designated cartel bidder, does not commit to a specific bid, and in particular may eventually bid 0 in the auction. The probability of every agent to be the collusion designer, is given by a probability vector, denoted σ, which is part of the game data. Right after the addressed agents respond to the offer, the auction takes place. Hence, the formed cartel, its representative in the auction, and the transfer payments agreed upon among the members of the cartel, define the state of the economy at the beginning of the auction.

Definition 3.2. A *state* s in the game is the tuple (C, B_l, d) where:

- $C \subset B$
- $B_l \in C$
- $d \in \mathbb{R}^n$ such that:
  - For all $1 \leq j \leq n$ there exists an integer m_j such that $d_j = m_j \epsilon$
  - For all $j \notin C$ it holds that $d_j = 0$
  - $\sum\limits_{j \in C} d_j = 0$

The interpretation is that C is a cartel, of which B_l is the representative, and d are transfer payments which the agents receive, if B_l wins the auction. For the simplicity of notations we refer to d as a vector of transfer payments to the members of the society as a whole. Transfer payments correspond to the money unit ϵ. Surely enough, transfer payments outside the cartel are 0. Within the cartel, transfer payments are balanced.

We denote s^0 the initial state of the economy, where there is no cartel, and no commitment to transfer payments. If the offer which the collusion designer makes is declined by any of the addressed agents, the economy stays in the state s^0. In this case, as no cartel is formed, all agents go to the auction as non-cooperating bidders, without any commitment to bids nor to transfer payments. Finally, the collusion designer may prefer to leave the economy at the initial state s^0, which we refer to as *negotiation status-quo.*

[14] We assume that if the cartel loses the auction then there is no motivation for further cooperation. In particular, no transfer payments are made in such a case.

Definition 3.3. Let s be a state. The set of *relevant bidders* in the state s, denoted $B(s)$, is given by:

- If $s = s^0$ then $B(s^0) = B$.
- Otherwise, let $s = (C, B_l, d)$, then $B(s) = \{B_l\} \cup (B \setminus C)$.

Consider B_i as the collusion designer, we say that a *proposal* is a state s such that either $s = s^0$, or $s = (C, B_l, d)$ and $B_i \in C$. The interpretation is that B_i may either choose the negotiation status-quo, or may alternatively propose to move the economy to a state s, by suggesting to form a cartel in which he is a member.

Given the assumptions detailed above, the *collusion game* is the following:

- Stage 1: A collusion designer, B_i, is chosen by a chance move according to σ.
- Stage 2: B_i makes a take-it-or-leave-it offer, to move the economy to the state s.
- Stage 3: If $s = s^0$ the game moves to the next stage. Otherwise, if $s = (C, B_l, d)$, all members of C, but the designer, signal sequentially whether they accept or reject the proposal[15]. At the end of this stage, if all agents accepted the proposal, then the economy moves to the new state s. Otherwise, if at least one agent rejected the proposal, then the economy stays at the primary stage s^0, where all agents are singletons.
- Stage 4: A first price auction takes place, with respect to the established state.

Let $s = (C, B_l, d)$ be the state of the economy when the auction takes place, and let $b(s)$ be a valid bidding vector with respect to the state s. For the sake of simplicity let us assume that there is a single highest positive bid, p, made by $B_w \in B(s)$. The case where there are several winners is treated similarly. The utility function, $u(s, b(s))$, is defined as follows:

[15] Assume that agents are addressed in an ascending order with respect to their indices. As we discuss subgame perfect Nash equilibria (SPNE), the analysis does not depend on the order in which agents are addressed and respond. In particular, the restriction to SPNE rules out equilibria where all agents reject because if another rejects all responses are equivalent. Equivalently, one can consider trembling hand perfect equilibria of the game, in which the agents respond simultaneously.

- If no agent wins the good in the auction, namely $b(s) = \overline{0}$, then the utility of the seller is 0, and every B_j gets $u_j(s, \overline{0}) = 0$
- Otherwise, the seller's utility is given by $u_S(s, b(s)) = p$.
- If $w = l$ then $u_j(s, b(s)) = \begin{cases} \pi_l + d_l - p & \text{if } j = l \\ \alpha_{lj} + d_j & \text{if } j \in C \setminus \{l\} \\ \alpha_{lj} & \text{if } j \notin C \end{cases}$
- If $w \neq l$ then $u_j(s, b(s)) = \begin{cases} \pi_w - p & \text{if } j = w \\ \alpha_{wj} & \text{if } j \neq w \end{cases}$

Finally, we wish to consider the following definition of efficiency, as we study the question of efficiency in equilibrium.

Definition 3.4. Let C be a cartel in a generic market. We say that B_i is the *efficient* member of C if $i = \arg\max_{j \in C}(\pi_j + \sum_{l \in C \setminus \{j\}} \alpha_{jl})$. We refer to the *efficient agent* as the efficient member of B.[16]

4 The zero-externality case

We shall start the game analysis by discussing the case where no agent exerts externalities on the others, namely, for all $i \neq j$, $\alpha_{ij} = 0$. We shall prove that in all Subgame Perfect Nash Equilibria (SPNE), with pure bidding strategies in the auction, full collusion emerges with probability one.

Let us emphasize that our results hold for the non-restricted case as well, where mixed strategies are allowed. However, the analysis one should follow is somewhat more complicated. For instance, in a generic market with externalities the winner in equilibrium of the first price auction is not uniquely determined (see, e.g., Jehiel and Moldovanu (1996)). Moreover, even in the case without externalities, there are non-generic markets, where both a bid leading to a tie between two agents, and a bid which yields one of them as a single winner, are in equilibrium. We note, however, that considering mixed strategies in the auction simplifies the discussion regarding the existence of SPNE in the collusion game. One may conclude the existence

[16]The efficient agent in every cartel C is unique due to genericity.

of an equilibrium of the auction game in mixed strategies, as bids are discrete, and continue in backward induction to deduce the existence of an SPNE of the collusion game.

We shall first define genericity in the 0-externality case.

Definition 4.1. A zero-externality market is considered to be *generic* if the following holds:

- All valuations are linearly independent with respect to the set of coefficients $\{-1, 0, 1\}$. Namely, for any set of coefficients $\{\delta_i\}_{i=1}^{n}$ which take values in $\{-1, 0, 1\}$, if not all coefficients are null then $\sum_{i=1}^{n} \delta_i \pi_i \neq 0$.
- Adding or subtracting up to $(n+2)\epsilon$ to any of the valuations maintains the linear independence. Namely, for any set of coefficients $\{\eta_i\}_{i=1}^{n}$ such that $-(n+2)\epsilon \leq \eta_i \leq (n+2)\epsilon$, the valuations $\{\pi_i + \eta_i\}_{i=1}^{n}$, are linearly independent with respect to the set of coefficients $\{-1, 0, 1\}$.

We start by understanding how an agreement made by agents affects the market description at the last stage of the game, when the auction takes place. The relation between an agreement and the market description is explained by transfer payments. For example, suppose that B_1 and B_2 form a cartel, where B_1 is the cartel bidder. In addition, suppose that they agree that if B_1 wins the good then B_2 gets a transfer of x, and B_1 gets a transfer of $-x$. When going to the auction and considering his bid, B_1 needs to take into account that if indeed he wins the good and consumes it, his payoff will not be his original valuation π_1, but his valuation fixed by his transfer payment, namely, $\pi_1 - x$. Therefore, the relevant valuation for B_1 while considering his bid should be updated according to the agreement he is part of.

The matrix of externalities is used both to determine what bids are in equilibrium in the auction, and to conclude agents' utilities when the good is consumed. Given a state, all cartel members but the cartel bidder, are committed to bid 0. Therefore, an equilibrium of a first price auction in this state is a function of the bids of the relevant bidders in this state only. Namely, the cartel bidder and fringe bidders, i.e., agents outside the cartel. Moreover, as bidding 0 cannot lead to winning the good and consuming it, the cartel members, but the cartel bidder, are not potential consumers. Hence, we should reduce the original matrix of valuations and externalities to a matrix composed of the valuations and externalities of relevant bidders only, with respect to the state in question. Hence, we start by erasing the rows and columns of the cartel members but the cartel bidder.

We continue by updating the valuation of the cartel bidder with respect to the agreed transfer payment. Since a valuation is defined as what an agent gets if he wins the auction and consumes the good, and the cartel bidder gets his transfer payment only if he indeed wins, we conclude that with respect to the agreement, the valuation of the cartel bidder is the sum of his original valuation and the transfer payment agreed upon. The valuations of the other relevant bidders stay as the original ones, as they are not involved in any agreement, and expect no transfer payments if winning. Finally, all externalities in the reduced matrix also stay as the original ones, as no relevant bidder is to get a transfer payment if another relevant bidder wins. This idea is formalized in the following lemma, which proof follows directly from definition 3.2.

Lemma 4.2. *Consider a generic zero-externality market. Let $s = (C, B_l, d)$ be the state of the economy when the auction takes place. For all $k, j \in B(s)$ denote $X_{kj}(s)$ the payoff to agent B_j if B_k consumes the good in state s. Then the matrix $X_{kj}(s)$ of dimension $|B(s)|$ is well defined and is given by:*

- *If $k = j = l$, $X_{kj}(s) = \pi_l + d_l$.*
- *If $k = j \neq l$, then $X_{kj}(s) = \pi_k$.*
- *If $k \neq j$, then $X_{kj}(s) = 0$.*

Example 4.3. Consider a generic 0-externality market with 3 potential buyers. Consider the state $s = (\{B_1, B_2\}, B_1, (-x, x))$. That is, B_1 and B_2 form a cartel, where B_1 is its bidder, and if B_1 eventually wins the good, he commits to pay x to B_2. Then $B(s) = \{B_1, B_3\}$, and $X(s)$ is given by the following matrix:

	1	3
1	$\pi_1 - x$	0
3	0	π_3

□

Finally, the discussed link between agents behavior in the auction and the agreements they are involved in is used in order to prove the formation of the grand-cartel in equilibrium in the absence of externalities.

The proof of the concluding proposition of this section is brought in appendix C. The intuition, however, is quite clear. Due to the absence of externalities, the agent with the highest valuation, is the efficient one. If the negotiation status-quo is followed, the efficient agent wins the auction and consumes the good. By forming the grand-cartel with the efficient agent as the cartel bidder, the collusion designer can extract the seller's surplus from the efficient agent as a transfer payment, since the latter is going to win the good for the price of ϵ. As the efficient agent wins the good in the negotiation status-quo in the first place, all other agents stay indifferent to the offer to form the grand-cartel with the efficient agent as the cartel bidder. Hence, the collusion designer need not compensate any of them, and the seller's surplus is his net gain. Any alternative offer to form a smaller cartel, preserves competition between potential buyer in the auction, which raises the winning price, and in turn decreases the surplus which the collusion designer can extract.

Proposition 4.4. *In a generic market without externalities the set of SPNE points of the collusion game is not empty. Moreover, full collusion, i.e. the all-bidder cartel, emerges with probability 1 in all SPNE points of the game.*

5 Formation of small cartels in the presence of externalities

We establish in this section that the presence of externalities may lead agents to form small cartels. In order to understand how an agreement to form a cartel affects the state of the economy we start by an analysis which is similar to the one in the previous section. We show that in every possible state there is a bid which is in equilibrium in a first price auction. In appendix D we prove that the set of SPNE points of the game in the presence of externalities is not empty. (As stated in the previous section, we consider pure bidding strategies in the auction stage, hence, the existence of SPNE is to be proved.) Finally, we demonstrate that there exists a market with externalities where partial collusion arises in SPNE with positive probability.

Lemma 5.1. *Consider a generic market with externalities. Let $s = (C, B_l, d)$ be the state of the economy when the auction takes place. For all $k, j \in B(s)$ denote $X_{kj}(s)$ the payoff to agent B_j if B_k consumes the good in state s. Then $X_{kj}(s)$ is well defined and is given by:*

- *If* $k = j = l$, *then* $X_{kj}(s) = \pi_l + d_l$.
- *If* $k = j \neq l$, *then* $X_{kj}(s) = \pi_k$.
- *If* $k \neq j$, *then* $X_{kj}(s) = \alpha_{kj}$.

The proof follows directly from definition 3.2.

The following example shows that the presence of externalities, may lead to a situation where the collusion designer is strictly better off forming a cartel smaller than the grand one. The intuition is that externalities may be so low, that some agents will demand a high compensation from the collusion designer in order to join the grand cartel. The designer in this case, is better off leaving those agents outside the cartel he forms, although he is risking a tougher competition in the auction by doing so. In the proof of the proposition we will use the following example.

Example 5.2. Consider the following 4-player market with externalities: $\pi_1 = 8, \pi_2 = \pi_3 = \pi_4 = 1; \alpha_{1j} = -2, \forall j \neq 1; \alpha_{2j} = 0, \forall j \neq 2; \alpha_{3j} = -8, \forall j \neq 3; \alpha_{4j} = -7, \forall j \neq 4.$[17]

8	-2	-2	-2
0	1	0	0
-8	-8	1	-8
-7	-7	-7	1

We claim that in this market, B_2 can gain more as a collusion designer by forming a cartel with B_1 only, rather than forming the grand cartel. The following proposition proves this claim formally, however, we would like to precede the formal discussion with an intuitive one.

If no cartel is formed and all agents go to the auction as non-cooperating bidders, i.e. negotiation status-quo, we consider an equilibrium bid as a result of which B_2 wins the auction paying the seller $p = 8$ for the good. (See corollary A.2, with $m = 1$.)

Considering the formation of the grand cartel, it is but natural, that the collusion designer is best off designating the efficient agent, B_1, as the

[17]As already stated, partial collusion may emerge in a 3-player market as well as in a non-symmetric setup. See section 2 for further details.

cartel bidder[18]. That is as the efficient agent maximizes the utility of the grand cartel if he consumes the good. However, if B_2, the collusion designer, wishes to do so, he needs to compensate B_3 and B_4, as the latter gains a lower externality if indeed the efficient agent, B_1, consumes the good, compared to the negotiation status-quo (e.g., $\alpha_{13} = -2 < 0 = \alpha_{23}$).

And indeed, by excluding B_3 and B_4 from the cartel (and by that avoiding the compensations they would claim), and forming a small cartel with B_1 only, B_2 gains a higher utility as detailed in the proof below. Note, that the latter is true although clearly by forming a small cartel, the price payed for the good in the auction rises, compared to an auction in which the grand cartel forms ($p = 3$ as opposed to $p = \epsilon$ respectively). It means that the collusion designer is better off experiencing a tougher competition in the auction, than compensating B_3 and B_4. $\square$

Proposition 5.3. *There exists a generic market with externalities, and an SPNE of the game in this market, in which a cartel smaller than the grand one forms with a positive probability.*

Proof. Consider the 4-player market with externalities in example 5.2[19]. We consider the following strategies of the agents:

- In the state s^0, agents bid $b(s^0) = (8 - 2\epsilon, 8, 8 - \epsilon, 8 - 2\epsilon)$. That is an equilibrium bid of the first price auction in the state s^0, according to corollary A.2 (with $m = 1$).

- For every state s, such that B_l is a single bidder in s, namely $B(s) = \{B_l\}$, B_l bids ϵ if $X(s) \geq \epsilon$, and 0 otherwise. According to proposition A.8 this is an equilibrium bid in this state.

- In the state $s^{12} = (\{B_1, B_2\}, B_1, d^{12})$, where $d_1^{12} = -d_2^{12} = -5$

[18]Note, however, that there are markets where the collusion designer may prefer to form the grand cartel with a bidder different than the efficient agent. This may happen, for example, if the designer demands a high transfer from the efficient agent, so that the efficient agent gains more if the seller keeps the good.

[19]There exist $\epsilon > 0$ and $\delta > 0$, both small enough, such that we can change the valuations and externalities in the neighborhood of δ, in order to achieve genericity of the market. Moreover, for small enough ϵ and δ the analysis in the proof holds for the generic market as well.

	1	3	4
1	3	-2	-2
3	-8	1	-8
4	-7	-7	1

we consider the bid $b(s^{12}) = (3, 0, 3-\epsilon, 3-2\epsilon)$, which is in equilibrium by corollary A.2.

- For every other state s, consider some equilibrium bid $b(s)$, which exists as established in appendix D (lemma D.1).

- Finally, with respect to the above described function $b(s)$, for every proposal made by B_i to move to the state $s = (C, B_l, d)$, every $B_j \in C \setminus \{B_i\}$ accepts the proposal if and only if $u_j(s, b(s)) \geq u_j(s^0, b(s^0))$.

As proved in appendix D (corollary D.3), there exists an SPNE of the game which includes these strategies. It is enough to demonstrate that in such an SPNE, the utility that B_2 derives as the collusion designer by proposing a cartel different than the grand one, is strictly greater than the utility he can derive by full collusion.

Consider first the initial state s^0. As stated above we consider the bid $b(s^0) = (8-2\epsilon, 8, 8-\epsilon, 8-2\epsilon)$. As a result of this bid B_2 wins the good, pays $p = 8$ to the seller and consumes. The utility vector of the agents is therefore $u(s^0, b(s^0)) = (\alpha_{21}, \pi_2 - p, \alpha_{23}, \alpha_{24}) = (0, -7, 0, 0)$.

Consider now the offer to form the grand cartel with the efficient agent as its representative, namely, to move to the state $s^{GC} = (B, B_1, d^{GC})$, where $d^{GC} = (-8+\epsilon, 4-\epsilon, 2, 2)$. If the offer is accepted then $X(s^{GC}) = \pi_1 + d_1^{GC} = 8 + (-8+\epsilon) = \epsilon$. Therefore, B_1 bids ϵ and wins the auction. He pays $p = \epsilon$ to the seller, and gets $\pi_1 - p + d_1^{GC} = 8 - \epsilon + (-8+\epsilon) = 0$. As $u_1(s^0, b(s^0)) = 0$, B_1 accepts the offer. In a similar way, if the offer is accepted, and B_1 wins the good and consumes, B_3 gains $\alpha_{13} + d_3^{GC} = -2 + 2 = 0$. As $u_3(s^0, b(s^0)) = 0$, B_3 accepts the offer. The same holds for B_4. To conclude, B_2 gains by proposing the discussed offer a utility of $\alpha_{12} + d_2^{GC^1} = -2 + 4 - \epsilon = 2 - \epsilon$. Clearly, B_2 cannot gain by proposing a higher transfer to any of the agents. On the other hand, by proposing a lower transfer to either B_3 or B_4, the offer will be rejected and the grand cartel will not form. Offering a lower transfer payment to B_1, will lead to a state where B_1 bids 0 in the auction, the good stays in the possession of the seller, and B_2 gains 0, which is strictly less than what he gains by proposing s^{GC}.

The same analysis can be repeated considering other possible bidders on behalf of the grand cartel, to conclude that by forming the grand cartel B_2 can gain at most $2 - \epsilon$ with respect to the considered strategy profile.

We shall now demonstrate an alternative offer to form a cartel with B_1 only, i.e., partial collusion, which is accepted by B_1 according to the considered strategy profile, and yields B_2 a strictly greater utility. Consider the proposal to move to the state $s^{12} = (\{B_1, B_2\}, B_1, d^{12})$, where $d_1^{12} = -d_2^{12} = -5$. $X(s^{12})$ is given by:

	1	3	4
1	3	-2	-2
3	-8	1	-8
4	-7	-7	1

where, $X(s^{12})_{11} = \pi_1 + d_1^{12} = 8 + (-5) = 3$. We consider the following equilibrium bid $b(s^{12}) = (3, 0, 3 - \epsilon, 3 - 2\epsilon)$. As a result of which B_1 wins the auction as a single winner, pays $p = 3$ to the seller, and consumes the good. He therefore gains, $X(s^{12})_{11} - p = 3 - 3 = 0$ which is equal to $u_1(s^0, b(s^0))$. Therefore, B_1 accepts the offer, and B_2 can gain $\alpha_{12} + d_2^{12} = 0 + 3 = 3$. Indeed, that is a greater utility than the one that can be achieved by full collusion, and therefore there is a positive probability that the grand cartel will not form in the considered SPNE. $\square$

6 Efficiency

In the previous section we demonstrated that partial collusion may arise in SPNE in the presence of externalities. In this section, we further show that the cartel bidder in SPNE in the presence of externalities is not necessarily the cartel's efficient member. This phenomenon is explained by a trade-off between the welfare of the cartel on one hand, which is maximized if the efficient member of the cartel consumes the good, and the price paid for the good on the other hand, which depends on the externalities that the cartel bidder exerts on fringe agents, i.e. agents outside the cartel.

Consider a 3-player market in which the only feasible cartel is $C = \{B_1, B_2\}$, and let B_1 be the collusion designer[20]. The following example

[20]Caillaud and Jehiel (1998) consider such an example where 2 European firms were considering a joint venture vis-a-vis a Japanese firm in the 1992 South-Korean high speed train tender. A joint venture in the context of this example between a European firm and

presents a setup in which B_2 is the cartel's efficient member, and at the same time introduces a major threat on the fringe agent B_3. A-priori in order to maximize the cartel's profits, it should be represented in the auction by B_2. Doing so, however, would increase dramatically the price that the cartel would need to pay in the auction if winning, as the fringe agent would be bidding aggressively in order to avoid the potential loss he might suffer if B_2 indeed wins. The collusion designer in this case is better off if the cartel is not represented by its efficient member, as the low price the cartel will pay compensates for the loss of potential welfare.

Example 6.1. Consider the following 3-player market with externalities: $\pi_1 = \pi_2 = 2, \pi_3 = 1, \alpha_{12} = -1, \alpha_{23} = -10, \alpha_{31} = \alpha_{32} = -2$, and all other externalities are null. Let B_1 be the collusion designer in the beginning of the second stage.

2	-1	0
0	2	-10
-2	-2	1

Consider a proposal to form the cartel $C = \{B_1, B_2\}$. B_2 is the efficient member of this cartel as $\pi_2 + \alpha_{21} > \pi_1 + \alpha_{12}$. Note that B_2 is a great threat to B_3 as $\alpha_{23} = -10$. Therefore, B_3 would be willing to place a high bid in order to prevent B_2 from winning the auction. Hence, if B_1 wishes to send B_2 to the auction as the cartel bidder, he has to commit to a high transfer payment to B_2, to enable him to overcome B_3's expected high bid. We shall demonstrate that B_1 can gain more by proposing himself as the cartel bidder, rather than proposing B_2.

Consider first the initial state s^0, where agents bid $b(s^0) = (4-2\epsilon, 4-\epsilon, 4)$. (This is an equilibrium bid in this state according to corollary A.2.) As a result of this bid, B_3 wins the good, pays $p = 4$ to the seller, and consumes the good. The utility vector of the agents is therefore $u(s^0, b(s^0)) = (\alpha_{31}, \alpha_{32}, \pi_3 - p) = (-2, -2, -3)$.

Consider now the proposal to move the economy to the state $s^1 = (C, B_1, d^1)$, where $d^1_1 = -d^1_2 = 1$. If B_2 accepts the offer then the economy moves to the state s^1. $X(s^1)$ is given by:

	1	3
1	3	0
3	-2	1

the Japanese one is less likely to be feasible.

where, $X(s^1)_{11} = \pi_1 + d_1^1 = 2 + 1 = 3$. We consider the following equilibrium bid $b(s^1) = (1, 0, 1 - \epsilon)$. (This is an equilibrium bid in this state according to corollary A.2.) As a result of which B_1 wins the auction as a single winner, pays $p = 1$ to the seller, and consumes the good. Therefore, B_2 gains $\alpha_{12} + d_2^1 = -1 + (-1) = -2$ which is equal to what he gains by rejecting the offer (i.e., $u_2(s^0, b(s^0))$). Hence, B_2 may accept the offer in SPNE, and let us consider a strategy profile in which he does. As a result, by proposing to move to s^1, B_1 gains $X(s^1)_{11} - p = 3 - 1 = 2$.

Let us now look at the alternative proposal to move the economy to the state $s^2 = (C, B_2, d^2)$, where the cartel C is represented by its efficient member B_2. If B_2 accepts the offer then the economy moves to the state s^2, where $X(s^2)$ is given by:

	2	3
2	$2 + d_2^2$	-10
3	-2	1

In order for B_2 to win the auction in state s^2 it must hold that $X(s^2)_{22} - X(s^2)_{32} > X(s^2)_{33} - X(s^2)_{23}$ (see corollary A.2), namely $2 + d_2^2 - (-2) > 1 - (-10)$, which is equivalent to $d_2^2 > 7$. We conclude that if B_1 proposes B_2 as the cartel bidder, he ends up with a negative utility (his transfer payment would be negative, and the externality that B_2 exerts on B_1 is null)[21]. Hence, B_1 can profit more by going to the auction himself as the cartel bidder, rather than sending the cartel's efficient member. □

The following proposition discusses inefficient collusion in the presence of externalities in the general case, namely, where any form of collusion is feasible. The proof appears in appendix E.

Proposition 6.2. *There exists a generic market with externalities, and there exists an SPNE of the collusion game in this market, where with positive probability, the cartel bidder is not the cartel's efficient member.*

7 Strategic non-participation

Jehiel and Moldovanu (1996) presented the idea of strategic non-participation in auctions. They considered an extended auction game, where in the first

[21] The analysis omits the case of a tie between B_2 and B_3 in the state s^2, and the case where the cartel forms and loses the auction. B_1 cannot gain a positive utility in any of these cases as well.

stage all agents decide simultaneously whether they wish to participate or not in the auction. At the next stage the auction takes place, and only agents who have decided to participate during the first stage take part in it. At the end of the auction, the winner consumes the good, gains his valuation and pays the proper price to the seller. Every other agent, whether he has participated or not in the auction, gains the corresponding externality according to the identity of the consumer.

They show that agents may be better off committing not to participate in the auction, providing the following intuition to explain this phenomenon. The absence of a specific agent in the auction may remove a potential threat he imposes on others. As a result, the bidding strategy of the participants may change, which in turn may lead to a different winner. This alternative winner may be better off for the non-participating agent, as an alternative consumer of the good, in terms of the externalities which the alternative winner imposes.

Our model provides agents with the possibility of non-participating, by joining a cartel in which they commit to place an irrelevant bid. The following proposition provides a link between the strategic non-participation presented in Jehiel and Moldovanu (1996) and the collusion game. We prove that in the collusion game, the collusion designer can gain strictly more than what he could have achieved by not participating à la Jehiel and Moldovanu. The intuition is, that by full collusion where the negotiation status-quo winner is the designated cartel bidder, the collusion designer can have all agents accepting to join for an ϵ transfer payment. In this way he is capable of extracting a net transfer approximately equal to the seller's surplus in negotiation status-quo. This net transfer is large enough to beat any externality, especially the externality he would have gained if not participating à la Jehiel and Moldovanu. As we wish to compare with non-participation of the collusion designer only, the following definitions suffice for the proposition we present.

Consider a generic market with externalities consisting of n potential buyers, and let B_i be a potential buyer. We say that the bidding vector $b^{NP_i} \in \mathbb{R}^n$ is an equilibrium bid of the first price auction in the market which corresponds to the non-participation of B_i, if $b_i^{NP_i} = 0$, and if omitting the i'th coordinate of b^{NP_i} yields a bidding vector in $\mathbb{R}^{n-1}$, which is an equilibrium bid in a first price auction held in the market derived from the original market by omitting the i'th row and column[22].

We denote $u_i(b^{NP_i})$ the utility of the non-participating agent. If B_w is the winner in a first price auction held in a market which corresponds to the non-participation of B_i, then $u_i(b^{NP_i}) = \alpha_{wi}$.

Finally, in order to be consistent with their framework of externalities, we will respect their restriction to non-positive externalities, namely, for all $i \neq j$, $\alpha_{ij} le 0$.

Proposition 7.1. *Consider a generic market with externalities, where all externalities are non-positive. Let B_i be the collusion designer at the second stage. Then in every SPNE of a sub-game of the collusion game where B_i is the collusion designer, he gains strictly more than what he could have gained by not participating à la Jehiel and Moldovanu.*

Namely, consider an SPNE of the sub-game of the collusion game where B_i is drawn to be the collusion designer, let s be the state in which the auction takes place in this SPNE, and let $b(s)$ be the corresponding bidding vector, then for every b^{NP_i}, a bidding vector which is an equilibrium of the first price auction in the market which corresponds to the non-participation of B_i, it holds that:

$$u_i(s, b(s)) > u_i(b^{NP_i})$$

Proof. Consider an SPNE of the sub-game of the collusion game where B_i is drawn to be the collusion designer. Denote $b(s)$ the function which maps every state s to an equilibrium bid of the first price auction in this SPNE. Finally, let b^{NP_i} be an equilibrium bid of the first price auction in the market which corresponds to the non-participation of B_i. Denote B_w the winner of the auction in state s^0 according to $b(s^0)$, and denote $B_{w'}$ the winner of the auction if B_i is not participating, according to b^{NP_i}. Due to genericity there is no tie in s^0, nor in the market which corresponds to the non-participation of B_i. Moreover, it is readily verified that the no-winner bid, $b = \overline{0}$, is not in equilibrium. Hence, by non-participating B_i gains $\alpha_{w'i}$.

Let us consider first the case where B_i is the winner in s^0, namely $B_w = B_i$. Consider the state where the grand cartel forms with B_i as the cartel bidder, namely $s^{GC^i} = (B, B_i, d^{GC^i})$, where $d_j^{GC^i} = \epsilon$, for all $j \neq i$, and $d_i^{GC^i} =$

[22] All other agents but B_i participate in the auction as single bidders. Note that a market derived from a generic market by omitting a row and a column is also generic. It follows that the bidding vector $b = \overline{0}$ is not an equilibrium of the auction in the derived market. Moreover, a bidding vector which leads to a tie is not in equilibrium.

$-(n-1)\epsilon$. If all agents accept the offer, then the grand cartel forms, and B_i is a single bidder. Genericity yields $X(s^{GC^i}) = \pi_i + d_i^{GC^i} = \pi_i - (n-1)\epsilon > \epsilon$. By proposition A.8, B_i wins the good in s^{GC^i} for the price of $p = \epsilon$. Therefore, every agents $B_j \neq B_i$ derives a utility of $\alpha_{ij} + d_j^{GC^i} = \alpha_{ij} + \epsilon$. The latter is strictly greater than his utility if he refuses the offer, α_{ij}. As lemma C.7 clearly holds also for a market with externalities, it follows that this offer is accepted in every SPNE of the considered sub-game. Thus, B_i gains in every SPNE at least $\pi_i + d_i^{GC^i} - p = \pi_i - n\epsilon$, which is strictly greater than $\alpha_{w'i}$.

Consider now the case $B_w \neq B_i$. The analysis is similar. Consider the state $s^{GC^w} = (B, B_w, d^{GC^w})$, where $d_j^{GC^w} = \epsilon$, for all $j \neq i, w$, $d_w^{GC^w} = -p(s^0) + 2\epsilon$, and $d_i^{GC^i} = p(s^0) - n\epsilon$. If the offer is rejected, every agent $B_j \neq B_i, B_w$ gains α_{wj}, whereas B_w gets $\pi_w - p(s^0)$. If all agents accept the offer, then the grand cartel forms, and B_w is a single bidder. We consider 2 different cases according to B_w's bid in SPNE at the state s^{GC^w}.

If B_w bids 0 in SPNE at the state s^{GC^w}, then the good stays in the possession of the seller and all agents gain a 0 utility. According to proposition A.8, bidding 0 in SPNE yields $\epsilon \geq \pi_w + d_w^{GC^w} = \pi_w - p(s^0) + 2\epsilon$, otherwise, B_w would profitably win the good for a minimal price of $p = \epsilon$. It therefore holds that $0 > \pi_w - p(s^0)$, hence, from lemma C.7, B_w accepts the considered offer in every such SPNE of the game. As all externalities are negative, for all $j \neq i, w$, $0 < \alpha_{wj}$, and therefore, all other agents accept the considered offer as well in every such SPNE of the game. We conclude that B_i can gain in such SPNE 0, which is strictly greater than $\alpha_{w'i}$.

Alternatively, being the only bidder in the auction, B_w bids ϵ in the state s^{GC^w}, and wins the good for the price of $p = \epsilon$. Hence, every agents $B_j \neq B_i, B_w$ derives a utility of $\alpha_{wj} + d_j^{GC^w} = \alpha_{wj} + \epsilon$, and B_w derives $\pi_w - p + d_w^{GC^w} = \pi_w - p(s^0) + \epsilon$. As all agents gain strictly more by accepting this offer than by rejecting it, it follows from lemma C.7, that this offer is accepted in every SPNE of the game, where B_w bids ϵ in s^{GC^w}. Thus, B_i gains in every such SPNE at least $\alpha_{wi} + d_i^{GC^w} = \alpha_{wi} + p(s^0) - n\epsilon$. By corollary A.2, it holds that $p(s^0) \geq \max_{j \neq w}(\pi_j - \alpha_{wj})$. Therefore, B_i's utility in every such SPNE is at least $\alpha_{wi} + \max_{j \neq w}(\pi_j - \alpha_{wj}) - n\epsilon \geq \alpha_{wi} + \pi_i - \alpha_{wi} - n\epsilon = \pi_i - n\epsilon$, which is strictly greater than $\alpha_{w'i}$.

$\square$

8 Model extension - Contingent transfers on a winner outside the cartel

In this section we consider a possible extension to the collusion game, which corresponds to the following motivation. Consider a market where the collusion designer is interested in the winning of a specific agent, a "preferred consumer", due to a high externality which this "preferred consumer" exerts on the collusion designer, for example. However, forming a cartel represented by this "preferred consumer", as the collusion game suggests, may not be feasible, as the "preferred consumer" may claim a high transfer. In such a market, the collusion designer might profit from the possibility to form a cartel with other agents who will also enjoy high externalities if this "preferred consumer" wins the auction. The designer may then extract some transfer payments from these agents, which are contingent on the winning of the "preferred consumer" who is not part of the formed cartel.

With respect to definition 3.2, a state is now extended to be the tuple (C, B_l, B_w, d). The interpretation is that, as before, the cartel C is represented in the auction by its member B_l, who is the only member who may make a positive bid in the auction. However, the transfer payments d are made among the members of the cartel C, if and only if B_w wins the auction. Restricting to $B_w = B_l$ yields the collusion game, discussed so far in this paper.

When the extended collusion game reaches a state s, where $B_l \neq B_w$, then in the matrix of the updated externalities $X(s)$, it is the term $X(s)_{wl}$ which is updated, to take the value $X(s)_{wl} = \alpha_{wl} + d_l$ (See lemma 5.1).

Note, that in a market without externalities, as agents have no reason to prefer one consumer over the other, there is no motivation to form a cartel with contingent transfers on a winner outside the cartel. The analysis in this case is analogous to the one presented in the collusion game, and indeed full collusion arises in SPNE (See proposition 4.4). We shall not go further to formalize the extended collusion game, but shall demonstrate its motivation instead.

Example 8.1. Consider the following 4-player market with externalities: $\pi_1 = \pi_2 = \pi_3 = \pi_4 = 1, \alpha_{12} = \alpha_{13} = \alpha_{14} = -1, \alpha_{21} = -5, \alpha_{34} = -4, \alpha_{41} = \alpha_{42} = \alpha_{43} = -6$, and all other externalities are null.

6	-1	-1	-1
0	1	0	0
5	0	1	-5
-1	-6	-6	1

Consider B_2 as the collusion designer. B_2 can gain more in the extended collusion game by forming a cartel with B_1, where the transfer are contingent on the winning of the agent they both prefer as a consumer, B_3. In status-quo, we consider a bid, as a result of which B_2 wins the good for the price of $p = 6$ (See corollary A.2). Hence, the utilities of the agents in negotiation status-quo are $u = (0, -5, 0, 0)$.

Following an analysis similar to the one presented in proposition 6.2 leads to the conclusion that in all SPNE points of the collusion game, B_2 cannot gain more than 3 (e.g., forming the grand cartel with the efficient agent, B_1, as the cartel bidder).

Consider now the following alternative proposal that B_2 may make in the extended collusion game. B_2 forms a cartel with B_1, where B_2 is the cartel bidder, and B_1 commits to pay B_2 a transfer of $d_2 = 5$ if B_3, their preferred consumer, wins the auction. Note that B_1 is ready to pay such a transfer due to "free-riding" if B_3 indeed wins. The externality which the latter exerts on B_1 is higher compared to the externality which is exerted on B_1 in the negotiation status-quo (i.e., $\alpha_{31} = 5 > 0 = \alpha_{21}$).

Such an agreement leads to a state s, corresponding to the following matrix of externalities $X(s)$:

	2	3	4
2	1	0	0
3	5	1	-5
4	-6	-6	1

where $X(s)_{32} = \alpha_{32} + d_2 = 0 + 5 = 5$, as B_2 gets a transfer payment from B_1 if indeed B_3 wins the auction and consumes the good. We consider a bid in this state, which results in the winning of B_3 for the price of $p = 6$ (See corollary A.2). Therefore, B_1 makes the transfer agreed upon to B_2, who ends up, after taking into account the corresponding externality which is exerted on him, with a utility of $\alpha_{32} + d_2 = 0 + 5 = 5$. This is more than he can gain in the collusion game as explained before. $\square$

As a concluding remark for this section we wish to note that the model might be further on extended, to consider agreements where agents commit to different transfer payments for different possible consumers. As mentioned earlier in this section, since in the absence of externalities agents have no reason to prefer one consumer over the other, one concludes that in such an extended model the behavior of agents in a market without externalities is similar to their behavior in the collusion game. Namely, full collusion will always appear in equilibrium (See proposition 4.4). Moreover, as the collusion game is a restricted case of such an extension, and if the grand cartel forms there is only one possible consumer (the cartel bidder), which narrows down the extended model to the collusion game again, one concludes that small cartels may form in equilibrium in the presence of externalities in an extended model as well.

9 Conclusion

We considered a first price auction in which the winner exerts direct external effects on losing bidders. We further specified a negotiation protocol according to which agents may form a bidding ring prior to the auction, where all bidders but the cartel representative commit to place an irrelevant bid in the auction.

We showed that in the absence of externalities bidders will form the all-inclusive cartel, which in turn eliminates competition in the auction, and allows winning the good for a minimal price. In the presence of external effects inefficiencies may arise. The collusion designer may find it profitable to form a small cartel excluding demanding bidders while risking tougher competition in the auction. Furthermore, we showed that the formed cartel may be better off designating an inefficient representative if the efficient cartel member constitutes a major threat on fringe bidders, as such a threat may lead to aggressive bids and a high winning price.

Finally, a comparison was made between strategic non-participation (Jehiel and Moldovanu (1996)) and strategic collusion, finding that the collusion designer is strictly better off forming an appropriate bidding ring than not participating in the auction at all.

A Appendix: First price auction equilibrium in the presence of externalities

The collusion game we study in this work has, generally speaking, two phases. It starts with a negotiation phase, in which agents are allowed to form a cartel, given some transfer payments among them. In the second phase, a first price auction takes place. The participating bidders are all players who are not part of the cartel, and the cartel bidder only. All members of the cartel other than its representative, are committed to bid 0. As only positive bids may win, a 0-bid in the auction is practically equivalent to non-participating.

Every SPNE point of the collusion game includes a bidding strategy for every possible state. Namely, a bidding strategy which corresponds to a market with externalities in which a certain cartel was formed with a certain representative, and a commitment to transfer payments (See definition 3.2). This bidding strategy is, in particular, a set of equilibrium points of the first price auction which takes place in the different states. The following discussion characterizes the pure equilibrium bids in a first price auction, which takes place in a market with externalities, not necessarily generic (Due to transfer payments, see example 4.3). The grand cartel may form, and in this case only one bidder actively participates in the auction. Hence, we will discuss also the case of a first price auction with a single bidder. We remind the reader that bids in the auction are discrete, and correspond to a given smallest money unit, denoted ϵ.

We start the analysis with the case of a single winner. In a first price auction, the winner bids in equilibrium just a bit above the second highest bid, formally, ϵ. Therefore, if the winner lowers his bid a bit, he is in tie with the second highest bidders. Thus, we give special attention to the number of second highest bids, in the equilibrium analysis.

Let us draw the reader's attention to the fact that the following necessary and sufficient condition is met in a generic market. Hence, in a generic market there is always an equilibrium of the first price auction. As corollary A.7 shows, this is not necessarily the case in a non-generic market.

Proposition A.1. *Let $n \geq 3$, and $m < n-1$. The following condition is necessary and sufficient for having an equilibrium bid $b = (b_1, b_2, ..., b_n)$ where B_i wins the auction as a single winner, and $B_{k_1}, B_{k_2}, \ldots, B_{k_m}$ are the second highest bidders. Namely, $b_i > b_{k_1} = b_{k_2} = ... = b_{k_m} > b_j$ for all $j \neq i, k_1, k_2, .., k_m$:*

$$\forall j \neq i \quad \pi_i - \frac{1}{m}\epsilon - \frac{1}{m}\sum_{l=1}^{m} \alpha_{k_l i} \geq \max\{\pi_j - \alpha_{ij}, 2\epsilon\}^{23}$$

Proof. Let us assume that such an equilibrium point exists. We will demonstrate that the condition holds. Note, that as b is an equilibrium bid, it holds that $b_{k_l} = b_i - \epsilon$, for all $1 \leq l \leq m$. In order for B_i to profit from winning the auction it must hold that $\frac{1}{m+1}(\pi_i - (b_i - \epsilon)) + \frac{1}{m+1}\sum_{l=1}^{m} \alpha_{k_l i} \leq \pi_i - b_i$, which is equivalent to $b_i \leq \pi_i - \frac{1}{m}\sum_{l=1}^{m} \alpha_{k_l i} - \frac{1}{m}\epsilon$. In words, B_i is better off bidding b_i at least as bidding $b_{k_l} = b_i - \epsilon$. The condition we bring here regarding a possible tie with B_{k_l} is stronger than the one corresponding to a situation where B_i bids below b_{k_l}. In addition, it is necessary that any agent B_j, where $j \neq i$, cannot profit from bidding b_i which means that $\frac{1}{2}(\pi_j - b_i) + \frac{1}{2}\alpha_{ij} \leq \alpha_{ij}$, which is equivalent to $b_i \geq \pi_j - \alpha_{ij}$. Again, it is a stronger condition than saying that B_j cannot profit from over-bidding B_i's bid. Finally, let $j \neq i, k_1, k_2, ..., k_m$. As $b_i > b_{k_1} = b_{k_2} = ... = b_{k_m} > b_j \geq 0$ it follows that $b_i \geq 2\epsilon$. Combining the three conditions on b_i yields, $\forall j \neq i$, $\pi_i - \frac{1}{m}\epsilon - \frac{1}{m}\sum_{l=1}^{m} \alpha_{k_l i} \geq \max\{\pi_j - \alpha_{ij}, 2\epsilon\}$, as required.

Let us assume now that the condition holds for some $B_i, B_{k_1}, B_{k_2} ... B_{k_m}$, all different. Then there exists p such that $\forall j \neq i$, $\pi_i - \frac{1}{m}\epsilon - \frac{1}{m}\sum_{l=1}^{m} \alpha_{k_l i} \geq p \geq \max\{\pi_j - \alpha_{ij}, 2\epsilon\}$, and p is a valid bid. Consider

[23] If $n = 2$ or $m = n-1$, the necessary and sufficient condition is, $\forall j \neq i \quad \pi_i - \frac{1}{m}\epsilon - \frac{1}{m}\sum_{l=1}^{m} \alpha_{k_l i} \geq \max\{\pi_j - \alpha_{ij}, \epsilon\}$.

the following bidding vector: $b_i = p$, $b_{k_1} = b_{k_2} = ... = b_{k_m} = p - \epsilon$, and for all $j \neq i, k_1, k_2, ..., k_m$, $b_j = p - 2\epsilon$. Then B_i wins the good as a single winner, pays p to the seller and gains $u_i = \pi_i - p$. Every other agent $B_j \neq B_i$ gains $u_j = \alpha_{ij}$. Clearly, B_i cannot gain more by raising his bid. If B_i lowers his bid to $p - \epsilon$, he wins with probability $\frac{1}{m+1}$ and gains $\frac{1}{m+1}(\pi_i - (p - \epsilon)) + \frac{1}{m+1}\sum_{l=1}^{m} \alpha_{k_l i}$. It is readily verified that this utility is at most $\pi_i - p$. If he lowers his bid further below $p - \epsilon$, he gains $\frac{1}{m}\sum_{l=1}^{m} \alpha_{k_l i}$ which is strictly less than $\pi_i - p$. For any $j \neq i$, it is clear that B_j cannot gain more by any bid lower than $b_i = p$. By bidding p, B_j gains $\frac{1}{2}(\pi_j - p) + \frac{1}{2}\alpha_{ij}$. It is readily verified that due to the condition this term is lower than $u_j = \alpha_{ij}$. The same holds for overbidding B_i. □

Corollary A.2. *Let $n \geq 3$, $m < n - 1$, and let $b = (b_1, b_2, ..., b_n)$ be a bidding vector where B_i makes the single highest bid and $B_{k_1}, B_{k_2}, \ldots, B_{k_m}$ make the second highest bid. Namely, $b_i > b_{k_1} = b_{k_2} = ... = b_{k_m} > b_j$ for all $j \neq i, k_1, k_2, .., k_m$. Then b is an equilibrium point of the first price auction if and only if:*

$$\forall j \neq i \quad \pi_i - \frac{1}{m}\epsilon - \frac{1}{m}\sum_{l=1}^{m} \alpha_{k_l i} \geq b_i \geq \max\{\pi_j - \alpha_{ij}, 2\epsilon\}^{24}$$

$$\forall 1 \leq l \leq m \quad b_{k_l} = b_i - \epsilon$$

The following analysis, characterizes equilibrium bids in which there is a tie between several bidders. The case were all bidders are in tie is handled separately.

Proposition A.3. *Let $2 \leq m < n$. The following are necessary and sufficient conditions for having an equilibrium bid $b = (b_1, b_2, ..., b_n)$ where there are m winners, denoted $B_{i_1}, B_{i_2}, \ldots, B_{i_m}$:*

$$\forall 1 \leq k \neq l \leq m \quad |(\pi_{i_k} - \frac{1}{m-1}\sum_{j=1, j\neq k}^{m} \alpha_{i_j i_k}) - (\pi_{i_l} - \frac{1}{m-1}\sum_{j=1, j\neq l}^{m} \alpha_{i_j i_l})| \leq \frac{m}{m-1}\epsilon$$

$$\forall 1 \leq k \leq m, \forall m < q \leq n \quad \max\{\epsilon, \pi_{i_q} - \frac{1}{m}\sum_{j=1}^{m} \alpha_{i_j i_q}\} \leq \pi_{i_k} - \frac{1}{m-1}\sum_{j=1, j\neq k}^{m} \alpha_{i_j i_k}$$

Proof. Let b be a bid which leads to m winners, denoted $B_{i_1}, B_{i_2}, \ldots, B_{i_m}$, and denote p the winning bid in b. Namely, $p = b_{i_1} = b_{i_2} = ... = b_{i_m} > b_j$ for all $j \neq i_1, i_2, ..., i_m$. Note that as p is a winning bid it holds that $p \geq \epsilon$. Then the utilities of the agents are:

$$u_{i_k} = \begin{cases} \frac{1}{m}[(\pi_{i_k} - p) + \sum_{j=1, j\neq k}^{m} \alpha_{i_j i_k}] & \text{if } 1 \leq k \leq m \\ \frac{1}{m}\sum_{j=1}^{m} \alpha_{i_j i_k} & \text{if } m < k \leq n \end{cases}$$

Let us assume that b is in equilibrium. We shall demonstrate that the conditions hold. Let $1 \leq k \leq m$. As b is in equilibrium, it holds that B_{i_k} cannot profit from neither overbidding nor underbidding p. Namely, $\pi_{i_k} - (p + \epsilon) \leq \frac{1}{m}[(\pi_{i_k} - p) + \sum_{j=1, j\neq k}^{m} \alpha_{i_j i_k}]$ and $\frac{1}{m-1}\sum_{j=1, j\neq k}^{m} \alpha_{i_j i_k} \leq \frac{1}{m}[(\pi_{i_k} - p) + \sum_{j=1, j\neq k}^{m} \alpha_{i_j i_k}]$ respectively. It follows that $\pi_{i_k} - \frac{m}{m-1}\epsilon - \frac{1}{m-1}\sum_{j=1, j\neq k}^{m} \alpha_{i_j i_k} \leq p \leq \pi_{i_k} - \frac{1}{m-1}\sum_{j=1, j\neq k}^{m} \alpha_{i_j i_k}$, which yields the first condition in the statement. Moreover, for all $m < q \leq n$ it holds that B_{i_q} cannot profit

[24] If $n = 2$ or $m = n - 1$, the necessary and sufficient conditions are, $\forall j \neq i \quad \pi_i - \frac{1}{m}\epsilon - \frac{1}{m}\sum_{l=1}^{m} \alpha_{k_l i} \geq b_i \geq \max\{\pi_j - \alpha_{ij}, \epsilon\}$, and $\forall 1 \leq l \leq m \quad b_{k_l} = b_i - \epsilon$.

from bidding p as well. Formally, $\frac{1}{m+1}[(\pi_{i_q} - p) + \sum_{j=1}^{m} \alpha_{i_j i_q}] \leq \frac{1}{m} \sum_{j=1}^{m} \alpha_{i_j i_q}$, which is equivalent to $p \geq \pi_{i_q} - \frac{1}{m} \sum_{j=1}^{m} \alpha_{i_j i_q}$. Together with the former conditions on p we get the second condition in the statement. Note, that the latter is stronger than demanding that B_{i_q} cannot profit from overbidding p.

We shall now demonstrate that the conditions are sufficient. It follows from the conditions that there exists $p \geq \epsilon$ such that,

$$\forall 1 \leq k \neq l \leq m \quad p \leq \pi_{i_k} - \frac{1}{m-1} \sum_{j=1, j \neq k}^{m} \alpha_{i_j i_k} \leq p + \frac{m}{m-1}\epsilon$$

$$\forall m < q \leq n \quad \pi_{i_q} - \frac{1}{m} \sum_{j=1}^{m} \alpha_{i_j i_q} \leq p$$

Consider the following bid, for all $1 \leq k \leq m$ $b_{i_k} = p$, and for all $m < q \leq n$ $b_{i_q} = p - \epsilon$. Following the same analysis as above, it is readily verified that b is in equilibrium. □

Corollary A.4. *Let $2 \leq m < n$, and let $b = (b_1, b_2, ..., b_n)$ be a bidding vector where $B_{i_1}, B_{i_2}, ..., B_{i_m}$ make the highest bid, denoted p. Namely, $p = b_{i_1} = b_{i_2} = ... = b_{i_m} > b_j$ for all $j \neq i_1, i_2, ..., i_m$. Then b is an equilibrium point of the first price auction if and only if:*

$$\forall 1 \leq k \leq m \quad \pi_{i_k} - \frac{m}{m-1}\epsilon - \frac{1}{m-1} \sum_{j=1, j \neq k}^{m} \alpha_{i_j i_k} \leq p \leq \pi_{i_k} - \frac{1}{m-1} \sum_{j=1, j \neq k}^{m} \alpha_{i_j i_k}$$

$$\forall m < q \leq n \quad p \geq \max\{\epsilon, \pi_{i_q} - \frac{1}{m} \sum_{j=1}^{m} \alpha_{i_j i_q}\}$$

In the special case where all agents are winners, namely $m = n$, we get the following characterization in a similar way.

Proposition A.5. *The following are necessary and sufficient conditions for having an equilibrium bid $b = (b_1, b_2, ..., b_n)$ where all buyers are winners. Namely, $b_1 = b_2 = ... = b_n \geq \epsilon$:*

$$\forall 1 \leq k \neq l \leq n \quad |(\pi_k - \frac{1}{n-1} \sum_{j=1, j \neq k}^{n} \alpha_{jk}) - (\pi_l - \frac{1}{n-1} \sum_{j=1, j \neq l}^{n} \alpha_{jl})| \leq \frac{n}{n-1}\epsilon$$

$$\forall 1 \leq k \leq n \quad \pi_k - \frac{1}{n-1} \sum_{j=1, j \neq k}^{n} \alpha_{jk} \geq \epsilon$$

Corollary A.6. *Let $b = (b_1, b_2, ..., b_n)$ be a bidding vector where all buyers make the same bid, denoted p. Then b is an equilibrium of the first price auction if and only if there exists $p \geq \epsilon$ such that:*

$$\forall 1 \leq k \leq n \quad \pi_k - \frac{n}{n-1}\epsilon - \frac{1}{n-1} \sum_{j=1, j \neq k}^{n} \alpha_{jk} \leq p \leq \pi_k - \frac{1}{n-1} \sum_{j=1, j \neq k}^{n} \alpha_{jk}$$

As a conclusion of the characterization that we have presented so far in this appendix, we get the following corollary, which demonstrates a non-generic market with externalities, in which none of the sufficient conditions is satisfied, namely, there is no equilibrium bid in the first price auction held in this market. Nevertheless, in the analysis of the collusion game, we prove that although many states of the economy correspond to non-generic markets, in every state of the collusion game, there exists an equilibrium bid in pure strategies in the first price auction held in the market corresponding to the state in question.

Corollary A.7. *In the following non-generic market there is no bid which is in equilibrium in the first price auction for a small enough* ϵ*:* $\pi_1 = 1, \pi_2 = 2, \pi_3 = 3, \alpha_{13} = -1, \alpha_{21} = -3, \alpha_{32} = -2$*, and all other externalities are null.*

1	0	-1
-3	2	0
0	-2	3

In order to complete the analysis, and as we consider a negotiation process before the auction during which a cartel may form , we need to consider the special case in which the grand cartel forms in the market, which yields a single bidder in the auction.

Proposition A.8. *Consider a market with externalities. If a single agent,* B_l*, goes to the auction in* $X(s) = X(s)_{ll}$[25] *then:*

- *If* $X(s)_{ll} > \epsilon$ *then* ϵ *is the only equilibrium bid for* B_l*.*
- *If* $X(s)_{ll} < \epsilon$ *then 0 is the only equilibrium bid for* B_l*.*
- *If* $X(s)_{ll} = \epsilon$ *then 0 and* ϵ *are the only equilibrium bids for* B_l*.*

Proof. If $X(s)_{ll} > \epsilon$ then by bidding ϵ, B_l wins the auction consumes the good and gets $X(s)_{ll} - \epsilon$. Clearly, he cannot gain more by raising his bid. By lowering his bid to 0, the good stays in the possession of the seller, and B_l gets $0 < X(s)_{ll} - \epsilon$.

If $X(s)_{ll} < \epsilon$ then by bidding 0, the good stays in the possession of the seller, and B_l gets 0. By raising his bid to ϵ, he wins the auction and gains $X(s)_{ll} - \epsilon < 0$.

Finally, if $X(s)_{ll} = \epsilon$ by bidding ϵ he wins and gains $X(s)_{ll} - \epsilon$, which is the same as what he gains if he bids 0. Clearly, by raising the bid he cannot gain more. $\square$

B Appendix: Weakly dominated strategies in first price auction

The following lemma characterizes the undominated bidding strategy in a first price auction which takes place in a generic market in the presence of externalities. The non-generic case follows a similar analysis.

Lemma B.1. *Consider a generic market with externalities. Denote* $\beta_i = \pi_i - \min \alpha_{ji} - \epsilon$*. Then any bid* $b_i > \beta_i$ *is weakly dominated by* β_i*, the bid 0 is dominated by the bid* ϵ*, and every positive bid* $0 < b_i \leq \beta_i$ *is undominated.*

Proof. Note that due to genericity $\beta_i > 0$. Let b_{-i} be a bid of all players but B_i, and let $b_i > \beta_i$, or equivalently, $b_i \geq \beta_i + \epsilon$. If b_i is not the highest bid in b, then player B_i achieves the same utility by bidding b_i as by bidding β_i. Let us assume that b_i is the highest bid. If B_i is a single winner in b then he gains a utility of $u_i = \pi_i - b_i \leq \pi_i - (\beta_i + \epsilon) = \min \alpha_{ji}$, which means that by lowering his bid to β_i he can only do better. If B_i is one among m winners, denoted $\{B_i, B_{j_1}, B_{j_2}, \ldots, B_{j_{m-1}}\}$, then his utility is

$$u_i = \frac{1}{m}(\pi_i - b_i) + \frac{1}{m}\sum_{k=1}^{m-1} \alpha_{j_k i} \leq \frac{1}{m}(\pi_i - (\beta_i + \epsilon)) + \frac{1}{m}\sum_{k=1}^{m-1} \alpha_{j_k i} =$$

$$\frac{1}{m}(\min \alpha_{ji}) + \frac{1}{m}\sum_{k=1}^{m-1} \alpha_{j_k i} \leq \frac{1}{m-1}\sum_{k=1}^{m-1} \alpha_{j_k i}$$

[25] See lemma 5.1.

the latter being what B_i gains by lowering his bid to β_i.

Consider now a zero bid made by B_i. If all other agents make a zero bid as well, B_i gains 0. By bidding ϵ instead he gains $\pi_i - \epsilon 0$, which is positive due to genericity. If alternatively the highest bid is positive, then by bidding ϵ, B_i cannot gain less.

Let $0 < b_i \leq \beta_i$, and denote $k = \arg\min \alpha_{ji}$. Consider the following bid of all the players but B_i. B_k bids $b_k = b_i - \epsilon$, and every B_j such that $j \neq i, k$ bids 0. B_i is a single winner and he derives a utility of $\pi_i - b_i$. It is clear that by raising his bid he gains strictly less. If $b_i > \epsilon$, then by lowering his bid he gains strictly less as $\pi_i - b_i > \alpha_{ki}$, and $b_k = b_i - \epsilon > 0$. Alternatively, $b_i = \epsilon$, and his utility is $\pi_i - \epsilon$ which is positive and therefore strictly greater than the utility he gets if lowering his bid to 0. □

The following example demonstrates a market with externalities in which every equilibrium bid involves a weakly dominated strategy of at least one of the agents.

Example B.2. Consider a 5-player market with externalities where: $\pi_1 = 5, \pi_2 = 1, \pi_3 = 6, \pi_4 = 7, \pi_5 = 8, \alpha_{21} = -5, \alpha_{35} = -1, \alpha_{43} = -3, \alpha_{54} = -2$, and all other externalities are null.

5	0	0	0	0
-5	1	0	0	0
0	0	6	0	-1
0	0	-3	7	0
0	0	0	-2	8

For a small enough ϵ, the only equilibrium bid is where B_1 is a single winner, bidding at least 8, and B_2 makes the second highest bid (See appendix A). Hence, B_2 bids at least $8 - \epsilon$, which is a weakly dominated strategy for him. □

C Appendix: Proof of full collusion in the absence of externalities

Throughout this appendix, we will assume, thus WLOG, that in a generic market without externalities, $\pi_1 > \pi_2 > ... > \pi_n$. Note, that due to genericity it holds in particular that for all $1 \leq i < n$, $\pi_i - \pi_{i+1} > (n+2)\epsilon$, as well as for all $1 \leq i \leq n$, $\pi_i > (n+2)\epsilon$.

We learn from Lemma 4.2, that indeed for any state s, $X(s)$ is a market without externalities, however, it is not necessarily generic (e.g., consider $x = \pi_1 - \pi_3$ in example 4.3). Nevertheless, as at most one valuation has changed, the updated matrix is generic, except maybe for a single valuation. We give special attention to the question of genericity in the update matrix, as non-genericity may lead to potential ties in the auction which follows[26].

Lemma C.1. *Consider a generic 0-externality market, and let $s = (C, B_l, d)$ be a state. If $dim(X(s)) \geq 2$, denote the 2 highest valuations in $X(s)$, $i_1 = \arg\max X(s)_{jj}$, and $i_2 = \arg\max_{j \neq i_1} X(s)_{jj}$. Then one of the followings holds:*

- *$X(s)$ is of dimension one.*
- *$dim(X(s)) \geq 2$, and $X(s)_{i_1 i_1} - X(s)_{i_2 i_2} > 2\epsilon$.*
- *$dim(X(s)) \geq 2$, $X(s)_{i_1 i_1} - X(s)_{i_2 i_2} \leq 2\epsilon$, and either $i_1 = l$ or $i_2 = l$.*

[26] In a non-generic market with externalities, there is not necessarily an equilibrium bid in a first price auction (see corollary A.7 in appendix A). This is not the case in a non-generic market without externalities, where there is always an equilibrium bid in the auction.

Proof. If the grand cartel forms in s, namely $C = B$, then by lemma 4.2 $X(s) = X(s)_{ll} = \pi_l + d_l$, and the first case in the statement holds. Let us assume then that $C \subsetneq B$. By lemma 4.2 for every $k \in B(s) \setminus \{B_l\}$, it holds that $X(s)_{kk} = \pi_k$. As the original market is generic, if $i_1, i_2 \neq l$ then the second case in the statement holds. Otherwise, either $i_1 = l$ or $i_2 = l$ and one of either the second or the third case in the statement holds. $\square$

In order to analyze the SPNE points of the collusion game in the absence of externalities, we now move on to discuss agents' equilibrium behavior in the auction given a state s. We distinguish between different scenarios, according to the different possible market types in which the auction takes place, as characterized by lemma C.1. The proofs follow from the analysis of equilibrium bids in a first price auction in a market with externalities which appears in appendix A, and are therefore omitted.

Lemma C.2. *Consider a generic zero externality market, and let $s = (B, B_l, d)$ be a state where the grand cartel forms. If $X(s)_{ll} > \epsilon$ then bidding ϵ is the only equilibrium strategy of the single bidder B_l in a first price auction in this market. If $X(s)_{ll} < \epsilon$ then bidding 0 is the only equilibrium strategy in a first price auction in this market. If $X(s)_{ll} = \epsilon$ then bidding either ϵ or 0 are the only equilibrium strategies in a first price auction in this market.*

Lemma C.3. *Consider a generic zero-externality market, and let s be a state. Assume that there are at least 2 potential buyers in s, namely $|B(s)| \geq 2$. Denote the 2 highest valuations in $X(s)$, $i_1 = \arg\max X(s)_{jj}$, and $i_2 = \arg\max_{j \neq i_1} X(s)_{jj}$. If $X(s)_{i_1 i_1} - X(s)_{i_2 i_2} > 2\epsilon$ then a bidding vector b in a first price auction held in this state, is in equilibrium if and only if B_{i_1} makes the single highest bid, namely, $b_{i_1} > b_j$ for all $j \neq i_1$. In addition, the price $p = b_{i_1}$ that the winner pays for the good, is in the interval $X(s)_{i_1 i_1} > p \geq \max\{\epsilon, X(s)_{i_2 i_2}\}$.*

Lemma C.4. *Consider a generic zero-externality market, and let s be a state. Assume that there are at least 2 potential buyers in s, namely $|B(s)| \geq 2$. Denote the 2 highest valuations in $X(s)$, $i_1 = \arg\max X(s)_{jj}$, and $i_2 = \arg\max_{j \neq i_1} X(s)_{jj}$. Assume that $X(s)_{i_1 i_1} - X(s)_{i_2 i_2} \leq 2\epsilon$ and let b be a bidding vector of a first price auction in this market.*

- *If $X(s)_{i_1 i_1} = X(s)_{i_2 i_2}$ then b is in equilibrium if and only if both B_{i_1} and B_{i_2} make the highest bid p. In addition, the price they pay for the good is in the interval $X(s)_{i_1 i_1} - 2\epsilon \leq p \leq X(s)_{i_1 i_1}$.*

- *If $X(s)_{i_1 i_1} > X(s)_{i_2 i_2}$ then b is in equilibrium if and only if either $B_{i_1 i_1}$ makes the single highest bid p where $X(s)_{i_2 i_2} \leq p < X(s)_{i_1 i_1}$, or both $B_{i_1 i_1}$ and $B_{i_2 i_2}$ make the highest bid p where $X(s)_{i_1 i_1} - 2\epsilon \leq p \leq X(s)_{i_2 i_2}$.*

As a corollary of lemma C.1, lemma C.2, lemma C.3, and lemma C.4 we conclude, that in every state there exists an equilibrium bid of the first price auction if held in this state. This is a step in order to establish the existence of an SPNE point of the collusion game without externalities. Note, however, that, as remarked before, one may conclude the existence of an equilibrium bid in every state of the collusion game without externalities, from the analysis in appendix A only. Nevertheless, lemma C.1, lemma C.2, lemma C.3, and lemma C.4 are necessary to the proof of proposition 4.4.

Corollary C.5. *Consider a generic zero-externality market. There exists a function, denoted $b(s)$, which maps every state s to an equilibrium bid of the first price auction in that state.*

The following corollary follows from lemma C.3 and lemma C.4, and will be useful in the proof of proposition 4.4, later in this appendix.

Corollary C.6. *Consider a generic zero-externality market, and let $s = (C, B_l, d)$, $C \subsetneq B$. If the representative of the cartel, B_l, wins the auction in s as a single winner or in a tie, then the price $p(s)$ that he pays to the seller satisfies $p(s) > n\epsilon$.*

Proof. Using the notations of lemma C.3 and lemma C.4, if B_l is a single winner, then $X(s)_{ll} > X(s)_{i_2i_2} > n\epsilon$, where the last inequality holds due to genericity. Again from lemma C.3 and lemma C.4 we learn that $p(s) \geq X(s)_{i_2i_2}$, which yields $p(s) > n\epsilon$.

If B_l wins in a tie, then from lemma C.4, $p \geq X(s)_{i_1i_1} - 2\epsilon \geq \max\{X(s)_{i_1i_1}, X(s)_{i_2i_2}\} - 2\epsilon > (n+2)\epsilon - 2\epsilon = n\epsilon$. □

The next step in the proof of the existence of an SPNE of the collusion game without externalities, is to study which offers agents accept and reject in SPNE. As denoted in corollary C.5, $b(s)$ is an arbitrary mapping of states to equilibrium bids of the first price auction in these states. In addition, given $b(s)$, we denote $p(s)$ the corresponding price that the winner pays to the auctioneer, namely, $p(s) = \max b_i(s)$. The following lemma follows directly from the definition of SPNE.

Lemma C.7. *Consider a generic market without externalities, and let $b(s)$ be an arbitrary selection of equilibrium bid of the first price auction for every state s. Let $s(C, B_l, d) \neq s^0$ be a proposal made by B_i at the second stage of the game. Denote $u^r = u(s^0, b(s^0))$ the vector of utilities if the considered proposal is rejected, and $u^a(s) = u(s, b(s))$ the vector of utilities if it is accepted. Then for all $j \in C \setminus \{B_i\}$:*

- *If $u^a_j(s) > u^r_j$ then B_j accepts the offer in every SPNE of the game.*
- *If $u^a_j(s) < u^r_j$ then B_j rejects the offer in every SPNE of the game.*

proof of proposition 4.4. Let $b(s)$ be an equilibrium point function of the first price auction. We shall demonstrate that for every collusion designer who is selected in the first stage, there exists a proposal to form the grand cartel, which strictly maximizes his utility, and is accepted by all agents. It follows that the grand cartel forms with probability one in every SPNE of the game, as claimed.

If the collusion designer makes an offer that gets rejected, then the economy stays in state s^0. Namely, all agents go to the auction as single non-cooperating bidders. It follows from lemma C.3 that in s^0, B_1 wins the good for a price of $\pi_2 \leq p(s^0) < \pi_1$. Hence, with respect to the previous notations,

$$u^r_j = \begin{cases} \pi_1 - p(s^0) & \text{if } j = 1 \\ 0 & \text{if } j \geq 2 \end{cases}$$

We start with the case where B_1, the agent with the highest valuation, is the collusion designer in the beginning of the second stage. Consider the proposal to form the grand cartel represented in the auction by B_1, with ϵ transfer payments to all agents. Namely, B_1 proposes to move to the state $s^{GC^1} = (B, B_1, d^{GC^1})$, where $d^{GC^1}_1 = -(n-1)\epsilon$, and for all $j \neq 1$, $d^{GC^1}_j = \epsilon$. It follows from genericity and lemma C.2 that in s^{GC^1} B_1 wins the good for a price of ϵ. As all transfer payments in s^{GC^1} are ϵ,

$$u^a_j(s^{GC^1}) = \begin{cases} \pi_1 - n\epsilon & \text{if } j = 1 \\ \epsilon & \text{if } j \geq 2 \end{cases}$$

Then from lemma C.7 B_1's proposal to move to the state s^{GC^1} is unanimously accepted, and B_1 gains $\pi_1 - n\epsilon$. We need to show that for any alternative proposal B_1 makes, namely to move to the state $s = (C, B_l, d)$ where $C \subsetneq B$, B_1 gains strictly less than $\pi_1 - n\epsilon$. If the alternative offer is rejected by at least one agent, then B_1 gains $u^r_1 = \pi_1 - p(s^0) \leq \pi_1 - \pi_2 < \pi_1 - n\epsilon$ due to genericity. If the alternative offer is accepted by all agents, then from lemma C.7, for all $B_j \in C \setminus \{B_1\}$, $u^a_j(s) \geq u^r_j$. We consider 2 different cases.

First, consider the case where the proposed representative is B_1 himself, namely, $B_l = B_1$. Then for all $B_j \in C \setminus \{B_1\}$, $d_j = u^a_j(s) \geq u^r_j = 0$. As transfers are balanced in C it follows that $d_1 \leq 0$. By corollary C.6, the price that B_1 pays in s if he wins the good, as a single or co-winner, satisfies $p(s) > n\epsilon$. Therefore, if B_1 wins the good as a single winner in s he gains $\pi_1 + d_1 - p(s) \leq \pi_1 - p(s) < \pi_1 - n\epsilon$. If B_1 wins the good with a second winner in s he gains $\frac{1}{2}(\pi_1 + d_1 - p(s))$ which is again strictly less than $\pi_1 - n\epsilon$. Finally, if B_1 loses in s he gets $0 < \pi_1 - n\epsilon$ due to genericity.

Consider now the case where $B_l \neq B_1$. As before, for all $B_j \in C \setminus \{B_1\}$, $u_j^a(s) \geq u_j^r$, since we assume that the offer is accepted. Therefore, for all $B_j \in C \setminus \{B_1, B_l\}$, $d_j = u_j^a(s) \geq u_j^r = 0$. If B_l wins in s he gains $\pi_l + d_l - p(s)$, whereas in the case of a tie in s with another agent he gains $\frac{1}{2}(\pi_l + d_l - p(s))$. If he loses in s he gains 0. If B_l loses in s then B_1 gains $0 < \pi_1 - n\epsilon$. If B_l wins in s, then $0 = u_l^r \leq u_l^a(s) \leq \pi_l + d_l$, as $p(s) > 0$. Therefore, $-d_l \leq \pi_l$, which yields $d_1 \leq \pi_l$. Thus, if B_l is a single winner in s, B_1 gains $d_1 \leq \pi_l < \pi_1 - n\epsilon$ due to genericity. In a similar way, in case of a tie in s, B_1 gains strictly less than $\pi_1 - n\epsilon$. We conclude that if B_1 is drawn in the first stage to be the collusion designer, he can achieve $\pi_1 - n\epsilon$ by forming the grand cartel, and can only gain strictly less by considering any alternative proposal to form a cartel which is not the grand one. Yet, however, as a function of the strategies of the others, if the offer to form the grand cartel with B_1 as its representative, is accepted by some $B_j \neq B_1$ for a 0-transfer payment, then B_1 would indeed offer them a 0-transfer payment, in order to maximize his utility. Clearly, by offering these lower transfer, B_1 can only gain more than $\pi_1 - n\epsilon$[27].

We repeat the analysis for the case where at the first stage some $B_i \neq B_1$ is drawn to be the collusion designer. Consider the proposal to form the grand cartel represented in the auction by B_1, with a transfer of $-p(s^0) + 2\epsilon$ to B_1, and ϵ transfer payments to all other agents. Namely, $s^{GC^i} = (B, B_1, d^{GC^i})$ where,

$$d_j^{GC^i} = \begin{cases} \epsilon & \text{if } j \neq 1, i \\ -p(s^0) + 2\epsilon & \text{if } j = 1 \\ p(s^0) - n\epsilon & \text{if } j = i \end{cases}$$

Hence, $X(s^{GC^i})_{11} = \pi_1 - p(s^0) + 2\epsilon \geq \pi_1 - \pi_2 + 2\epsilon$. Therefore, genericity yields $X(s^{GC^i})_{11} > \epsilon$. By lemma C.2 we conclude that in s^{GC^i} B_1 wins the good for a price of ϵ. Therefore,

$$u_j^a(s^{GC^i}) = \begin{cases} d_j & \text{if } j \neq 1, i \\ \pi_1 - \epsilon + d_1 & \text{if } j = 1 \\ d_i & \text{if } j = i \end{cases} = \begin{cases} \epsilon & \text{if } j \neq 1, i \\ \pi_1 - p(s^0) + \epsilon & \text{if } j = 1 \\ p(s^0) - n\epsilon & \text{if } j = i \end{cases}$$

Therefore, B_i's proposal s^{GC^i} is unanimously accepted, and B_i gains $p(s^0) - n\epsilon$. We need to prove that for any alternative proposal $s = (C, B_l, d)$, $C \subsetneqq B$, that B_i may make, he gains strictly less than $p(s^0) - n\epsilon$. If B_i proposes to stay in s^0, or if the alternative offer is rejected then B_i gains $u_i^r = 0 < p(s^0) - n\epsilon$. It suffices to assume then that the alternative offer is accepted. From lemma C.7, it holds in this case that for all $B_j \in C \setminus \{B_i\}$, $u_j^a(s) \geq u_j^r$. We consider 2 different cases.

First, consider the case where the proposed representative is B_1, namely, $B_l = B_1$. Then for all $B_j \in C \setminus \{B_1, B_i\}$, $d_j = u_j^a(s) \geq u_j^r = 0$. Let us look at the 3 possible outcomes of the auction in s. If B_1 loses then B_i gains $0 < p(s^0) - n\epsilon$. If B_1 wins the auction as a single winner, then $u_1^a(s) = \pi_1 - p(s) + d_1$. As the offer is accepted, it holds that $\pi_1 - p(s^0) = u_1^r \leq u_1^a(s) = \pi_1 - p(s) + d_1$, and therefore, $d_1 \geq p(s) - p(s^0)$. As all other transfer payments are non-negative we conclude that $d_i \leq p(s^0) - p(s)$. So B_i gains in this case $d_i \leq p(s^0) - p(s) < p(s^0) - n\epsilon$, where the last inequality follows from corollary C.6. Finally, if B_1 wins the auction in s in a tie with another agent then $u_1^a(s) = \frac{1}{2}(\pi_1 - p(s) + d_1)$, which now yields $d_1 \geq p(s) - 2p(s^0) + \pi_1$. In such a case B_i gains $\frac{1}{2}d_i \leq -\frac{1}{2}d_1 \leq -\frac{1}{2}(p(s) - 2p(s^0) + \pi_1) = p(s^0) - \frac{1}{2}\pi_1 - \frac{1}{2}p(s) < p(s^o) - n\epsilon$, where the last inequality follows from genericity, and from corollary C.6.

Consider now the case where $B_l \neq B_1$. First, let us assume that $B_l = B_i$. It holds that for all $B_j \in C \setminus \{B_1, B_i\}$, $d_j = u_j^a(s) \geq u_j^r = 0$. If B_i loses the auction in s then he gets $0 < p(s^0) - n\epsilon$. It is therefore sufficient to assume that, either B_i wins in s as a single winner, or B_i wins in a tie with another agent. If $B_1 \notin C$, then $1 = \arg\max_{j \neq l} X(s)_{jj}$. By lemma C.3 and lemma C.4, it must hold that $\pi_i + d_i = X(s)_{ii} \geq X(s)_{11} - 2\epsilon = \pi_1 - 2\epsilon$. It yields $d_i \geq (\pi_1 - \pi_i) - 2\epsilon > 0$. This stands in a contradiction

[27] If B_1 is the collusion designer then in the different equilibrium points of the game, he gains a utility of $\pi_1 - (k+1)\epsilon$, where $0 \leq k \leq n-1$ is the number of agents who reject forming the grand cartel with B_1 as its representative for a 0-transfer payment, in the equilibrium in question.

to the fact that transfers to all agents but B_i are non-negative. It thus suffices to consider the case $B_1 \in C$. We distinguish between two possible outcomes in the auction in s according to the bid $b(s)$. If B_i wins in s as a single winner, then as $i \neq 1$ it holds that $u_1^a(s) = d_1$. Furthermore, as $u_1^r = \pi_1 - p(s^0)$, and the offer to move to state s is accepted in the considered case, we conclude that $d_1 \geq \pi_1 - p(s^0)$. We conclude that B_i gains $d_i \leq -d_1 \leq -(\pi_1 - p(s^0)) < p(s^0) - n\epsilon$. It therefore holds that B_i's utility in this case is strictly less than $p(s^0) - n\epsilon$. Finally, the analysis in the case where B_i wins in a tie in s is similar.

Finally, assume that $B_l \neq B_i$. As before, it holds that for all $B_j \in C \backslash \{B_1, B_i, B_l\}$, $d_j = u_j^a(s) \geq u_j^r = 0$. If B_l loses the auction in s then B_i gets $0 < p(s^0) - n\epsilon$. It is therefore sufficient to assume that, either B_l wins in s as a single winner, or B_l wins in a tie with another agent. If $B_1 \notin C$, then $1 = \arg\max_{j \neq l} X(s)_{jj}$. By lemma C.3 and lemma C.4, it must hold that $\pi_l + d_l = X(s)_{ll} \geq X(s)_{11} - 2\epsilon = \pi_1 - 2\epsilon$. Hence, $d_l \geq (\pi_1 - \pi_l) - 2\epsilon > 0$. As transfer for all the other agents are non-negative, it follows that $d_i < 0$, which means a negative final utility to B_i in this case. It thus suffices to consider the case $B_1 \in C$. We distinguish between two possible outcomes in the auction in s according to the bid $b(s)$. If B_l wins in s as a single winner, then $u_l^a(s) = \pi_l - p(s) + d_l$. As $u_l^r = 0$, and the offer to move to state s is accepted in the considered case, we conclude that $d_l \geq p(s) - \pi_l$. As $l \neq 1$ it holds that $u_1^a(s) = d_1$. Furthermore, as $u_1^r = \pi_1 - p(s^0)$, and the offer to move to state s is accepted in the considered case, we conclude that $d_1 \geq \pi_1 - p(s^0)$. We conclude that B_i gains $d_i \leq -d_l - d_1 \leq -(p(s) - \pi_l) - (\pi_1 - p(s^0)) \leq p(s^0) + (\pi_l - \pi_1) < p(s^0) - n\epsilon$. It therefore holds that B_i's utility in this case is strictly less than $p(s^0) - n\epsilon$. Finally, the analysis in the case where B_l wins in a tie in s is similar.

Thus, as in the previous case, we conclude that if B_i, where $i \neq 1$, is drawn in the first stage to be the collusion designer, he can achieve $p(s^0) - n\epsilon$ by forming the grand cartel, and can only gain strictly less by considering any alternative proposal to form a cartel which is not the grand one. As we remarked in the previous case, as a function of the strategies of the others, if the offer to form the grand cartel with B_1 as its representative, is accepted by some $B_j \neq B_i, B_1$ for a 0-transfer payment, or by B_1 for a transfer of $-p(s^0) + \epsilon$, then B_i would indeed offer these agents a 0-transfer payment in order to maximize his utility. Offering these lower transfer payments can gain B_i only more than $p(s^0) - n\epsilon$ (Up to $p(s^0) - \epsilon$ in the SPNE point which is best to B_i as the collusion designer).

As for every state s the considered bidding strategy $b(s)$ is in equilibrium of a first price auction in this state, and as for every possible collusion designer, there exists an offer to form the grand cartel, which strictly maximizes his utility, and is accepted by all agents in SPNE, the set of SPNE points of the game is not empty, and in every SPNE point of the game the grand cartel forms with probability 1 as claimed. ☐

D Appendix: Existence of SPNE of the collusion game in the presence of externalities

As previously stated in the paper, we restrict agents to pure strategies in the auction. That is in order to simplify the equilibrium bid analysis in a first price auction in a market with externalities. Note however, that following this approach prevents us from using Nash's (1951) result regarding the existence of equilibrium point in strategic form games with complete information. Therefore, the existence of SPNE of the collusion game is to be proved explicitly.

In addition to proving the existence of SPNE points of the collusion game in the presence of externalities, we also discuss in the following appendix some features of such SPNE points, which we use in the proof of proposition 5.3 to demonstrate partial collusion in the presence of externalities.

Lemma D.1. *Consider a generic market with externalities and let $s = (C, B_l, d)$ be a state, such that $|B(s)| \geq 2$. There exists an equilibrium bid of a first price auction in $X(s)$ in which the good does not stay in the possession of the seller.*

Proof. Denote $(i_1, k_1) = \arg_{(j,m)} \max_{m \neq j} (\pi_j - \alpha_{mj})$. Such a pair is unique due to genericity. If there exists a pair of indices $g \neq h$ such that

$$X(s)_{gg} - X(s)_{hg} > X(s)_{jj} - X(s)_{gj} \quad \forall j \neq g \tag{D.0.1}$$

then by proposition A.1 there is an equilibrium bid in the state s. Otherwise, let us consider two different cases. Let us first assume that $l \neq i_1$. By lemma 5.1 we learn that the only difference between $X(s)$ and the original matrix $X(s^0)$ may be the term $X(s)_{ll}$. Together with genericity, we conclude that $X(s)_{ll} - X(s)_{i_1 l} \geq X(s)_{i_1 i_1} - X(s)_{k_1 i_1}$, otherwise the pair (i_1, k_1) would be satisfying equation D.0.1. If $l \neq k_1$, then by the definition of (i_1, k_1) it holds that for all $j \neq l$, $X(s)_{i_1 i_1} - X(s)_{k_1 i_1} = \pi_{i_1} - \alpha_{k_1 i_1} > \pi_j - \alpha_{lj} = X(s)_{jj} - X(s)_{lj}$. It follows that for all $j \neq l$, $X(s)_{ll} - X(s)_{i_1 l} > X(s)_{jj} - X(s)_{lj}$. The pair (l, i_1) maintains equation D.0.1, in contradiction to the case assumption. Hence $l = k_1$. If $X(s)_{ll} - X(s)_{i_1 l} > X(s)_{i_1 i_1} - X(s)_{k_1 i_1}$, then again as for all $j \neq l$ it holds that $X(s)_{i_1 i_1} - X(s)_{k_1 i_1} \geq X(s)_{jj} - X(s)_{lj}$, we get a contradiction to the case assumption with the pair (l, i_1). We conclude therefore that $X(s)_{ll} - X(s)_{i_1 l} = X(s)_{i_1 i_1} - X(s)_{k_1 i_1}$, and as $l = k_1$, we get $X(s)_{ll} - X(s)_{i_1 l} = X(s)_{i_1 i_1} - X(s)_{l i_1}$. In addition, by the definition of (i_1, k_1), for all $q \neq i_1, k_1$ it holds that $X(s)_{i_1 i_1} - X(s)_{k_1 i_1} \geq X(s)_{qq} - \frac{1}{2}(X(s)_{i_1 q} + X(s)_{k_1 q})$. Therefore, according to proposition A.3, there is an equilibrium bid in s, where B_{i_1} and B_l win the good in tie.

The case $l = i_1$ is handled in a similar way. As we assume that equation D.0.1 does not hold, and as $|B(s)| \geq 2$, there exists $i_3 \neq i_1$ such that $X(s)_{i_3 i_3} - X(s)_{i_1 i_3} \geq X(s)_{i_1 i_1} - X(s)_{k_1 i_1}$. Denote $(i_2, k_2) = \arg_{(j,m)} \max_{m \neq j, j \neq i_1} (X(s)_{jj} - X(s)_{mj})$, such a pair is unique due to genericity, lemma 5.1, and the case assumption $l = i_1$. It follows that for every $j \neq i_1, i_2$, $X(s)_{i_2 i_2} - X(s)_{k_2 i_2} > X(s)_{jj} - X(s)_{i_2 j}$. Therefore, on one hand it holds that $X(s)_{i_2 i_2} - X(s)_{k_2 i_2} \leq X(s)_{i_1 i_1} - X(s)_{i_2 i_1} \leq X(s)_{i_1 i_1} - X(s)_{k_1 i_1}$, where the first inequality is a result of the assumption that equation D.0.1 does not hold, and the second inequality follows from the definition of (i_1, k_1). On the other hand it holds that, $X(s)_{i_2 i_2} - X(s)_{k_2 i_2} \geq X(s)_{i_3 i_3} - X(s)_{i_1 i_3} \geq X(s)_{i_1 i_1} - X(s)_{k_1 i_1}$, where the first inequality follows from the definition of (i_2, k_2), and the second from the definition of i_3. Thus, we get an equality. Therefore, by genericity, lemma 5.1, and as $i_1 \neq i_2, i_3$ we conclude that $k_1 = i_2 = i_3$, and $k_2 = i_1$. It holds therefore, that $X(s)_{i_3 i_3} - X(s)_{i_1 i_3} = X(s)_{i_1 i_1} - X(s)_{i_3 i_1}$. Moreover, as in the previous case, by the definition of (i_2, k_2), for all $q \neq i_1, i_2$ it holds that $X(s)_{i_2 i_2} - X(s)_{k_2 i_2} \geq X(s)_{qq} - \frac{1}{2}(X(s)_{i_1 q} + X(s)_{i_2 q})$. According to proposition A.3 there is an equilibrium bid in s, where B_{i_1} and B_{i_2} win the good in tie. □

Proposition D.2. *The set of SPNE points of the collusion game in the presence of externalities is not empty.*

Proof. Let $b(s)$ be a selection of equilibrium bids of the first price auction, where the good does not stay in the possession of the seller, for every state s, such that $|B(s)| \geq 2$. According to lemma D.1, and lemma A.8, such a selection exists. For every proposal $s = (C, B_l, d)$ made by B_i, let $B_j \in C \setminus \{B_i\}$ accept the offer if and only if $u_j(s, b(s)) \geq u_j(s^0, b(s^0))$. It suffices to demonstrate that for every agent B_i who is selected in the first stage to be the collusion designer, there exists an offer which maximizes his utility. Let B_i be the selected designer. If he makes an offer that gets rejected, or by proposing s^0, he gains $u_i(s^0, b(s^0))$. If he offers to move to a state $s = (C, B_l, d)$, the offer is accepted and B_l does not win the auction, he gains a value in the set $\{\alpha_{ki}\}_{k \neq i} \cup \{0\}$. Both scenarios yield a finite set of potential utilities for B_i.

Therefore, it suffices to demonstrate that the set of offers which may be profitable for B_i (i.e., can gain him more than his worst externality, $\min\{0, \min_{k \neq i} \alpha_{ki}\}$), may be accepted by the addressed agents, and where B_l indeed wins the auction in the state s, is finite. If so, then there exists an offer in this set which maximizes B_i's utility under these assumptions, and therefore there exists an offer which maximizes his utility in general.

Due to the fact that transfer payments are ϵ discrete and balanced, it suffices to demonstrate that for every agent B_j there exists a threshold $\underline{d_j}$, such that $B_j \in C \setminus \{B_i\}$ rejects any offer where $d_j < \underline{d_j}$, and B_i does not propose an offer where $d_i < \underline{d_i}$. We claim that if B_l wins as a single winner then the required threshold is given by $\underline{d_j} = \min_{k \neq j} \alpha_{kj} - \pi_j$. The analysis for the case where B_l wins in a tie is similar. Indeed, if $B_j \in C \setminus \{B_i\}$ accepts an offer to move to a state s where B_l wins as a single winner, he gains at most $\pi_j + d_j$. If $d_j < \underline{d_j} = \min_{k \neq j} \alpha_{kj} - \pi_j$, then he gains strictly less than $\pi_j + \underline{d_j} = \pi_j + \min_{k \neq j} \alpha_{kj} - \pi_j = \min_{k \neq j} \alpha_{kj}$. He can profitably deviate by refusing the offer, and gain at least $\min_{k \neq j} \alpha_{kj}$, as in s^0 the good does not stay in the possession of the seller. In the same manner, if B_i offers to move to a state s where B_l wins as a single winner, he gains at most $\pi_i + d_i$. Following the same calculation, if $d_i < \underline{d_i}$ he gains strictly less than $\min_{k\neq i} \alpha_{ki}$, where by deviating and proposing s^0 he gains at least $\min_{k\neq i} \alpha_{ki}$. $\square$

Corollary D.3. *Consider a generic market with externalities, and let $b(s)$ be a function which maps every state to an equilibrium bid of the first price auction in this state. There exists an SPNE of the collusion game in which agents bid in the auction in every state s according to $b(s)$, and for every proposal $s = (C, B_l, d)$ made by B_i, every $B_j \in C \setminus \{B_i\}$ accepts the offer if and only if $u_j(s, b(s)) \geq u_j(s^0, b(s^0))$.*

E Appendix: Proof of non-efficiency in the presence of externalities

Proof of proposition 6.2. Consider the following 5-player market with externalities: $\pi_1 = \pi_2 = \pi_5 = 1, \pi_3 = 6, \pi_4 = 2, \alpha_{14} = \alpha_{15} = \alpha_{51} = \alpha_{52} = \alpha_{53} = \alpha_{54} = -1, \alpha_{21} = -4, \alpha_{24} = 2, \alpha_{25} = 1, \alpha_{34} = -10, \alpha_{42} = -20$, and all other externalities are null. Let B_1 be the collusion designer in the beginning of the second stage.

1	0	0	-1	-1
-4	1	0	2	1
0	0	6	-10	0
0	-20	0	2	0
-1	-1	-1	-1	1

This market is clearly non-generic, but we can change the valuations and externalities a little to get a generic market in which the same analysis holds. (See also footnote 5.3.) σ is a probability vector, such that B_1 is the collusion designer with a positive probability, $\sigma_1 > 0$. We consider the following strategies of the agents:

- In the state s^0, agents bid $b(s^0) = (5-\epsilon, 6, 6-2\epsilon, 6-\epsilon, 2-\epsilon)$. That is an equilibrium bid of the first price auction in the state s^0, according to corollary A.2.

- In the state $s^1 = (\{B_1, B_2, B_3\}, B_1, d^1)$, where $d^1_1 = 5, d^1_2 = -5, d^1_3 = 0$

	1	4	5
1	6	-1	-1
4	0	2	0
5	-1	-1	1

  we consider the bid $b(s^1) = (3, 0, 0, 3-\epsilon, 2-\epsilon)$, which is in equilibrium by corollary A.2.

- In every state $s \neq s^0, s^1$, such that $|B(s)| \geq 2$ and there exists a unique pair of indices (i,k) such that
  $$(i,k) = \arg_{(j,m)} \max_{m\neq j\in B(s)} (X(s)_{jj} - X(s)_{mj})$$
  we consider an equilibrium bid $b(s)$, where B_i wins the auction, and pays $p(s) = \max_{j\neq i\in B(s)} (X(s)_{jj} - X(s)_{ij})$. Such an equilibrium bid exists according to corollary A.2.

- For every other state s, consider some equilibrium bid $b(s)$, which exists by lemma D.1, and lemma A.8.

- Finally, with respect to the above described function $b(s)$, for every proposal $s = (C, B_l, d)$ made by B_i, every $B_j \in C \setminus \{B_i\}$ accepts the offer if and only if $u_j(s, b(s)) \geq u_j(s^0, b(s^0))$.

According to corollary D.3, there exists an SPNE of the game which respects these strategies. We will demonstrate that the proposal to move to state s^1 maximizes B_1's utility as the collusion designer with respect to the considered strategy profile. As B_3 is the efficient member of the cartel $\{B_1, B_2, B_3\}$ which forms in the state s^1, and B_1 is the cartel bidder in this state, the argument follows.

Consider first the initial state s^0. As stated above we consider the bid $b(s^0) = (5-\epsilon, 6, 6-2\epsilon, 6-\epsilon, 2-\epsilon)$. As a result of this bid, B_2 wins the good, pays $p = 6$ to the seller, and consumes the good. The utility vector of the agents is therefore $u(s^0, b(s^0)) = (\alpha_{21}, \pi_2 - p, \alpha_{23}, \alpha_{24}, \alpha_{25}) = (-4, -5, 0, 2, 1)$.

Consider now the proposal to move the economy to the state s^1, where $s^1 = (\{B_1, B_2, B_3\}, B_1, d^1)$, and $d_1^1 = 5, d_2^1 = -5, d_3^1 = 0$. If B_2 and B_3 accept the offer then the economy moves to the state s^1, where $X(s^1)$ is given by:

	1	4	5
1	6	-1	-1
4	0	2	0
5	-1	-1	1

where, $X(s^1)_{11} = \pi_1 + d_1^1 = 1+5 = 6$. We consider the following equilibrium bid $b(s^1) = (3, 0, 0, 3-\epsilon, 2-\epsilon)$. As a result of which B_1 wins the auction as a single winner, pays $p = 3$ to the seller, and consumes the good. Therefore, B_2 gains $\alpha_{12} + d_2^1 = -5$, and B_3 gains $\alpha_{13} + d_3^1 = 0$, which is equal to what they gain by rejecting the offer. Hence, B_2 and B_3 accept the offer in the considered strategy profile. As a result, by proposing to move to s^1, B_1 gains $X(s^1)_{11} - p = 6 - 3 = 3$. It is clear, that by proposing higher transfers to B_2 or B_3, B_1 gains less. Moreover, lowering any of the discussed transfer payments will yield a rejection. The conclusion is that by forming a cartel with B_2 and B_3, which is represent by B_1, the latter can gain at most 3. It suffices then to demonstrate that every other proposal will gain him strictly less than 3.

Note first that in s^0 he gains -4, which is indeed strictly less than 3. Therefore, we should consider only offers which may be accepted in the discussed strategy profile. Moreover, note that for all $j \neq 1$, $\alpha_{j1} < 3$. It yields, that we should not consider offers where B_1 proposes to move to a state $s = (C, B_l, d)$, if B_l does not win the auction in s according to the considered $b(s)$.

We shall continue by discussing 5 different case, according to the possible identity of the designated cartel bidder, B_l. Let us start with the case where $B_l = B_1$. In order to have $B_j \neq B_1$ to join a cartel C with respect to the considered strategy profile, it must hold that by accepting the offer, B_j gains at least as much as he does by rejecting it. As we consider only offers where $B_l = B_1$ wins if the offer is accepted, B_j accepts if and only if $\alpha_{1j} + d_j \geq u_j(s^0, b(s^0))$, where d_j is the proposed transfer. Equivalently, B_j accepts the offer to join a cartel of which B_1 is the representative, if and only if $d_j \geq u_j(s^0, b(s^0)) - \alpha_{1j}$. More specifically, B_2 will accept such an offer if and only if $d_2 \geq -5 - 0 = -5$, B_3 if and only if $d_3 \geq 0 - 0 = 0$, B_4 if and only if $d_4 \geq 2 - (-1) = 3$, and finally B_5 if and only if $d_5 \geq 1 - (-1) = 2$. Since transfers inside the cartel are balanced, $\pi_1 = 1$, and the price paid in the auction in order to win the good is positive, it follows that by forming a cartel C with himself as the cartel bidder, B_1 can gain at most $\pi_1 - p(C, B_1, d) - \sum\limits_{j \in C, j \neq 1} d_j < 1 - \sum\limits_{j \in C, j \neq 1} d_j$. In order to gain at least 3, B_1 should therefore consider one of the following cartels only: $\{B_1, B_2\}$,$\{B_1, B_2, B_4\}$,$\{B_1, B_2, B_3, B_4\}$,$\{B_1, B_2, B_5\}$,$\{B_1, B_2, B_3, B_5\}$. Consider first a proposal to form the cartel $\{B_1, B_2\}$. It corresponds to the updated matrix of externalities $X(s)$:

	1	3	4	5
1	$1 + d_1$	0	-1	-1
3	0	6	-10	0
4	0	0	2	0
5	-1	-1	-1	1

As calculated above, $d_1 = -d_2 \leq 5$, and therefore, with respect to the considered strategy profile, B_4 wins in this state and not B_1. Hence, such a proposal cannot yield B_1 a utility greater than 3. The same analysis can be repeated for proposals to form the cartels $\{B_1, B_2, B_4\}$ and $\{B_1, B_2, B_5\}$ with B_1 as the representative.

Consider now a proposal to form the cartel $\{B_1, B_2, B_3, B_4\}$ with B_1 as its representative. It corresponds to the updated matrix of externalities $X(s)$:

	1	5
1	$1 + d_1$	-1
5	-1	1

As calculate above, $d_1 = -d_2 - d_3 - d_4 \leq 5 - 0 - 3 = 2$. If $d_1 > 0$, B_1 wins in this state with respect to the considered strategy profile, and pays $p(s) = 2$. He therefore gains $X(s)_{11} - p(s) = 1 + d_1 - 2 \leq 1 + 2 - 2 = 1$, which is strictly less than 3. If $d_1 = 0$ then there is a tie in this state with respect to the considered strategy profile. By corollary A.6, the price is at least $2 - 2\epsilon$, hence, B_1 gains strictly less than 3. Eventually if $d_1 < 0$, B_2 wins. The same analysis can be repeated for a proposal to form the cartel $\{B_1, B_2, B_3, B_5\}$ with B_1 as the cartel bidder.

Consider now the case where the designated cartel bidder is $B_l = B_2$. As calculated in the previous case, in order to have an agent to join a cartel, B_1 should offer him a transfer payment which will guarantee him a utility at least as high as the utility that he will get if he declines the offer with respect to the considered strategy profile. As B_2 wins the auction and consumes the good in the state s^0, it is clear that no money can be extracted from B_3, B_4 and B_5 in order to form a cartel represented by B_2. As B_2 gains -5 in s^0, and his valuation is $\pi_2 = 1$, B_1 would not be able to extract more than 6 from B_2. Hence, whatever cartel B_1 forms with B_2 as the cartel bidder, if B_2 indeed wins the auction in the new state, B_1 gains $\alpha_{21} + d_1$, which is at most $-4 + 6 = 2$.

Consider the case $B_l = B_3$. Following the same analysis, B_1 cannot extract more than 6 from B_3, and can extract at most 5 from B_2. On the other hand, B_4 will demand at least 12 in order to participate, and B_5 will demand at least 1. It follows that it suffices to consider the following cartels: $\{B_1, B_3\}$,$\{B_1, B_2, B_3\}$,$\{B_1, B_3, B_5\}$,$\{B_1, B_2, B_3, B_5\}$. Consider a proposal to form the cartel $\{B_1, B_3\}$. It corresponds to the updated matrix of externalities $X(s)$:

	2	3	4	5
2	1	0	2	1
3	0	$6 + d_3$	-10	0
4	-20	0	2	0
5	-1	-1	-1	1

Note that in order to gain at least 3, as $\alpha_{31} = 0$, it follows that $d_1 \geq 3$, hence, $d_3 = -d_1 \leq -3$. Therefore, with respect to the considered strategy profile, it is B_2 who wins the auction, and not B_3. Hence, such a proposal cannot yield B_1 a utility greater than 3. The same analysis can be repeated for proposals to form the other relevant cartels.

Consider the case $B_l = B_4$. B_1 cannot extract any money from B_4 as his valuation is equal to the externality he gains in s^0. He will need to compensate B_2 for his participation by at least 15, and would not be able to extract any money from B_3 and B_5. As $\alpha_{41} = 0$, whatever cartel B_1 forms with B_4 as its representative, he would not be able to gain a positive utility. The analysis in the last possible case $B_l = B_5$ is similar.

The conclusion is that the proposal to move the economy to the state s^1 is maximizing B_1's utility with respect to the considered strategy profile. Therefore, there exists an SPNE in this market, in which with a positive probability B_1 is the collusion designer, he offers to form a cartel with a representative who is not its efficient member, and this offer is accepted. $\square$

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