PURE EQUILIBRIA IN NON-ANONYMOUS LARGE GAMES

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ABSTRACT. Recent literature shows that pure approximate Nash equilibria exist in anonymous and continuous large finite games. Here we study continuous but non-anonymous games. Call the *impact* of a game to the maximal difference in some player's payoff when one other player changes his strategy. We prove that small impact is exactly what guarantees existence of pure approximate equilibria. That is, we show that there is a threshold (which depends on the number of players and strategies in the game) such that pure approximate equilibria exist whenever the impact is less than this threshold. Further, whenever the impact is larger than the threshold there are arbitrarily large games with no pure approximate equilibria.

1. INTRODUCTION

There is a growing interest in properties of large (with many players) but finite non–cooperative games. The work that initiated this study is [7], where it is shown that Nash equilibria of incomplete information large games that satisfy certain anonymity and continuity conditions are robust in the following sense: Even after types and strategies of all players are realized and revealed to everyone, with high probability

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no player would be able to gain much by deviating to another strategy[1]. Restricting attention to the special case of complete information games, this result implies that the realized strategy profile of a mixed equilibrium is with high probability an approximate pure equilibrium in such games. Several subsequent papers present similar results under various assumptions on the spaces of types and strategies available to the players, and on properties of the payoff functions in the game. Examples for such works include[2] [3], [4], [5], [6] and [11].

The above 'self–purification' property implies of course that approximate pure equilibria exist in anonymous and continuous games. In this paper we ask a more modest question: What properties of large games guarantee existence of approximate pure equilibrium[3]? Thus, we are interested in a weaker property of the equilibria set; and as a consequence we are able to show that a larger class of games has this property.

An assumption common to all of the above works is that the payoff function of each player is continuous in the *distribution* induced by the opponents' strategy profile. In other words, for every player i and strategy a_i, if the proportion of players that choose each strategy is close under two strategy profiles a_{-i} and a'_{-i}, then the payoffs $f_i(a_i, a_{-i})$ and $f_i(a_i, a'_{-i})$ are close. This continuity condition already implies that the game is anonymous - the payoff to player i does not change if two

[1]The result in [7] is in fact stronger since it shows that the equilibria of such games are robust in other ways as well.
[2]There are also a couple of earlier works on related issues. See [9] and [2].
[3]The concept of mixed strategy is often criticized as having little appeal in practical situations (see [10] for a discussion). Many attempts were made to justify this concept and to identify classes of games where pure equilibrium does exist.

players j and k switch their strategies. Thus, anonymity follows from continuity and the two cannot be separated.

Here, on the other hand, we use a weaker notion of continuity which doesn't imply anonymity. Namely, we call the *impact* of a game to the maximal difference in some player's payoff when one other player changes his strategy. Smaller impact corresponds to 'more continuity'. For instance, the condition of continuity in the induced distribution employed by the previous literature implies that the impact is of the order of $1/n$, where n is the number of players in the game.

Our main result (Theorem 3.1) shows that any game with impact less than $\epsilon/\sqrt{8n\log(2mn)}$ admits a pure ϵ–equilibrium, where n and m are the numbers of players and strategies for each player, respectively. We emphasize that no anonymity condition is required for this result[4]. Further, we show (Theorem 3.4) that this bound is almost tight in the following sense: There are arbitrarily large games with $m = 2$ strategies for each player and impact less than $60/\sqrt{n}$ with no pure $1/3$–equilibrium.

The proof of the positive result follows the same idea as in [7]: Start with a possibly mixed equilibrium and randomly (according to the equilibrium distribution) choose a pure strategy profile. [7] shows that with high probability you will get an approximate equilibrium; we show that there is a positive probability to get an approximate equilibrium, which is enough to establish existence. For the proof that the bound is tight we construct a non-anonymous version of matching pennies with

[4]In a companion paper we show that a much stronger result holds for anonymous games.

$n = 2k$ players[5]: There are k males and k females and each one of them should choose between the actions $+1$ and -1. For each pair (i, j) of a female and a male, either i wants to match j (and j to mismatch i) or i wants to mismatch j (and j to match i). The payoff to each player is the sum across all these 'small games' times some constant δ. We use the probabilistic method to show that it is possible to determine who want to match and who want to mismatch in such a way that no approximate equilibrium exist when δ is in the order of $1/\sqrt{n}$.

2. Setup

An n–player game in normal form is given by finite sets $\{A_i\}_{i=1}^n$ of *strategies* and by *payoff functions* $\{f_i : A \to [-1, 1]\}_{i=1}^n$, where $A = \prod_{i=1}^n A_i$ is the set of *strategy profiles.* A *mixed strategy* for player i is a probability distribution over A_i. Each f_i is extended linearly to profiles of mixed strategies. Nash equilibrium and ϵ–Nash equilibrium (in pure or mixed strategies) are defined as usual.

For each i, we view the product space $A_{-i} = \prod_{j \neq i} A_i$ as a metric space, with the metric $\mathrm{d}(a'_{-i}, a''_{-i}) = \#\{1 \leq j \leq n \,:\, j \neq i,\ a'_j \neq a''_j\}$.

2.1. **Definition.** The *impact* of a game is given by

$$\max\{|f_i(a_i, a'_{-i}) - f_i(a_i, a''_{-i})|\},$$

where the maximum ranges over all players i, all strategies $a_i \in A_i$ and all pairs a'_{-i}, a''_{-i} of opponents' strategy profiles such that $\mathrm{d}(a'_{-i}, a''_{-i}) = 1$.

[5]An anonymous version of this game is used as an example in [7].

Games with impact δ have the property that a player's payoff does not change by more than δ when one opponent changes her strategy. This implies that for each i and each $a_i \in A_i$ the function $f_i(a_i, \cdot)$ is δ–Lipshitz on A_{-i}.

We denote by $L(n, m, \delta)$ the set of games with n players, at most m strategies for every player and impact of at most δ.

3. Main results

In this section we state and prove the main results of the paper. We start with the positive result of existence of pure approximate equilibrium in games with small impact. Then we show that the bound is (almost) tight.

3.1. **Theorem.** *Let $\epsilon > 0$. Then every game in $L(n, m, \delta)$ for $\delta = \epsilon/\sqrt{8n\log(2mn)}$ admits a pure ϵ–equilibrium.*

Proof. Consider a game in $L(n, m, \delta)$ with $\delta = \epsilon/\sqrt{8n\log(2mn)}$. Let $(\mu_1, \ldots, \mu_n)$ be a mixed strategy Nash equilibrium of the game. Thus, each μ_i is a probability distribution over A_i and

$$\text{support}(\mu_i) \subseteq \arg\max_{a_i \in A_i} \int f_i(a_i, \tau)\mu_{-i}(\mathrm{d}\tau), \tag{1}$$

where $\mu_{-i} = \prod_{j \neq i} \mu_j$.

For every player i and every strategy $h \in A_i$ let $E_{i,h} \subseteq A$ be the set of all strategy profiles a such that, if player i plays h against a_{-i} his payoff is roughly the same as his expected payoff when he plays h and the opponents play their Nash equilibrium strategy:

$$E_{i,h} = A_i \times \left\{ a_{-i} \in A_{-i} \colon \left| f_i(h, a_{-i}) - \int f_i(h,\tau)\mu_{-i}(\mathrm{d}\tau) \right| \leq \epsilon/2 \right\}.$$

From Proposition A.1 in the Appendix and the choice of δ it follows that

$$\mu(E_{i,h}^c) \leq 2\exp(-\epsilon^2/8(n-1)\delta^2) < 1/nm$$

for every player i and every $h \in A_i$. Since there are at most mn such events $E_{i,h}$, it follows that $\mu(\cap E_{i,h}) > 0$. Let a^* be a strategy profile such that $a^* \in \text{support}(\mu)$ and $a^* \in \cap E_{i,h}$. We claim that a^* is an ϵ–equilibrium. Indeed, for every player i and every deviation $d \in A_i$ one has

$$f_i(d, a^*_{-i}) \leq \int f_i(d,\tau)\mu_{-i}(\mathrm{d}\tau) + \epsilon/2 \leq \int f_i(a^*_i,\tau)\mu_{-i}(\mathrm{d}\tau) + \epsilon/2 \leq f_i(a^*, a^*_{-i}) + \epsilon$$

where the first inequality follows from the fact that $a^* \in E_{i,d}$, the second from (1) and the third from the fact that $a^* \in E_{i,a_i^*}$. □

Since our main interest is in games with many players, we would like to think of the strategy set as fixed and increase the number of players to infinity. The following immediate corollary of Theorem 3.1 establishes the required rate of convergence for this case.

3.2. **Corollary.** *Fix m and consider a family of games with m strategies for each player. Let $\delta : \mathbb{N} \to [0,1]$ be such that $\delta(n) = o(1/\sqrt{n \log n})$. Then for every $\epsilon > 0$, there is N such that every n player game with $n > N$ and impact smaller than $\delta(n)$ admits a pure ϵ-equilibrium.*

3.3. *Remark.* Theorem 3.1 (and Corollary 3.2) is true even if the payoff functions are not restricted to get values in $[-1, 1]$. Indeed, the proof does not use this fact at all.

The tightness of the bound is demonstrated by the following theorem.

3.4. **Theorem.** *For every even n large enough there is a game in $L(n, 2, 60/\sqrt{n})$ with no pure 1/3–equilibrium.*

Proof. Let the number of players be $n = 2k$ and let the set of strategies for each player be $\{+1, -1\}$. We divide the players into two groups of k players, females and males, and denote their strategy profiles by $\bar{x} = (x_1, \ldots, x_k)$ and $\bar{y} = (y_1, \ldots, y_k)$ respectively, viewed as row vectors. Fix some constant $\delta > 0$. We consider games that can be described by a $k \times k$ matrix $M = \{m_{ij}\}$ with entries ± 1. The payoff for female i is $K(u_i)$ where $K(t) = t$ for $|t| \leq 1$ and $K(t) = t/|t|$ for $|t| > 1$, and u_i, the *untruncated payoff of female* i, is given by

$$u_i(\bar{x}, \bar{y}) = \delta x_i \sum_j m_{ij} y_j = \delta x_i \cdot (M\bar{y}^T)_i.$$

The payoff for male j is given by $K(v_j)$ where the untruncated payoff of male j is given by

$$v_j(\bar{x}, \bar{y}) = -\delta y_j \sum_i m_{ij} x_i = -\delta y_j \cdot (\bar{x}M)_j.$$

Notice that the impact of every such game is 2δ. Also, from the definitions of the untruncated payoff it follows that

$$\sum_i u_i(\bar{x},\bar{y}) = -\sum_j v_j(\bar{x},\bar{y}) \tag{2}$$

for every profile $(\bar{x},\bar{y})$.

By Lemma B.1 below, for every sufficiently large k and $\delta = 20/\sqrt{k} < 30/\sqrt{n}$ there exits a $k \times k$ matrix M with the property

$$\#\left\{1 \le j \le k \;:\; |(\bar{x}M)_j| > \frac{1}{\delta}\right\} > k/3 \tag{3}$$

for every strategy profile $\bar{x}$ of the females.

Fix a strategy profile $\bar{x}$ for the females and let $\bar{y}$ be a profile such that all the males play 1/3–best response to $\bar{x}$. From the definition of v_j it follows that $v_j > 0$ whenever $|v_j| > 1/6$ (since by changing his strategy a player inverts the sign of his untrancated payoff). Therefore, by (3) it follows that

$$\sum_j v_j(\bar{x},\bar{y}) > k/3 * 1 + (2k/3) * (-1/6) > k/6.$$

By (2), it follows that $u_i(\bar{x},\bar{y}) < -1/6$ for some female i. Since every player inverts the sign of her payoff by changing strategy it follows that female i does not 1/3 best-respond to $\bar{y}$. $\square$

4. Varying strategy sets

The results of the previous section focus on the case where the strategy sets are fixed while the number of players is increasing. One may also be interested in the case where strategy sets are allowed to grow with the number of players. This is the case studied in the current section. First, we establish a trivial existence result where the bound on the impact is independent of the number of strategies in the game.

4.1. **Theorem.** *Let $\epsilon > 0$ and $m \in \mathbb{N}$ be arbitrary. Then every game in $L(n, m, \delta)$ for $\delta = \epsilon/2n$ admits a pure ϵ–equilibrium.*

Proof. Let $a \in A$ be an arbitrary strategy profile and let a^* be a strategy profile such that a_i^* is a best response to a_{-i} for every player i. Then a^* is an ϵ–equilibrium. Indeed, for every player i and every deviation $d \in A_i$ one has

$$f_i(d, a_{-i}^*) \leq f_i(d, a_{-i}) + (n-1)\epsilon/2n \leq$$

$$f_i(a_i^*, a_{-i}) + (n-1)\epsilon/2n \leq f_i(a_i^*, a_{-i}^*) + 2(n-1)\epsilon/2n < f_i(a_i^*, a_{-i}^*) + \epsilon,$$

where the first and third inequalities follow from the Lipschitz property of f_i and the second inequality follows from the definition of a^*. □

The following theorem shows that, for games with unbounded strategy sets, the bound in Theorem 4.1 is the best possible (up to a constant).

4.2. **Theorem.** *For every even n there is a game in $L(n, 2^{n/2}, 1/n)$ with no pure $1/8$–equilibrium.*

Proof. Let the number of players be even $n = 2k$, and let the strategy set of each player be $\{+1, -1\}^k$. Like in the previous proof, we divide the players into two groups, females and males, and denote their strategy profiles by $\bar{x} = (x_1, \ldots, x_k)$ and $\bar{y} = (y_1, \ldots, y_k)$ respectively. The strategy of female i is given by the vector $(x_i[j])_{j=1}^k$, and similarly the strategy of male j is given by the vector $(y_j[i])_{i=1}^k$.

The payoff to female i is

$$u_i(\bar{x},\bar{y}) = \frac{1}{4k}\sum_j x_i[j]\cdot y_j[i],$$

and the payoff to male j is

$$v_j(\bar{x},\bar{y}) = -\frac{1}{4k}\sum_i x_i[j]\cdot y_j[i].$$

The game has impact $2/4k = 1/n$ but no pure $1/8$–equilibrium: For every strategy profile of the opponents, every player can guarantee $1/4$. Therefore, in every $1/8$–equilibrium every player should get at least $1/8$. But this is impossible since the sum of the payoffs is 0 in every profile. $\square$

Appendix A. Concentration of measure

We collect here some facts that we use in the proofs.

A.1. **Proposition.** [8, Corollary 1.17] *Let $A_1,\dots,A_n$ be finite sets and let $\mu=\mu_1\times\cdots\times\mu_n$ be a product probability measure over $A=\prod_i A_i$. Let $F: A\to\mathbb{R}$ be a real valued function such that $|F(a)-F(a')|\leq 1$ whenever $a,a'\in A$ and $\#\{i|a_i\neq a_i'\}=1$. Then for every $r>0$*

$$\mu\left(\left\{a\colon F(a)\geq \int F\,d\mu + r\right\}\right)\leq e^{-r^2/2n}.$$

A.2. *Remark.* (i) If the Lipschitz constant for F is δ (instead of 1 as in the above formulation) then by considering the function F/δ the bound on the probability becomes $e^{-r^2/2n\delta^2}$.

(ii) By applying the same bound to $-F$ one gets

$$\mu\left(\left\{a\colon \left|F(a)-\int F\mathrm{d}\mu\right|\geq r\right\}\right)\leq 2e^{-r^2/2n}.$$

For the case in which $A_i=\{0,1\}$ for every i, $\mu_1=\cdots=\mu_n$ are coin tosses with probability p for success, and $F(a_1,\dots,a_n)=\sum_{i=1}^n a_i$ the constant in the exponent can be improved. This is Chernoff bound:

A.3. **Proposition.** [1, Theorem A.1.4] *Let $X_1,\dots,X_n$ be i.i.d. with $\mathbb{P}(X_i=1)=1-\mathbb{P}(X_i=0)=p$. Then*

$$\mathbb{P}(X_1+\cdots+X_n\geq pn+r)\leq e^{-2r^2/n}.$$

Appendix B. Unbalancing lights

B.1. **Lemma.** *For sufficiently large k there exists $k\times k$ matrix M with entries in $\{+1,-1\}$ such that*

$$\#\left\{1\leq j\leq k\ :\ |(\bar{x}M)_j|>\frac{\sqrt{k}}{20}\right\}>k/3 \tag{4}$$

for every row vector $\bar{x}$ of length k with entries in $\{+1,-1\}$.

The lemma has an interesting interpretation: Consider an array of $k\times k$ lights, each can be either on or off. Assume that for every row there is a switch, such that if the switch of row i is pulled then all the lights in that row are switched (from on to off or off to on). Then the lemma says that there exists some initial configuration of the lights such that whatever switches are performed on the rows, many columns will be unbalanced (i.e. will have much more lights on than off or vice versa). Alon and Spencer [1, Section 2.5] use the probabilistic method to prove that for every initial configuration it is possible to switch lights to unbalance the matrix. We turn the probabilistic method 'on its head' to prove that there exists some initial configuration for which any switching will result in an unbalanced matrix. The argument follows the proof of lower bound in the classical discrepancy problem [1, Section 13.4]

Proof. Fix k and let $M_{k\times k}=\{m_{ij}\}$ where m_{ij} are independent random signs. For a fixed $\bar{x}$, the entries of $z=\bar{x}\cdot M$ are $i.i.d$, and each z_j is distributed like the sum of k independent random signs. Thus, by the central limit theorem

$$\mathbb{P}\left(|z_j|\leq\frac{\sqrt{k}}{20}\right)\xrightarrow{k\to\infty}\frac{1}{\sqrt{2\pi}}\int_{-1/20}^{1/20}e^{-\tau^2/2}\mathrm{d}\tau<1/25$$

Let $E_{\bar{x}}$ be the event that (4) is not satisfied. Then $E_{\bar{x}}$ is the event that there are more than $2k/3$ successes in k independent trials with probability for success smaller than $1/25$. From Chernoff inequality A.3 we get

$$\mathbb{P}(E_{\bar{x}})\leq\exp\left(-2k\left(2/3-1/25\right)^2\right)<1/2^k.$$

Since there are only 2^k possible $\bar{x}$-s, it follows that $P\left(\cup_{\bar{x}}E_{\bar{x}}\right)<1$. Therefore, for some choice of M none of the $E_{\bar{x}}$ occurs, as desired. $\square$

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