Repeated Games with Time-Inconsistent Preferences[1]

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Abstract

I examine when and how results from the theory of repeated games hold in a model with time-inconsistent preferences. Two results which emerge are that Nash reversion can be used to support beneficial cooperation whenever the sum of the discount factors is sufficiently large, and that the two most well-known folk theorems hold for large classes of discount functions that have a parameter that can be adjusted to make the future more important. I also identify conditions for when the players always want to commit to their equilibrium strategy even though they are ranking outcome paths inconsistently.

1 Introduction

The purpose of this paper is to examine when and how results from the theory of repeated games hold in a model with time-inconsistent preferences which allows time-consistent exponential discounting as a particular case. Instead of having players discount their instantaneous utility t periods into the future with the discount factor δ^t, there is a discount function f such that the players discount the value of their instantaneous utility function t periods into the future with the discount factor $f(t)$. The only assumption made about the discounting process is that f is nonnegative and summable: $\sum_{t=1}^{\infty} f(t) < +\infty$.

Time-inconsistency arises for example when the discount function is "quasi-exponential" and $f(t) = \beta\delta^t$.[3] If $\beta < 1$, then this discount function is such that the decision maker is less willing to postpone pleasure from today to tomorrow than from a period far into the future to the period after that. That is, the discount rate decreases as the time before payoffs are realized grows longer. Experimental studies in economics often suggest that individuals behave as if they discount in this way, and evolutionary explanations for why this may be the case have been provided by Dasgupta and Maskin (2005) and Wärneryd (2006).

So far, most studies on how time-inconsistent players play repeatead games have focused on the case where there is one decision maker in a game with himself. In this setting Vieille and Weibull (2009) have studied the ability or inability of a time-inconsistent decision maker to avoid temptation and shown that time-inconsistency can give rise to multiple equilibria with different payoffs even though there is only one decision maker. Also when Krusell and Smith (2003) examine how quasi-exponential discounting affects the consumption-savings problem, their main finding is one of indeterminacy of equilibrium savings rules.

Since time-inconsistency makes even a game with a single decision maker complicated, one might expect that repeated games with several time-inconsistent players are not amenable to

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[3] This discount function is also often called *quasi-geometric* or *quasi-hyperbolic.* The terminology quasi-hyperbolic emphasizes that the discount function, for some parameter values, is similar to a hyperbolic discount function. The terminology quasi-exponential (quasi-geometric) is used to emphasize that the discount function allows exponential (geometric) discounting as a particular case.

analysis. Fortunately that is not the case. Chade, Prokopovych and Smith (2008) analyze a model of repeated games with quasi-exponential discounting and are able to characterize the set of equilibrium payoffs. All results derived by Chade et al. apply in the more general model I use in this paper. Chade et al. also introduce a new equilibrium concept for repeated games with time-inconsistency. In section 4, I use this equilibrium concept to examine when the conflict between current and future selves of a player disappears in equilibrium.

The reason that repeated games with time-inconsistent preferences can be analyzed much in the same way as usual is that even with time-inconsistent prefences each subgame of the infinitely repeated game is still identical to the game itself. This stationarity is sufficient to make it possible to perform the same basic analysis as that usually done for repeated games with exponential discounting. It is almost business as usual and for large classes of discount functions Nash reversion and folk theorems continue to work as usual. Furthermore, the theory on discounted games developed by Abreu (1988) does not require exponential discounting. Abreu's results guarantee the existence of optimal penal codes and show that any subgame perfect equilibrium outcome can be supported by simple history independent equilibrium strategies. The stationary mentioned above is sufficient for this analysis and time-consistency is not required. Abreu (1988) concludes with: "Analogues to the theorems established here ought to appear in any model with discounting and a "repeated" strucure." I write down the anticipated analogues that hold with time-inconsistent preferences in section 5.

Using subgame perfect Nash equilibrium as the solution concept I ask the following two questions: (a) When can the players use Nash reversion to support mutually beneficial cooperation? (b) Suppose that the values of the discount function f depend on some parameter α, what must be assumed about the discount function f to make a folk theorem work?

The answer to the first question is given in proposition 2: Nash reversion can be used to support mutually beneficial cooperation whenever the sum of the discount factors is sufficiently large.

The second questions needs to be clarified before any answer can be described. Loosely speaking, standard folk theorems say that when δ is close to 1, then almost any outcome is possible in equilibrium. The question here is: when is it the case that almost any outcome is possible equilibrium when the parameter α of the discount function is adjusted properly? This question does not have one answer, it depends on which folk theorem that is considered and in particular it depends on the strategies used by the players. In a subgame perfect equilibrium it has to be optimal for a player to follow his strategy after any history of play in the previous periods. Therefore, what needs to be assumed about the discount function to make sure that some given strategy is a subgame perfect equilibrium will depend on the strategy.

I consider only the possibility of generalizing two simple and well-known folk theorems: one theorem with just two players and no player-specific punishments, and one theorem with n players and player-specific punishments. I identify for each theorem a condition for the discount function f such that if it is satisfied, then the folk theorem works as usual. These conditions are found as condition 1 and condition 2 in section 3.2 and they turn out to be surprisingly simple. The answer to question (b) I arrive at is thus: The two player folk theorem works as usual for any discount function that satisfies condition 1; the $n-$player folk theorem works as usual for any discount function that satisfies condition 2. Since exponential discounting meets both condition 1 and condition 2, the original versions of the folk theorems are implied by proposition 3 and 4 in section 3.2 which give the formal version of this answer.

The rest of this paper is organized as follows. Section 2 studies Nash reversion for the particular case of quasi-exponential discounting. Section 3 introduces the general model and uses it to study Nash reversion and folk theorems. Section 4 uses a new equilibrium concept introduced by Chade, Prokopovych and Smith (2008) to examine when the conflict between current and future selves of the players disappear. Section 5 states the analogues of Abreu's (1988) results for time-inconsistent preferences. Section 6 concludes.

2 Repeated Games with Quasi-Exponential Discounting

All conceptual difficulties with time-inconsistent preferences are present when the discount function f is quasi-exponential and $f(t) = \beta\delta^t$. This section introduces a model of repeated games with this discount function and uses it to study Nash reversion strategies. The step to general discount functions, which is taken in the next section, will then be small.

Model and Notation. There is a strategic game $G = \langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle$, where $N = \{1, \ldots, n\}$ is a finite set of players, A_i is the set of actions available to player i and $u_i : \times_{j\in N} A_j \to \mathbb{R}$ is a utility function describing the preferences of player i. The set A is defined by $A = \times_{i\in N} A_i$, and an action profile $a = (a_i)_{i\in N}$ from A will be referred to as an *outcome*. For any $i \in N$ the notation A_{-i} is used for the set $\times_{j\in N\setminus\{i\}} A_j$, and given a profile of actions $a_{-i} \in A_{-i}$ and an action $a_i \in A_i$, (a_i, a_{-i}) denotes the outcome $(a_j)_{j\in N}$. To ensure that all maximum and minimum values to be defined are well-defined we assume that A_i is a compact subset of a Euclidean space and that u_i is continuous for all $i \in N$.

The stage game G is repeated infinitely many times. For each $i \in N$ there is a collection $(i_t)_{t=0}^{\infty}$ of "$i-$players". Player i_t controls the period t action of the $i-$players. Let A^∞ be the collection of all sequences $(a^t)_{t=0}^{\infty}$ with $a^t \in A$ for all $t \in \mathbb{N}$. Elements $(a^t)_{t=0}^{\infty}$ of A^∞ will be referred to as *outcome paths*. For each $i \in N$ the preferences of the $i-$players on A^∞ are described by a collection of utility functions $(U_{it})_{t=0}^{\infty}$. Given an outcome path $(a^s)_{s=0}^{\infty} = (a^s)$ from A^∞, the utility of player i_τ is $U_{i\tau}((a^s))$. The function $U_{i\tau}$ is defined by

$$U_{i\tau}\left((a^s)\right) = u_i(a^\tau) + \sum_{t=1}^{\infty} \beta\delta^t u_i(a^{\tau+t})$$

for all $i \in N$, all $\tau \in \mathbb{N}$ and all $(a^s) \in A^\infty$, where $\delta \in (0, 1)$ and $\beta \geq 0$ are given constants.

The solution concept adopted is that of subgame perfect equilibrium. A formal description of histories, strategy profiles and subgame perfect equilibrium is now provided. The notation introduced for strategy profiles and histories is not used until section 4, where the discussion requires some notation for these concepts. Only the intuitive interpretation of the model discussed in a paragraph below is required to state, discuss and prove all results in section 2 and 3.

Put $A^0 = \{\emptyset\}$ and let $H = \cup_{t=0}^{\infty} A^t$ so that H is the set of histories for the repeated game and the empty set $\emptyset$ represents the empty history. A strategy for player i_t is a description of how player i_t plans to act in the game and specifies one action for each history of play $h \in A^t$ leading up to period t. That is, a strategy of player i_t is a function $g_{it} : A^t \to A_i$. A strategy profile is a collection of strategies which specifies for each $i \in N$, and each $t \in \mathbb{N}$ one strategy for player i_t. One compact way to describe this is to define a strategy profile as a function $g : H \to A$. If $h \in A^t$, then the i'th coordinate of $g(h)$, written $g_i(h)$, is the action that the strategy profile suggest for player i_t after history h. Since each player i_t only controls one action, a strategy profile g is a subgame perfect equilibrium if and only if the following statement is true for all $i \in N$, all $t \in \mathbb{N}$ and all histories $h \in A^t$ of play up to period t :

Given that the history of play has been h, and that all other current and future players will play as suggested by the strategy profile g, it is optimal for player i_t to also use the action that the strategy profile g suggests.

Relation to the canonical model of exponential discounting. With $\beta = 1$, we get exponential discounting. The preferences of the $i-$players are then consistent in the sense that if two outcome paths (a^t) and (b^t) from A^∞ are such that $a^t = b^t$ for all $t < \tau$, then player i_τ prefers (a^t) to (b^t) if and only if player i_0 prefers (a^t) to (b^t). In this case one can replace the sequence of $i-$players with just player i_0, and by the one-shot deviation principle for the canonical model of exponential discounting this has no effect on the analysis. Therefore, if we use subgame perfect equilibrium as solution concept for the repeated game, then the model presented above allows the canonical model of exponential discounting as a special case.

On the contrary, if $\beta < 1$, then it can happen that player i_τ prefers (a^t) to (b^t) while player i_0 ranks the outcome paths in the opposite way even though $a^t = b^t$ for all $t < \tau$. In this sense the preferences of the $i-$players are time-inconsistent.

Interpretation. A game consists of a collection of players, strategy sets and payoffs, with one strategy set and one payoff function for each player. This is true both for a one-shot game and a repeated game viewed as a game in its own right. There is no need to step away from this framework to analyze repeated games with time-inconsistent preferences. Time-inconsistent preferences become interesting only if the decision maker lacks ability to commit. If there is a single decision maker, then the decision maker lacks the ability to commit to a plan of actions. If there are several players, then the players lack the ability to commit to a strategy. The period t-decision will be made in period t, and the decision maker will choose it in accordance with the preferences that are relevant at period t. To make this explicit, this paper uses a model with a collection $\{i_t : i \in N, t \in \mathbb{N}\}$ of players, where player i_t controls the period t action of the $i-$players, and where player i_t has the relevant preferences on outcome paths. With this model, we are in the usual game theoretic setting: there are players, strategy sets and payoffs, and each player maximizes a well-defined utility function. Then there is no need to introduce any new solution concept, we can stick to subgame perfect equilibrium.

2.1 Nash Reversion

For this subsection, assume that there exists a Nash equilibrium $a^* \in A$ of the stage game. A Nash reversion strategy is a strategy where the players agree to play the outcome path $(a^t)_{t=0}^\infty$ but if in any period τ a player deviates, then the stage game Nash equilibrium a^* is to be played in all future periods $t = \tau + 1, \tau + 2, \ldots$, no matter what happens after the deviation. Here only Nash reversion strategies that call for players to repeat the same strategy in each period are considered. That is, we assume that for some $a \in A$ the players initially agree to play $a^t = a$ in each period t. Before stating the proposition to follow, which answers the question of when this is a subgame perfect equilibrium, it will be convenient to have a notation for the highest possible payoff for player i given the actions of the other players. For each $i \in N$, define $\hat{u}_i : A \to \mathbb{R}$ by

$$\hat{u}_i(a) = \max_{a_i' \in A_i} u_i(a_i', a_{-i})$$

so that $\hat{u}_i(a)$ is the highest possible payoff for player i given that the other players use the actions a_{-i}.

Proposition 1. *The Nash reversion strategy that calls for playing $a \in A$ in each period is a subgame perfect equilibrium if and only if*

$$u_i(a)(1-\delta+\beta\delta) \geq (1-\delta)\hat{u}_i(a) + \delta\beta u_i(a^*)$$

for all $i \in N$.

Proof. It is sufficient to show that after any history of play no player has an incentive to deviate. Suppose we are in period τ. If the play during periods $1, 2, \ldots, \tau - 1$ has been a, then player i_τ is called to play a_i. Deviation gives the immediate payoff $\hat{u}_i(a)$ but means that the Nash equilibrium will be played in all future periods. Hence it is optimal for player i_τ to conform if and only if

$$u_i(a) + \sum_{t=1}^{\infty} \beta\delta^t u_i(a) \geq \hat{u}_i(a) + \sum_{t=1}^{\infty} \beta\delta^t u_i(a^*).$$

Computing the sums and multiplying by $1 - \delta$ gives the specified condition.

If any player has deviated then the players are called to play the stage game Nash equilibrium forever and have no incentive to deviate. □

Consider an action profile $a \in A$ that the players prefer to the Nash equilibrium: $u_i(a) > u_i(a^*)$ for all $i \in N$. All players are better of if it is possible to use punishment to support the play of a instead of the Nash equilibrium a^*. The following corollary answers the question of when this is possible.

Corollary 1. *Let $a \in A$ be such that $u_i(a) > u_i(a^*)$ for all $i \in N$. The Nash reversion strategy that calls for playing $a \in A$ in each period is a subgame perfect equilibrium if and only if*

$$\left(u_i(a) - u_i(a^*)\right)\beta\delta \geq (1-\delta)\left(\hat{u}_i(a) - u_i(a)\right)$$

for all $i \in N$. Hence for any $\beta > 0$ there is a $\underline{\delta} \in (0,1)$ such that the Nash reversion strategy that calls for playing $a \in A$ in each period is a subgame perfect equilibrium if $\delta > \underline{\delta}$.

Proof. Follows from Proposition 1 and that $(1-\delta) \to 0$ as $\delta \to 1$. □

So for any choice of $\beta > 0$ the conclusion that a sufficiently high δ makes cooperation that is beneficial relative to a Nash equilibrium possible still holds. The value of β affects the lowest possible choice of $\underline{\delta}$ in Corollary 1.

To illustrate the effect of β on the possibilities of cooperation in a simple setting, suppose for a moment that the stage game has two players playing a prisoner's dilemma with payoffs $0, 1, 2, 3$. That is, cooperation gives both players the payoff 2, the Nash equilibrium gives both players the payoff 1, and deviating from cooperation gives the deviator a payoff of 3. By Corollary 1, cooperation can be sustained using a Nash reversion strategy precisely when

$$(2-1)\beta\delta \geq (1-\delta)(3-2),$$

or equivalently when

$$\beta\frac{\delta}{1-\delta} \geq 1.$$

Solving the equation $\beta\underline{\delta}/(1-\underline{\delta}) = 1$ for $\underline{\delta}$ as a function of β thus gives the lowest possible value of δ necessary to support cooperation as $\underline{\delta} = 1/(\beta+1)$. Streich and Levy (2007) provide an extensive discusson on the possibilities of cooperating in a prisoner's dilemma with different discount functions.

3 Repeated Games with Time-Inconsistency

Model and Notation. Nothing is changed with respect to the stage game $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$. The following changes are made regarding the preferences of player i_τ on the set A^∞ of outcome paths. Instead of constants β and δ to describe time preferences there is a *discount function* $f : \mathbb{N} \to \mathbb{R}$ such that the function $U_{i\tau}$ is defined by

$$U_{i\tau}\left((a^s)\right) = u_i(a^\tau) + \sum_{t=1}^{\infty} f(t) u_i(a^{\tau+t})$$

for all $i \in N$, all $\tau \in \mathbb{N}$ and all $(a^s) \in A^\infty$.

The function f is assumed to be nonnegative and summable: $\sum_{t=1}^{\infty} f(t) < +\infty$. The results derived below are not sensitive to the assumption that all players discount with the same discount function f. Analogous propositions hold when there is a collection $(f_i)_{i \in N}$ of discount functions and the $i-$players discount with the discount function f_i. This is discussed further in the final remark of section 5.

The solution concept adopted is that of subgame perfect equilibrium.

3.1 Nash Reversion

For this subsection, assume that there exists a Nash equilibrium $a^* \in A$ of the stage game. The Nash reversion strategies of section 2 are just as easy to study with an arbitrary discount function as with a quasi-exponential or exponential discount function, and we have the following result:

Proposition 2. *The Nash reversion strategy that calls for playing $a \in A$ in each period is a subgame perfect equilibrium if and only if*

$$(u_i(a) - u_i(a^*)) \sum_{t=1}^{\infty} f(t) \geq \hat{u}_i(a) - u_i(a)$$

for all $i \in N$.

Proof. The specified condition is equivalent with

$$u_i(a) + \sum_{t=1}^{\infty} f(t) u_i(a) \geq \hat{u}_i(a) + \sum_{t=1}^{\infty} f(t) u_i(a^*)$$

for all $i \in N$ which ensures that there is no incentive to deviate. □

Suppose that the value $f(t)$ depends on the parameter α and write $f(t; \alpha)$ for the value of f at t to denote this dependence on the parameter α. Suppose furthermore that there is a α^* such that

$$\lim_{\alpha \to \alpha^*} \sum_{t=1}^{\infty} f(t; \alpha) = +\infty.$$

That is, suppose that by moving α towards α^* we can make the sum $\sum_{t=1}^{\infty} f(t; \alpha)$ arbitrarily large. Then Proposition 2 shows that any outcome $a \in A$ that all players prefer to a^* can be played repeatedly in a subgame perfect equilibrium for some parameter values α. All that is necessary is that α is chosen close enough to α^* so that the sum $\sum_{t=1}^{\infty} f(t;\alpha)$ is large enough. The result of Corollary 1 is a special case of this with $\alpha = \delta$ and $\alpha^* = \delta^* = 1$.

The more general conclusion of Proposition 2 is that the possibilities of cooperating using Nash reversion to support the play of a constant outcome $a \in A$ in each period depend only on the sum $\sum_{t=1}^{\infty} f(t)$, and how $f(t)$ varies over time is completely irrelevant. That all relevant information is contained in the sum $\sum_{t=1}^{\infty} f(t)$ is a consequence of the fact that the strategy studied is that of constant play of a given outcome with Nash reversion as punishment. With a timevarying outcome path and other punishments the sum no longer contains all information needed to evaluate if a strategy is a subgame perfect equilibrium. In the next subsection, where folk theorems for general discount functions are provided, we will however see that even when more complex strategies are used the sum of the discount factors is still an informative measure of the players patience.

3.2 Folk Theorems

The goal of this section is to examine the possibilities of generalizing well-known folk theorems that assume the discount function $f(t) = \delta^t$ to allow for more general discount functions. Two folk theorems are considered, one where the stage game has just two players where the strategy profile is such that the two players minmax each other in the punishment phase, and one theorem for n players where the strategy profile has player-specific punishment phases and players are rewarded for punishing another player. Before the theorems can be stated some definitions are needed. For all $i \in N$, let v_i denote the minmax payoff of player i :

$$v_i = \min_{a_{-i} \in A_{-i}} \max_{a_i \in A_i} u_i(a_i, a_{-i}).$$

An outcome $a \in A$ such that $u_i(a) > v_i$ for all $i \in N$ will be called a *strictly individually rational outcome of* $G = \langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle$. A stage game G is usually said to *allow player-specific punishments* if it is possible given any strictly individually rational outcome $a^* \in A$ to find a collection $(a(i))_{i\in N}$ of strictly individually rational outcomes such that $u_i(a^*) > u_i(a(i))$ and $u_i(a(j)) > u_i(a(i))$ for all $i \in N$ and all $j \in N \backslash \{i\}$. The following two folk theorems for repeated games with exponential discounting are simple special cases of folk theorems due to Fudenberg and Maskin (1986) which also cover the case of correlated randomization.

Folk Theorem 1. *Suppose that* $N = \{1, 2\}$. *Let* a^* *be a strictly individually rational outcome of the stage game* G. *Then there exists* $\underline{\delta} \in (0, 1)$ *such that for all* $\delta \in (\underline{\delta}, 1)$ *there is a subgame perfect equilibrium of the* $\delta-$*discounted infinitely repeated game of* G *that generates the outcome path* (a^t) *in which* $a^t = a^*$ *for all* $t \in \mathbb{N}$.

Folk Theorem 2. *Let* a^* *be a strictly individually rational outcome of the stage game* G. *Assume that there is a collection* $(a(i))_{i\in N}$ *of strictly individually rational outcomes of* G *such that* $u_i(a^*) > u_i(a(i))$ *and* $u_i(a(j)) > u_i(a(i))$ *for all* $i \in N$ *and all* $j \in N\backslash\{i\}$. *Then there exists* $\underline{\delta} \in (0, 1)$ *such that for all* $\delta \in (\underline{\delta}, 1)$ *there is a subgame perfect equilibrium of the* $\delta-$*discounted infinitely repeated game of* G *that generates the outcome path* (a^t) *in which* $a^t = a^*$ *for all* $t \in \mathbb{N}$.

The propositions show that for a given a strictly individually rational outcome a^* we can find a neighborhood $\mathcal{O}$ of 1 such that for all $\delta \in \mathcal{O} \cap [0, 1)$ there is a subgame perfect equilibrium of the $\delta-$discounted infinitely repeated game in which a^* is played in each period. We now consider more general discount functions. Suppose that the values of the discount function f depend on some parameter α which belongs to a subset Ω of $\mathbb{R}^L$ for some $L \geq 1$. To denote this dependence, write $f(t;\alpha)$ for the value of f at t. Exponential discounting occurs when $f(t;\delta) = \delta^t$. We ask the following question:

- ☒ *When can we find for each strictly individually rational outcome $a^* \in A$ an open subset $\mathcal{O}$ of the parameter space Ω such that for all $\alpha \in \mathcal{O}$ repeated play of a^* is a possible outcome path in a subgame perfect equilibrium?*

It is clear that for folk theorems to work, the players have to care a lot about the future. For δ discounting we have that

$$\lim_{\delta \to 1^-} \sum_{t=1}^{\infty} f(t;\delta) = \lim_{\delta \to 1^-} \frac{\delta}{1-\delta} = +\infty,$$

so that the complete future can be made arbitrarily important compared to the current period which has weight 1. A natural guess is thus that for a folk theorem to work for an arbitrary parameterized discount function $f(t;\alpha)$ we have to have for some α^* that

$$\lim_{\alpha \to \alpha^*} \sum_{t=1}^{\infty} f(t;\alpha) = +\infty.$$

This guess is a good start but we need to put some additional structure on f. Exactly which additional structure that is needed depends on the nature of the folk theorem and in particular which strategies the players are assumed to be able to play. For the two folk theorems considered here the two following different conditions are appropriate. Condition 1 is for a generalization of the two player folk theorem and Condition 2 is for the $n-$player folk theorem.

Condition 1. *There exists α^* in the closure of the parameter space Ω with $\alpha^* \notin \Omega$ such that*

$$\lim_{\alpha \to \alpha^*} f(t;\alpha) \geq 1 \text{ for all } t \in \mathbb{N}.$$

Condition 2. *There exists α^* in the closure of the parameter space Ω with $\alpha^* \notin \Omega$ such that*

(i) $\lim_{\alpha \to \alpha^*} \sum_{t=1}^{\infty} f(t;\alpha) = +\infty$;

(ii) there exists a real number $r > 0$ such that $f(t;\alpha) \leq r$ for all $t \in \mathbb{N}$ and all $\alpha \in \Omega$.

When comparing the two conditions it should be noted that the boundedness requirement in part (ii) of Condition 2 is a very mild assumption. We expect this to be satisfied with $r = 1$ for normal discount functions. The conditions have the following intuitive interpretation: Condition 2 means that the future can be made arbitrarily important; Condition 1 means not only that the future be made arbitrarily important, but also that the players can be made to care almost as much about any future period as they care about the current period. This interpretation emphasizes that Condition 1 is more restrictive. When $f(t,\alpha)$ is close to 1 for many t, then the sum $\sum f(t)$ is necessarily large. This argument is formalized as Lemma 1 in the appendix which shows that Condition 1 implies part (i) of Condition 2.

The extra structure on the discount function that Condition 1 guarantees is needed to generalize the two-player folk theorem because the two-player folk theorem does not assume that player-specific punishments can be created where one player is rewarded for punishing the other player. Instead the players minmax each other in the punishment phase, and the only thing stopping players from deviating in the punishment phase is the threat of an additional period of punishment some time in the future. Condition 1 is sufficient for making the punishment phase feel while having the threat of an additional punishment period at some point in the future matter so that the threat of punishment is credible.

Proposition 3 and 4 below are generalizations of the two folk theorems above. They are generalizations because the discount function f with $f(t;\delta) = \delta^t$ satisfies both Condition 1 and Condition 2 with $\Omega = [0, 1)$ and $\delta^* = 1$, and the only thing that is changed is that the hypothesis of exponential discounting is replaced by a weaker hypothesis of a discount function satisfying Condition 1 or Condition 2.

Proposition 3. *Suppose that $N = \{1, 2\}$ and that f satisfies Condition 1. Let a^* be a strictly individually rational outcome of the stage game G. Then there exists a neighborhood $\mathcal{O}$ of α^* such that for all $\alpha \in \mathcal{O} \cap \Omega$ there is a subgame perfect equilibrium of the $f(\cdot\,;\alpha)-$discounted infinitely repeated game of G that generates the outcome path (a^t) in which $a^t = a^*$ for all $t \in \mathbb{N}$.*

Proposition 4. *Suppose that f satisfies Condition 2. Let a^* be a strictly individually rational outcome of the stage game G. Assume that there is a collection $(a(i))_{i\in N}$ of strictly individually rational outcomes of G such that $u_i(a^*) > u_i(a(i))$ and $u_i(a(j)) > u_i(a(i))$ for all $i \in N$ and all $j \in N\backslash\{i\}$. Then there exists a neighborhood $\mathcal{O}$ of α^* such that for all $\alpha \in \mathcal{O} \cap \Omega$ there is a subgame perfect equilibrium of the $f(\cdot\,;\alpha)-$discounted infinitely repeated game of G that generates the outcome path (a^t) in which $a^t = a^*$ for all $t \in \mathbb{N}$.*

Proofs. If Condition 1 is met, and if α is close to α^*, then it is possible to choose $T \in \mathbb{N}$ such that $\sum_{t=1}^{T} f(t;\alpha)$ is large and $f(T;\alpha)$ is close to 1. If Condition 2 is met, and if α is close to α^*, then it is possible to choose $T \in \mathbb{N}$ such that $\sum_{t=1}^{\infty} f(t;\alpha)$ is large and the fraction $\sum_{t=1}^{T} f(t;\alpha)/\sum_{t=1}^{\infty} f(t;\alpha)$ lies almost anywhere on the open interval $(0, 1)$. This is sufficient to construct a subgame perfect equilibrium to support the outcome path (a^t) under the different hypothesises. A complete proof for each theorem can be found in the appendix. □

These results can be applied. Suppose that we are studying players that play a game repeatedly with any summable nonnegative discount function f whose values depend on some parameters. We could for example have $f(t) = \delta^t$, or $f(t) = \beta\delta^t$, or $f(t) = (1+\alpha t)^{\frac{-\gamma}{\alpha}}$, or $f(t) = \beta\delta^t(1+\alpha t)^{\frac{-\gamma}{\alpha}}$, or any other summable nonnegative discount function. We can then check if Condition 2 is met and if Condition 1 is met. If Condition 2 is met, then we can conclude that for some values of the parameters almost any outcome is possible in the repeated game if the stage game allows player-specific punishments. If Condition 1 is met, then we can conclude that even if the stage game does not allow player-specific punishments, if there are just two players, then for some values of the parameters almost any outcome is possible in the repeated game.

For an example of this procedure, suppose that f is quasi-exponential, $f(t;\beta,\delta) = \beta\delta^t$. The parameter space can be taken to be $\Omega = \{(\beta,\delta) \in \mathbb{R}^2 : 0 \le \beta \le 2,\ 0 < \delta < 1\}$. Let (β^*, δ^*) be any point in $\mathbb{R}^2$ with $\beta^* \in (0, 2]$ and $\delta^* = 1$. Then (β^*, δ^*) is in the closure of Ω, and furthermore we have that $\lim_{(\beta,\delta)\to(\beta^*,\delta^*)} \sum f(t;\beta,\delta) = +\infty$. Since part (ii) of Condition 2 is also met with $r = 2$, we have a folk theorem for quasi-exponential discounting: if the stage game allows player-specific punishments, then almost any outcome is possible in the repeated game when (β,δ) is close to (β^*, δ^*). To get another folk theorem, let (β^*, δ^*) be any point in $\mathbb{R}^2$ with $\beta^* \in [1, 2]$ and $\delta^* = 1$. Then (β^*, δ^*) is in the closure of Ω and furthermore we have that $\lim_{(\beta,\delta)\to(\beta^*,\delta^*)} f(t;\beta,\delta) = \beta^* \ge 1$

for all $t \in \mathbb{N}$. Thus, if the stage game only has two players, then almost any outcome is possible in the repeated game when (β, δ) is close to (β^*, δ^*).

For another example of this procedure, suppose that f is hyperbolic, $f(t; \alpha, \gamma) = (1 + \alpha t)^{\frac{-\gamma}{\alpha}}$. The parameter space can be taken to be $\Omega = \{(\alpha, \gamma) \in \mathbb{R}^2 : \alpha > 0, \gamma > \alpha\}$. Let (α^*, γ^*) be any point in $\mathbb{R}^2$ with $\gamma^* = \alpha^* \geq 0$. Then (α^*, γ^*) is in the closure of Ω, and furthermore we have that $\lim_{(\alpha,\gamma)\to(\alpha^*,\gamma^*)} \sum f(t; \alpha, \gamma) = +\infty$. Since part (ii) of Condition 2 is also met with $r = 1$, we have a folk theorem for hyperbolic discounting. To get another folk theorem, let $(\alpha^*, \gamma^*) = (0, 0)$. Then (α^*, γ^*) is in the closure of Ω and furthermore we have that $\lim_{(\alpha,\gamma)\to(\alpha^*,\gamma^*)} f(t; \alpha, \gamma) = 1$ for all $t \in \mathbb{N}$. Thus, if the stage game only has two players, then almost any outcome is possible in the repeated game when (α, γ) is close to (α^*, γ^*). Arguments for the last two convergence claims are given as Claim 1 in the appendix.

In this way Proposition 3 and 4 generate the standard folk theorems for exponential discounting, and furthermore folk theorems for quasi-exponential and hyperbolic discounting. More importantly, the discussion in this section illustrates that when discounting is exponential, $f(t) = \delta^t$, and when δ approaches 1, then the decision maker is made more patient in two distinct ways. Firstly, the sum $\sum f(t)$ approaches infinity; and secondly, $f(t)$ tends to 1 for each $t \in \mathbb{N}$. Other discount functions may have a parameter that can be adjusted to make the future more important in the first way but not the second. The $n-$player folk theorem requires only the first type of patience, that the entire weight of the future can be made arbitrarily large compared to the current period which has weight 1. Associated with this type of patience is a clear numerical measure of the patience associated with the discount function f, namely the sum $\sum f(t)$. In section 3.1 we saw that this is also the relevant type of patience for Nash Reversion. The other type of patience, that the weight of periods far into the future is almost as large as the weight for the current period, has no obvious numerical measure. If f is nonincreasing, then $f(1)$ is an upper bound on f for $t \geq 1$, and therefore $f(1)$ is one indicator of this type of patience. At least we know if $f(1)$ is small that the decision maker is not patient in this sense.

4 Time-Consistent Equilibria

Chade, Prokopovych and Smith (2008) introduce a new equilibrium concept for repeated games with time-inconsistent preferences called *sincere subgame perfect equilibrium.* Sincere subgame perfect equilibrium is the equilibrium concept that results if player i can control his period $t-$action at period 0, at period 1, at period $2, \ldots,$ at period $t-1$, and at period t. That is, in the plainest possible english, sincere subgame perfect equilibrium is the equilibrium concept that results if a player controls today what he is going to do tomorrow, and then he controls it tomorrow again. In a sincere subgame perfect equilibrium, player i is allowed to evaluate the optimality of his period τ action with his current preferences at all periods $t \leq \tau$. In real life an action can not be controlled at all periods $t \leq \tau$ at the same time, so there is no obvious reason to expect that players will play sincere subgame perfect equilibria.

However, being a sincere subgame perfect equilibrium is an interesting property of a subgame perfect equilibrium. As noted by Chade et al., the sincere subgame perfect equilibria are those equilbria where time-inconsistency is not an issue for any player and, given the strategies of the other players, each player always wants to commit to his strategy. There is no conflict between current and future selves of any player: the players i_t and i_s want the same thing. The players might not even notice that they are evaluating outcome paths in an inconsistent way, since they always want to commit to their strategy anyway time-inconsistency never becomes an issue. For this reason I suggest that what Chade et al. call sincere subgame perfect equilibria should be called *time-consistent subgame perfect equilibria*. From now on, I will only the term time-consistent subgame perfect equilibrium.

Chade et al. find that if the discount function is quasi-exponential with $\beta < 1$, then not only do time-consistent subgame perfect equilibria typically exist, but any subgame perfect equilibrium that satisfies a certain punishment property is a time-consistent subgame perfect equilibrium (Theorem 2, Chade et al.). Here this punishment property will be called the *weak punishment property* to distinguish it from another stronger punishment property. The weak punishment property is that a one-shot deviation never increases the discounted sum of instantaneous payoffs during all future periods. Chade et al. have thus identified a crucial difference between repeated games with several time-inconsistent players and repeated games with only one time-inconsistent player: with several players a strategy profile which the players always want to commit to can be created by having the other players punish a deviator. In the particular case of quasi-exponential discounting with $\beta < 1$, this force is so strong that time-inconsistency does not have to be an issue along any equilibrium outcome path (Corollary 1, Chade et al.).

In subsection 4.2, we will see that the full force of the result in Chade et al. is a mathematical curiosity of the quasi-exponential discount function. The quasi-exponential discount function is the only discount function for which it is true that the weak punishment property is sufficient to make the conflict between current and future selves disappear. First, subsection 4.1 develops a related result about the relation between subgame perfect equilibria and time-consistent subgame perfect equilibria which applies for fewer strategy profiles but for more general discount functions.

4.1 Punishment and Time-Consistency

The goal of this subsection is to state and discuss Proposition 5 below which provides sufficient conditions on the discount function and the strategy profile for when the conflict between current and future selves of each player disappears in equilibrium. Some additional notation is required for this section before a time-consistent subgame-perfect equilibrium can be defined. For all $i \in N$ and all $t \in \mathbb{N}$, extend the function U_{it} so that it is defined also for strategy profiles in the natural way. That is, if $(a^s) \in A^\infty$ is the outcome path induced by the strategy profile g, then $U_{it}(g) = U_{it}((a^s))$. A strategy profile g and a history h naturally defines a strategy profile $g^{|h}$ that describes the behavior of the players in the game after history h. The function $g^{|h}$ is defined by setting $g^{|h}(h') = g(h, h')$ for any $h' \in H$, where (h, h') is the history constructed by listing all outcomes in h followed by those in h'.

Definition. *A strategy profile g is a* ***time-consistent subgame perfect equilibrium*** *if for all histories $h \in H$, the strategy profile $g^{|h}$ constitutes a Nash equilibrium in the repeated game where the players $(i_t)_{t=0}^\infty$ are replaced by just one player i who controls the action of the $i-$players in each period. That is, a strategy profile g is a time-consistent subgame perfect equilibrium if for all $i \in N$, all $h \in H$ and all $\tilde{g}_i : H \to A_i$, we have that $U_{i0}(g_i^{|h}, g_{-i}^{|h}) \geq U_{i0}(\tilde{g}_i, g_{-i}^{|h})$.*

This definition is such that a strategy profile g is a time-consistent subgame perfect equilibrium precisely when it is what Chade, Prokopovych and Smith (2008) call a sincere subgame perfect equilibrium. The definition is written in a way which is intended to emphasize that the equilibrium concept identifies strategy profiles where a player always wants to commit to the proposed strategy: player i_0 is happy not only with the action $g_i^{|h}(\emptyset)$, but also with the complete strategy for the repeated game described by $g_i^{|h}$. Any time-consistent subgame perfect equilibrium is a subgame perfect equilibrium, so time-consistency is a refinement criterion for subgame perfect equilibria. If $f(t) = \delta^t$, then the converse also holds and any subgame perfect equilibrium is a time-consistent subgame perfect equilibrium as the preferences of the $i-$players are then consistent.

The relevant condition which we will have to impose on the discount function is that the ratio $f(t)/f(t+1)$ is nonincreasing in t for $t \in \mathbb{N}$. A discount function which satisfies this property will be called present biased.

Definition. *A discount function with $f(0) = 1$ is* ***present biased*** *if the ratio $f(t)/f(t+1)$ is nonincreasing in t for $t \in \mathbb{N}$.*

This definition is such that a discount function is present biased when the discount rate between two adjacent periods decreases as the time before the periods are reached grows longer. This is because if we are in period τ, then the discount rate between period $\tau+t$ and $\tau+t+1$ is precisely $f(t)/f(t+1)$. As mentioned in the introduction, empirical studies in economics often suggest that individuals discount in this way and thus are less willing to postpone pleasure from today to tomorrow than from a period far into the future to the period after that. This property is satisfied for example by the quasi-exponential discount function with $\beta \leq 1$, and the hyperbolic discount function $f(t) = (1 + \alpha t)^{-\gamma/\alpha}$.

Present biasedness should not be confused with impatience. That is, decreasing discount rates should not be confused with high discount rates. When the discount rates are decreasing, then the discount rates are initially higher than they will be later. This does not imply that the discout rates are high on average, it doesn't even imply that discount rates are initially high. As an illustration of this, consider the hyperbolic discount function $f(t) = (1 + \alpha t)^{-\gamma/\alpha}$. In section 3.2, we saw that when (α, γ) is close to $(0, 0)$, then the sum $\sum f(t)$ is high and furthermore $f(1)$ is close to 1. That is, then the discount rates are low on average and furthermore discount rates are never high. Despite this, the discount function is present biased.

Above, $f(1)$ was used as a measure of patience. This was discussed in section 3.2, but we can now note that $f(1)$ works particularly well as a measure of patience for present biased discount functions. The reason is that when f is present biased, then $f(t) \geq f(1)^t$ for each $t \in \mathbb{N}$. So, when f is present biased, then f is unambigously at least as patient as the exponential discount function with $\delta = f(1)$. This line of reasoning can also be used to show that if $f(\cdot\,;\alpha)$ is present biased for each $\alpha \in \Omega$, then Condition 1 is met if and only if $\lim_{\alpha \to \alpha^*} f(1;\alpha) = 1$. A complete argument for this claim is given as Claim 2 in the appendix.

A strategy profile satisfies the *strong punishment property* if it is true after any history of play and for all players that a deviation does not increase the payoff in any future period. One way to write exactly this, but formally, is as follows.

Definition. *The strategy profile g satisfies the* ***strong punishment property*** *if for all $i \in N$, all $\tau \in \mathbb{N}$ and all histories $h \in A^\tau$, if $\tilde{g}$ is a strategy profile such that $\tilde{g} = g$ on $\cup_{t=1}^{\infty} A^{\tau+t}$, and $\tilde{g}_j(h) = g_j(h)$ for all $j \in N\backslash\{i\}$, and if $(a^t)_{t=0}^{\infty}$ and $(b^t)_{t=0}^{\infty}$ are the outcome paths in A^∞ induced by $g^{|h}$ and $\tilde{g}^{|h}$ respectively, then*

$$u_i(a^t) \geq u_i(b^t)$$

for all $t \geq 1$.

We now have all the tools to state the main result of this section.

Proposition 5. *Suppose that the discount function f is present biased. Then any strategy profile g that is a subgame perfect equilibrium and satisfies the strong punishment property is a time-consistent subgame perfect equilibrium.*

Proof. In the appendix. $\square$

The interpretation of this is straightforward and similar to the interpretation of the result in Theorem 2 of Chade et al. which requires a weaker punishment property but the discount function $f(t) = \beta\delta^t$. Because the strategy profile satisfies the strong punishment property, a deviation lowers the payoffs in all future periods. Because the discount function is present biased, the punishment for the $i-$players for a deviation from player i_τ will be at least as bad relative to the one period gain in period τ for player i_t, $t < \tau$, as it is for player i_τ. It follows that as long as player i_τ is happy with his period τ action after each history, then so is player i_t, $t < \tau$. There is no conflict between current and future selves of player i in this case.

The most basic examples of strategy profiles that satisfy the strong punishment property are the Nash reversion strategies discussed in section 2 and 3. It follows from Proposition 5 that if the discount function is present biased, then the equilibria found are not only subgame perfect, they are time-consistent and the players always want to commit to the proposed strategy. The strategy used to prove the two-player folk theorem also satisfies the strong punishment property. Putting Proposition 3 and 5 together thus provides time-consistent subgame perfect equilbria if the discount function satisfies Condition 1 and is present biased. The strategy profile used to prove the $n-$player folk theorem does not generally satisfy the strong punishment property, it does so precisely when all players prefer to punish any other player to being punished themselves.

4.2 The Weak Punishment Property and Quasi-Exponential Discounting

A strategy profile will be said to satisfy the *weak punishment property* if the discounted sum of all future instantaneous payoffs never increases after a deviation.[4] This is weaker than the strong punishment property that requires that the instantaneous payoff does not increase in any future period after a devation. As mentioned in the introduction to this section, it is not possible to replace the strong punishment property in Proposition 5 with the weak punishment property. We know however from Chade, Prokopovych and Smith (2008) that with the particular discount function $f(t) = \beta\delta^t$, $\beta \leq 1$, this replacement is possible. This subsection explores the reason for this, and it turns out that the function $f(t) = \beta\delta^t$, $\beta \leq 1$, is characterized by the fact that this replacement is possible. The result that the weak punishment property is sufficient to make the conflict between current and future selves disappear under quasi-exponential discounting is thus not robust to even the slightest perturbation of the discounting process. Before arriving at this result, we need the following proposition which characterizes the quasi-exponential discount function in a more obvious way.

Proposition 6(a). *Suppose that the discount function f is such that $f(2) \geq f(1)^2$, and that for all positive integers $k \geq 1$, and all real numbers x_k, x_{k+1}, if $f(k)x_k + f(k+1)x_{k+1} \geq 0$, then $f(k+1)x_k + f(k+2)x_{k+1} \geq 0$. Then there exists constants $\beta \in [0,1]$ and $\delta \in (0,1)$ such that $f(t) = \beta\delta^t$ for all $t \geq 1$.*

[4]For a precise definition, replace $u_i(a^t) \geq u_i(b^t)$ for all $t \geq 1$ with $\sum_{t=1}^{\infty} f(t)u_i(a^t) \geq \sum_{t=1}^{\infty} f(t)u_i(b^t)$ in the definition of the strong punishment property.

Proposition 6(b). *Suppose that* $f(t) = \beta\delta^t$ *for some* $\beta \in [0,1]$ *and some* $\delta \in (0,1)$, *and let* $(x_t)_{t=0}^{\infty}$ *be a bounded sequence of real numbers such that* $\sum_{t=1}^{\infty} f(t)x_t \geq 0$ *and* $\sum_{t=1}^{\infty} f(t)x_t \geq x_0$. *Then* $\sum_{t=1}^{\infty} f(T+t)x_t \geq f(T)x_0$ *for all* $T \in \mathbb{N}$.

Proof. Consider part (a) of the proposition. If $f(k) = 0$ for some postive integer k, then the specified property holds only if $f(k') = 0$ for all positive integers k'. Assume therefore that we are in the case where $f(k) > 0$ for all positive integers k. Let $k \geq 1$ be a given positive integer. Put $x_k = 1/f(k)$, $x_{k+1} = -1/f(k+1)$. If the property is to hold, then we have to have $f(k+1)/f(k) \geq f(k+2)/f(k+1)$. Put $x_k = -1/f(k)$, $x_{k+1} = 1/f(k+1)$. If the property is to hold, then we have to have $f(k+1)/f(k) \leq f(k+2)/f(k+1)$. Combining the two inequalities gives $f(k+1)/f(k) = f(k+2)/f(k+1)$. Since $k \geq 1$ was arbitrary, this holds for all $k \geq 1$. Put $\delta = f(2)/f(1)$. Then $f(k) = f(1)\delta^{k-1}$ for all $k \geq 1$. So the discount function f satisfies $f(t) = \beta\delta^t$ with $\beta = f(1)/\delta$. Since $f(2)/f(1) = \delta \geq f(1)$, we have $\beta \leq 1$. That $\delta \in (0,1)$ follows from that f is summable: $\sum f(t) < +\infty$.

It is straightforward to verify part (b) of the proposition. □

Part (a) of Proposition 6 characterizes the quasi-exponential discount function with $\beta \leq 1$ because by part (a) this discount function is the only discount function that could possibly satisfy the specified condition, and the quasi-exponential discount function with $\beta \leq 1$ does satisfy this condition.

Part (b) of Proposition 6 captures exactly why the weak punishment works in a particular way for the discount function $f(t) = \beta\delta^t$. To see this, suppose that player i_T is given the choice between two different streams of instantaneous utilities during periods $T, T+1, \ldots$, say stream 1 and stream 2, and let $(y_t)_{t=T}^{\infty}$ be the stream of real numbers that you get by taking the difference between these two streams. That is, y_t is the period t value of stream 1 minus the period t value of stream 2. Put $x_t = y_{T+t}$ for all $t \in \mathbb{N}$.

If stream 1 is the instantaneous utilities along an outcome path of a subgame perfect equilibrium which satisfies the weak punishment property, and if stream 2 is the stream associated with a deviation from player i_T, then we have that $\sum_{t=1}^{\infty} f(t)x_t \geq 0$ and $\sum_{t=1}^{\infty} f(t)x_t \geq -x_0$; player i_T can not increase his continuation payoff or his complete payoff by any deviation. By part (b) of Proposition 6, this implies that $\sum_{t=1}^{\infty} f(T+t)x_t \geq -f(T)x_0$. The meaning of this is that player i_0 agrees with player i_T about the ranking of the streams 1 and 2. The conflict between player i_0 and player i_T disappears. This is a complete argument for half of the claim of Proposition 7 below. The other half of the proof of Proposition 7, which relies on part (a) of Proposition 6, can be found in the appendix.

Proposition 7. *Let a discount function* f *be given and consider the following statement: For any stage game* G, *if a strategy profile* g *is a subgame perfect equilibrium of the repeated game and satisfies the weak punishment property, then* g *is a time-consistent subgame perfect equilibrium. This statement is true if and only if there exists constants* $\beta \in [0,1]$ *and* $\delta \in (0,1)$ *such that* $f(t) = \beta\delta^t$ *for all* $t \geq 1$.

Proposition 7 is the promised result: quasi-exponential discounting with $\beta \leq 1$ is characterized by the fact that the conflict between current and future selves of the decision maker disappears as soon as the weak punishment property is satisfied.

5 Abreu's Theorems for Repeated Games with Discounting

Abreu (1988) shows that although strategy profiles allow players to make their actions contingent on the entire history of play, much of this potential strategic complexity is redundant. Abreu thinks of a strategy profile as an outcome path and punishment paths for deviations from the outcome path or any punishment path. A *simple strategy profile* is then a strategy profile such that a unilateral deviation of a given player in any period and after any history leads to the same punishment. Therefore a simple strategy profile in a game with n players is completely described by specifying $n+1$ outcome paths: one initial path and one punishment path for each player. Abreu then establishes that optimal penal codes which give the worst possible punishment for a player exist, and that therefore every outcome path of a subgame perfect equilibrium is the outcome path of a subgame perfect equilibrium which is simple. We will see here that as anticipated by Abreu, the theorems provided by Abreu really are theorems for repeated games with discounting, and not just theorems for repeated games with exponential discounting. Before stating the analogues of Abreu's theorems for the present model, we go through the necessary new concepts and notation.

Definition. *Let $Q_0 \in A^\infty$, and let $Q_i \in A^\infty$ for $i \in N$. The* ***simple strategy profile*** $\sigma(Q_0, Q_1, \ldots, Q_n)$ *specifies: (i) play Q_0 until some player deviates singly from Q_0; (ii) for any $j \in N$, play Q_j if the j'th player deviates singly from Q_i, $i = 0, 1, \ldots, n$, where Q_i is an ongoing previously specified path; continue with Q_i if no deviations occur or if two or more players deviate simultaneously.*

Examples of simple strategies are the Nash reversion strategies of section 2 and 3 and the strategies used to prove the folk theorems of section 3.

Let A^P be the subset of A^∞ that contains all outcome paths that are induced by some subgame perfect equilibrium strategy profile. That is, A^P consists of all outcome paths that are possible in a subgame perfect equilibrium. Applying Abreu's analysis to the present model gives the following result.

Proposition 8. *Suppose that there exists at least one subgame perfect equilibrium strategy profile. Then there exists a collection $(Q_1, Q_2, \ldots, Q_n)$ of outcome paths from A^P, called an optimal penal code, with the following properties:*

(i) *For $i \in N$, if $(a^t)_{t=0}^\infty = Q_i$, then*

$$\sum_{t=0}^{\infty} f(t+1)u_i(a^t) = \inf \left\{ \sum_{t=0}^{\infty} f(t+1)u_i(b^t) : (b^t)_{t=0}^\infty \in A^P \right\}.$$

(ii) *Let $Q \in A^\infty$. Then $Q \in A^P$ if and only if the simple strategy profile $\sigma(Q, Q_1, Q_2, \ldots, Q_n)$ is a subgame perfect equilibrium.*

Proof. The proposition can be proven by following Abreu (1988) and adapting the proof to account for time-inconsistency and an arbitrary summable discount function. The adaptation to time-inconsistency is present when the infimum is taken in (i) above. The infimum which gives the worst possible punishment for the current $i-$player is the infimum of the number $\sum_{t=0}^{\infty} f(t+1)u_i(b^t)$ as $(b^t)_{t=0}^\infty$ varies over all subgame perfect equilibrium paths. The resulting real number is not the lowest possible payoff to player i_0 in a subgame perfect equilibrium or the lowest possible payoff for any other $i-$player. It is the lowest possible equilibrium payoff during all future periods; that is, all periods except the current one. A complete proof can be found in the appendix. $\square$

The existence result in Proposition 8 is abstract in the sense that the proof does not provide any method to construct the optimal penal codes. This lack of an explicit optimal penal code is not caused by time-inconsistency, the same is true for exponential discounting.

Since $Q_i \in A^P$ for $i \in N$, it follows from part (ii) of Proposition 8 that the simple strategy $\sigma(Q_i, Q_1, Q_2, \ldots, Q_n)$ is subgame perfect for each $i \in N$. Because $\sigma(Q_i, Q_1, Q_2, \ldots, Q_n)$ is subgame perfect for each $i \in N$, we have the following useful corollary of Proposition 8.

Corollary 2. *Suppose that there exists at least one subgame perfect equilibrium strategy profile, and put* $\underline{v}_i = \inf\left\{\sum_{t=0}^{\infty} f(t+1)u_i(b^t) : (b^t)_{t=0}^{\infty} \in A^P\right\}$ *for* $i \in N$. *Then the outcome path* $\left(a^t\right)_{t=0}^{\infty}$ *from* A^{∞} *is a subgame perfect equilibrium path if and only if*

$$u_i(a^{\tau}) + \sum_{t=1}^{\infty} f(t)u_i(a^{\tau+t}) \geq u_i(b_i, a_{-i}^{\tau}) + \underline{v}_i \tag{1}$$

for all $i \in N$, *all* $b_i \in A_i$, *and all* $\tau \in \mathbb{N}$.

The numbers $(\underline{v}_i)_{i\in N}$ associated with an optimal penal code thus describe the collection A^P of all subgame perfect equilibrium paths through the inequality (1). This result in turn can be used to say something about the subset of $\mathbb{R}^n$ which contains all payoffs that occur in the repeated game for the players $(i_0)_{i\in N}$ along some subgame perfect equilibrium path: this set is compact. The argument is given as Claim 3 in the appendix. Therefore the function U_{i0} which gives the payoff to player i_0 in the repeated game can be maximized and minimized on A^P. That is, for player i_0 there is a most preferred and a least preferred subgame perfect equilibrium path.

Final Remark. The results derived above are not sensitive to the assumption that all players discount with the same discount function f. Analogous propositions hold when there is a collection $(f_i)_{i\in N}$ of discount functions and the $i-$players discount with the discount function f_i.

In this alternative setting, the Nash Reversion result in Proposition 2 holds when f is replaced by f_i in the inequalities of the proposition. Folk theorems like those in Proposition 3 and 4 can be constructed by requiring that $(f_i, \Omega_i, \alpha_i^*)$ meets the specified condition for each $i \in N$. Then there exists a collection $(\mathcal{O}_i)_{i\in N}$ of sets, where $\mathcal{O}_i$ is an open subset of Ω_i, such that if $\alpha_i \in \mathcal{O}_i$ for each $i \in N$, then the repeated game with discount functions $(f_i(\,\cdot\,;\alpha_i))_{i\in N}$ has the wanted property. In Proposition 5 about the relation between punishment and time-consistency, the assumption that f is present biased can be replaced with the assumption that f_i is present biased for each $i \in N$. Proposition 8 and Corollary 2 hold when f is replaced with f_i everywhere. The subset of $\mathbb{R}^n$ which contains all payoffs that occur in the repeated game for the players $(i_0)_{i\in N}$ along some subgame perfect equilibrium path remains compact.

6 Conclusion

Even with time-inconsistent prefences each subgame of the infinitely repeated game is still identical to the game itself. This stationarity is sufficient to make it possible to perform the same basic analysis as that usually done for repeated games with exponential discounting. This paper explores to which extent the theory of repeated games with exponential discounting can be generalized to a theory of repeated games with discounting. The question raised is: *If something is true for exponential discounting, is it then true for other discount functions sufficiently similar to exponential discounting?* Given the empirical relevance of present biased non-exponential discounting, the answer to the question is important for the applicability of the results that the theory of repeated games has delivered.

For the topics considered here the results are as follows: Nash reversion can be used to support mutually beneficial cooperation whenever the sum of the discount factors is sufficiently large. The two most well-known folk theorems hold not just for exponential discounting, but also for large classes of parameterized discount functions that meet Condition 1 or Condition 2 and thus have a parameter that can be adjusted to make the future more important in an appropriate way. There always exists optimal penal codes, and any subgame perfect equilibrium outcome path is the outcome path of a subgame perfect equilibrium strategy profile which is simple.

Hence, for these topics the answer to the question is a clear yes, but how similar to exponential discounting the discount function has to be varies greatly. While the answer is yes for these topics, there may well be other results which hold only for exponential discounting. The presented model can be used to examine how sensitive results are to the assumption of exponential discounting.

In the process of deriving these results, it becomes clear that when discounting is exponential, $f(t) = \delta^t$, and when δ approaches 1, then the decision maker is made more patient in two distinct ways. Firstly, the sum $\sum f(t)$ approaches infinity; and secondly, $f(t)$ tends to 1 for each $t \in \mathbb{N}$. In general these two types of patience are distinct and other discount functions may have a parameter that can be adjusted to make the future more important in the first way but not the second. This leads to two different measures of the patience associated with a discount function, $\sum f(t)$ and $f(1)$. The second measure, $f(1)$, works particularly well for present biased discount functions.

Apart from repeating old questions in a more general setting, with time-inconsistent preferences it is also possible to ask entirely new questions like when there is a conflict between current and future selves of a player. If the discount function is present biased so that players are more willing to postpone pleasure in the future than today, then the conflict between current and future selves disappears in an equilibrium strategy profile where the players punish deviatiors so that the strong punishment property is satisfied. Then a player always wants to commit to his strategy and the step from time-consistent to time-inconsistent preferences becomes blurry not only for the analyst, but also for the players.

7 Appendix

The following lemma is used in the proof of Proposition 3.

Lemma 1. *If a discounting function f satisfies Condition 1, then for any real number M there exists $T \in \mathbb{N}$ and a neighborhood $\mathcal{O}$ of α^* with the following property: for all $\alpha \in \mathcal{O} \cap \Omega$ we have that $\sum_{t=1}^{T} f(t;\alpha) > M$.*

Proof. Let M be any given real number. Let $T \in \mathbb{N}$ be such that $T > M$. There exists $\varepsilon > 0$ such that if $f(t;\alpha) > 1-\varepsilon$ for all $t \in \{1,2,\ldots,T\}$, then $\sum_{t=1}^{T} f(t;\alpha) > M$. For example, choosing $\varepsilon = (T-M)/T$ works. For each $t \in \{1,2,\ldots,T\}$, let $\mathcal{O}_t$ be a neighborhood of α^* such that $f(t;\alpha) > 1-\varepsilon$ for all $\alpha \in \mathcal{O}_t \cap \Omega$. Such neighborhoods exists since $\lim_{\alpha\to\alpha^*} f(t;\alpha) \geq 1$. Define $\mathcal{O}$ by $\mathcal{O} = \cap_{t=1}^{T} \mathcal{O}_t$. This neighborhood has the desired property. $\square$

Since $\sum_{t=1}^{\infty} f(t;\alpha) \geq \sum_{t=1}^{T} f(t;\alpha)$, it follows from Lemma 1 that Condition 1 implies that $\lim_{\alpha\to\alpha^*} \sum_{t=1}^{\infty} f(t;\alpha) = +\infty$ as claimed.

Proof of Proposition 3. Fix a strictly individually rational action profile $a^* \in A$. Put

$$\bar{v} = \max_{i\in N, a\in A} u_i(a)$$

so that $\bar{v}$ is the highest payoff available to any player in the stage game. In particular, $\bar{v}$ is at least as good as any deviation for any player. For $i = 1,2$, let $p_{-i} \in A_{-i}$ be one of the solutions to the problem of minmaxing player i. Let p be the action profile $p = (p_{-2}, p_{-1}) \in A_1 \times A_2$. When p is played both players are minmaxing the other player. Recall that v_i is the minmax payoff for player i. That is, v_i is the maximum payoff for player i when player j plays p_{-i}. Since a^* is strictly individually rational we thus have that

$$u_i(a^*) > v_i \geq u_i(p)$$

for $i = 1,2$. The strategy constructed using these actions is described by the following instructions and the addition that play begins in Phase I.

Phase I Play a^*. Remain in Phase I unless a single player deviates. If a single player deviates, go to Phase II.

Phase II Play p in each period. Continue for T periods as long as play is always p or both players deviate from p. After T periods go to Phase I. If a single player deviates, restart Phase II.

The rest of the proof is divided into two steps. In step 1, two conditions which are sufficient for the strategy to be a subgame perfect equilibrium are derived. In step 2, the restrictions on the discount function in Condition 1 are used to show that these conditons can be met.

Step 1. We go through each phase and give sufficient conditions for it to be optimal for player i to conform to the proposed strategy. In Phase I, it is optimal for player i to conform if

$$u_i(a^*) + \sum_{t=1}^{\infty} f(t;\alpha)u_i(a^*) \geq \bar{v} + \sum_{t=1}^{T} f(t;\alpha)v_i + \sum_{t=T+1}^{\infty} f(t;\alpha)u_i(a^*),$$

which can be written as

$$\sum_{t=1}^{T} f(t;\alpha)\left(u_i(a^*) - v_i\right) \geq \bar{v} - u_i(a^*).$$

Since $u_i(a^*) - v_i > 0$ for $i = 1, 2$, it follows that there exists a real number m_1 such that if

$$\sum_{t=1}^{T} f(t;\alpha) > m_1, \tag{2}$$

then it is optimal for both player 1 and player 2 to conform in Phase I.

Consider now Phase II. As both players are minmaxing each other, an optimal deviation in this phase gives player i the current payoff v_i. Thus in Phase II with $T' \leq T$ periods of punishment remaining it is optimal for player i to conform if

$$u_i(p) + \sum_{t=1}^{T'-1} f(t;\alpha)u_i(p) + \sum_{t=T'}^{\infty} f(t;\alpha)u_i(a^*) \geq v_i + \sum_{t=1}^{T} f(t;\alpha)u_i(p) + \sum_{t=T+1}^{\infty} f(t;\alpha)u_i(a^*). \tag{3}$$

Since $u_i(p) < u_i(a^*)$, the left hand side of (3) is at least as large as

$$u_i(p) + \sum_{t=1}^{T-1} f(t;\alpha)u_i(p) + \sum_{t=T}^{\infty} f(t;\alpha)u_i(a^*).$$

Thus (3) is satisfied if

$$u_i(p) - v_i + f(T;\alpha)\left(u_i(a^*) - u_i(p)\right) \geq 0.$$

Since $u_i(a^*) - v_i > 0$ for $i = 1, 2$, it follows that there exists a real number $m_2 \in (0, 1)$ such that if

$$f(T;\alpha) > m_2, \tag{4}$$

then it is optimal for both player 1 and player 2 to conform in Phase II. The conclusion from this step is that if T and α are such that (2) and (4) hold, then the proposed strategy is a subgame perfect equilibrium. For the next step we will need only that $m_2 < 1$.

Step 2. To complete the proof, we have to show that it is possible to find a neighborhood $\mathcal{O}$ of α^* with the following property: for each $\alpha \in \mathcal{O} \cap \Omega$ there exists $T_\alpha \in \mathbb{N}$ such that with $T = T_\alpha$, (2) and (4) hold.

By Lemma 1 there is an integer T and a neighborhood $\mathcal{O}'$ of α^* such that for each $\alpha \in \mathcal{O}' \cap \Omega$ we have that $\sum_{t=1}^{T} f(t;\alpha) > m_1$. Let $\mathcal{O}$ be a neighborhood of α^* that is a subset of $\mathcal{O}'$ and such that $f(T;\alpha) > m_2$ for all $\alpha \in \mathcal{O} \cap \Omega$. Such a neighborhood $\mathcal{O}$ exists since $\lim_{\alpha \to \alpha^*} f(T;\alpha) \geq 1 > m_2$. The constructed neighborhood $\mathcal{O}$ of α^* has the required properties and for all $\alpha \in \mathcal{O} \cap \Omega$ we may set $T_\alpha = T$.

To see this, let $\alpha \in \mathcal{O} \cap \Omega$. Since $\mathcal{O}$ is a subset of $\mathcal{O}'$, we have that $\sum_{t=1}^{T} f(t;\alpha) > m_1$. Also $\mathcal{O}$ was chosen such that we have $f(T;\alpha) > m_2$. That is, (2) and (4) are satisfied. $\square$

The following lemmas are used in the proof of Proposition 4 below.

Lemma 2. *If a discounting function f satisfies Condition 2, then for all $\varepsilon > 0$ there exists a neighborhood $\mathcal{O}$ of α^* such that for all $\alpha \in \mathcal{O} \cap \Omega$ and all $t \in \mathbb{N}$ we have that $f(t;\alpha)/\sum_{s=1}^{\infty} f(s;\alpha) < \varepsilon$.*

Proof. By (i) there is a neighborhood $\mathcal{O}$ of α^* such that $\sum_{s=1}^{\infty} f(s;\alpha) > r/\varepsilon$ for all $\alpha \in \mathcal{O} \cap \Omega$. Hence, by (ii), we have that $f(t;\alpha)/\sum_{s=1}^{\infty} f(s;\alpha) < r\varepsilon/r = \varepsilon$ for all $t \in \mathbb{N}$ and all $\alpha \in \mathcal{O} \cap \Omega$. □

Lemma 3. *If a discounting function f satisfies Condition 2, then for all $\mu \in (0,1)$ and all $\varepsilon > 0$ there exists a neighborhood $\mathcal{O}$ of α^* with the following property: for each $\alpha \in \mathcal{O} \cap \Omega$ there exists $T_\alpha \in \mathbb{N}$ such that*

$$\mu - \varepsilon < \frac{\sum_{t=1}^{T_\alpha} f(t;\alpha)}{\sum_{t=1}^{\infty} f(t;\alpha)} \le \mu.$$

Proof. By Lemma 2 there is a neighborhood $\mathcal{O}$ of α^* such that $f(t;\alpha)/\sum_{s=1}^{\infty} f(s;\alpha) < \min\{\mu, \varepsilon\}$ for all $\alpha \in \mathcal{O} \cap \Omega$ and all $t \in \mathbb{N}$. This neighborhood has the desired property.

To see this, fix an arbitrary $\alpha \in \mathcal{O} \cap \Omega$. We have that

$$\frac{f(t;\alpha)}{\sum_{s=1}^{\infty} f(s;\alpha)} \le \mu$$

for all $t \in \mathbb{N}$, so in particular this holds with $t = 1$. Hence there exists at least one $T \in \mathbb{N}$ such that

$$\frac{\sum_{t=1}^{T} f(t;\alpha)}{\sum_{t=1}^{\infty} f(t;\alpha)} \le \mu, \tag{5}$$

namely $T = 1$. Since $\mu < 1$, we have that

$$\sum_{t=1}^{T} f(t;\alpha) > \mu \sum_{t=1}^{\infty} f(t;\alpha)$$

for all T sufficient large. Hence it is possible to define T_α as the largest positive integer such that (5) is satisfied with $T = T_\alpha$. Then

$$\frac{\sum_{t=1}^{T_\alpha+1} f(t;\alpha)}{\sum_{t=1}^{\infty} f(t;\alpha)} = \frac{\sum_{t=1}^{T_\alpha} f(t;\alpha)}{\sum_{t=1}^{\infty} f(t;\alpha)} + \frac{f(T_\alpha+1,\alpha)}{\sum_{t=1}^{\infty} f(t;\alpha)} > \mu$$

which, since $f(T_\alpha+1,\alpha)/\sum_{t=1}^{\infty} f(t;\alpha) < \varepsilon$, implies that

$$\frac{\sum_{t=1}^{T_\alpha} f(t;\alpha)}{\sum_{t=1}^{\infty} f(t;\alpha)} > \mu - \varepsilon. \quad \square$$

Lemma 3 shows that if Condition 2 is satisfied, and if α is close to α^*, then it is possible to place the fraction

$$\frac{\sum_{t=1}^{T} f(t;\alpha)}{\sum_{t=1}^{\infty} f(t;\alpha)}$$

almost anywhere on the interval $(0,1)$ by choosing T appropriately. In our application T is the length of the punishment phase and Lemma 3 shows that the fraction of the sum of the discount factors that belong to the punishment phase can be chosen almost arbitrarily from the interval $(0,1)$.

Proof of Proposition 4. Let $a^* \in A$ be a strictly individually rational outcome of G. Assume that there is a collection $(a(i))_{i \in N}$ of strictly individually rational outcomes of G such that for all $i \in N$ and all $j \in N \backslash \{i\}$ we have that $u_i(a^*) > u_i(a(i))$ and $u_i(a(j)) > u_i(a(i))$. Put

$$\bar{v} = \max_{i \in N, a \in A} u_i(a)$$

so that $\bar{v}$ is the highest payoff available to any player in the stage game. In particular, $\bar{v}$ is at least as good as any deviation for any player. For any $i \in N$, let $p_{-i}(i) \in A_{-i}$ be one of the solutions to the problem of minmaxing player i. Let $p_i(i)$ be a best reply of player i to $p_{-i}(i)$, and write $p(i) = (p_i(i), p_{-i}(i))$. The outcomes $(a(i), p(i))_{i \in N}$ and a^* are such that

$$\begin{aligned} u_i(a^*) &> u_i(a(i)) > u_i(p(i)) = v_i, \quad \text{and} \\ u_i(a(j)) &> u_i(a(i)) > u_i(p(i)) = v_i, \end{aligned}$$

for all $i \in N$ and all $j \in N \backslash \{i\}$.

The strategy profile constructed using the outcomes $a^*, (a(i))_{i \in N}$ and $(p(i))_{i \in N}$ is described by the following instructions and the addition that play begins in Phase I.

Phase I Play a^*. Remain in Phase I unless a single player deviates. If a single player j deviates, go to Phase II$_j$.

Phase II$_j$ Play $p(j)$ in each period. Continue for T periods as long as play is always $p(j)$ or at least two players deviate from $p(j)$. After T periods go to Phase III$_j$. If a single player i deviates, go to Phase II$_i$.

Phase III$_j$ Play $a(j)$. Remain in Phase III$_j$ unless a single player deviates. If a single player i deviates go to Phase II$_i$.

This means that the players begin by playing a^*. If player j deviates during any phase, then a punisment phase begins where the deviator j is punished for T periods with his minmax payoff. If there are no deviations during the punishment phase, then a punisher $i \in N \backslash \{j\}$ gets rewarded for not deviating with $u_i(a(j)) > u_i(a(i))$.

The rest of the proof is divided into two steps. In step 1, two conditions which are sufficient for the strategy to be a subgame perfect equilibrium are derived. In step 2, the restrictions on the discount function in Condition 2 are used to show that these conditons can be met.

Step 1. We go through each phase and give sufficient conditions for it to be optimal for player i to conform to the proposed strategy. In Phase I it is optimal for player i to conform if

$$u_i(a^*) + \sum_{t=1}^{\infty} f(t;\alpha) u_i(a^*) \geq \bar{v} + \sum_{t=1}^{T} f(t;\alpha) v_i + \sum_{t=T+1}^{\infty} f(t;\alpha) u_i(a(i)). \tag{6}$$

To rewrite this expression on a form that will be useful later, define M and λ by

$$\begin{aligned} M(\alpha) &= \sum_{t=1}^{\infty} f(t;\alpha), \\ \lambda(T,\alpha) &= \frac{\sum_{t=1}^{T} f(t;\alpha)}{\sum_{t=1}^{\infty} f(t;\alpha)}, \end{aligned}$$

so that M is the total sum of the discount factors whereas λ gives the proportion of the sum of the discount factors that belongs to the first T periods. Then, since $u_i(a^*) > u_i(a(i))$, (6) holds if

$$\frac{u_i(a^*) - \bar{v}}{M(\alpha)} + \lambda(T,\alpha) \cdot [u_i(a^*) - v_i] \geq 0. \tag{7}$$

In Phase III$_j$ with $i \neq j$ it is optimal for player i to conform if

$$\frac{u_i(a(j)) - \bar{v}}{M(\alpha)} + \lambda(T,\alpha) \cdot [u_i(a(j)) - v_i] + (1 - \lambda(T,\alpha)) \cdot [u_i(a(j)) - u_i(a(i))] \geq 0. \tag{8}$$

In Phase III$_i$ it is optimal for player i to conform if

$$\frac{u_i(a(i)) - \bar{v}}{M(\alpha)} + \lambda(T,\alpha) \cdot [u_i(a(i)) - v_i] \geq 0. \tag{9}$$

Since $u_i(a^*) > u_i(a(i))$ and $u_i(a(j)) > u_i(a(i))$ for all $i \in N$ and all $j \in N \backslash \{i\}$, we have that (9) implies (8) and (7) so we may forget about (8) and (7). Since $u_i(a(i))$ is strictly greater than v_i for all $i \in N$, it follows that there exists real numbers m_1 and m_2, with $m_2 > 0$, such that if

$$\frac{m_1}{M(\alpha)} + \lambda(T,\alpha) \cdot m_2 > 0, \tag{10}$$

then it is optimal for all players $i \in N$ to conform in Phase I and Phase III$_j$ for all $j \in N$. The number m_2 could for example be taken as the smallest of the finitely many positive numbers $u_i(a(i)) - v_i$ when i varies over N. Similarly m_1 could be taken as the minimum of $u_i(a(i)) - \bar{v}$ as i varies over N.

It remains to consider the punishment phase, Phase II. It is clear that for the player being punished there is no incentive to deviate: deviating does not raise the payoff in any period. Consider therefore Phase II$_j$ with $j \neq i$ where player i is punishing player j and suppose that there are $T' \leq T$ periods of punisment left. Conforming is optimal for player i if

$$u_i(p(j)) + \sum_{t=1}^{T'-1} f(t;\alpha) u_i(p(j)) + \sum_{t=T'}^{\infty} f(t;\alpha) u_i(a(j)) \geq \bar{v} + \sum_{t=1}^{T} f(t;\alpha) v_i + \sum_{t=T+1}^{\infty} f(t) u_i(a(i)). \tag{11}$$

The left hand side of (11) is larger than

$$u_i(p(j)) + \sum_{t=1}^{T} f(t;\alpha) \min\{u_i(p(j)), u_i(a(j))\} + \sum_{t=T+1}^{\infty} f(t) u_i(a(j)).$$

Hence (11) holds if

$$\frac{u_i(p(j)) - \bar{v}}{M(\alpha)} + \lambda(T,\alpha) \cdot [\min\{u_i(p(j)), u_i(a(j))\} - v_i] + (1 - \lambda(T,\alpha)) \cdot [u_i(a(j)) - u_i(a(i))] \geq 0. \tag{12}$$

Since $u_i(a(j)) - u_i(a(i)) > 0$ for all $i \in N$ and all $j \in N \backslash \{i\}$, it follows that there exists real numbers m_3, m_4 and m_5, with $m_5 > 0$, such that if

$$\frac{m_3}{M(\alpha)} + \lambda(T,\alpha) \cdot m_4 + (1 - \lambda(T,\alpha)) \cdot m_5 > 0, \tag{13}$$

then it is optimal for all players $i \in N$ to conform in Phase II$_j$ for all $j \in N$. The number m_5 could for example be taken as the smallest of the finitely many positive numbers $u_i(a(j)) - u_i(a(i))$ where i varies over N and j varies over $N \backslash \{i\}$. The conclusion from this step is that if α and T are such that (10) and (13) hold, then the proposed strategy is a subgame perfect equilibrium. For the next step we will need only that $m_2 > 0$ and $m_5 > 0$.

Step 2. To complete the proof, we have to show that it is possible to find a neighborhood $\mathcal{O}$ of α^* with the following property: for each $\alpha \in \mathcal{O} \cap \Omega$ there exists $T_\alpha \in \mathbb{N}$ such that with $T = T_\alpha$ (10) and (13) do hold. Let $\mu \in (0,1)$ be such that

$$\mu' \cdot m_4 + (1-\mu') \cdot m_5 > 0$$

for all $\mu' \in [\mu/2, \mu]$. Since $m_5 > 0$, all that is required is that μ is chosen small enough. Since the interval $[\mu/2, \mu]$ is closed, the function

$$\mu' \mapsto \mu' \cdot m_4 + (1-\mu') \cdot m_5$$

can be minimzed on $[\mu/2, \mu]$. Let this minimum, which is strictly greater than 0, be achieved at $\bar{\mu} \in [\mu/2, \mu]$. Recall that $M(\alpha) = \sum_{t=1}^{\infty} f(t;\alpha)$. By part (i) of Condition 2, we have that $\lim_{\alpha \to \alpha^*} \sum_{t=1}^{\infty} f(t;\alpha) = +\infty$. Since $m_2 > 0$ and $\bar{\mu} \cdot m_4 + (1-\bar{\mu}) \cdot m_5 > 0$, it follows that there exists a neighborhood $\mathcal{O}'$ of α^* such that

$$\begin{aligned} \frac{m_1}{M(\alpha)} + \frac{\mu}{2} \cdot m_2 &> 0, \\ \frac{m_3}{M(\alpha)} + \bar{\mu} \cdot m_4 + (1-\bar{\mu}) \cdot m_5 &> 0, \end{aligned}$$

for all $\alpha \in \mathcal{O}' \cap \Omega$. This neighborhood $\mathcal{O}'$ of α^* is such that

$$\frac{m_1}{M(\alpha)} + \mu' \cdot m_2 > 0, \tag{14}$$

$$\frac{m_3}{M(\alpha)} + \mu' \cdot m_4 + (1-\mu') \cdot m_5 > 0, \tag{15}$$

for all $\alpha \in \mathcal{O}' \cap \Omega$ and all $\mu' \in [\mu/2, \mu]$. By Lemma 3, there exists a neighborhood $\mathcal{O}''$ of α^* with the following property: for each $\alpha \in \mathcal{O}'' \cap \Omega$ there exists $T_\alpha \in \mathbb{N}$ such that

$$\frac{\mu}{2} < \frac{\sum_{t=1}^{T_\alpha} f(t;\alpha)}{\sum_{t=1}^{\infty} f(t;\alpha)} = \lambda(\alpha, T_\alpha) \leq \mu.$$

Put $\mathcal{O} = \mathcal{O}' \cap \mathcal{O}''$. This neighborhood has the desired property.

To see this, fix any $\alpha \in \mathcal{O} \cap \Omega$. Since $\mathcal{O}$ is a subset of $\mathcal{O}''$, we can find an integer T_α such that $\lambda(\alpha, T_\alpha) \in [\mu/2, \mu]$. Since $\mathcal{O}$ is a subset of $\mathcal{O}'$, the inequalities (14) and (15) are satisfied for all $\mu' \in [\mu/2, \mu]$. Then in particular they are satisfied with $\mu' = \lambda(\alpha, T_\alpha)$. That is, (10) and (13) do hold with $T = T_\alpha$. □

Proof of Proposition 5. Suppose that the discount function f is present biased. Assume for a contradiction that $\tilde{g}$ is a strategy profile that is a subgame perfect equilibrium and satisfies the strong punishment property, but $\tilde{g}$ is not a time-consistent subgame perfect equilibrium. Then there exists some history $\tilde{h} \in H$ such that $\tilde{g}^{|\tilde{h}}$ is not a Nash equilibrium of the repeated game with just the players $\{i_0 : i \in N\}$. Put $g = \tilde{g}^{|\tilde{h}}$. Since g is not a Nash equilibrium of the repeated game with players $\{i_0 : i \in N\}$, there exists some $i \in N$ and some $g'_i : H \to A_i$ such that $U_{i0}(g_i, g_{-i}) < U_{i0}(g'_i, g_{-i})$. For each $k \in \mathbb{N}$, define g_i^k by

$$g_i^k(h) = \begin{cases} g'_i(h) & \text{if } h \in \bigcup_{t=0}^{k} A^t, \\ g_i(h) & \text{if } h \in \bigcup_{t=k+1}^{\infty} A^t, \end{cases}$$

so that g_i^k agrees with g_i' for small histories and agrees with g_i for large histories. Since u_i is continuous and A_i is compact, u_i is bounded. Thus there exists a constant M such that

$$\left|U_{i0}(g_i', g_{-i}) - U_{i0}(g_i^k, g_{-i})\right| \leq M \sum_{t=k+1}^{\infty} f(t). \tag{16}$$

Since f is summable, $\sum_{t=k+1}^{\infty} f(t) \to 0$ as $k \to \infty$, and thus it follows from (16) and $U_{i0}\left(g_i, g_{-i}\right) < U_{i0}(g_i', g_{-i})$ that $U_{i0}\left(g_i, g_{-i}\right) < U_{i0}(g_i^k, g_{-i})$ for all k sufficiently large. Fix an integer k such that $U_{i0}\left(g_i, g_{-i}\right) < U_{i0}(g_i^k, g_{-i})$. Let $(a^t) \in A^\infty$ be the outcome path induced by the strategy profile (g_i^{k-1}, g_{-i}) and let $(b^t) \in A^\infty$ be the outcome path induced by the strategy profile (g_i^k, g_{-i}). Put $h = (a^t)_{t=0}^{k-1} = (b^t)_{t=0}^{k-1}$. Since g is a subgame perfect equilibrium, player i_k can not improve upon g by any deviation after history h. Thus

$$U_{ik}(g_i^{k-1}, g_{-i}) = u_i(a^k) + \sum_{t=1}^{\infty} f(t) u_i(a^{k+t}) \geq u_i(b^k) + \sum_{t=1}^{\infty} f(t) u_i(b^{k+t}) = U_{ik}(g_i^k, g_{-i}). \tag{17}$$

We have that $(g_i^{k-1}, g_{-i})^{|h} = g^h$, and that $(g_i^k, g_{-i}) = g$ on $\cup_{t=1}^{\infty} A^{k+t}$. Since g satisfies the strong punishment property, it follows that $u_i\left(a^{k+t}\right) \geq u_i(b^{k+t})$ for all $t \geq 1$. Since f is present biased, an induction argument shows that $f(k)f(t) \leq f(k+t)$ for all $t \geq 1$. Since $u_i\left(a^{k+t}\right) - u_i(b^{k+t}) \geq 0$ for all $t \geq 1$, it follows from this and (17) that

$$f(k) u_i(a^k) + \sum_{t=1}^{\infty} f(k+t) u_i(a^{k+t}) \geq f(k) u_i(b^k) + \sum_{t=1}^{\infty} f(k+t) u_i(b^{k+t}).$$

The left hand side of this equality gives the payoff to player i_0 during periods $t = k, k+1, \ldots$, for the strategy profile (g_i^{k-1}, g_{-i}), and the right hand side gives the payoff to player i_0 during periods $t = k, k+1, \ldots$, for the strategy profile (g_i^k, g_{-i}). Thus, since the profiles agree on histories of length smaller than k and $(a^t)_{t=0}^{k-1} = (b^t)_{t=0}^{k-1}$, we have that $U_{i0}(g_i^{k-1}, g_{-i}) \geq U_{i0}\left(g_i^k, g_{-i}\right)$.

Since g is a subgame perfect equilibrium, none of the players $(i_t)_{t=0}^{k-1}$ can improve upon g by any deviation. Repeating the same argument therefore gives the chain of inequalities

$$U_{i0}\left(g_i, g_{-i}\right) \geq U_{i0}\left(g_i^0, g_{-i}\right) \geq \cdots \geq U_{i0}(g_i^{k-2}, g_{-i}) \geq U_{i0}(g_i^{k-1}, g_{-i}) \geq U_{i0}(g_i^k, g_{-i})$$

which contradicts that $U_{i0}\left(g_i, g_{-i}\right) < U_{i0}(g_i^k, g_{-i})$. □

Proof of Proposition 7. From Chade et al. (2008), or the discussion in the paragraph preceeding Proposition 7, we know that if f satisfies $f(t) = \beta\delta^t$ for some $\beta \in [0,1]$ and some $\delta \in (0,1)$, then the statement is true.

Suppose that f does not satisfy $f(t) = \beta\delta^t$ for some $\beta \in [0,1]$ and some $\delta \in (0,1)$. By Proposition 6(a), we either have $f(2) < f(1)^2$, or there exists $k \geq 1$ and real numbers x_k, x_{k+1} such that $f(k)x_k + f(k+1)x_{k+1} \geq 0$ and $f(k+1)x_k + f(k+2)x_{k+1} < 0$.

In the first case, $f(2) < f(1)^2$, consider the streams $x = (0,0,1,0,0,\ldots)$ and $y = (0, f(1), 0, 0, \ldots)$ of instantaneous utilities for the 1−players. Player 1_0 gets the payoff $f(2)$ for the $x-$stream, and the payoff $f(1)^2$ for the $y-$stream, so he strictly preferes the $y-$stream. Player 1_1 gets the payoff $f(1)$ for the $x-$stream, and the payoff $f(1)$ for the $y-$stream, so he is indifferent. Suppose that there exists a subgame perfect equilibrium strategy profile where player 1_1 is supposed to choose the $x-$stream over the $y-$stream. A deviation from player 1_1 would change the continuation payoff from $f(1)$ to 0, so the strategy profile satisfies the weak punishment property. Player 1_1 is happy with choosing the $x-$stream over the $y-$stream. Player 1_0 on the other hand would like to reverse this choice, so the strategy profile is not a time-consistent subgame perfect equilibrium.

In the second case, let $k \geq 2$ be a positive integer, and let $x = (x_t)_{t=0}^{\infty}$ be a sequence of real numbers such that $x_t = 0$ for all t except $t = k$ and $t = k+1$, $\sum_{t=1}^{\infty} f(t)x_{t+1} \geq 0$, and $\sum_{t=1}^{\infty} f(t)x_t < 0$. Suppose that there exists a subgame perfect equilibrium strategy profile where player 1_1 is supposed to choose the stream x of instantaneous utilities for the $1-$players over a stream of zeros in each period. A deviation from player 1_1 would change the continuation payoff from $\sum_{t=1}^{\infty} f(t)x_{t+1} \geq 0$ to 0, so the strategy profile satisfies the weak punishment property. Player 1_1 is happy with choosing the $x-$stream over a stream of zeros. Player 1_0 on the other hand would like to reverse this choice because $\sum_{t=1}^{\infty} f(t)x_t < 0$, so the strategy profile is not a time-consistent subgame perfect equilibrium.

It remains only to construct a stage game G and a subgame perfect equilibrium strategy profile where player 1_1 has the choice between two given streams, and where all other $1-$players are powerless. This can be done by letting the stage game have two players, where player 2 is indifferent between all outcomes of the stage game and can in effect control the instantaneous payoffs available to player 1 in the stage game in each period. The actions of player 2 in a subgame perfect equilibrium can then be made contingent on the action of player 1_1 but unaffected by the actions of all other $1-$players. □

Notation for the proof of Proposition 8. Let Σ denote the collection of all strategy profiles for the repeated game. Let $\Sigma^p \subset \Sigma$ denote the collection of all subgame perfect equilibrium strategy profiles. Given a strategy profile $g \in \Sigma$, let $Q(g) \in A^\infty$ be the outcome path induced by g. Finally, let A^P be the subset of A^∞ that contains all outcome paths $(a^t)_{t=0}^{\infty}$ such that $(a^t) = Q(g)$ for some $g \in \Sigma^p$. That is, A^P contains all outcome paths that are subgame perfect equilibrium paths.

The following lemmas are used in the proof of Proposition 8.

Lemma 4. *Put* $\underline{v}_i = \inf\left\{\sum_{t=0}^{\infty} f(t+1)u_i(b^t) : (b^t)_{t=0}^{\infty} \in A^P\right\}$ *for all* $i \in N$. *Suppose* $(a^t) \in A^P$. *Then for all* $\tau \in \mathbb{N}$, *all* $i \in N$, *and all* $\tilde{a}_i \in A_i$,

$$u_i(a^\tau) + \sum_{t=1}^{\infty} f(t)u_i(a^{\tau+t}) \geq u_i(\tilde{a}_i, a_{-i}^\tau) + \underline{v}_i. \tag{18}$$

Proof. Let the strategy profile $g \in \Sigma^P$ be such that $Q(g) = (a^t)$, and let $(b^t) \in A^\infty$ be the punishment path specified by g for a deviation of player i_τ from a_i^τ to $\tilde{a}_i \in A$ after history $\left(a^t\right)_{t=0}^{\tau-1}$. Then $\sum_{t=0}^{\infty} f(t+1)u_i(b^t) \geq \underline{v}_i$ by definition of $\underline{v}_i$. Suppose for a contradiciton that (18) does not hold. Then

$$u_i(a^\tau) + \sum_{t=1}^{\infty} f(t)u_i(a^{\tau+t}) < u_i(\tilde{a}_i, a_{-i}^\tau) + \sum_{t=0}^{\infty} f(t+1)u_i(b^t),$$

and hence player i_τ has a profitable deviation from g. This contradicts $g \in \Sigma^P$. □

Lemma 5. *Let* $\varepsilon > 0$ *and* $i \in N$ *be given. There exists* $T \in \mathbb{N}$ *such that*

$$\left|\sum_{t=T}^{\infty} f(t)u_i(a^t) - \sum_{t=T}^{\infty} f(t)u_i(b^t)\right| < \varepsilon$$

for all outcome paths $(a^t)_{t=0}^{\infty}$, $(b^t)_{t=0}^{\infty}$ *from* A^{∞}.

Proof. Since u_i is bounded, and since f is nonnegative, there is a constant M such that

$$\left|\sum_{t=T}^{\infty} f(t)u_i(a^t) - \sum_{t=T}^{\infty} f(t)u_i(b^t)\right| \leq M \sum_{t=T}^{\infty} f(t)$$

for all $T \in \mathbb{N}$ and all $(a^t), (b^t)$ from A^{∞}. Since f is summable, we also have that $\sum_{t=T}^{\infty} f(t) < \varepsilon/M$ for all T large enough. $\square$

Since u_i is continuous, Lemma 5 shows that for any two outcome paths $(a^t), (b^t)$ from A^{∞}, if a^t is close to b^t for all $t < T$, then the number $\left|\sum_{t=1}^{\infty} f(t)u_i(a^t) - \sum_{t=1}^{\infty} f(t)u_i(b^t)\right|$ is small no matter how far apart a^t and b^t are for $t \geq T$. This is how Lemma 5 is used in the proof of Proposition 8 to show that various function from A^{∞} to $\mathbb{R}$ are continuous when A^{∞} is endowed with the product topology.

Proof of part (i) of Proposition 8. The proof follows Abreu (1988) with adaptations to fit the present model with time-inconsistency of an unspecified form.

Step 1 (Defining Q_i for $i \in N$). Fix some $i \in N$. Since A^P is nonempty, and since u_i is bounded, the number $\underline{v}_i = \inf\left\{\sum_{t=0}^{\infty} f(t+1)u_i(b^t) : (b^t)_{t=0}^{\infty} \in A^P\right\}$ is a well-defined real number. Define the function $v : A^{\infty} \to \mathbb{R}$ by

$$(a^t)_{t=0}^{\infty} \mapsto \sum_{t=0}^{\infty} f(t+1)u_i(a^t).$$

Let $(Q_i^{\eta})_{\eta=0}^{\infty}$ be such that $Q_i^{\eta} \in A^P$ for all $\eta \in \mathbb{N}$ and $\lim_{\eta\to\infty} v(Q_i^{\eta}) = \underline{v}_i$. Let A^{∞} be endowed with the product topology. By Tychonoff's theorem, A^{∞} is compact since A is compact, and it follows that the sequence (Q_i^{η}) in A^{∞} has a convergent subsequence. Assume without loss of generality that (Q_i^{η}) converges. By Lemma 5, the function v is continuous. Put $Q_i = \lim_{\eta\to\infty} Q_i^{\eta}$. If $(a^t) = Q_i$, then it follows from $\lim_{\eta\to\infty} v(Q_i^{\eta}) = \underline{v}_i$ and continuity of v that $\sum_{t=0}^{\infty} f(t+1)u_i(a^t) = v(Q_i) = \underline{v}_i$.

Step 2 ($\boldsymbol{\sigma(Q_i, Q_1, Q_2 \ldots, Q_n) \in \Sigma^P}$ for $\boldsymbol{i \in N}$). Suppose $\sigma(Q_i, Q_1, Q_2, \ldots, Q_n) \notin \Sigma^P$ for some $i \in N$. Then there is some $\tau \in \mathbb{N}$, some $j \in N$ and some $\tilde{a}_j \in A_j$ such that for player j_τ it is profitable to deviate to $\tilde{a}_j$ after some history $h \in A^{\tau}$. Let $k \in N$ be such that after the history h, the simple strategy profile $\sigma(Q_i, Q_1, Q_2, \ldots, Q_n)$ proscribes play along the outcome path Q_k, and let $(a^t) = Q_k$. Then the profitable deviation for player j_τ means that

$$u_j(a^s) + \sum_{t=1}^{\infty} f(t)u_j(a^{s+t}) < u_j(\tilde{a}_j, a_{-j}^s) + \underline{v}_j \tag{19}$$

for some $s \in \mathbb{N}$. The number s indicates how far into the outcome path Q_k that play is when player j_τ deviates. By Lemma 5, the function

$$\begin{array}{rcl} A^{\infty} & \to & \mathbb{R} \\ (b^t)_{t=0}^{\infty} & \mapsto & u_j(b^s) + \sum_{t=1}^{\infty} f(t)u_j(b^{s+t}) - u_j(\tilde{a}_j, b_{-j}^s) \end{array}$$

is continuous, and since $(a^t) = \lim_{\eta\to\infty} Q_k^\eta$, it therefore follows from (19) that if η is large enough, and if $Q_k^\eta = (b^t)$, then

$$u_j(b^s) + \sum_{t=1}^{\infty} f(t) u_j(b^{s+t}) < u_j(\tilde{a}_j, b_{-j}^s) + \underline{v}_j. \tag{20}$$

By Lemma 4, the inequality (20) contradicts that $Q_k^\eta = (b^t) \in A^P$.

Step 3 (Conclusion). By step 2, for the outcome paths $(Q_1, Q_2, \ldots, Q_n)$ defined in step 1 we have that $\sigma(Q_i, Q_1, Q_2, \ldots, Q_n)$ is subgame perfect for $i \in N$. Hence $Q_i \in A^P$. Also by step 1, if $(a^t) = Q_i$, then $\sum_{t=0}^{\infty} f(t+1) u_i(a^t) = \underline{v}_i$. Thus the collection $(Q_1, Q_2, \ldots, Q_n)$ of outcome paths does satisfy property (i) of Proposition 6. □

Proof of part (ii) of Proposition 8. Let $(Q_1, Q_2, \ldots, Q_n)$ be defined as in step 1 above. If $\sigma(Q, Q_1, Q_2 \ldots, Q_n)$ is a subgame perfect equilibrium, then Q is the outcome path of a subgame perfect equilibrium. To prove the converse, suppose that $(a^t) = Q$ is a subgame perfect equilibrium path. Then by Lemma 4, it is true for all $\tau \in N$, all $i \in N$, and all $\tilde{a}_i \in A_i$ that

$$u_i(a^\tau) + \sum_{t=1}^{\infty} f(t) u_i(a^{\tau+t}) \geq u_i(\tilde{a}_i, a_{-i}^\tau) + \underline{v}_i.$$

Hence player i_τ does not wish to deviate from a_i^τ if the punishment for this will be $\underline{v}_i$. Since, by Step 2 above, we also have that $\sigma(Q_i, Q_1, Q_2 \ldots, Q_n)$ is a subgame perfect equilibrium for all $i \in N$, it follows that $\sigma(Q, Q_1, \ldots, Q_n)$ is a subgame perfect equilibrium. □

Claim 1. *Put* $\Omega = \{(\alpha, \gamma) \in \mathbb{R}^2 : \alpha > 0, \gamma > \alpha\}$, *and let* $f : \mathbb{N} \times \Omega \to \mathbb{R}$ *be defined by* $f(t; \alpha, \gamma) = (1 + \alpha t)^{-\gamma/\alpha}$. *Then for any* $(\alpha^*, \gamma^*) \in \mathbb{R}^2$ *with* $\alpha^* \geq 0$ *and* $\gamma^* = \alpha^*$, *we have that* $\lim_{(\alpha,\gamma)\to(\alpha^*,\gamma^*)} \sum f(t; \alpha, \gamma) = +\infty$. *If* $\alpha^* = \gamma^* = 0$, *then also* $\lim_{(\alpha,\gamma)\to(\alpha^*,\gamma^*)} f(t; \alpha, \gamma) = 1$ *for all* $t \in \mathbb{N}$.

Proof. Suppose that $\alpha^* = \gamma^* > 0$, and let M be a given real number. There exists $x > 1$ and $y > \alpha^*$ such that $\sum_{t=1}^{\infty} \frac{1}{(1+yt)^x} > M$. There also exists a neighborhood $\mathcal{O}$ of (α^*, γ^*) such that if $(\alpha, \gamma) \in \mathcal{O} \cap \Omega$, then $\gamma/\alpha \in (1, x)$ and $\alpha < y$. Hence $\sum_{t=1}^{\infty} f(t; \alpha, \gamma) > M$ for all $(\alpha, \gamma) \in \mathcal{O} \cap \Omega$, and thus $\lim_{(\alpha,\gamma)\to(\alpha^*,\gamma^*)} \sum f(t; \alpha, \gamma) = +\infty$.

Suppose that $\alpha^* = \gamma^* = 0$ and fix any $t \geq 1$. Let $\varepsilon > 0$ be given. Let $x > 0$ be such that $e^{-xt} > 1 - \varepsilon$. Since $\lim_{z\to 0}(1+z)^{\frac{1}{z}} = e$, we have that $\lim_{\alpha\to 0} f(t; \alpha, x) = e^{-xt} > 1 - \varepsilon$. Let $\mathcal{O}$ be a neighborhood of (α^*, γ^*) such that if $(\alpha, \gamma) \in \mathcal{O} \cap \Omega$, then $\gamma < x$ and $f(t; \alpha, x) > 1 - 2\varepsilon$. This neighborhood $\mathcal{O}$ is such that if $(\alpha, \gamma) \in \mathcal{O} \cap \Omega$, then $f(t; \alpha, \gamma) > 1 - 2\varepsilon$. Hence $\lim_{(\alpha,\gamma)\to(\alpha^*,\gamma^*)} f(t; \alpha, \gamma) = 1$. Now unfix t. By Lemma 1, it follows that also $\lim_{(\alpha,\gamma)\to(\alpha^*,\gamma^*)} \sum f(t; \alpha, \gamma) = +\infty$. □

Claim 2. *Let* α^* *belong to the closure of* Ω *and satisfy* $\alpha^* \notin \Omega$, *and let* $f : \mathbb{N} \times \Omega \to \mathbb{R}$ *be such that* $f(\cdot\,; \alpha)$ *is a present biased discount function for each* $\alpha \in \Omega$. *Then Condition 1 is met by* f *and* α^* *if and only if* $\lim_{\alpha\to\alpha^*} f(1; \alpha) = 1$.

Proof. Since $f(\cdot\,; \alpha)$ is present biased for each $\alpha \in \Omega$, the function f has to be bounded from above by 1 on $\mathbb{N} \times \Omega$. Otherwise there is some $\alpha \in \Omega$ such that the discount function $f(\cdot\,; \alpha)$ is not summable. Hence, if Condition 1 is met by f, then $\lim_{\alpha\to\alpha^*} f(1; \alpha) = 1$. For the converse statement, suppose that $\lim_{\alpha\to\alpha^*} f(1; \alpha) = 1$. Fix any $t \geq 1$, and let $\varepsilon > 0$ be given. Let $\delta \in (0, 1)$ be such that $\delta^t > 1 - \varepsilon$. Since $\lim_{\alpha\to\alpha^*} f(1; \alpha) = 1$, there is a neighborhood $\mathcal{O}$ of α^* such that

$f(1,\alpha) > \delta$ for all $\alpha \in \mathcal{O} \cap \Omega$. Since $f(\cdot\,;\alpha)$ is present biased for all $\alpha \in \Omega$, we have that $f(t;\alpha) \geq f(1;\alpha)^t$ for all $\alpha \in \Omega$. Hence $f(t;\alpha) \geq f(1;\alpha)^t > \delta^t > 1 - \varepsilon$ for all $\alpha \in \mathcal{O} \cap \Omega$, and thus $\lim_{\alpha \to \alpha^*} f(t;\alpha) = 1$. □

Claim 3. *Let $U_0 : A^\infty \to \mathbb{R}^n$ be the combined payoff function for the players $(i_0)_{i \in N}$; that is, let U_{i0} be the i'th coordinate function of U_0 for $i = 1, \ldots, n$. Then $U_0(A^P)$ is a compact subset of $\mathbb{R}^n$.*

Proof. Endow A^∞ with the product topology. We first establish that $A^\infty \backslash A^P$ is an open subset of A^∞. Let $(a^t) \in A^\infty$ be such that $(a^t) \notin A^P$. By Corollary 2 we have that

$$u_i(a^\tau) + \sum_{t=1}^{\infty} f(t) u_i(a^{\tau+t}) < u_i(\tilde{a}_i, a^\tau_{-i}) + \underline{v_i} \tag{21}$$

for some $i \in N$, some $\tilde{a}_i \in A_i$, and some $\tau \in \mathbb{N}$. By Lemma 5, there then exists a neighborhood $\mathcal{O}$ of (a^t) such that (21) holds when (a^t) is replaced by any $(b^t) \in \mathcal{O}$. Thus, by Corollary 2, if $(b^t) \in \mathcal{O}$, then $(b^t) \notin A^P$. Hence $A^\infty \backslash A^P$ is open.

Since $A^\infty \backslash A^P$ is open, A^P is closed. Since A^∞ is compact, it follows that A^P is compact. By Lemma 5, U_0 is continuous. Since A^P is compact and U_0 is continuous, $U_0(A^P)$ is compact. □

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